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Research Article

The Stability of Periodic Orbits for Kawahara Equation

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Abstract: In the presented work, we investigate the Kawahara equation, a non-integrable partial differential equation that generalizes the classical Korteweg-de Vries (KdV) equation by including a fifth-order derivative. Our study encompasses two key contributions. First, we analyze the traveling wave solutions of the Kawahara equation, focusing on cases where the associated ordinary differential equation (ODE) exhibits complex characteristic exponents. Using Hamiltonian dynamics, we explore the geometry of invariant manifolds arising from non-real characteristic exponents. Second, we extend this analysis by introducing a control-theoretic perspective, modifying the ODE to include a control term. This approach leads to the design and experimental validation of a novel linear observer, demonstrating its stability and potential applications in systems governed by periodic coefficients.

***Keywords*:** high-gain observer, Nonlinear waves, Floquét multipliers, periodic solutions, Kawahara equation

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1. Introduction

The Kawahara equation

$$u_t + uu_x + \mu u_{xxx} + u_{xxxxx} = 0,$$

is a model for weakly nonlinear, dispersive waves for which the third and fifth order dispersion terms both appear. The Kawahara equation, linearized about a background $u = a$, is the constant coefficient PDE given by

$$u_t + au_x + \mu u_{xxx} + u_{xxxxx} = 0, \tag{1}$$

where $a,\ \mu \in \mathbb{R}$ are constants. The Kawahara equation is generally a non-integrable equation. However, in some specific cases, one can solve the equation. The equation (1) accepts periodic solutions, according to [1]. One distinguishes three

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cases $\mu = 0,\ \mu > 0$ and $\mu < 0$. By [2], we can re-scale (1) as $u \to bu,\ t \to b^{-5/4}t,\ x \to b^{-1/4}x$, so we reduce to consider just the cases $\mu = 0,\ -1,\ +1$. Numerical computations of solitary wave solutions of (1) that asymptote to a constant a at infinity were first implemented by Kawahara [3], who observed that the structure of solutions depends on the choice of the parameter μ. For $\mu = 0$ or $\mu = -1$, solitary wave solutions that decay to the constant a at ∞ exist for any velocity $c < a$, whereas for $\mu = +1$, solitary wave solutions can only be found for $c < -1/4 + a$. Further details of solitary wave solutions can be found in the articles [4–7]. The Kawahara equation is a generalization of the KdV equation however possesses properties rather different from the canonical KdV equation

$$u_t + uu_x + \mu u_{xxx} = 0,\ \mu = \pm 1, \tag{2}$$

where the soliton solution is a wave of elevation if $\mu = +1$ and a wave of depression if $\mu = -1$. The KdV equation is integrable, and its solutions can be found using the inverse scattering transform [8]. However, the Kawahara equation is not integrable and a general characterization of solutions is not available, see [9–11].

In this article we present two results concerning the traveling waves solutions of the Kawahara equation. First we give a geometric description of the perturbations of the periodic orbit when we have a complex non-real multiplier. This study is complementary to [2, 12, 13] from this view point. We use some tools and methods from the Hamiltonian dynamics [14]. We prove several stability results for the traveling wave solutions of the Kawahara equation employing Floquét theory of periodic ODE. The second result is to design feedback control system associated to the Kawahara equation. Our results in this case are both theoretical and numerical. We explain how the theory of observers in system theory can be used to study the stability of traveling waves. We provide several numerical simulations and results for the observer in this case. The latter motivation is completely new to the literature of Kawahara and KdV type equations.

Traveling wave solutions of the Kawahara equation have the form $u(x,\ t) = f(\xi),\ \xi = x - ct$, where f is a real smooth function defining the profile of the wave and the speed c, also called the velocity of the traveling wave. Substituting and integrating once over the variable ξ we obtain

$$-cf + \frac{1}{2}f^2 + \mu f'' + f^{(4)} = C, \tag{3}$$

where C is a constant of integration. Multiplying (3) by f and integrating once more leads to the Hamiltonian,

$$\begin{aligned} &H_\mu(y,\ z,\ p,\ q) = pq - \frac{\mu}{2}q^2 - \frac{1}{2}z^2 + \frac{1}{6}y^3 - \frac{c}{2}y^2 - Cy, \\ &y = f,\ z = f'',\ p = f''' + \mu f',\ q = f'. \end{aligned} \tag{4}$$

We obtain the first-order Hamiltonian system

$$y = \partial H/\partial p,\ p = -\partial H/\partial y,\ z = \partial H/\partial q,\ q = -\partial H/\partial z, \tag{5}$$

The 4-th order ODE (3) can be written as a first-order linear system in $\mathbb{R}^4$, with coordinates $(f,\ f',\ f'',\ f''')$.

The Hamiltonian is constant on each solution so that the level sets $H = \text{Const}$ are 3-dimensional hypersurfaces in $\mathbb{R}^4$, and each periodic solution is topologically a circle in one of these surfaces. Thus we may write the equation (3) as a bilinear equation in the form,

$$\dot{\phi} = A.\phi,\ A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ c - x_1 & 0 & -\mu & 0 \end{bmatrix},\ \phi = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix},\ x_1 := f, \tag{6}$$

One can also look at the equation (6) using the matrix flow map $\Phi(\xi)$ (see [2]), and consider,

$$\dot{\Phi} = A.\Phi,\ \Phi(0) = I_4, \tag{7}$$

where I_4 is the 4×4 identity matrix. Then, $M = \Phi(2\pi)$ defines the monodromy matrix, whose four eigenvalues $\{\lambda_i\}$ are the Floquét multipliers of the periodic orbit ϕ. As mentioned before, the equation (6) defines a family of Hamiltonian vector fields (5) in $\mathbb{R}^4$ parametrized by μ, with Hamiltonian H_μ given by the (4). The monodromy matrix M has an eigenvalue $\lambda = +1$ with algebraic multiplicity two. The remaining two nontrivial Floquét multipliers are reciprocal $\lambda,\ 1/\lambda$ of one another. If a nontrivial Floquét multiplier is real, the solution is called hyperbolic, and if it is non-real (i.e., $|\lambda| = 1,\ \lambda \neq \pm 1$) the solution is called elliptic. These properties follow from the fact that the monodromy matrix of an autonomous Hamiltonian system is symplectic [14], see also [9, 15, 16].

