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Research Article

Uniqueness of the Adjoint Problem in the Lagrange Sense for a Non-Local Boundary Problem with Complex Parameter

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Abstract: The article is devoted to the problem of finding conjugate boundary conditions for a nonlocal conditional problem for a homogeneous ordinary differential equation of the second order, the coefficients of which depend on a complex parameter. The conjugate boundary conditions known in the literature, associated with arbitrary constants, are investigated, and the issue of eliminating arbitrariness in boundary conditions is considered.

Keywords: boundary value problem with coefficients depending on a complex parameter, adjoint mixed problem, non-local condition

MSC: 34B10

1. Introduction

The Fourier and Laplace transform methods are widely used to solve problems in mathematical physics. The main idea of the Fourier method is to transform the mixed problem to suitable Cauchy and boundary value problem with complex parameter-dependent and to construct the solution of the main problem from their solutions in a certain way. Unlike the Fourier method, when the Laplace transform is applied to differential equations that depend on two variables, one of the variables in the equation is eliminated, and it becomes a boundary condition problem for the ordinary differential equation [1–4].

The application of the Laplace transform to find the function $u(x, t)$ satisfying the equation

$$\sum_{i+j\leq 2} c_{ij}(x)\frac{\partial^{i+j}u(x, t)}{\partial x^i \partial t^j} = 0$$

with second order continuous derivatives with respect to both variables defined in the domain $D = \{a < x < b, t > 0\}$, and verifying the initial conditions

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$$u(x,\, 0) = \varphi(x),$$

$$\left.\frac{\partial u(x,\, t)}{\partial t}\right|_{t=0} = \psi(x),$$

and the boundary conditions

$$\sum_{j=0}^{1} \left(\alpha_{\nu j} \frac{\partial^j u(a,\, t)}{\partial x^j} + \beta_{\nu j} \frac{\partial^j u(b,\, t)}{\partial x^j} \right) = \gamma_\nu(t), \quad (\nu = 1,\, 2)$$

within the domain, leads to the solution of the following boundary value problem

$$c_{20}(x) \frac{d^2}{dx^2} \widetilde{u}(x,\, \rho) + (\rho c_{11}(x) + c_{10}(x)) \frac{d}{dx} \widetilde{u}(x,\, \rho) + \left(\rho^2 c_{02}(x) + \rho c_{01}(x) + c_{00}(x)\right) \widetilde{u}(x,\, \rho) = F(x,\, \rho),$$

$$\sum_{j=0}^{1} \left(\alpha_{\nu j} \frac{d^j \widetilde{u}(a,\, \rho)}{dx^j} + \beta_{\nu j} \frac{d^j \widetilde{u}(b,\, \rho)}{dx^j} \right) = \widetilde{\gamma}_\nu(\rho), \quad (\nu = 1,\, 2).$$

Here,

$$F(x,\, \rho) = (\rho\ c_{02}(x) + c_{01}(x))\, \varphi(x) + c_{11}(x) \varphi'(x) + c_{02}(x) \psi(x),$$

$\widetilde{u}(x,\, \rho)$ and $\widetilde{\gamma}_\nu(\rho)$ are Laplace transforms of functions $u(x,\, t)$ and $\gamma_\nu(t)$, respectively. If we assume that $c_{20}(x) = 1$ and $F(x,\, \rho) = 0$ then get the following problem. Problems type of (1), (2) are considered independently of the solution of the mixed problem written for partial differential equations of the second order given above. This is included only to indicate the source from which (1), (2) appeared.

Let us consider the following problem

$$\ell[y] \equiv y^{''}(x) + [\lambda\ a_{11}(x) + a_{10}(x)]\, y^{'}(x) + \left[\lambda^2 a_{02}(x) + \lambda\ a_{01}(x) + a_{00}(x)\right] y(x) = 0,\ \ a < x < b \tag{1}$$

$$\ell_\nu[y] \equiv \alpha_{\nu 0} y(a) + \beta_{\nu 0} y(b) + \alpha_{\nu 1} y^{'}(a) + \beta_{\nu 1} y^{'}(b) = 0,\ \ \nu = 1,\, 2. \tag{2}$$

Here $\lambda \in \mathbb{C}$ is a complex parameter, $a_{ij}(x)$, $(i,\ j = 0,\, 1,\, 2)$ are generally complex-valued continuous functions, and $\alpha_{\nu j}$, $\beta_{\nu j}$ $(\nu = 1,\, 2,\ j = 0,\, 1)$ are complex constants. The boundary conditions (2) are linearly independent, that is

$$\operatorname{rank} \begin{pmatrix} \alpha_{10} & \alpha_{11} & \beta_{10} & \beta_{11} \\ \alpha_{20} & \alpha_{21} & \beta_{20} & \beta_{21} \end{pmatrix} = 2. \tag{3}$$

It's known from a course of ordinary differential equations the finding solutions posed problem tie in with familiar of the Lagrange formula [2, 5–7]. In order to it is necessary to create an adjoint problem corresponding to input equation. For the system of first order linear differential equation with of the coefficients in which are summable function of the variable x in finite interval (a, b) has been investigated by Rasulov [2]. More detailed information about the Lagrange formula can be found in [5, 8].

