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Research Article

Some Common Fixed-Point Theorem on Complex-Valued Metric Spaces and Its Application to Solve Urysohn Integral Equations

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Abstract: In this paper, a new and generalized contraction principles are proved on complex-valued metric space. By adopting a suitable hypothesis on sequence converging in complex-valued metric space new contractions are established for proving the common fixed-point theorem. Moreover, a rational contractive condition is improved in the complex-valued metric spaces. The obtained results through theoretical study are verified by solving the solution of the nonlinear system of Urysohn integral equations.

***Keywords*:** fixed-point theory, complex-valued metric spaces, common fixed-point, nonlinear system, theoretical study

MSC: 47H09, 47H10

1. Introduction

Fixed-point theory is an essential concept in analysis. It is possible to express many mathematical problems that come from different scientific fields as fixed-point problems, which require the determination of a function's fixed-point. The presence of a solution to the initial problem can be ensured by using fixed-point theorems, which provide adequate conditions under which a fixed-point for a particular function exists. Algebraic, order theoretic, or topological characteristics of the mapping or its domain are all involved in a number of necessary or sufficient requirements for the presence of such points. It extended the research on economics, control theory, differential equations, optimization problems and so on. By using fixed-point theory [1–3] recently, many authors have studied the qualitative theory of dynamical properties such as existence, controllability, stability, optimal control, etc., for more details [4–6]. In mathematics, the Banach fixed-point theorem serves as a crucial technique in the theory of metric spaces; it ensures both the existence and uniqueness of fixed-points for specific self-maps of metric spaces and offers a constructive method for finding the fixed-points. In this direction, many authors are interested in developing this area of research, and some related findings are given in [7–9]. The theorem is named for Stefan Banach, who originally proposed it in 1922. Common fixed-points on almost generalized contractive mappings [10], rational expressions on cone metric spaces [2], fixed-point theorems by altering distances between the points [11] have been well established.

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Real-valued spaces can be naturally extended by complex-valued metric spaces, which are very important in many domains that deal with functional and complex analysis [3]. The presence of solutions in a variety of mathematical models can be demonstrated with the use of common fixed-point theorems in such spaces, especially when numerous mappings are involved [3, 12, 13]. Powerful tools for resolving theoretical and applied mathematical problems can be obtained by modifying classical fixed-point conclusions to the complex context. In many fields, including pure and applied mathematics, engineering, and science, complex-valued metric spaces are essential [13, 14]. Complex-valued metric spaces were first proposed by Azam et al. [15] in 2011, who also showed common fixed-point theorems that satisfy rational contractive mapping. A common fixed-point of rational inequalities has been proposed in [11]. A unified common fixed-point theorem has been studied in [16] by using implicit relations. On the ordered complex partial metric space, a contractive condition of rational expression has been established in [13]. Some common fixed-point theorems on complex-valued metric space for dislocated metric spaces [14], generalized contractive type [17] have been analyzed. On the complex-valued metric spaces, some common fixed-point theorems satisfying particular rational expressions have been proved in [18].

Based on the above discussion, there is no new work reported on generalized common fixed-point theorems in complex-valued metric spaces. Motivated by these analyses and their applications to the integral equations, in this paper, the authors study some common fixed-point theorem and their applications in complex-valued metric spaces. Also, the proposed results are new and generalize the existing results from the literature.

Key Contributions of the Article:

1. A new generalized rational contraction mapping is proposed to demonstrate a common fixed-point in complex-valued metric spaces.

2. We additionally established the fixed-point in the corollary using rational contraction mapping.

3. We use the new rational contraction to validate the statement that the system of Urysohn integral equations has just a unique simple solution.

2. Preliminaries

The basic definitions and notions are as follows:

Consider the C, complex number set and $\Xi_1,\ \Xi_2 \in C$. Let the partial order $\preceq$ on C are defined as $\Xi_1 \preceq \Xi_2$ iff $\mathscr{R}(\Xi_1) \preceq \mathscr{R}(\Xi_2), I(\Xi_1) \preceq I(\Xi_2)$. If $\Xi_1 \preceq \Xi_2$, then the following conditions are satisfied:

(i) $\mathscr{R}(\Xi_1) = \mathscr{R}(\Xi_2), I(\Xi_1) \prec I(\Xi_2)$,

(ii) $\mathscr{R}(\Xi_1) \prec \mathscr{R}(\Xi_2), I(\Xi_1) = I(\Xi_2)$,

(iii) $\mathscr{R}(\Xi_1) \prec \mathscr{R}(\Xi_2), I(\Xi_1) \prec I(\Xi_2)$,

(iv) $\mathscr{R}(\Xi_1) = \mathscr{R}(\Xi_2), I(\Xi_1) = I(\Xi_2)$.

In particular, $\Xi_1 \precnsim \Xi_2$ if $\Xi_1 \neq \Xi_2$ and (i), (ii), and (iii) are all satisfied. We may write as $\Xi_1 \prec \Xi_2$ if only (iii) is satisfied. We notice the following conditions also:

(a) If $0 \preceq \Xi_1 \precnsim \Xi_2$, then $|\Xi_1| \prec |\Xi_2|$,

(b) If $\Xi_1 \preceq \Xi_2$ and $\Xi_2 \prec \Xi_3$ then $\Xi_1 \prec \Xi_3$,

(c) If $a,\ b \in \mathscr{R}$ and $a \preceq b$ then $a\Xi \preceq b\Xi$ for each $\Xi \in C$.

Definition 1 [16] Let Y be a non void set and the function $\mathfrak{A}:\ Y \times Y \to C$ satisfying the following conditions:

(i) $\theta \preceq \mathfrak{A}(\mathfrak{w},\ p)$ for each $\mathfrak{w},\ p \in Y$ and $\mathfrak{A}(\mathfrak{w},\ p) = \theta$ iff $\mathfrak{w} = p$,

(ii) $\mathfrak{A}(\mathfrak{w},\ p) = \mathfrak{A}(p,\ \mathfrak{w})$ for each $\mathfrak{w},\ p \in Y$,

(iii) $\mathfrak{A}(\mathfrak{w},\ p) \preceq \mathfrak{A}(\mathfrak{w},\ r) + \mathfrak{A}(r,\ p)$ for each $\mathfrak{w},\ p,\ r \in Y$.

Then, the function $\mathfrak{A}$ is called complex-valued metric space and the pair $(Y,\ \mathfrak{A})$ is known as complex-valued metric space.

Example 2 [15] Let $Y = C$ be a collection of complex numbers and the function is $\mathfrak{A}:\ Y \times Y \to C$ by $\mathfrak{A}(\Xi_1,\ \Xi_2) = e^{ip}|\Xi_1 - \Xi_2|$ where each $p \in R$. Then, $(Y,\ \mathfrak{A})$ is a complex-valued metric space.