It is a general fact in Hamiltonian dynamics, that the integral curves of the equation (6) lie on level surfaces of the Hamiltonian H_μ for each μ. When the Hessian of the Hamiltonian is non-degenerate at the equilibrium point, one may use the first-order differential of the matrix equation as a geometric approximation to study the asymptotic behavior of the integral curves. The first order differential of the equation (6) satisfies a symplectic symmetry of $\mathbb{R}^4$ in suitable coordinates. When the Floquét multipliers are all real, these coordinates can be represented using the formal solutions of the linearized equation. Using Floquét theory one deduces that these solutions fill up the stable and unstable manifolds when considering associated eigenspaces.

In this work, we make two significant contributions to the study of the Kawahara equation. First, we provide a comprehensive geometric analysis of perturbations in the periodic traveling wave solutions by leveraging advanced tools from Hamiltonian dynamics and Floquét theory. This analysis offers precise characterizations of stability conditions under various Floquét multiplier regimes, complementing and extending prior studies. Second, we introduce a novel feedback control framework for the Kawahara equation, employing observer-based system theory to stabilize specific traveling wave solutions. Through a combination of theoretical results and numerical simulations, we demonstrate the efficacy of this approach in managing non-integrable dispersive systems. These contributions not only enhance the theoretical understanding of the Kawahara equation but also pave the way for its practical control and application in complex physical systems.

1.1 *Applications to control theory*

For motivations toward control theory, we shall consider a modified equation obtained by perturbation of the former ODE (6) adding a term containing an input function namely $u(t)$. The input term affects the different parameters of the Kawahara equation. One can interpret this as solving an ODE with periodic coefficients when some extra environmental conditions affecting the system. The control theory for systems with periodic coefficients is interesting on its own. The effect of the input function can appear for various reasons relevant to our system or experiments [17–19]. We formulate a high gain observer for a control extension of equation (6), based on a result of [20, 21] and discuss the asymptotic stability of its corresponding error. A high-gain observer is a kind of equation in feedback control where the gain parameter of the system should be chosen high to achieve stability. We also study the stabilization via the associated closed loop system. A goal here is to look for conditions that make the solutions convergent or asymptotically bounded [22–25].

1.2 *Problem set up*

We explain two problems in the following.

1.2.1 *Periodic orbits with non-real multipliers*

As it was mentioned, the Hamiltonian system (6) has one of the characteristic multipliers equal to one and it appears (always) with multiplicity two at the critical point. The other two appear in pairs $(\lambda,\ 1/\lambda)$. One distinguishes three different cases. When the Floquét multipliers $\lambda,\ 1/\lambda$ are real the periodic orbit has both a two-dimensional unstable and a two-dimensional stable manifolds. In the case of positive multipliers the invariant manifold is topologically a cylinder, whereas in the case of negative multipliers, it is topologically a Mobius strip [4, 5]. In both of these cases, the stable or unstable manifolds can be studied using the formal solutions obtained by Floquét theory. The same question can be asked when there are non-real multipliers.

Problem 1 What is the geometric description of the invariant manifolds of the equation (6) in the case of non-real multipliers? How the stability of periodic orbits can be described in this case? Describe the geometry of invariant manifolds when non-real multipliers exist.

In Case of non-real comples multpliers there do not exist stable and unstable submanifolds. However one still can discuss the stability of periodic orbits.As mentioned before this case is a remaining open case for equation (6).

1.2.2 *Design of observer*

In theory of feedback control systems we add some control input term to the matrix equation (6), in the following form

$$\dot{\phi} = A.\phi + uB\phi, \quad B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_1 & 0 & a_3 & 0 \end{bmatrix}, \quad \phi = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \tag{8}$$

$$\psi = x_1,$$

where the entries a_i are constants and affect the terms of (6) by its weights. The equation (8) is designed by a perturbation of a linear approximation of (6) using an input function. The function $u = u(t)$ is the input control, and we interpret it as a way we can affect the specific terms in the Kawahara equation. The function ψ is called the output of the system. In other words, we are going to evaluate the outcome of the system (8) through the first coordinate. The second problem one may address is the following.

Problem 2 Design a linear observer for the linear part of equation (8), i.e. when A is replaced by its first order linear approximation. Provide an estimated bound for the error dynamics of the associated observer. Answer the same question for the general nonlinear equation (8).

Observers are auxiliary equations used to study the error dynamics in feedback input-output systems. Their design is not unique and depends to the expectations in the experiment. The intent to apply tools from feedback systems to the equation obtained from Kawahara equation allows to study systematically the concepts such as stability and error dynamics.

2. Linear differential equations with periodic coefficients

Consider a system of n linear ordinary differential equations written in the matrix form as

$$\dot{x} = A(t).\, x,\ A(t) = [a_{ij}(t)]_{n\times n},\ x^t = (x_1 \ldots x_n),\ t \in \mathbb{R}, \tag{9}$$

where $x_k = x_k(t)$ are complex-valued functions, and $a_{ij}(t+T) = a_{ij}(t)$ for any $i,\ j$. The number $T > 0$ is called the period of the coefficients of the system (9). We assume that the functions $a_{ij}(t)$ are defined for $t \in \mathbb{R}$ and are measurable and Lebesgue integrable on $[0,\ T]$, with the periodicity condition holding almost everywhere, i.e., that is, $A(t+T) = A(t)$. A solution of (9) is a vector function $x = x(t)$ whose components are absolutely continuous and satisfy (9) almost everywhere. The following theorem is key in the study of differential equation (9).