The main idea of constructing the Lagrange formula is as follows. Let is considered boundary value problem for n. order ordinary differential equation with n boundary conditions. It is required obtaining of the adjoint equation and adjoint boundary conditions to input problem. Obtainig adjoint equation carry out, as usually, by means of multiplies original equation required number differentiable function and integration by parts derived expression over domain of definition of solution.

In order to construct adjoint conditions according to general theory of the ordinary differential equation the first set boundary conditions are added to n new boundary conditions with arbitrary coefficients that we get system composed of $2n$ in number of algebraical equations. Further, considering boundary values in (2) as unknowns those unknown are found from the extended system of algebraic equations. Substituting founded values in bilinear form we get adjoint boundary conditions in which be contained arbitrarily constants. Thus, adjoint boundary conditions dependent from arbitrary constants [1, 9–12].

The aim of this work is to eliminate the arbitrariness present in the adjoint boundary conditions of the conditions. For the simplicity we consider the case $n = 2$.

Lagrange formula: Now let us establish the Lagrange formula for problem (1), (2). For this, if we multiply Equation (1) by $\bar{z}(x)$ and integrate on the interval $[a, b]$, we get

$$(\ell y, z) \equiv \int_a^b \ell y \bar{z}(x) dx = B(y, \bar{z})|_a^b + (y, \ell^* z). \tag{4}$$

Here, $z(x)$ is complex-valued continuous functions, $\bar{z}(x)$ adjoint to $z(x)$ function, and

$$B(y, \bar{z})|_a^b = y'(b)\bar{z}(b) - y'(a)\bar{z}(a) - y(b)\bar{z}'(b) + y(a)\bar{z}'(a)$$

$$+ [\lambda\, a_{11}(b) + a_{10}(b)]\, y(b)\, \bar{z}(b) - [\lambda\, a_{11}(a) + a_{10}(a)]\, y(a)\, \bar{z}(a) \tag{5}$$

is bilinear expression. The differential expression

$$\ell^* z = z''(x) - \left[\left(\bar{\lambda}\, \bar{a}_{11}(x) + \bar{a}_{10}(x)\right)\, z(x)\right]' + \left[\bar{\lambda}^2 \bar{a}_{20}(x) + \bar{\lambda}\, \bar{a}_{21}(x) + \bar{a}_{22}(x)\right]\, z(x). \tag{6}$$

is called the adjoint expression corresponding to the Equation (1).

Conjugate problem: Now let us describe the domain of the adjoint operator or the boundary condition of the adjoint problem. Appending two arbitrary expressions to the given boundary conditions (2) we have four equalities as

$$\ell_\nu[y] \equiv \alpha_{\nu 0} y(a) + \beta_{\nu 0} y(b) + \alpha_{\nu 1} y'(a) + \beta_{\nu 1} y'(b), \quad \nu = 1, 2, 3, 4 \tag{7}$$

namely, when $\nu = 1$ and $\nu = 2$, the boundary conditions coincide to conditions (2) that is $\ell_\nu[y] = 0$, and when $\nu = 3$ and $\nu = 4$ then $\ell_\nu[y]$ are expressions containing arbitrary coefficients.

Assume that

$$\Delta = \begin{vmatrix} \alpha_{10} & \beta_{10} & \alpha_{11} & \beta_{11} \\ \alpha_{20} & \beta_{20} & \alpha_{21} & \beta_{21} \\ \alpha_{30} & \beta_{30} & \alpha_{31} & \beta_{31} \\ \alpha_{40} & \beta_{40} & \alpha_{41} & \beta_{41} \end{vmatrix} \neq 0. \tag{8}$$