Example 3 [13] Let $Y = C$ be a collection of complex numbers and the function is defined as $\mathfrak{A}: Y \times Y \to C$ by $\mathfrak{A}(\Xi_1, \Xi_2) = 3i|\Xi_1 - \Xi_2|$ for each $\Xi_1, \Xi_2 \in Y$. Then, $(Y, \mathfrak{A})$ is a complex-valued metric space.

Definition 4 [19] Let $(Y, \mathfrak{A})$ be a complex-valued metric space. Then, the following conditions are satisfied:

(i) Let the element $\mathfrak{w} \in Y$ be an interior point of the set $O \subseteq Y$ if $\exists\ \theta \prec \mathfrak{w} \in C, B(\mathfrak{w}, a) = \{p \in Y: \mathfrak{A}(\mathfrak{w}, p) \prec a\} \subseteq O$,

(ii) Let the element $\mathfrak{w} \in Y$ be a limit point of O if for every $\theta \prec a \in C,\ B(\mathfrak{w}, a) \cap (O - Y) \neq \phi$,

(iii) Let $O \subseteq Y$ be an open if each element of $\mathfrak{w}$ is an interior point of O,

(iv) Let $O \subseteq Y$ is closed if each limit point of $\mathfrak{w}$ belongs to O,

(v) Let the Hausdorff topology τ on Y be a sub-basis in a family of $F = \{B(\mathfrak{w}, a): \mathfrak{w} \in Y, \theta \prec a\}$.

Definition 5 [13] Let $(Y, \mathfrak{A})$ be a complex-valued metric space. Then, $\{\mathfrak{w}_n\}$ a sequence in Y for $\mathfrak{w} \in Y$, we have

(i) For each $a \in C$ with $\theta \prec a$ find an $N \exists$ for every $n \succ N$, $\mathfrak{A}(\mathfrak{w}_n, \mathfrak{w}) \prec a$ then $\{\mathfrak{w}_n\}$ is convergent, $\{\mathfrak{w}_n\}$ converges to $\mathfrak{w}$ and $\mathfrak{w}$ is the limit point of $\{\mathfrak{w}_n\}$,

(ii) If each $a \in C$, $\theta \prec a$ find N there exists for all $n \succ N$, $\mathfrak{A}(\mathfrak{w}_n, \mathfrak{w}_{n+m}) \prec c$, where $m \in N$ then $\{\mathfrak{w}_n\}$ called as a Cauchy sequence,

(iii) There is convergence for each Cauchy sequence in Y, then $(Y, \mathfrak{A})$ is complex-valued metric spaces which is complete.

Lemma 6 [13] Let $(Y, \mathfrak{A})$ be a complex-valued metric space and $\{\mathfrak{w}_n\}$ be sequence in Y. Then, $\{\mathfrak{w}_n\}$ convergent to $\mathfrak{w}$ if and only if $|\mathfrak{A}(\mathfrak{w}_n, \mathfrak{w})| \to \theta$ as $n \to +\infty$.

Lemma 7 [16] Let $(Y, \mathfrak{A})$ be a complex-valued metric space. Let $\{\mathfrak{w}_n\}$ be a sequence in Y, then $\{\mathfrak{w}_n\}$ is a Cauchy sequence if and only if $|\mathfrak{A}(\mathfrak{w}_n, \mathfrak{w}_{n+m})| \to \theta$ as $n, m \to +\infty$.

Definition 8 [16] The self mappings V and K of a non void set Y. Then, we have

(i) Let $\mathfrak{w} \in Y$ be an element which is a fixed-point of K if $K\mathfrak{w} = \mathfrak{w}$,

(ii) Let $\mathfrak{w} \in Y$ be an element which is a coincidence point of V and K if $V\mathfrak{w} = K\mathfrak{w}$ and $w = V\mathfrak{w} = K\mathfrak{w}$ which is the point where V and K coincide,

(iii) Let $\mathfrak{w} \in Y$ be a point which is the point where V and K coincide if $\mathfrak{w} = V\mathfrak{w} = K\mathfrak{w}$.

3. Main results

We establish a rational contractive condition in the complex-valued metric spaces and implement those condition to apply the Urysohn integral equations.

Theorem 9 Let $(Y, \mathfrak{A})$ be a complete complex-valued metric space. Let $V, K: Y \to Y$ if there is a function $\chi, \xi: Y \to [0, 1) \ni$ for each $\mathfrak{w}, p \in Y$ and the following conditions hold:

(i) $\chi(V\mathfrak{w}) \leq \chi(\mathfrak{w})$ and $\xi(V\mathfrak{w}) \leq \xi(\mathfrak{w})$,

(ii) $\chi(K\mathfrak{w}) \leq \chi(\mathfrak{w})$ and $\xi(K\mathfrak{w}) \leq \xi(\mathfrak{w})$,

(iii) $(\chi + \xi)(\mathfrak{w}) \leq 1$,

(iv) $\mathfrak{A}(V\mathfrak{w}, Kp) \leq \chi(\mathfrak{w})\mathfrak{A}(\mathfrak{w}, p) + \xi(\mathfrak{w}) \left[\dfrac{\mathfrak{A}(Kp, p)\mathfrak{A}(p, V\mathfrak{w}) + \mathfrak{A}(\mathfrak{w}, V\mathfrak{w})\mathfrak{A}(\mathfrak{w}, Kp)}{1 + \mathfrak{A}(\mathfrak{w}, p)}\right]$.

Then, V and K has an unique common fixed-point.

Proof. Assume that $\mathfrak{w}_0$ a arbitrary point in Y. Since, $V(Y) \subseteq Y$ and $K(Y) \subseteq Y$, now the sequence $\{\mathfrak{w}_\mathfrak{x}\}$ in Y $\ni$ $\mathfrak{w}_{2\mathfrak{x}+1} = V\mathfrak{w}_{2\mathfrak{x}}$ and $\mathfrak{w}_{2\mathfrak{x}+2} = K\mathfrak{w}_{2\mathfrak{x}+1}$ for each $\mathfrak{x} \geq 0$.