Theorem 1 [26, 27] Suppose that $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$, are given. Then, a solution $x(t)$ satisfying the condition $x(t_0) = x_0$ exists and is unique.

The equation (9) can be described equivalently by its transition matrix. A matrix $X(t)$, of order n, with absolutely-continuous entries is called the evolution matrix, (or transition matrix) of (9) if almost everywhere on $\mathbb{R}$ one has:

$$\dot{X} = A(t)X,\ X(0) = I, \tag{10}$$

where I is the unit $n \times n$ matrix. The transition matrix $X(t)$ satisfies the relation $X(t+T) = X(t)X(T),\ t \in \mathbb{R}$.

Definition 1 We call the matrix $X(T)$ the monodromy matrix. Its eigenvalues λ_j are called the multipliers of (9). The equation

$$\det[X(T) - \lambda I] = 0 \tag{11}$$

is called the characteristic equation of equation (9).

The transition matrix provides an equivalent way to present the solutions of (9). To every eigenvector v_0 of the monodromy matrix with multiplier λ_0 corresponds a solution $x^{(0)}(t) = X(t)v_0$ of (9) satisfying the condition $x^{(0)}(t+T) = \lambda_0 x^{(0)}(t)$. The following theorem explains the relation between the eigenvalues of the matrix $X(T)$ and the solutions of (9), specifically.

Theorem 2 (Floquét-Lyapunov theorem, [26, 27]) The transition matrix of (9) with T-periodic matrix $A(t)$ can be represented in the form

$$X(t) = F(t)e^{tK}, \tag{12}$$

where K is a constant matrix and $F(t)$ is an absolutely-continuous matrix function, periodic with period T, non-singular for all $t \in \mathbf{R}$, and such that $F(0) = I$. Conversely, if $F(t)$ and K are matrices with the given properties, then the matrix (12) is the transition matrix of (9) with T-periodic matrix $A(t)$.

The matrix K called the indicator matrix, and the matrix function $F(t)$ in the representation (12) are not uniquely determined. If the coefficients $a_{ij}(t)$ are real in (12), then $X(t)$ is a real matrix, but $F(t)$ and K are complex matrices, generally. Having the information of the multipliers of (9) we can write down the solutions in a formal explicit form as follows.

Theorem 3 (Floquét's theorem, [26, 27]) The Fundamental system of solutions of equation (9) splits into subsets of the form

$$x^{(1)}(t) = e^{\lambda t} u_1(t),$$

$$x^{(2)}(t) = e^{\lambda t}[t u_1(t) + u_2(t)],$$

$$\ldots$$

$$x^{(m)}(t) = e^{\lambda t}\left[\frac{t^{m-1}}{(m-1)!} u_1(t) + \cdots + t u_{m-1}(t) + u_m(t)\right],$$

where the $u_j(t)$ are absolutely continuous T-periodic complex-valued functions corresponding to an $(m \times m)$-cell of the Jordan form of K.

The relation between the linear equation $\dot{y} = K.y$ and (9) is structural. In fact, formula (12) implies the following.

Theorem 4 (Lyapunov's theorem, [26, 27]) The equation (9) is reducible to the equation $\dot{y} = K.y$ by means of the change of variable $x = F(t)y$. Let $\lambda_1 \ldots \lambda_n$ be the multipliers of (9) and let K be an arbitrary indicator matrix, that is, $e^{TK} = X(T)$. Then, we have $e^{T\rho_j} = \lambda_j$, $j = 1 \ldots n$.

The eigenvalues $\rho_1 \ldots \rho_n$ of K are called the characteristic exponents of (9). The characteristic exponent ρ can also be defined as the complex number for which (9) has a solution that is representable in the form $x(t) = e^{\rho t} u(t)$, where $u(t)$ is a T-periodic vector-valued function.

We consider two types of generalizing (9). The first one is the equation

$$\dot{x} = A(t)x + f(t), \tag{13}$$

where $A(t)$ and $f(t)$ are both periodic of period T, has periodic solutions when the orthogonality relations $\langle f,\ z_j \rangle = \int_0^T f\overline{z_j} = 0,\ j = 1,\ 2,\ \ldots$ between the function f and the solutions of the adjoint equation $\dot{z} = A^*(t)z$ holds. Under this orthogonality condition, an arbitrary solution of (13) can be written as

$$x(t) = x_0(t) + a_1 y_1(t) + \cdots + a_n y_n(t),$$

where $y_i(t)$ are solutions of the linear part $\dot{y} = A(t)y$. We have the following estimate on the periodic solutions of (13).

Theorem 5 [26, 27] There is a constant $C > 0$, independent of $f(t)$, such that

$$|x(t)| \leq C\left(\int_0^T |f(s)|^2 ds\right)^{1/2},\ t \in [0,\ T]. \tag{14}$$

Another relevant equation to (9) is a bifurcation equation in the form

$$\dot{x} = A(t,\ \varepsilon)x, \tag{15}$$

The multipliers for the new equation can be approximated by the multipliers of the equation (9). One writes

$$A(t,\ \varepsilon) = A_0(t) + A_1(t)\varepsilon + A_2(t)\varepsilon^2 + \ldots, \tag{16}$$

where we assume the series is convergent in the norm. Then the characteristic exponents of the equation (15) can be written as

$$\rho(\varepsilon) = \rho_0 + \rho_1\varepsilon + O(\varepsilon^{1+1/q_1}), \tag{17}$$

where ρ_1 are between the eigenvalues of the matrix $[\langle A_1^{m_j-m_i} a_i,\ b_j\rangle]$, where a_i and b_j are normalized eigenvectors of A_0 and A_0^*, and where m_i's are the multiplicities of corresponding eigenvalues. The number q_1 is the multiplicity of ρ_1. We refer the interested reader to [26–28] for details.