Considering $y(a), y(b), y'(a), y'(b)$ as unknowns in the system of algebraic Equation (7), we apply Cramer's method to this system and find the unknowns as

$$\begin{aligned}
y(a) &= \frac{1}{\Delta} \begin{vmatrix} 0 & \beta_{10} & \alpha_{11} & \beta_{11} \\ 0 & \beta_{20} & \alpha_{21} & \beta_{21} \\ \ell_3[y] & \beta_{30} & \alpha_{31} & \beta_{31} \\ \ell_4[y] & \beta_{40} & \alpha_{41} & \beta_{41} \end{vmatrix} = \frac{1}{\Delta}\left(\Delta^{(3,1)}\ell_3[y] + \Delta^{(4,1)}\ell_4[y]\right), \\
y(b) &= \frac{1}{\Delta} \begin{vmatrix} \alpha_{10} & 0 & \alpha_{11} & \beta_{11} \\ \alpha_{20} & 0 & \alpha_{21} & \beta_{21} \\ \alpha_{30} & \ell_3[y] & \alpha_{31} & \beta_{31} \\ \alpha_{40} & \ell_4[y] & \alpha_{41} & \beta_{41} \end{vmatrix} = \frac{1}{\Delta}\left(\Delta^{(3,2)}\ell_3[y] + \Delta^{(4,2)}\ell_4[y]\right), \\
y'(a) &= \frac{1}{\Delta} \begin{vmatrix} \alpha_{10} & \beta_{10} & 0 & \beta_{11} \\ \alpha_{20} & \beta_{20} & 0 & \beta_{21} \\ \alpha_{30} & \beta_{30} & \ell_3[y] & \beta_{31} \\ \alpha_{40} & \beta_{40} & \ell_4[y] & \beta_{41} \end{vmatrix} = \frac{1}{\Delta}\left(\Delta^{(3,3)}\ell_3[y] + \Delta^{(4,3)}\ell_4[y]\right), \\
y'(b) &= \frac{1}{\Delta} \begin{vmatrix} \alpha_{10} & \beta_{10} & \alpha_{11} & 0 \\ \alpha_{20} & \beta_{20} & \alpha_{21} & 0 \\ \alpha_{30} & \beta_{30} & \alpha_{31} & \ell_3[y] \\ \alpha_{40} & \beta_{40} & \alpha_{41} & \ell_4[y] \end{vmatrix} = \frac{1}{\Delta}\left(\Delta^{(3,4)}\ell_3[y] + \Delta^{(4,4)}\ell_4[y]\right).
\end{aligned} \tag{9}$$

Here, the expressions $\Delta^{(i,j)}$ are cofactors of the entries in the intersection of row i and column j of the determinant (8).

$$\begin{aligned}
&\frac{1}{\Delta}\left(\Delta^{(3,4)}\ell_3[y] + \Delta^{(4,4)}\ell_4[y]\right)\overline{z}(b) - \frac{1}{\Delta}\left(\Delta^{(3,3)}\ell_3[y] + \Delta^{(4,3)}\ell_4[y]\right)\overline{z}(a) - \frac{1}{\Delta}\left(\Delta^{(3,2)}\ell_3[y] + \Delta^{(4,2)}\ell_4[y]\right)\overline{z}'(b) \\
&+ \frac{1}{\Delta}\left(\Delta^{(3,1)}\ell_3[y] + \Delta^{(4,1)}\ell_4[y]\right)\overline{z}'(a) + [\lambda\, a_{11}(b) + a_{10}(b)]\frac{1}{\Delta}\left(\Delta^{(3,2)}\ell_3[y] + \Delta^{(4,2)}\ell_4[y]\right)\overline{z}(b) \\
&- [\lambda\, a_{11}(a) + a_{10}(a)]\frac{1}{\Delta}\left(\Delta^{(3,1)}\ell_3[y] + \Delta^{(4,1)}\ell_4[y]\right)\overline{z}(a) = 0.
\end{aligned}$$

Since $\ell_3[y]$ and $\ell_4[y]$ are arbitrary in the last equation, we can decompose it into two boundary conditions as follows:

$$\overline{\Delta}^{(3,4)}z(b)-\overline{\Delta}^{(3,3)}z(a)+\overline{\Delta}^{(3,1)}z'(a)$$

$$-\overline{\Delta}^{(3,2)}z'(b)+\left(\overline{\lambda}\,\overline{a}_{11}(b)+\overline{a}_{10}(b)\right)\overline{\Delta}^{(3,2)}z(b)-\left(\overline{\lambda}\overline{a}_{11}(a)+\overline{a}_{10}(a)\right)\overline{\Delta}^{(3,1)}z(a)=0,$$

$$\overline{\Delta}^{(4,4)}z(b)-\overline{\Delta}^{(4,3)}z(a)-\overline{\Delta}^{(4,2)}z'(b)+\overline{\Delta}^{(4,1)}z'(a)$$

$$+\left[\overline{\lambda}\,\overline{a}_{11}(b)+\overline{a}_{10}(b)\right]\overline{\Delta}^{(4,2)}z(b)-\left[\overline{\lambda}\,\overline{a}_{11}(a)+\overline{a}_{10}(a)\right]\overline{\Delta}^{(4,1)}z(a)=0.$$