Therefore, the hypothesis becomes

$$
\begin{aligned}
\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+2}) &= \mathfrak{A}(V\mathfrak{w}_{2\mathfrak{r}},\ K\mathfrak{w}_{2\mathfrak{r}+1}) \\
&\preceq \chi(\mathfrak{w}_{2\mathfrak{r}})\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+1}) + \xi(\mathfrak{w}_{2\mathfrak{r}}) \\
&\quad \times \left[\frac{\mathfrak{A}(K\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+1})\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ V\mathfrak{w}_{2\mathfrak{r}}) + \mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ V\mathfrak{w}_{2\mathfrak{r}})\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ K\mathfrak{w}_{2\mathfrak{r}+1})}{1+\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+1})}\right] \\
&= \chi(\mathfrak{w}_{2\mathfrak{r}})\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+1}) + \xi(\mathfrak{w}_{2\mathfrak{r}}) \\
&\quad \times \left[\frac{\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+1})\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+1}) + \mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+1})\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+2})}{1+\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+1})}\right] \\
&= \chi(\mathfrak{w}_{2\mathfrak{r}})\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+1}) + \xi(\mathfrak{w}_{2\mathfrak{r}})\left[\frac{\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+1})\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+2})}{1+\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+1})}\right] \\
&\preceq \chi(\mathfrak{w}_{2\mathfrak{r}})\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+1}) \\
&\quad + \xi(\mathfrak{w}_{2\mathfrak{r}})\left[\frac{\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+1})[\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+1}) + \mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+2})]}{1+\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+1})}\right] \\
&\preceq \chi(\mathfrak{w}_{2\mathfrak{r}})\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+1}) + \xi(\mathfrak{w}_{2\mathfrak{r}})[\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+1}) + \mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+2})] \\
&= [\chi(\mathfrak{w}_{2\mathfrak{r}}) + \xi(\mathfrak{w}_{2\mathfrak{r}})]\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+1}) + \xi(\mathfrak{w}_{2\mathfrak{r}})\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+2})] \\
&= [\chi(K\mathfrak{w}_{2\mathfrak{r}-1}) + \xi(K\mathfrak{w}_{2\mathfrak{r}-1})]\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+1}) + \xi(K\mathfrak{w}_{2\mathfrak{r}-1})\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+2})] \\
&\preceq [\chi(\mathfrak{w}_{2\mathfrak{r}-1}) + \xi(\mathfrak{w}_{2\mathfrak{r}-1})]\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+1}) + \xi(\mathfrak{w}_{2\mathfrak{r}-1})\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+2})] \\
&= [\chi(K\mathfrak{w}_{2\mathfrak{r}-2}) + \xi(K\mathfrak{w}_{2\mathfrak{r}-2})]\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+1}) + \xi(K\mathfrak{w}_{2\mathfrak{r}-2})\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+2})] \\
&\preceq [\chi(\mathfrak{w}_{2\mathfrak{r}-2}) + \xi(\mathfrak{w}_{2\mathfrak{r}-2})]\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+1}) + \xi(\mathfrak{w}_{2\mathfrak{r}-2})\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+2})] \\
&\preceq [\chi(\mathfrak{w}_0) + \xi(\mathfrak{w}_0)]\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+1}) + \xi(\mathfrak{w}_0)\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+2})]
\end{aligned}
$$

which implies that

$$
\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+2}) \preceq \left(\frac{\chi(\mathfrak{w}_0) + \xi(\mathfrak{w}_0)}{1-\xi(\mathfrak{w}_0)}\right)\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+1}).
$$

Similarly, we proceed like that

$$\begin{aligned}
\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+3}) &= \mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+3},\ \mathfrak{w}_{2\mathfrak{r}+2}) \\
&= \mathfrak{A}(V\mathfrak{w}_{2\mathfrak{r}+2},\ K\mathfrak{w}_{2\mathfrak{r}+1}) \\
&\preceq \chi(\mathfrak{w}_{2\mathfrak{r}+2})\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+1}) \\
&\quad + \xi(\mathfrak{w}_{2\mathfrak{r}+2})\left[\frac{\mathfrak{A}(K\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+1})\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ V\mathfrak{w}_{2\mathfrak{r}+2}) + \mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ V\mathfrak{w}_{2\mathfrak{r}+2})\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ K\mathfrak{w}_{2\mathfrak{r}+1})}{1+\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+1})}\right] \\
&\preceq \chi(\mathfrak{w}_{2\mathfrak{r}+2})\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+1}) \\
&\quad + \xi(\mathfrak{w}_{2\mathfrak{r}+2})\left[\frac{\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+1})\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+3}) + \mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+3})\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+2})}{1+\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+1})}\right] \\
&= \chi(\mathfrak{w}_{2\mathfrak{r}+2})\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+1}) + \xi(\mathfrak{w}_{2\mathfrak{r}+2})\left[\frac{\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+1})\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+3})}{1+\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+1})}\right] \\
&\preceq \chi(\mathfrak{w}_{2\mathfrak{r}+2})\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+1}) + \xi(\mathfrak{w}_{2\mathfrak{r}+2})\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+3}) \\
&\preceq \chi(\mathfrak{w}_{2\mathfrak{r}+2})\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+1}) + \xi(\mathfrak{w}_{2\mathfrak{r}+2})[\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+2}) + \mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+3})] \\
&\preceq [\chi(\mathfrak{w}_{2\mathfrak{r}+2}) + \xi(\mathfrak{w}_{2\mathfrak{r}+2})]\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+2}) + \xi(\mathfrak{w}_{2\mathfrak{r}+2})[\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+3})] \\
&= [\chi(K\mathfrak{w}_{2\mathfrak{r}+1}) + \xi(K\mathfrak{w}_{2\mathfrak{r}+1})]\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+2}) + \xi(K\mathfrak{w}_{2\mathfrak{r}+1})[\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+3})] \\
&\preceq [\chi(\mathfrak{w}_{2\mathfrak{r}}) + \xi(\mathfrak{w}_{2\mathfrak{r}})]\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+2}) + \xi(\mathfrak{w}_{2\mathfrak{r}})[\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+3})] \\
&\vdots \\
&\preceq [\chi(\mathfrak{w}_0) + \xi(\mathfrak{w}_0)]\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+2}) + \xi(\mathfrak{w}_0)[\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+3})].
\end{aligned}$$

Therefore, we get

$$\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+3}) \preceq \frac{\chi(\mathfrak{w}_0)+\xi(\mathfrak{w}_0)}{1-\xi(\mathfrak{w}_0)}\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+2}).$$

Let us choose $\lambda = \dfrac{\chi(\mathfrak{w}_0) + \xi(\mathfrak{w}_0)}{1 - \xi(\mathfrak{w}_0)}$

$$\begin{aligned}\mathfrak{A}(\mathfrak{w}_n,\ \mathfrak{w}_{n+1}) &\preceq \lambda\mathfrak{A}(\mathfrak{w}_{n-1},\ \mathfrak{w}_n)\\ &\preceq \lambda^2\mathfrak{A}(\mathfrak{w}_{n-2},\ \mathfrak{w}_{n-1})\\ &\vdots\\ &\preceq \lambda^n\mathfrak{A}(\mathfrak{w}_0,\ \mathfrak{w}_1).\end{aligned}$$