A geometric way to interpret the multipliers associated with the periodic solutions is through the Poincaré map of a transversal section at some point of the orbit. We briefly review this concept below.

Poincaré map: Let X be a vector field on a smooth manifold M with integral given by $F : D \subset M \times \mathbb{R} \to M$, and ϕ a closed orbit of X with period T. A **local transversal section** of X at $x_0 \in M$ is a submanifold $S \subset M$ of codimension one with $x_0 \in S$ and for all $s \in S,\ X(s)$ is not contained in T_*S. A **Poincare map** of ϕ is a mapping $\Theta : W_0 \to W_1$, where:

1. $W_0,\ W_1 \subset S$ are open neighborhoods of $x_0 \in S$, and Θ is a diffeomorphism;
2. There is a continuous function $\delta :\ W \to \mathbb{R}$ such that for all $s \in W_0,\ (s,\ T - \delta(s)) \subset D$, and $\Theta(s) = F(s,\ T - \delta(s))$;
3. If $t \in (0,\ T - \delta(s))$, then $F(s,\ t) \neq W_0$.

By uniqueness of Θ up to local conjugacy, $T_{x_0}\Theta'$ is similar to $T_{x_0}\Theta$, for any other Poincaré map Θ' on a local transversal section at $x_1 \in \phi$. Therefore, the eigenvalues of $T_{x_0}\Theta$ are independent of $x_0 \in \phi$ and the particular section S at x_0. Thus, if ϕ is a closed orbit of X, the characteristic multipliers of X at ϕ are the eigenvalues of $T_{x_0}\Theta$, for any Poincaré map Θ at any $x_0 \in \phi$. Because $F_T^*(X(x_0)) = X(x_0)$, so $T_{x_0}F$, always has an eigenvalue 1 corresponding to the eigenvector $X(x_0)$. The $(n-1)$ remaining eigenvalues (if $dim(M) = n$) are in fact the characteristic multipliers of X at ϕ [14].

Hamiltonian case: Assume $(M,\ w)$ is a **symplectic** manifold, and X_H is the **Hamiltonian** vector field with Hamiltonian H. Then characteristic exponents of X_H at x_0 are defined as the eigenvalues of the linear mapping $X_H'(x_0) \in L(T_{x_0}M,\ T_{x_0}M)$, where $T_{x_0}M$ is symplectic with the form $w(x_0)$. The characteristic exponents of X_H at a critical point $x_0 \in M$ occur in pairs of the same multiplicity of $(\rho,\ -\rho)$. Thus if ρ is a characteristic exponent, so are $\bar{\rho},\ -\rho,\ -\bar{\rho}$, all having the same multiplicity. The exponent zero always has even multiplicity. Moreover, the stable and unstable manifolds of the critical point $x_0 \in M$ have the same dimension, and the center manifold is even dimensional.

Corollary 1 [14] The characteristic multipliers of X_H at a closed orbit $\phi \subset M$ occur in pairs $(\lambda,\ \lambda^{-1})$ of the same multiplicity. If λ is a characteristic multiplier, so are $\bar{\lambda},\ \lambda^{-1},\ \bar{\lambda}^{-1}$ all having the same multiplicity. The multiplier one always occurs with an odd multiplicity of at least one.

Assume the characteristic multipliers are $(1,\ e^{\pm i\rho_1},\ \ldots,\ e^{\pm i\rho_{n-1}})$ for a closed orbit, or $(e^{\pm i\rho_1},\ \ldots,\ e^{\pm i\rho_n})$ for a critical point, where $\rho_i \in [0,\ 2\pi)$, then the real numbers $\{\rho_i/2\pi\}$ are called the frequencies. When the multipliers at a critical point are real one can naturally describe the asymptotic behavior of the solutions. In this case, the description of stable and unstable manifolds is clear. However, when the multipliers are not real, there are no stable and unstable manifolds. The following theorem gives a description of invariant manifolds at the critical point.

Theorem 6 (Lyapunov subcenter stability, [14]) Assume $(M,\ w)$ is a symplectic manifold and $x_0 \in M$ be a critical point of X, where the principal characteristic exponents at x_0 are linearly independent over $\mathbf{Z}$. Then, if $i\rho$ is a characteristic exponent of $x_0,\ (\rho \in \mathbf{R})$, there is a two-dimensional submanifold M_ρ with $x_0 \in M_\rho$ such that:

(i) $T_{x_0}M_\rho$ is the eigenspace corresponding to the characteristic exponents $i\rho$ and $-i\rho$;

(ii) M_ρ is an invariant submanifold of X;

(iii) M_ρ is a union of closed orbits ϕ_r such that there is a diffeomophism $f : M_\rho \to D_1$ (D_1 is the disk of radius one in $\mathbf{R}^2$) with $f(\phi_r)$ a circle of radius r about $0 = f(x_0)$. Moreover, if τ_r denotes the period of ϕ_r, then $lim_{r\to 0}\tau_r = 2\pi/\rho$.

Theorem 6 states a special case of (orbital) $o^{\pm}$-stablity of the periodic orbit ϕ in the sense of the definition below.

Definition 2 We say that the solitary wave $e^{-i\omega t}\phi$ is orbitally stable if for any $\varepsilon > 0$ there is $\delta > 0$ such that for any v_0 with $\|\phi - v_0\| < \delta$ there is a solution $v(t)$ defined for all $t \geq 0$ such that $v(0) = v_0$, and such that this solution satisfies

$$\sup_{t \gg 0} \inf_{s \in \mathbf{R}} \|v(t) - e^{is}\phi\| < \varepsilon. \tag{18}$$

Assume (V, w) is a symplectic vector space and let $H = H_2 + H_3 + \cdots + H_n + \dots$ be a formal power series on V, where H_n is a homogeneous polynomial of degree n.