The last two equations can be expressed as follows

$$\begin{vmatrix} \alpha_{10} & \beta_{10} & \alpha_{11} & \beta_{11} \\ \alpha_{20} & \beta_{20} & \alpha_{21} & \beta_{21} \\ z'(a)-\left(\overline{\lambda}\,\overline{a}_{11}(a)+\overline{a}_{10}(a)\right)z(a) & -z'(b)+\left(\overline{\lambda}\,\overline{a}_{11}(b)+\overline{a}_{10}(b)\right)z(b) & -z(a) & z(b) \\ \alpha_{40} & \beta_{40} & \alpha_{41} & \beta_{41} \end{vmatrix}=0, \tag{10}$$

$$\begin{vmatrix} \alpha_{10} & \beta_{10} & \alpha_{11} & \beta_{11} \\ \alpha_{20} & \beta_{20} & \alpha_{21} & \beta_{21} \\ \alpha_{30} & \beta_{30} & \alpha_{31} & \beta_{31} \\ z'(a)-\left(\overline{\lambda}\,\overline{a}_{11}(a)+\overline{a}_{10}(a)\right)z(a) & -z'(b)+\left(\overline{\lambda}\,\overline{a}_{11}(b)+\overline{a}_{10}(b)\right)z(b) & -z(a) & z(b) \end{vmatrix}=0. \tag{11}$$

Since the coefficients $\alpha_{\nu j}$, $\beta_{\nu j}$ ($\nu=3,\,4,\; j=1,\,2,\,3,\,4$) in the expressions of the 4×4 determinants above are arbitrary constants, we get the following from (10)

$$\begin{vmatrix} \beta_{10} & \alpha_{11} & \beta_{11} \\ \beta_{20} & \alpha_{21} & \beta_{21} \\ -z'(b)+\left(\overline{\lambda}\overline{a}_{11}(b)+\overline{a}_{10}(b)\right)z(b) & -z(a) & z(b) \end{vmatrix}=0, \tag{12}$$

$$\begin{vmatrix} \alpha_{10} & \alpha_{11} & \beta_{11} \\ \alpha_{20} & \alpha_{21} & \beta_{21} \\ z'(a)-\left(\overline{\lambda}\overline{a}_{11}(a)+\overline{a}_{10}(a)\right)z(a) & -z(a) & z(b) \end{vmatrix}=0, \tag{13}$$

$$\begin{vmatrix} \alpha_{10} & \beta_{10} & \beta_{11} \\ \alpha_{20} & \beta_{20} & \beta_{21} \\ z'(a)-\left(\overline{\lambda}\overline{a}_{11}(a)+\overline{a}_{10}(a)\right)z(a) & -z'(b)+\left(\overline{\lambda}\overline{a}_{11}(b)+\overline{a}_{10}(b)\right)z(b) & z(b) \end{vmatrix}=0, \tag{14}$$

$$\begin{vmatrix} \alpha_{10} & \beta_{10} & \alpha_{11} \\ \alpha_{20} & \beta_{20} & \alpha_{21} \\ z^{'}(a) - \left(\overline{\lambda}\overline{a}_{11}(a) + \overline{a}_{10}(a)\right) z(a) & -z^{'}(b) + \left(\overline{\lambda}\overline{a}_{11}(b) + \overline{a}_{10}(b)\right) z(b) & -z(a) \end{vmatrix} = 0. \tag{15}$$

Analogical results get from (11) too

$$\begin{vmatrix} \beta_{10} & \alpha_{11} & \beta_{11} \\ \beta_{20} & \alpha_{21} & \beta_{21} \\ -z^{'}(b) + \left(\overline{\lambda}\overline{a}_{11}(b) + \overline{a}_{10}(b)\right) z(b) & -z(a) & z(b) \end{vmatrix} = 0, \tag{16}$$

$$\begin{vmatrix} \alpha_{10} & \alpha_{11} & \beta_{11} \\ \alpha_{20} & \alpha_{21} & \beta_{21} \\ z^{'}(a) - \left(\overline{\lambda}\overline{a}_{11}(a) + \overline{a}_{10}(a)\right) z(a) & -z(a) & z(b) \end{vmatrix} = 0, \tag{17}$$

$$\begin{vmatrix} \alpha_{10} & \beta_{10} & \beta_{11} \\ \alpha_{20} & \beta_{20} & \beta_{21} \\ z^{'}(a) - \left(\overline{\lambda}\overline{a}_{11}(a) + \overline{a}_{10}(a)\right) z(a) & -z^{'}(b) + \left(\overline{\lambda}\overline{a}_{11}(b) + \overline{a}_{10}(b)\right) z(b) & z(b) \end{vmatrix} = 0, \tag{18}$$

$$\begin{vmatrix} \alpha_{10} & \beta_{10} & \alpha_{11} \\ \alpha_{20} & \beta_{20} & \alpha_{21} \\ z^{'}(a) - \left(\overline{\lambda}\overline{a}_{11}(a) + \overline{a}_{10}(a)\right) z(a) & -z^{'}(b) + \left(\overline{\lambda}\overline{a}_{11}(b) + \overline{a}_{10}(b)\right) z(b) & -z(a) \end{vmatrix} = 0, \tag{19}$$

As it is seen that the four expressions (12)-(15) and (16)-(19) derived from of (10) and (11) are coincide. This mean that these four expressions (12)-(15) are linearly dependent.