Consider a natural number m and n with $m \succ n$, for each $n \in N$, we have

$$\begin{aligned}\mathfrak{A}(\mathfrak{w}_n,\ \mathfrak{w}_m) &\preceq \mathfrak{A}(\mathfrak{w}_n,\ \mathfrak{w}_{n+1}) + \mathfrak{A}(\mathfrak{w}_{n+1},\ \mathfrak{w}_{n+2}) + \cdots + \mathfrak{A}(\mathfrak{w}_{m-1},\ \mathfrak{w}_m)\\ &\preceq \lambda^n\mathfrak{A}(\mathfrak{w}_0,\ \mathfrak{w}_1) + \lambda^{n+1}\mathfrak{A}(\mathfrak{w}_0,\ \mathfrak{w}_1) + \cdots + \lambda^{m-1}\mathfrak{A}(\mathfrak{w}_0,\ \mathfrak{w}_1)\\ &= (\lambda^n + \lambda^{n+1} + ... + \lambda^{m-1})\mathfrak{A}(\mathfrak{w}_0,\ \mathfrak{w}_1)\\ &\preceq \left(\frac{\lambda^n}{1-\lambda}\right)\mathfrak{A}(\mathfrak{w}_0,\ \mathfrak{w}_1).\end{aligned}$$

Therefore, we have

$$|\mathfrak{A}(\mathfrak{w}_n,\ \mathfrak{w}_m)| \preceq \left(\frac{\lambda^n}{1-\lambda}\right)|\mathfrak{A}(\mathfrak{w}_0,\ \mathfrak{w}_1)|.$$

Since $\lambda \in [0,\ 1)$, letting the $m,\ n \to 0$ limit shows that the $\{\mathfrak{w}_n\}$ is a Cauchy sequence. Hence, Y is complete, there is a point $a \in Y \ni \mathfrak{w}_n \to a$ as $n \to +\infty$.

To show that $Va = a$. Now,

$$\begin{aligned}\mathfrak{A}(a,\ Va) &\preceq \mathfrak{A}(a,\ \mathfrak{w}_{2\mathfrak{r}+2}) + \mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ Va)\\ &= \mathfrak{A}(a,\ \mathfrak{w}_{2\mathfrak{r}+2}) + \mathfrak{A}(K\mathfrak{w}_{2\mathfrak{r}+1},\ Va)\\ &= \mathfrak{A}(a,\ \mathfrak{w}_{2\mathfrak{r}+2}) + \mathfrak{A}(Va,\ K\mathfrak{w}_{2\mathfrak{r}+1})\\ &\preceq \mathfrak{A}(a,\ \mathfrak{w}_{2\mathfrak{r}+2}) + \chi(a)\mathfrak{A}(a,\ \mathfrak{w}_{2\mathfrak{r}+1})\end{aligned}$$

$$+\xi(a)\left[\frac{\mathfrak{A}(K\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+1})\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ Va)+\mathfrak{A}(a,\ Va)\mathfrak{A}(a,\ K\mathfrak{w}_{2\mathfrak{r}+1})}{1+\mathfrak{A}(a,\ \mathfrak{w}_{2\mathfrak{r}+1})}\right]$$

$$=\mathfrak{A}(a,\ \mathfrak{w}_{2\mathfrak{r}+2})+\chi(a)\mathfrak{A}(a,\ \mathfrak{w}_{2\mathfrak{r}+1})$$

$$+\xi(a)\left[\frac{\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+1})\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ Va)+\mathfrak{A}(a,\ Va)\mathfrak{A}(a,\ \mathfrak{w}_{2\mathfrak{r}+2})}{1+\mathfrak{A}(a,\ \mathfrak{w}_{2\mathfrak{r}+1})}\right]$$

which implies that

$$|\mathfrak{A}(a,\ Va)|\preceq|\mathfrak{A}(a,\ \mathfrak{w}_{2\mathfrak{r}+2})|+\chi(a)|\mathfrak{A}(a,\ \mathfrak{w}_{2\mathfrak{r}+1})|$$

$$+\xi(a)\left[\frac{|\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+1})||\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ Va)|+|\mathfrak{A}(a,\ Va)||\mathfrak{A}(a,\ \mathfrak{w}_{2\mathfrak{r}+2})|}{|1+\mathfrak{A}(a,\ \mathfrak{w}_{2\mathfrak{r}+1})|}\right].$$

As $\mathfrak{r}\to\infty$ we have $|\mathfrak{A}(a,\ Va)|=0$ which shows that $\mathfrak{A}(a,\ Va)=0$, Hence, we get $Va=a$. Similarly, we get that $Ka=a$. Therefore, a is the common fixed-point of V and K.

Next claim that a is a unique common fixed-point of functions V and K.

Let us choose another common fixed-point a_1 that is $a_1=Va_1=Ka_1$. It follows from

$$\mathfrak{A}(a,\ a_1)=\mathfrak{A}(Va,\ Ka_1)$$

$$\preceq\chi(a)\mathfrak{A}(a,\ a_1)+\xi(a)\left[\frac{\mathfrak{A}(Ka_1,\ a_1)\mathfrak{A}(a_1,\ Va)+\mathfrak{A}(a,\ Va)\mathfrak{A}(a,\ Ka_1)}{1+\mathfrak{A}(a,\ a_1)}\right]$$

$$=\chi(a)\mathfrak{A}(a,\ a_1)$$

$$\preceq\chi(a)|\mathfrak{A}(a,\ a_1)|.$$

Since $\chi(a)\in[0,\ 1)$, we have $|\mathfrak{A}(a,\ a_1)|=0$. Thus, $a=a_1$ and hence a is only unique common fixed-point of V and K. □

Theorem 10 Let $(Y,\ \mathfrak{A})$ be a complete complex-valued metric space. Let $V,\ K:\ Y\to Y$ if there is a function $\chi,\ \xi:\ Y\to[0,\ 1)\ni$ for each $\mathfrak{w},\ p\in Y$ and the following conditions hold:

(i) $\chi(V\mathfrak{w})\leq\chi(\mathfrak{w})$ and $\xi(V\mathfrak{w})\leq\xi(\mathfrak{w})$,

(ii) $\chi(K\mathfrak{w})\leq\chi(\mathfrak{w})$ and $\xi(K\mathfrak{w})\leq\xi(\mathfrak{w})$,

(iii) $(\chi+\xi)(\mathfrak{w})\leq 1$,

(iv) $\mathfrak{A}(V\mathfrak{w},\ Kp)\leq\chi(\mathfrak{w})[\mathfrak{A}(\mathfrak{w},\ p)+\mathfrak{A}(\mathfrak{w},\ Kp)+\mathfrak{A}(p,\ V\mathfrak{w})]+\xi(\mathfrak{w})\left[\frac{\mathfrak{A}(Kp,\ p)\mathfrak{A}(p,\ V\mathfrak{w})+\mathfrak{A}(\mathfrak{w},\ V\mathfrak{w})\mathfrak{A}(\mathfrak{w},\ Kp)}{1+\mathfrak{A}(\mathfrak{w},\ p)}\right]$.