Definition 3 [14] The formal power series H is said to be in Birkhoff normal form if $L_{X_{H_2}} H = 0$ that is, $L_{X_{H_2}} H_n = 0$ for all $n > 2$. The flow $X_H(x,\ y)$ is given by $\exp(t\ \mathrm{ad}_H)\begin{pmatrix} x \\ y \end{pmatrix}$, where $(x,\ y)$ are symplectic coordinate functions on $(V,\ w)$ and $ad_H f = \{f,\ H\}$ (the symbol $\{.,\ .\}$ is the Poison bracket) for $f \in C^\infty(V,\ \mathbf{R})$. Note that $\mathrm{ad}_H x_i = \partial H/\partial y_i$ and $\mathrm{ad}_H y_i = -\partial H/\partial x_i$, $ad_H(x,\ y) = X_H(x,\ y)$.

Theorem 7 (Birkhoff normal form theorem [14, 29]) Assume $H = H_2 + H_3 + \dots$ is a Hamiltonian on a symplectic vector space. Then, there exists a change of coordinates, which transforms H to a Birkhoff normal form.

3. Traveling waves in Kawahara equation

We propose to describe the stability of the solutions of the parametric equation

$$\dot{\phi} = A.\phi,\ A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ c - x_1 & 0 & -\mu & 0 \end{bmatrix},\ \ \phi = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \tag{19}$$

based on the analysis of its Floquét multipliers. Due to the Hamiltonian structure of the ODE, each periodic solution has a Floquét multiplier $\lambda = 1$, of geometric multiplicity one and algebraic multiplicity 2, and two further multipliers λ and $1/\lambda$, with $\lambda \neq 1$ except at bifurcation points. These values are also eigenvalues of the monodromy matrix.

We shall consider a more general framework, that classifies and reconsiders the equation (19) under some axiomatic assumptions. Any other ODE satisfying the hypothesis will show similar dynamics. The equation (19) falls in the model of an equation of type

$$v_x = f(v,\ \mu),\ v \in \mathbb{R}^4,\ \mu \in \mathbb{R}, \tag{20}$$

where f is some smooth function with the following properties mentioned as a hypothesis. The differential equation (20) presents a family of differential equations parametrized by (depending to) $\mu \in \mathbf{R}$.

Assumption 1 There exists a linear map $\mathscr{R} : \mathbb{R}^4 \to \mathbb{R}^4$ with $\mathscr{R}^2 = 1$ and $\dim \mathrm{Fix}(\mathscr{R}) = 2$, where $\mathrm{Fix}(\mathscr{R})$ is the set of fixed points of $\mathscr{R}$, so that $f(\mathscr{R}v,\ \mu) = -\mathscr{R}f(v,\ \mu)$ for all $(v,\ \mu)$.

The map $\mathscr{R}$ is a reflection of $\mathbb{R}^4$, where f is invariant under its action. In the case of (19), the symmetry $\mathscr{R}$ will be a symplectic reflection of the Hamiltonian systems f parametrized by μ. It follows that the function $\mathscr{R}v(-x)$ satisfies (20) when $v(x)$ does.

Assumption 2 The origin $v = 0$ is a hyperbolic equilibrium of (20). Furthermore, $f_v(0,\ \mu)$ has two eigenvalues with strictly negative real parts and two eigenvalues with strictly positive real parts.

This assumption is generic, and follows from the relation $\mathscr{R} f_v(v,\ \mu) = -f_v(\mathscr{R}v,\ \mu)\mathscr{R}$. The assumption 1 together with 2 implies that the stable and unstable manifolds of $f_v(0,\ \mu)$ are Lagrangians.

Assumption 3 There is a smooth function $H : \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}$ with $H(\mathscr{R}v,\ \mu) = H(v,\ \mu)$ and $\langle H_v(v,\ \mu),\ f(v,\ \mu)\rangle = 0$ for all $(v,\ \mu)$.

The function $H_\mu = H(-,\ \mu)$ is a Hamiltonian for the system corresponding to μ. We normalize H so that $H(0,\ \mu) = 0$. The Assumption 3 indicates that we shall consider a smooth family of Hamiltonian vector fields $f(v,\ \mu)$ with Hamiltonians $H(v,\ \mu)$, depending smoothly on the parameter μ.

Assumption 4 A periodic orbit $\phi(x,\ \mu)$ with non-zero minimal period $l(\mu)$ exists with the following properties

(i) The family $\phi(x,\ \mu)$ depends smoothly on μ,

(ii) $\phi(x,\ \mu)$ is symmetric w.r.t $\mathscr{R}$,

(iii) $\phi(x,\ \mu)$ has zero energy: $H(\phi(x,\ \mu),\ \mu) = 0$, $H_u(\phi(x,\ \mu)) \neq 0$, for all x,

(iv) $\phi(x,\ \mu)$ is hyperbolic, so has two Floquét multiplier at one, and no others on the unit circle.

We shall assume that each of the Hamiltonian systems $H(\mu)$ has a periodic solution $\phi(x,\ \mu)$ depending smoothly on μ, that are preserved by $\mathscr{R}$, and appear as a hyperbolic equilibrium of the Hamiltonian system defined by H_μ. Moreover, one can assume that ϕ lies in the zero-level energy surface. Rescaling the time we can assume that all minimal periods $l(\mu)$ are equal to 2π.

In order to study the asymptotic properties of the solutions near the equilibrium we shall consider the first derivative of the function f with respect to v as a linear approximation. Thus, we consider the equation

$$v_x = f_v(\phi(x,\ \mu),\ \mu)v. \tag{21}$$

In other words the asymptotic behavior of the solutions for the two Hamiltonian systems (20) and (21) is similar. It is not difficult to see that when $v(x)$ is a solution to (21) so is $\mathscr{R}v(-x)$. The following lemma justifies this fact.