We will now show that arbitrary two of the expressions from (12)-(15) are not linearly dependent, i.e., arbitrary three of the those in (12)-(15) are linearly dependent. To this end first calculate the first three determinants (12)-(14):

$$\begin{vmatrix} \beta_{10} & \alpha_{11} & \beta_{11} \\ \beta_{20} & \alpha_{21} & \beta_{21} \\ -z^{'}(b) + \left(\overline{\lambda}\overline{a}_{11}(b) + \overline{a}_{10}(b)\right) z(b) & -z(a) & z(b) \end{vmatrix}$$

$$= -(\alpha_{11}\beta_{21} - \alpha_{21}\beta_{11}) z^{'}(b) + (\alpha_{11}\beta_{21} - \alpha_{21}\beta_{11}) \left(\overline{\lambda}\overline{a}_{11}(b) + \overline{a}_{10}(b)\right) z(b)$$

$$+ (\beta_{21}\beta_{10} - \beta_{20}\beta_{11}) z(a) + (\alpha_{21}\beta_{10} - \alpha_{11}\beta_{20}) z(b), \tag{20}$$

$$\begin{vmatrix} \alpha_{10} & \alpha_{11} & \beta_{11} \\ \alpha_{20} & \alpha_{21} & \beta_{21} \\ z'(a) - \left(\overline{\lambda}\overline{a}_{11}(a) + \overline{a}_{10}(a)\right) z(a) & -z(a) & z(b) \end{vmatrix}$$

$$= (\alpha_{11}\beta_{21} - \alpha_{21}\beta_{11}) z'(a) - (\alpha_{11}\beta_{21} - \alpha_{21}\beta_{11}) \left(\overline{\lambda}\overline{a}_{11}(a) + \overline{a}_{10}(a)\right) z(a)$$

$$+ (\beta_{21}\alpha_{10} - \alpha_{20}\beta_{11}) z(a) + (\alpha_{21}\alpha_{10} - \alpha_{11}\alpha_{20}) z(b), \tag{21}$$

$$\begin{vmatrix} \alpha_{10} & \beta_{10} & \beta_{11} \\ \alpha_{20} & \beta_{20} & \beta_{21} \\ z'(a) - \left(\overline{\lambda}\overline{a}_{11}(a) + \overline{a}_{10}(a)\right) z(a) & -z'(b) + \left(\overline{\lambda}\overline{a}_{11}(b) + \overline{a}_{10}(b)\right) z(b) & z(b) \end{vmatrix}$$

$$= (\beta_{21}\alpha_{10} - \alpha_{20}\beta_{11}) z'(b) + (\beta_{10}\beta_{21} - \beta_{20}\beta_{11}) z'(a) - (\beta_{21}\alpha_{10} - \beta_{11}\alpha_{20}) \left(\overline{\lambda}\overline{a}_{11}(b) + \overline{a}_{10}(b)\right) z(b)$$

$$- (\beta_{10}\beta_{21} - \beta_{20}\beta_{11}) \left(\overline{\lambda}\overline{a}_{11}(a) + \overline{a}_{10}(a)\right) z(a) + (\beta_{20}\alpha_{10} - \alpha_{20}\beta_{10}) z(b). \tag{22}$$

Multiplying (20)-(22) by the constants A, B, and C, which are not all zero at the same time, and let us equal of this linear combination to zero we get

$$-A(\alpha_{11}\beta_{21} - \alpha_{21}\beta_{11}) z'(b) + A(\alpha_{11}\beta_{21} - \alpha_{21}\beta_{11}) \left(\overline{\lambda}\overline{a}_{11}(b) + \overline{a}_{10}(b)\right) z(b)$$

$$+A(\beta_{21}\beta_{10} - \beta_{20}\beta_{11}) z(a) + A(\alpha_{21}\beta_{10} - \alpha_{11}\beta_{20}) z(b)$$

$$+B(\alpha_{11}\beta_{21} - \alpha_{21}\beta_{11}) z'(a) - B(\alpha_{11}\beta_{21} - \alpha_{21}\beta_{11}) \left(\overline{\lambda}\overline{a}_{11}(a) + \overline{a}_{10}(a)\right) z(a)$$