Then, V and K has an unique common fixed-point.

Proof. Assume $\mathfrak{w}_0$ an arbitrary point in Y. Since $V(Y)\subseteq Y$ and $K(Y)\subseteq Y$, now the sequence $\{\mathfrak{w}_k\}$ in $Y\ni\mathfrak{w}_{2\mathfrak{r}+1}=V\mathfrak{w}_{2\mathfrak{r}}$ and $\mathfrak{w}_{2\mathfrak{r}+2}=K\mathfrak{w}_{2\mathfrak{r}+1}$ for each $k\geq 0$. Therefore, the hypothesis becomes

$$
\begin{aligned}
\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+2}) &= \mathfrak{A}(V\mathfrak{w}_{2\mathfrak{r}},\ K\mathfrak{w}_{2\mathfrak{r}+1}) \\
&\preceq \chi(\mathfrak{w}_{2\mathfrak{r}})[\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+1}) + \mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ K\mathfrak{w}_{2\mathfrak{r}+1}) + \mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ V\mathfrak{w}_{2\mathfrak{r}})] \\
&\quad + \xi(\mathfrak{w}_{2\mathfrak{r}})\left[\frac{\mathfrak{A}(K\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+1})\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ V\mathfrak{w}_{2\mathfrak{r}}) + \mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ V\mathfrak{w}_{2\mathfrak{r}})\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ K\mathfrak{w}_{2\mathfrak{r}+1})}{1 + \mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+1})}\right] \\
&= \chi(\mathfrak{w}_{2\mathfrak{r}})[\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+1}) + \mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+2}) + \mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+1})] \\
&\quad + \xi(\mathfrak{w}_{2\mathfrak{r}})\left[\frac{\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+1})\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+1}) + \mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+1})\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+2})}{1 + \mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+1})}\right] \\
&= \chi(\mathfrak{w}_{2\mathfrak{r}})[\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+1}) + \mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+2})] + \xi(\mathfrak{w}_{2\mathfrak{r}})\left[\frac{\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+1})\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+2})}{1 + \mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+1})}\right] \\
&\preceq \chi(\mathfrak{w}_{2\mathfrak{r}})[2\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+1}) + \mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+2})] \\
&\quad + \xi(\mathfrak{w}_{2\mathfrak{r}})\left[\frac{\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+1})[\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+1}) + \mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+2})]}{1 + \mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+1})}\right] \\
&\preceq \chi(\mathfrak{w}_{2\mathfrak{r}})[2\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+1}) + \mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+2})] + \xi(\mathfrak{w}_{2\mathfrak{r}})[\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+1}) + \mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+2})] \\
&= [2\chi(\mathfrak{w}_{2\mathfrak{r}}) + \xi(\mathfrak{w}_{2\mathfrak{r}})]\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+1}) + [\xi(\mathfrak{w}_{2\mathfrak{r}}) + \chi(\mathfrak{w}_{2\mathfrak{r}})]\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+2})] \\
&= [2\chi(K\mathfrak{w}_{2\mathfrak{r}-1}) + \xi(K\mathfrak{w}_{2\mathfrak{r}-1})]\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+1}) + [\xi(K\mathfrak{w}_{2\mathfrak{r}-1}) + \chi(K\mathfrak{w}_{2\mathfrak{r}-1})]\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+2})] \\
&\preceq [2\chi(\mathfrak{w}_{2\mathfrak{r}-1}) + \xi(\mathfrak{w}_{2\mathfrak{r}-1})]\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+1}) + [\xi(\mathfrak{w}_{2\mathfrak{r}-1}) + \chi(\mathfrak{w}_{2\mathfrak{r}-1})]\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+2})] \\
&= [2\chi(K\mathfrak{w}_{2\mathfrak{r}-2}) + \xi(K\mathfrak{w}_{2\mathfrak{r}-2})]\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+1}) + [\xi(K\mathfrak{w}_{2\mathfrak{r}-2}) + \chi(K\mathfrak{w}_{2\mathfrak{r}-2})]\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+2}) \\
&\preceq [2\chi(\mathfrak{w}_{2\mathfrak{r}-2}) + \xi(\mathfrak{w}_{2\mathfrak{r}-2})]\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+1}) + [\xi(\mathfrak{w}_{2\mathfrak{r}-2}) + \chi(\mathfrak{w}_{2\mathfrak{r}-2})]\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+2}) \\
&\ \vdots \\
&\preceq [2\chi(\mathfrak{w}_0) + \xi(\mathfrak{w}_0)]\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+1}) + [\xi(\mathfrak{w}_0) + \chi(\mathfrak{w}_0)]\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+2})]
\end{aligned}
$$

which implies that

$$\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+2}) \preceq \left(\frac{2\chi(\mathfrak{w}_0)+\xi(\mathfrak{w}_0)}{1-[\xi(\mathfrak{w}_0)+\chi(\mathfrak{w}_0)]}\right)\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}},\ \mathfrak{w}_{2\mathfrak{r}+1}).$$