Lemma 1 [12, 13] If $v(x) = e^{\rho x}p(x)$ is a solution of (21) for some $\rho \in \mathbb{C}$ and some 2π-periodic function $p(x)$, then so is $\mathscr{R}v(-x)$. In particular, if ρ is a Floquét exponent so is $-\rho$.

In the case that multipliers are real there is a simple explanation of the unstable and stable manifolds near the periodic orbits. In fact, one can use the formal solutions of the (21) to explain the coordinates on these submanifolds. The following theorem explains specific coordinates, namely Fenichel coordinates, which are useful in the case of real multipliers.

Proposition 1 [12, 13] Assume the Floquét multipliers of the equation (21) satisfying assumptions 1-4 are real numbers. Then, here exist coordinates, such that the equation (21) restricted to the zero energy level can be given as follows

$$\begin{aligned}
v_x^c &= 1 + h^c(v,\ \mu)v^s v^u, \\
v_x^s &= -[\rho(\mu) + h_1^s(v,\ \mu)v^s + h_2^s(v,\ \mu)v^u]v^s, \\
v_x^u &= [\rho(\mu) + h_1^u(v,\ \mu)v^s + h_2^u(v,\ \mu)v^u]v^u,
\end{aligned} \tag{22}$$

where $v = (v^c,\ v^s,\ v^u) \in S^1 \times I \times I/\sim$, and $(v^c,\ v^s,\ v^u) \sim (v^c + 2\pi,\ -v^s,\ -v^u)$, $S^1 = [0,\ 4\pi]/\sim$. The symmetry R reverses the solution as $\mathscr{R}(v^c,\ v^s,\ v^u) = (-v^c,\ v^u,\ v^s)$.

We sketch an outline of the idea of the proof, (see [12, 13] for a detailed discussion). One can define the sections

$$\Sigma_{\text{in}} = S^1 \times \{v^s = \delta\} \times I,\ \Sigma_{\text{out}} = S^1 \times I \times \{v^u = \delta\}. \tag{23}$$

The tangent spaces of strongly stable and unstable fibers of $\phi(x,\ \mu)$ are spanned by $p^s(x,\ \mu)$ and $p^u(x,\ \mu)$, respectively. Omitting higher-order correction terms of the invariant manifolds of the periodic orbit suggests defining the transformation

$$v = (v^c,\ v^s,\ v^u) \in S \times I \times I \longmapsto u = \phi(v^c,\ \mu) + v^s p^s(v^c,\ \mu) + v^u p^u(v^c,\ \mu), \tag{24}$$

which maps $(v^c,\ 0,\ 0)$ to the periodic orbit and $(v^c,\ v^s,\ 0)$, and $(v^c,\ 0,\ v^u)$ onto the tangent spaces of the stable and unstable manifolds. Moreover, the calculation

$$\begin{aligned} \mathscr{R}u &= \mathscr{R}\phi(v^c,\ \mu) + v^s \mathscr{R}p^s(v^c,\ \phi) + v^u \mathscr{R}p^u(v^c,\ \phi) \\ &= \phi(-v^c,\ \mu) + v^s p^u(-v^c,\ \phi) + v^u p^s(-v^c,\ \phi) \end{aligned} \tag{25}$$

shows that $\mathscr{R}u$ corresponds to $\mathscr{R}v = (-v^c,\ v^u,\ v^s)$. The coordinates $(v^c,\ v^s,\ v^u)$ are called Fenichel coordinates [30]. Using the Fenichel coordinates we can solve the equation (22), as it is explained in the following lemma.

Lemma 2 (Exchange lemma) [30] Consider the system of differential equations

$$\begin{aligned} v_x^c &= \delta(\phi_0 + h_c(v^c,\ v^s,\ v^u)v^s v^u), \\ v_x^s &= -h^s(v^c,\ v^s,\ v^u)v^s, \\ v_x^u &= h^u(v^c,\ v^s,\ v^u)v^u, \end{aligned} \tag{26}$$

where $(v^c,\ v^s,\ v^u) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ and $\phi_0 = [1,\ 0,\ 0]$. We assume that the functions h^s and h^u are smooth and uniformly bounded and bounded away from 0. Let $\Sigma_1 = \{v^s = r\}$ and $\Sigma_2 = \{v^u = r\}$. Then for small fixed r, sufficiently large x, and any sufficiently small δ and any z there exists a unique solution $p(x)$ with flight time x from Σ_1 to Σ_2 such that

$$\begin{aligned} p(0) &= (z,\ r,\ a_1 e^{-\rho_u x}) + O(e^{-\rho x}(0,\ e^{-\rho_u x},\ 0)), \\ p(x) &= (z + rx\phi_0,\ a_2 e^{-\rho_s x},\ r) + O(e^{-\rho x}(e^{-\rho_s x},\ 0,\ r)). \end{aligned} \tag{27}$$

An application of the Lemma 2 is as follows. Consider the vector field $v_x^c = \delta\phi_0,\ v_x^s = -\rho_s v^s,\ v_x^u = \rho_u v^u$, The exchange lemma justifies using the simple model of the first-order approximation in (26). A comprehensive description and proof of the exchange lemma can be found in [30]. The Lemma 2 plays a crucial role in the bifurcation theory of differential equations.

Theorem 8 [12, 13] Assume equation (21) has only real multipliers, and satisfies the assumptions 1-4 mentioned above. Then, one can solve the equation in the new coordinates (22) as follows. There exists positive constants L_0 and η such that for $L > L_0$ and $\phi_0 \in S^1$ there is a unique solution $v(x)$, also denoted $v(x,\ \phi)$, defined for $x \in [-L,\ L]$ so that $v(-x) \in \Sigma_{\text{in}},\ v(x) \in \Sigma_{\text{out}},\ v^c(0) = \phi_0$. Furthermore, we have

$$v(-x) = \left(\phi_0 + O(e^{-\eta x})\ ,\ r\ ,\ re^{-2\rho(\mu)x}(1 + O(e^{-\eta x}))\right),$$

$$v(x) = \left(\phi_0 + O(e^{-\eta x})\ ,\ re^{-2\rho(\mu)x}(1 + O(e^{-\eta x}))\ ,\ r\right), \tag{28}$$

$$v(0) = \left(\phi_0\ ,\ re^{-2\rho(\mu)x}(1 + O(e^{-\eta x}))\ ,\ re^{-2\rho(\mu)x}(1 + O(e^{-\eta x}))\right).$$

The analysis of the stability of solutions of the equation (21) divides into three cases, which we mention now.