$$+B(\beta_{21}\alpha_{10} - \alpha_{20}\beta_{11}) z(a) + B(\alpha_{21}\alpha_{10} - \alpha_{11}\alpha_{20}) z(b)$$

$$+C(\beta_{21}\alpha_{10} - \alpha_{20}\beta_{11}) z'(b) + C(\beta_{10}\beta_{21} - \beta_{20}\beta_{11}) z'(a) - C(\beta_{21}\alpha_{10} - \beta_{11}\alpha_{20}) \left(\overline{\lambda}\overline{a}_{11}(b) + \overline{a}_{10}(b)\right) z(b)$$

$$-C(\beta_{10}\beta_{21} - \beta_{20}\beta_{11}) \left(\overline{\lambda}\overline{a}_{11}(a) + \overline{a}_{10}(a)\right) z(a) + C(\beta_{20}\alpha_{10} - \alpha_{20}\beta_{10}) z(b) = 0.$$

The last expression rewritten as follows:

$$[-A(\alpha_{11}\beta_{21}-\alpha_{21}\beta_{11})+C(\beta_{21}\alpha_{10}-\ \alpha_{20}\beta_{11})]z'(b)$$

$$+[B(\alpha_{11}\beta_{21}-\alpha_{21}\beta_{11})+C(\beta_{10}\beta_{21}-\beta_{20}\alpha_{11})]z'(a)$$

$$+\left\{A\left[(\alpha_{11}\beta_{21}-\alpha_{21}\beta_{11})\left(\overline{\lambda}\overline{a}_{11}(b)+\overline{a}_{10}(b)\right)+(\alpha_{21}\beta_{10}-\alpha_{11}\beta_{20})\right]+B(\alpha_{21}\alpha_{10}-\alpha_{11}\alpha_{20})\right.$$

$$\left.+C\left[(\beta_{20}\alpha_{10}-\alpha_{20}\beta_{10})-(\beta_{21}\alpha_{10}-\beta_{11}\alpha_{20})\left(\overline{\lambda}\overline{a}_{11}(b)+\overline{a}_{10}(b)\right)\right]\right\}z(b)$$

$$+\left\{A(\beta_{21}\beta_{10}-\beta_{20}\beta_{11})+B\left[(\beta_{21}\alpha_{10}-\alpha_{20}\beta_{11})-(\alpha_{11}\beta_{21}-\alpha_{21}\beta_{11})\left(\overline{\lambda}\overline{a}_{11}(a)+\overline{a}_{10}(a)\right)\right]\right.$$

$$\left.-C(\beta_{10}\beta_{21}-\beta_{20}\beta_{11})\left(\overline{\lambda}\overline{a}_{11}(a)+\overline{a}_{10}(a)\right)\right\}z(a)=0.$$

From here we get

$$[-A(\alpha_{11}\beta_{21}-\alpha_{21}\beta_{11})+C(\beta_{21}\alpha_{10}-\alpha_{20}\beta_{11})]=0,$$

$$[B(\alpha_{11}\beta_{21}-\alpha_{21}\beta_{11})+C(\beta_{10}\beta_{21}-\beta_{20}\alpha_{11})]=0,$$

$$A\left[(\alpha_{11}\beta_{21}-\alpha_{21}\beta_{11})\left(\overline{\lambda}\overline{a}_{11}(b)+\overline{a}_{10}(b)\right)+(\alpha_{21}\beta_{10}-\alpha_{11}\beta_{20})\right]+B(\alpha_{21}\alpha_{10}-\alpha_{11}\alpha_{20})$$

$$+C\left[(\beta_{20}\alpha_{10}-\alpha_{20}\beta_{10})-(\beta_{21}\alpha_{10}-\beta_{11}\alpha_{20})\left(\overline{\lambda}\overline{a}_{11}(b)+\overline{a}_{10}(b)\right)\right]=0,$$

$$A(\beta_{21}\beta_{10}-\ \beta_{20}\beta_{11})+B\left[(\beta_{21}\alpha_{10}-\alpha_{20}\beta_{11})-(\alpha_{11}\beta_{21}-\alpha_{21}\beta_{11})\left(\overline{\lambda}\overline{a}_{11}(b)+\overline{a}_{10}(b)\right)\right]$$

$$-\ C(\beta_{10}\beta_{21}-\ \beta_{20}\beta_{11})\left(\overline{\lambda}\overline{a}_{11}(a)+\overline{a}_{10}(a)\right)=0.$$

If

$$\delta=\alpha_{11}\beta_{21}-\alpha_{21}\beta_{11}\neq 0,$$

then

$$A=\frac{\alpha_{10}\beta_{21}-\alpha_{20}\beta_{11}}{\alpha_{11}\beta_{21}-\alpha_{21}\beta_{11}}C,\qquad B=-\frac{\beta_{10}\beta_{21}-\ \beta_{20}\beta_{11}}{\alpha_{11}\beta_{21}-\alpha_{21}\beta_{11}}C.$$

Taking account of obtained expressions for A and B, its seen that the coefficients of $z'(b)$ and $z'(a)$ in the previous two expressions are equal to zero.