Similarly, we proceed like that

$$\begin{aligned}
\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+3}) &= \mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+3},\ \mathfrak{w}_{2\mathfrak{r}+2}) \\
&= \mathfrak{A}(V\mathfrak{w}_{2\mathfrak{r}+2},\ K\mathfrak{w}_{2\mathfrak{r}+1}) \\
&\preceq \chi(\mathfrak{w}_{2\mathfrak{r}+2})[\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+1})+\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ K\mathfrak{w}_{2\mathfrak{r}+1})+\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ V\mathfrak{w}_{2\mathfrak{r}+2})]+\xi(\mathfrak{w}_{2\mathfrak{r}+2}) \\
&\quad \times \left[\frac{\mathfrak{A}(K\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+1})\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ V\mathfrak{w}_{2\mathfrak{r}+2})+\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ V\mathfrak{w}_{2\mathfrak{r}+2})\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ K\mathfrak{w}_{2\mathfrak{r}+1})}{1+\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+1})}\right] \\
&= \chi(\mathfrak{w}_{2\mathfrak{r}+2})[\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+1})+\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+2})+\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+3})] \\
&\quad +\xi(\mathfrak{w}_{2\mathfrak{r}+2})\left[\frac{\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+1})\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+3})+\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+3})\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+2})}{1+\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+1})}\right] \\
&= \chi(\mathfrak{w}_{2\mathfrak{r}+2})[\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+1})+\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+3})]+\xi(\mathfrak{w}_{2\mathfrak{r}+2}) \\
&\quad \times \left[\frac{\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+1})\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+3})}{1+\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+1})}\right] \\
&\preceq \chi(\mathfrak{w}_{2\mathfrak{r}+2})[\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+1})+\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+3})]+\xi(\mathfrak{w}_{2\mathfrak{r}+2})\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+3}) \\
&\preceq \chi(\mathfrak{w}_{2\mathfrak{r}+2})[2\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+1})+\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+3})] \\
&\quad +\xi(\mathfrak{w}_{2\mathfrak{r}+2})[\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+2})+\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+3})] \\
&= [2\chi(\mathfrak{w}_{2\mathfrak{r}+2})+\xi(\mathfrak{w}_{2\mathfrak{r}+2})]\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+1})+[\xi(\mathfrak{w}_{2\mathfrak{r}+2})+\chi(\mathfrak{w}_{2\mathfrak{r}+2})]\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+3}) \\
&= [2\chi(K\mathfrak{w}_{2\mathfrak{r}+1})+\xi(K\mathfrak{w}_{2\mathfrak{r}+1})]\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+1})+[\xi(K\mathfrak{w}_{2\mathfrak{r}+1})+\chi(K\mathfrak{w}_{2\mathfrak{r}+1})]\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+3}) \\
&\preceq [2\chi(\mathfrak{w}_{2\mathfrak{r}+1})+\xi(\mathfrak{w}_{2\mathfrak{r}+1})]\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+1})+[\xi(\mathfrak{w}_{2\mathfrak{r}+1})+\chi(\mathfrak{w}_{2\mathfrak{r}+1})]\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+3}) \\
&= [2\chi(K\mathfrak{w}_{2\mathfrak{r}})+\xi(K\mathfrak{w}_{2\mathfrak{r}})]\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+1})+[\xi(K\mathfrak{w}_{2\mathfrak{r}})+\chi(K\mathfrak{w}_{2\mathfrak{r}})]\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+3})
\end{aligned}$$

$$\preceq [2\chi(\mathfrak{w}_{2\mathfrak{r}}) + \xi(\mathfrak{w}_{2\mathfrak{r}})]\,\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2}, \mathfrak{w}_{2\mathfrak{r}+1}) + [\xi(\mathfrak{w}_{2\mathfrak{r}}) + \chi(\mathfrak{w}_{2\mathfrak{r}})]\,\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2}, \mathfrak{w}_{2\mathfrak{r}+3})$$

$$\vdots$$

$$\preceq [2\chi(\mathfrak{w}_0) + \xi(\mathfrak{w}_0)]\,\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2}, \mathfrak{w}_{2\mathfrak{r}+1}) + [\xi(\mathfrak{w}_0) + \chi(\mathfrak{w}_0)]\,\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2}, \mathfrak{w}_{2\mathfrak{r}+3})]$$

which implies that

$$\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+3}) \preceq \left[\frac{2\chi(\mathfrak{w}_0) + \xi(\mathfrak{w}_0)}{1 - [\xi(\mathfrak{w}_0) + \chi(\mathfrak{w}_0)]}\right]\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+2}).$$

Let us choose

$$\lambda = \frac{2\chi(\mathfrak{w}_0) + \xi(\mathfrak{w}_0)}{1 - [\xi(\mathfrak{w}_0) + \chi(\mathfrak{w}_0)]}$$

$$\begin{aligned}
\mathfrak{A}(\mathfrak{w}_n,\ \mathfrak{w}_{n+1}) &\preceq \lambda\mathfrak{A}(\mathfrak{w}_{n-1},\ \mathfrak{w}_n)\\
&\preceq \lambda^2\mathfrak{A}(\mathfrak{w}_{n-2},\ \mathfrak{w}_{n-1})\\
&\ \vdots\\
&\preceq \lambda^n\mathfrak{A}(\mathfrak{w}_0,\ \mathfrak{w}_1).
\end{aligned}$$

Consider a natural number m and n with $m \succ n$, for each $n \in N$, we have

$$\begin{aligned}
\mathfrak{A}(\mathfrak{w}_n,\ \mathfrak{w}_m) &\preceq \mathfrak{A}(\mathfrak{w}_n,\ \mathfrak{w}_{n+1}) + \mathfrak{A}(\mathfrak{w}_{n+1},\ \mathfrak{w}_{n+2}) + \ldots.. + \mathfrak{A}(\mathfrak{w}_{m-1},\ \mathfrak{w}_m)\\
&\preceq \lambda^n\mathfrak{A}(\mathfrak{w}_0,\ \mathfrak{w}_1) + \lambda^{n+1}\mathfrak{A}(\mathfrak{w}_0,\ \mathfrak{w}_1) + \ldots. + \lambda^{m-1}\mathfrak{A}(\mathfrak{w}_0,\ \mathfrak{w}_1)\\
&= (\lambda^n + \lambda^{n+1} + \ldots. + \lambda^{m-1})\mathfrak{A}(\mathfrak{w}_0,\ \mathfrak{w}_1)\\
&\preceq \left(\frac{\lambda^n}{1-\lambda}\right)\mathfrak{A}(\mathfrak{w}_0,\ \mathfrak{w}_1).
\end{aligned}$$

Therefore, we get

$$|\mathfrak{A}(\mathfrak{w}_n,\ \mathfrak{w}_m)| \preceq \left(\frac{\lambda^n}{1-\lambda}\right)|\mathfrak{A}(\mathfrak{w}_0,\ \mathfrak{w}_1)|.$$

Since $\lambda \in [0,\ 1)$, letting the limit as $m,\ n \to 0$ which gives that the $\{\mathfrak{w}_n\}$ is a Cauchy sequence. Therefore, Y is complete, there is a point $a \in Y \ni \mathfrak{w}_n \to a$ as $n \to +\infty$.