Case 1 The periodic solution $\phi(x,\ \mu)$ has two positive nontrivial Floquét multiplier $e^{\pm 2\pi\rho(\mu)}$ with $\rho(\mu) > 0$ for all μ.

Proposition 2 [12, 13] In case 1 the equation (21) has two nontrivial solutions of the form

$$v^s(x) = e^{-\rho(\mu)x} p^s(x,\ \mu),\ v^u(x) = e^{\rho(\mu)x} p^u(x,\ \mu). \tag{29}$$

Therefore because of symmetry with respect to $\mathscr{R}$ we have $p^u(x,\ \mu) = \mathscr{R} p^s(-x,\ \mu)$. In particular, the local stable and unstable manifolds $W^s(\phi,\ \mu)$ and $W^u(\phi,\ \mu)$ of periodic orbits are diffeomorphic to an annulus.

Using Finchel coordinates [12, 13],

$$v = (v^c,\ v^s,\ v^u) \in S^1 \times I \times I,\ I = [-r,\ r] \tag{30}$$

the stable and unstable manifolds of ϕ correspond to $\{v^u = 0\}$ and $\{v^s = 0\}$. Generically, $W^s(0,\ \mu)$ will intersect Σ_{out} in a one-dimensional curve, and intersections of $W^s(0,\ \mu)$ with $W^u(\phi(x,\ \mu),\ \mu)$ correspond to front solutions. Under some mild non-degeneracy conditions, it was shown in [13] that snaking of symmetric pulses will occur whenever Γ can be represented as a graph $\mu = z(\phi)$ for $\phi \in S^1$. This requires that for each fixed phase ϕ, there is a parameter value μ for which the strong unstable finger of $\phi(x,\ \mu)$ intersects the stable manifold of the origin, [12, 13].

Case 2 The periodic orbit $\phi(x,\ \mu)$ has two non-trivial negative Floquét multipliers $-e^{\pm 2\pi\rho(\mu)}$.

We have a similar result of [12, 13].

Proposition 3 [12, 13] In case 2 the equation (21) has two nontrivial solutions

$$v^s(x) = e^{-\rho(\mu)x} p^s(x,\ \mu),\ v^u(x) = e^{\rho(\mu)x} p^u(x,\ \mu) = e^{\rho(\mu)x} \mathscr{R} p^s(-x,\ \mu), \tag{31}$$

where $p^s(x,\ \mu)$ and $p^u(x,\ \mu)$ are real-valued and 4π-periodic in x and satisfy $p^s(x + 2\pi,\ \mu) = -p^s(x,\ \mu)$ and $p^u(x + 2\pi,\ \mu) = -p^u(x,\ \mu)$.

We can see that the stable and unstable manifolds of the periodic orbit are Mobius bands. When following a solution along the periodic orbit, the trajectory will be on the other side of the manifold after 2π time units due to the half-twist. This is reflected in the rotation of the vectors p^u and p^s as they move along the periodic orbit. We can restrict to $v^s > 0$ and $v^u > 0$. The sets $\{v : v^s = 0\}$ and $\{v : v^c = \phi,\ v^s = 0\}$ correspond, respectively, to the unstable manifold of $\phi(x,\ \mu)$ and the strong unstable fibers $W^{uu}(\phi(x,\ \mu),\ \mu)$, and analogously for the stable manifold and the strong stable fibers. The variables v^c parametrize a double cover of the periodic orbit $\phi(x,\ \mu)$. The sections Σ_{in} and Σ_{out} are defined via

$$\Sigma_{\text{in}} = S^1 \times \{v^s = r\} \times I, \ \ \Sigma_{\text{out}} = S^1 \times I \times \{v^u = r\}.$$

One can track solutions that enter a neighborhood of the periodic orbit near a back that lies in the intersection of the unstable manifold $W^u(0,\ \mu)$ of the equilibrium and a strong stable fiber $W^{ss}(\phi(x,\ \mu),\ \mu)$ of the periodic orbit and leave the neighborhood near a front that lies in the intersection of a strong unstable fiber $W^{uu}(\phi(x,\ \mu),\ \mu)$ and the stable manifold $W^s(0,\ \mu)$. Note that, compared to the orientable setup, the only change is that we use $S^1 = [0,\ 4\pi]/\sim$ instead of $S^1 = [0,\ 2\pi]/\sim$ [12, 13].

Case 3 The periodic orbit $\phi(x,\ \mu)$ has two non-real complex Floquét multipliers $e^{\pm i\rho(\mu)}$.

This case appears as one of our main results and we will attend to that in the next section.

4. Stability in case of non-real multipliers-main result

In this section, we attend to the last case mentioned as Case 3 in the previous section, which appears as our new contributions. As mentioned if non-real complex multipliers appear at a periodic orbit, the formalism for the coordinates v^s and v^u will break down in this case. Thus, we must choose other tools that can be applied to the non-real multiplier case. Still, the formal solutions can be given by Floquét theory. We explain the geometric picture for the invariant manifolds using some classical methods in Hamiltonian dynamics [9].