Now look to coefficients of $z(b)$ and $z(a)$. Substituting founded expressions for both A and B instead of third and fourth equality we have

$$\frac{\beta_{21}\alpha_{10}-\alpha_{20}\beta_{11}}{\alpha_{11}\beta_{21}-\alpha_{21}\beta_{11}}\left[(\alpha_{11}\beta_{21}-\ \alpha_{21}\beta_{11})\left(\overline{\lambda}\overline{a}_{11}(b)+\overline{a}_{10}(b)\right)+(\alpha_{21}\beta_{10}-\alpha_{11}\beta_{20})\right]$$

$$-\frac{\beta_{10}\beta_{21}-\beta_{20}\beta_{11}}{\alpha_{11}\beta_{21}-\alpha_{21}\beta_{11}}(\alpha_{21}\alpha_{10}-\alpha_{11}\alpha_{20})$$

$$+\left[(\beta_{20}\alpha_{10}-\alpha_{20}\beta_{10})-(\beta_{21}\alpha_{10}-\beta_{11}\alpha_{20})\left(\overline{\lambda}\overline{a}_{11}(b)+\overline{a}_{10}(b)\right)\right]=0,$$

and

$$A(\beta_{21}\beta_{10}-\beta_{20}\beta_{11})+B\left[(\beta_{21}\alpha_{10}-\alpha_{20}\beta_{11})-(\alpha_{11}\beta_{21}-\ \alpha_{21}\beta_{11})\left(\overline{\lambda}\overline{a}_{11}(b)+\overline{a}_{10}(b)\right)\right]$$

$$-C(\beta_{10}\beta_{21}-\beta_{20}\beta_{11})\left(\overline{\lambda}\overline{a}_{11}(a)+\overline{a}_{10}(a)\right)=\frac{(\beta_{21}\alpha_{10}-\alpha_{20}\beta_{11})}{\alpha_{11}\beta_{21}-\alpha_{21}\beta_{11}}(\beta_{21}\beta_{10}-\beta_{20}\beta_{11})$$

$$-\frac{(\beta_{10}\beta_{21}-\beta_{20}\beta_{11})}{\alpha_{11}\beta_{21}-\alpha_{21}\beta_{11}}(\beta_{21}\alpha_{10}-\alpha_{20}\beta_{11})$$

$$-\frac{\beta_{10}\beta_{21}-\beta_{20}\beta_{11}}{\alpha_{11}\beta_{21}-\alpha_{21}\beta_{11}}(\alpha_{11}\beta_{21}-\alpha_{21}\beta_{11})\left(\overline{\lambda}\overline{a}_{11}(b)+\overline{a}_{10}(b)\right)$$

$$+(\beta_{10}\beta_{21}-\beta_{20}\beta_{11})\left(\overline{\lambda}\overline{a}_{11}(a)+\overline{a}_{10}(a)\right)=0$$

This shows that the coefficients of $z(b)$ and $z(a)$ are equal to zero independently of C. It is shown that, the first three expressions (12)-(14) are linearly dependent.

Now let us show that expressions (12), (13) and (14) are linearly dependent. Let us calculate the determinant (15);

$$\begin{vmatrix} \alpha_{10} & \beta_{10} & \alpha_{11} \\ \alpha_{20} & \beta_{20} & \alpha_{21} \\ z'(a) - \left(\overline{\lambda}\overline{a}_{11}(a) + \overline{a}_{10}(a)\right) z(a) & -z'(b) + \left(\overline{\lambda}\overline{a}_{11}(b) + \overline{a}_{10}(b)\right) z(b) & -z(a) \end{vmatrix}$$

$$= (\beta_{10}\alpha_{21} - \alpha_{11}\beta_{20}) z'(a) - (\beta_{10}\alpha_{21} - \alpha_{11}\beta_{20}) \left(\overline{\lambda}\overline{a}_{11}(a) + \overline{a}_{10}(a)\right) z(a)$$

$$+ (\alpha_{21}\alpha_{10} - \alpha_{20}\alpha_{11}) z'(b) - (\alpha_{10}\beta_{20} - \beta_{10}\alpha_{20}) z(a)$$

$$- (\alpha_{10}\alpha_{21} - \alpha_{20}\alpha_{11}) \left(\overline{\lambda}\overline{a}_{11}(b) + \overline{a}_{10}(b)\right) z(b). \tag{23}$$