To show that $Va = a$. Now,

$$\begin{aligned}
\mathfrak{A}(a,\ Va) &\preceq \mathfrak{A}(a,\ \mathfrak{w}_{2\mathfrak{r}+2}) + \mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ Va)\\
&= \mathfrak{A}(a,\ \mathfrak{w}_{2\mathfrak{r}+2}) + \mathfrak{A}(K\mathfrak{w}_{2\mathfrak{r}+1},\ Va)\\
&= \mathfrak{A}(a,\ \mathfrak{w}_{2\mathfrak{r}+2}) + \mathfrak{A}(Va,\ K\mathfrak{w}_{2\mathfrak{r}+1})\\
&\preceq \mathfrak{A}(a,\ \mathfrak{w}_{2\mathfrak{r}+2}) + \chi(a)[\mathfrak{A}(a,\ \mathfrak{w}_{2\mathfrak{r}+1}) + \mathfrak{A}(a,\ K\mathfrak{w}_{2\mathfrak{r}+1}) + \mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ Va)]\\
&\quad + \xi(a)\left[\frac{\mathfrak{A}(K\mathfrak{w}_{2\mathfrak{r}+1},\ \mathfrak{w}_{2\mathfrak{r}+1})\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ Va) + \mathfrak{A}(a,\ Va)\mathfrak{A}(a,\ K\mathfrak{w}_{2\mathfrak{r}+1})}{1+\mathfrak{A}(a,\ \mathfrak{w}_{2\mathfrak{r}+1})}\right]\\
&= \mathfrak{A}(a,\ \mathfrak{w}_{2\mathfrak{r}+2}) + \chi(a)[\mathfrak{A}(a,\ \mathfrak{w}_{2\mathfrak{r}+1}) + \mathfrak{A}(a,\ \mathfrak{w}_{2\mathfrak{r}+2}) + \mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ Va)]\\
&\quad + \xi(a)\left[\frac{\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+1})\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ Va) + \mathfrak{A}(a,\ Va)\mathfrak{A}(a,\ \mathfrak{w}_{2\mathfrak{r}+2})}{1+\mathfrak{A}(a,\ \mathfrak{w}_{2\mathfrak{r}+1})}\right]
\end{aligned}$$

which implies that

$$\begin{aligned}
|\mathfrak{A}(a,\ Va)| &\preceq |\mathfrak{A}(a,\ \mathfrak{w}_{2\mathfrak{r}+2})| + \chi(a)[|\mathfrak{A}(a,\ \mathfrak{w}_{2\mathfrak{r}+1})| + |\mathfrak{A}(a,\ \mathfrak{w}_{2\mathfrak{r}+2})| + |\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ Va)]|\\
&\quad + \xi(a)\left[\frac{|\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+2},\ \mathfrak{w}_{2\mathfrak{r}+1})||\mathfrak{A}(\mathfrak{w}_{2\mathfrak{r}+1},\ Va)| + |\mathfrak{A}(a,\ Va)||\mathfrak{A}(a,\ \mathfrak{w}_{2\mathfrak{r}+2})|}{|1+\mathfrak{A}(a,\ \mathfrak{w}_{2\mathfrak{r}+1})|}\right].
\end{aligned}$$

As $\mathfrak{r} \to \infty$ we have $|\mathfrak{A}(a,\ Va)| = 0$ which shows that $\mathfrak{A}(a,\ Va) = 0$. Hence, we get $Va = a$. Similarly, we have $Ka = a$. It follows that a is the common fixed-point of V and K.

Next, to claim that a is a unique common fixed-point of the functions V and K. Let us choose another common fixed-point a_1 that is $a_1 = Va_1 = Ka_1$. It follows from

$$\mathfrak{A}(a,\, a_1) = \mathfrak{A}(Va,\, Ka_1)$$

$$\preceq \chi(a)[\mathfrak{A}(a,\, a_1) + \mathfrak{A}(a,\, Ka_1) + \mathfrak{A}(a_1,\, Va)] + \xi(a)\left[\frac{\mathfrak{A}(Ka_1,\, a_1)\mathfrak{A}(a_1,\, Va) + \mathfrak{A}(a,\, Va)\mathfrak{A}(a,\, Ka_1)}{1 + \mathfrak{A}(a,\, a_1)}\right]$$

$$= \chi(a)\mathfrak{A}(a,\, a_1)$$

$$\preceq \chi(a)|\mathfrak{A}(a,\, a_1)|.$$

Since $\chi(a) \in [0,\, 1)$, we have $|\mathfrak{A}(a,\, a_1)| = 0$. Thus, $a = a_1$ and hence a is only unique common fixed-point of V and K. □

Example 11 Let $Y = [0,\, 1]$. Assume that $(Y,\, \mathfrak{A})$ a complete complex-valued metric space. The functions $V,\, K : Y \to Y$ and $\chi,\, \xi : Y \to [0,\, 1) \ni$ defined as $\mathfrak{A}(\mathfrak{w},\, p) = [(\mathfrak{w} - p) + i(\mathfrak{w} - p)]$ for every $\mathfrak{w},\, p \in Y$, then it can be easily verify that $(Y,\, \mathfrak{A})$ is a complex-valued metric space. By assuming $V\mathfrak{w} = \dfrac{\mathfrak{w}}{3}$, $Kp = \dfrac{p}{3}$ for every $\mathfrak{w},\, p \in Y$, one can easily verify that the maps $V,\, K$ satisfying Theorem 3.1. Hence, unique common fixed-point is 0 in V and K.

Corollary 12 Assume that $(Y,\, \mathfrak{A})$ a complete complex-valued metric space. Let $V,\, K : Y \to Y$ and if the following inequality hold:

$$\mathfrak{A}(V\mathfrak{w},\, Kp) \preceq \alpha\mathfrak{A}(\mathfrak{w},\, p) + \beta\left[\frac{\mathfrak{A}(Kp,\, p)\mathfrak{A}(p,\, V\mathfrak{w}) + \mathfrak{A}(\mathfrak{w},\, V\mathfrak{w})\mathfrak{A}(\mathfrak{w},\, Kp)}{1 + \mathfrak{A}(\mathfrak{w},\, p)}\right]$$

for each $\mathfrak{w},\, p \in Y$ where $\alpha,\, \beta$ are positive reals with $\alpha + \beta \prec 1$. Then, V and K has an unique common fixed-point.

Proof. Using Theorem 3.1, one can prove the above result by taking $\chi(\mathfrak{w}) = \alpha$ and $\xi(\mathfrak{w}) = \beta$. □

Corollary 13 Let $(Y,\, \mathfrak{A})$ be a complete complex-valued metric spaces. The two functions $V : Y \to Y$ and $\chi,\, \xi : Y \to [0,\, 1) \ni$ for each $\mathfrak{w},\, p \in Y$ satisfying the following:

(i) $\chi(V\mathfrak{w}) \preceq \chi(\mathfrak{w})$ and $\xi(V\mathfrak{w}) \preceq \xi(\mathfrak{w})$,

(ii) $(\chi + \xi)(\mathfrak{w}) \preceq 1$,

(iii) $\mathfrak{A}(V\mathfrak{w},\, Vp) \preceq \chi(s)\mathfrak{A}(\mathfrak{w},\, p) + \xi(\mathfrak{w})\left[\dfrac{\mathfrak{A}(Vp,\, p)\mathfrak{A}(p,\, V\mathfrak{w}) + \mathfrak{A}(\mathfrak{w},\, V\mathfrak{w})\mathfrak{A}(\mathfrak{w},\, Vp)}{1 + \mathfrak{A}(\mathfrak{w},\, p)}\right]$.

Then, V has unique fixed-point.