Remultiply Equations (20), (21) and (23) by A_1, B_1 and C_1, then the sum is also equal to zero

$$[- A_1 (\alpha_{11}\beta_{21} - \alpha_{21}\beta_{11}) + C_1 (\alpha_{21}\alpha_{10} - \alpha_{20}\alpha_{11})] z'(b)$$

$$+ [B_1 (\alpha_{11}\beta_{21} - \alpha_{21}\beta_{11}) + C_1 (\beta_{10}\alpha_{21} - \alpha_{11}\beta_{20})] z'(a)$$

$$+ \Big\{ A_1 (\alpha_{11}\beta_{21} - \alpha_{21}\beta_{11}) \left(\overline{\lambda}\overline{a}_{11}(b) + \overline{a}_{10}(b)\right) + A_1 (\alpha_{21}\beta_{10} - \alpha_{11}\beta_{20})$$

$$+ B_1 (\alpha_{21}\alpha_{10} - \alpha_{11}\alpha_{20}) - C_1 (\alpha_{10}\alpha_{21} - \alpha_{20}\alpha_{11}) \left(\overline{\lambda}\overline{a}_{11}(b) + \overline{a}_{10}(b)\right) \Big\} z(b)$$

$$+ \Big\{ A_1 (\beta_{21}\beta_{10} - \beta_{20}\beta_{11}) + B_1 \left[(\beta_{21}\alpha_{10} - \alpha_{20}\beta_{11}) - (\alpha_{11}\beta_{21} - \alpha_{21}\beta_{11}) \left(\overline{\lambda}\overline{a}_{11}(b) + \overline{a}_{10}(b)\right) \right]$$

$$- C_1 (\alpha_{10}\beta_{20} - \beta_{10}\alpha_{20}) + (\beta_{10}\alpha_{21} - \alpha_{11}\beta_{20}) \left(\overline{\lambda}\overline{a}_{11}(a) + \overline{a}_{10}(a)\right) \Big\} z(a).$$

Again, under the following conditions

$$A_1 = C_1 \frac{(\alpha_{21}\alpha_{10} - \alpha_{20}\alpha_{11})}{(\alpha_{11}\beta_{21} - \alpha_{21}\beta_{11})}, \qquad B_1 = -C_1 \frac{(\beta_{10}\alpha_{21} - \alpha_{11}\beta_{20})}{(\alpha_{11}\beta_{21} - \alpha_{21}\beta_{11})}$$

the coefficients of the expressions $z'(b)$, $z'(a)$, $z(b)$ and $z(a)$ are equal to zero. Thus, expressions (12)-(14) and (12), (13), (15) are linearly dependent.

Now we show that the two relations in (10) are linearly independent. The conditions in (12) and (13) are written in the following form

$$(\beta_{21}\beta_{10} - \beta_{20}\beta_{11}) z(a) + \Delta_1 \cdot z(b) - \delta \cdot z'(b) = 0, \tag{24}$$

$$\Delta_2 \cdot z(a) + (\alpha_{21}\alpha_{10} - \alpha_{11}\alpha_{20}) z(b) + \delta \cdot z^{'}(a) = 0. \tag{25}$$

Here, $\Delta_1 = (\alpha_{21}\beta_{10} - \alpha_{11}\beta_{20}) + \delta \cdot \left(\overline{\lambda}\overline{a}_{11}(b) + \overline{a}_{10}(b)\right)$, and

$$\Delta_2 = (\beta_{21}\alpha_{10} - \alpha_{20}\beta_{11}) - \delta \cdot \left(\overline{\lambda}\overline{a}_{11}(a) + \overline{a}_{10}(a)\right).$$

The rank of the matrix of coefficients of $z'(b)$, $z'(a)$, $z(b)$, and $z(a)$ is

$$\text{rank} \begin{pmatrix} (\beta_{21}\beta_{10} - \beta_{20}\beta_{11}) & \Delta_1 & 0 & -\delta \\ \Delta_2 & (\alpha_{21}\alpha_{10} - \alpha_{11}\alpha_{20}) & \Delta & 0 \end{pmatrix} = 2.$$

That is, the boundary conditions (12) and (13) are not linearly dependent. Thus, we have proved that (24), (25) are the only adjoint problem corresponding to the boundary problem (2).

2. Conclusions

In the article, for the first time, for a scalar ordinary differential equation of the second order that coefficients of which depend on the complex parameter λ with a nonlocal boundary condition is obtained a general and free from arbitrariness conjugate boundary condition.

Author contributions

N. Aliyev drew attention to the issue. B. Sinsoysal researched relevant literature and assisted in the research process. M. Rasulov and B. Sinsoysal together did all the theoretical work and implemented the process of writing the article. All authors participated in the revision of the manuscript and approved the final submission. All authors contributed equally to this work.

Conflict of interest

The authors declare no competing financial interest.

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