Proof. By utilizing Theorem 3.1, one can prove the result with assuming $V = K$. □

Corollary 14 Assume that $(Y,\, \mathfrak{A})$ a complete complex-valued metric space and the function $V : Y \to Y$ if the condition hold:

$$\mathfrak{A}(V\mathfrak{w},\, Vp) \preceq \alpha\mathfrak{A}(\mathfrak{w},\, p) + \beta\left[\frac{\mathfrak{A}(Vp,\, p)\mathfrak{A}(p,\, V\mathfrak{w}) + \mathfrak{A}(\mathfrak{w},\, V\mathfrak{w})\mathfrak{A}(\mathfrak{w},\, Vp)}{1 + \mathfrak{A}(\mathfrak{w},\, p)}\right]$$

for each $\mathfrak{w},\, p \in Y$ where $\alpha,\, \beta$ are positive reals with $\alpha + \beta \prec 1$. Then, V has a unique fixed-point.

Proof. By using Corollary 3.3, one can prove this result with $\chi(\mathfrak{w}) = \alpha$ and $\xi(\mathfrak{w}) = \beta$. □

4. Applications

The system of Urysohn integral equations has only a uinque common solution. By using Theorem 3.1, we solve the following Urysohn integral equations:

Theorem 15 Let $Y = C([x,\ y],\ R^n)$ where $[x,\ y] \subset R^+$ and $\mathfrak{A}:\ Y \times Y \to C$ is define by

$$\mathfrak{A}(s,\ p) = max_{t\in[x,\ y]}||s(\curlywedge) - p(\curlywedge)||_\infty \sqrt{1+x^2}e^{itan^{-1}x}.$$

Consider the Urysohn integral equations

$$s(\curlywedge) = \int_x^y K_1(\curlywedge,\ v,\ s(v))dv + g(\curlywedge) \tag{1}$$

$$s(\curlywedge) = \int_x^y K_2(\curlywedge,\ v,\ s(v))dv + h(\curlywedge) \tag{2}$$

where $\curlywedge \in [x,\ y] \subset R^+$ and $s,\ g,\ h \in Y$.

Consider $K_1,\ K_2:\ [x,\ y] \times [x,\ y] \times R^n \to R^n$ are $\ni F_o,\ G_o \in Y$ for each $s \in Y$, where

$$F_o(\curlywedge) = \int_x^y K_1(\curlywedge,\ v,\ s(v))dv$$

and

$$G_o(\curlywedge) = \int_o^p K_2(\curlywedge,\ v,\ s(v))dv$$

for each $\curlywedge \in [x,\ y]$.

If there are two mappings $\chi,\ \xi:\ Y \to [0,\ 1) \ni$ for each $s,\ p \in Y$

(i) $\chi(F_o + g) \preceq \chi(s)$ and $\xi(F_o + g) \preceq \xi(s)$,

(ii) $\chi(G_o + h) \preceq \chi(s)$ and $\xi(G_o + h) \preceq \xi(s)$,

(iii) $(\chi + \xi)(s) \preceq 1$,

(iv) $||F_o(\curlywedge) - G_p(\curlywedge) + g(\curlywedge) - h(\curlywedge)||\sqrt{1+x^2}e^{itan^{-1}x} \preceq \chi(s)A(s,\ p)(\curlywedge) + \xi(s)B(s,\ p)(\curlywedge)$, where

$$A(s,\ p)(\curlywedge) = ||s(\curlywedge) - p(\curlywedge)||_\infty \sqrt{1+o^2}e^{i\tan^{-1}o},$$

$$B(s,\ p) = \frac{||F_o(\curlywedge) + g(\curlywedge) - s(\curlywedge)||_\infty ||G_p(\curlywedge) + h(\curlywedge) - p(\curlywedge)||_\infty}{1 + \mathfrak{A}(s,\ p)} \sqrt{1+x^2}e^{i\tan^{-1}x};$$

then the system of integral equations (1) and (2) have unique common solution.

Proof. Easy to verify that $(Y,\ \mathfrak{A})$ is a complex-valued metric space. The two mappings (which are defined already in Theorem 3.1) $V,\ K:\ Y \to Y$ by $Vs(F_o + g)$ and $Ks(G_o + h)$. Then,

$$\mathfrak{A}(Vs,\ Kp) = max_{t\in[x,\ y]}||F_o(\curlywedge) - G_p(\curlywedge) + g(\curlywedge) - h(\curlywedge)||\sqrt{1+x^2}e^{i\tan^{-1}x}$$

$$\mathfrak{A}(s,\ Vs) = max_{t\in[x,\ y]}||F_o(\curlywedge) + g(\curlywedge) - s(\curlywedge)||\sqrt{1+x^2}e^{i\tan^{-1}x}$$

and

$$\mathfrak{A}(p,\ Kp) = max_{t\in[x,\ y]}||G_p(\curlywedge) + g(\curlywedge) - s(\curlywedge)||\sqrt{1+x^2}e^{i\tan^{-1}x}.$$

To seen easily that for each $s,\ p\in Y$, we have
(i) $\chi(Vs) \preceq \chi(s)$ and $\xi(Vs) \preceq \xi(s)$,
(ii) $\chi(Ks) \preceq \chi(s)$ and $\xi(Ks) \preceq \xi(s)$,
(iii) $\mathfrak{A}(Vs,\ Kp) \preceq \chi(s)\mathfrak{A}(s,\ p) + \xi(s)\left[\frac{\mathfrak{A}(Kp,\ p)\mathfrak{A}(p,\ Vs) + \mathfrak{A}(s,\ Vs)\mathfrak{A}(s,\ Kp)}{1+\mathfrak{A}(s,\ p)}\right]$.

By Theorem 3.1, we get that V and K has a common fixed-point. So, there exists a unique point $s\in Y \ni s = Vs = Ko$. Now, we have $s = Vs = F_o + g$ and $s = Ks = G_o + h$, that is

$$s(\curlywedge) = \int_x^y K_1(\curlywedge,\ v,\ s(v))dv + g(\curlywedge)$$

and

$$s(\curlywedge) = \int_x^y K_2(\curlywedge,\ v,\ s(v))dv + h(\curlywedge).$$

Thus, from (1) and (2) the Urysohn integral have a unique common fixed-point. □

5. Conclusion and future scope

In this paper, a generalization about the rational contraction mapping has been proved for common fixed-point results. By using the Urysohn integral equation, we have verified the existence of a unique common fixed-point. By utilizing these contraction mappings analysis, one can analyze qualitative theory and provide applications of fractional-order dynamical systems in the near future. Also, Rao et al. [1] introduced the complex-valued b-metric spaces and proved the common fixed-point theorems which are interesting to study as an open question for our rational contraction mapping under this complex-valued b-metric spaces and also to prove application in Urysohn integral equations.

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Conflict of interest

The authors declare no competing financial interest.

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