Proposition 4 In Case 3, the equation (21) has two nontrivial solutions

$$v^{(1)}(x) = e^{-i\rho(\mu)x} p^{(1)}(x,\ \mu),\ v^{(2)}(x) = e^{i\rho(\mu)x} p^{(2)}(x,\ \mu),\ \rho(\mu) > 0, \tag{32}$$

with purely imaginary characteristic exponents, where $p^{(1)}(x,\ \mu)$ and $p^{(2)}(x,\ \mu)$ are real-valued and 2π-periodic in x, and satisfy

$$\mathscr{R} e^{i\rho(\mu)x} p^{(2)}(x,\ \mu) = e^{-i\rho(\mu)x} p^{(1)}(x,\ \mu). \tag{33}$$

Proof. This follows from Floquét theory, specifically Theorem 3 in section 2.

The following theorem explains the geometry of the dynamics around a periodic orbit when the orbit has non-real multipliers. The proof employs standard techniques from Hamiltonian dynamics.

Theorem 9 There is a parametrized family of the periodic orbits associated with the non-real exponents that fill up a 2-dimensional C^1-manifold tangent to the eigenspaces corresponding to the characteristic exponents $\pm i\rho$. Moreover, the Poincaré map at the periodic orbit ϕ namely, Θ, is $o^{\pm}$-stable within $\mathbf{R}^4$. The geometry near the periodic orbit is so that the orbit is surrounded by torus-like invariant manifolds.

Proof. The proof of the first claim of the theorem is a standard blow-up technique in Hamiltonian dynamics [14]. We sketch the idea. In this part of the theorem, we restrict ourselves to the eigenspaces associated with the two complex conjugate characteristic exponents $\pm i\rho$. The restricted system is also Hamiltonian in 2-dimension. It follows by the Theorem 6, the periodic orbit ϕ is orbitally stable within its energy surface in 2-dimension. The harder part is to show that $\phi \subset \mathbf{R}^4$ is $o^{\pm}$-stable within $\mathbf{R}^4$, we consider a local transversal section S for ϕ in $\mathbf{R}^4$. Define $\Gamma = \bigcup_\delta \{\phi_\delta; -\varepsilon < \delta < \varepsilon\}$ to be a cylinder of closed orbits through $\phi = \phi_0$, where $H(\phi_\delta) = \delta$. Let Σ_δ, be the level surface containing ϕ_δ, and $S_\delta := S \bigcap \Sigma_\delta$, to be a local transversal section for ϕ_δ. We can arrange $\Theta|_{S_\delta} = \Theta_\delta$ as a Poincaré map on S_δ. The derivatives of Θ_δ, of all orders, are continuous functions of δ. The eigenvalues of Θ_δ, are of the form $e^{\pm i\rho(\delta)}$ for small δ. Thus, ϕ_δ, is $o^{\pm}$-stable for $|\delta|$ small enough. It follows that, ϕ is also $o^{\pm}$-stable, because ϕ is compact. Let $x_0 \in \phi(x,\ \mu)$ and $(V,\ \Psi)$ be a symplectic chart at x_0 so that we have $\Psi : V \longrightarrow \mathbb{R}^4,\ v \mapsto (\delta(v),\ q,\ \varepsilon(v),\ p)$ where $\delta(u)$ is the time along the orbit since the Poincaé section S, and $\varepsilon(v) = \delta_0 - H(v)$, where $H(x_0) = \delta_0,\ \phi \subset \Sigma_0,\ S_\delta = S \cap \Sigma_\delta$. Let $\Theta : (q,\ \varepsilon,\ p) \longmapsto (Q,\ \varepsilon,\ P)$ be the Poincaré map in this chart and set $\pi : S \to S_\delta,\ (0,\ q,\ \varepsilon,\ p) \longmapsto (q,\ p)$. Denote $\psi := \pi \circ \Theta - \pi$ in this chart presentation. One can arrange all these set up so that the map $D\psi : T_{x_0}S \to T_{x_0}S$ is still surjective; i.e. the two maps $\pi,\ \pi \circ \Theta$ have transversal intersections. By the implicit function theorem, there are charts $\alpha_1 : U_1 \times V_1 \to U' \times V',\ \alpha_2 : W \to U'$ where $U_1 \times V_1 \subset S_\delta \times (\delta_0 - \varepsilon,\ \delta_0 + \varepsilon)$ such that $\alpha_2 \circ \psi \circ \alpha_1^{-1}((q,\ p),\ \delta) = (q,\ p)$ and α_2 is a local representative of the Poincaré map. This process defines a local cylinder. Clearly $\bigcup_{\delta \in [0,\ T]} \{\phi_\delta\}$, ($T$ is the period of $\phi(x,\ \mu)$) is diffeomorphic to a torus.

We can construct an analog of the suggested transformation (24) in this case.

Theorem 10 There is an open subset U of $\mathbb{R}^2$ and a smooth mapping $F : U \to \mathbb{R}^2$ of the form

$$F(z) = ze^{i(\rho + \eta|z|^2)} + R(z),\ F(0) = 0,\ |R(z)| \le K|z|^4, \tag{34}$$

for some constant K, such that $DF(0)$ has eigenvalues $e^{\pm i\rho}$, and the total flow map of (21) can be given as $(v^c,\ z) \longmapsto u = \phi(v^c,\ \mu) + F(z)$ in suitable coordinates. The $\mathscr{R}$-symmetry is given as

$$\mathscr{R}u = \phi(-v^c,\ \mu) + \mathscr{R}F(z), \tag{35}$$

where $\mathscr{R}$ switches the Floquét exponents $\pm i\rho$ within their eigenspaces.

Proof. We can assume that the local transversal section at a point x_0 in the periodic orbit ϕ is diffeomorphic to an open subset of $U \subset \mathbb{R}^2$. Then, the claim of the theorem is an application of Birkhoff's normal form for the restriction of the Hamiltonian system on a small open U. Because this restriction is also Hamiltonian, we write the associated Hamiltonian as $H_U = H_2 + H_3 + \ldots$ where H_j is homogeneous and $\deg(H_j) = j$. Then, its flow map is written as

$$F(z) = \exp(t\ \mathrm{ad}_H)(z) = \ ze^{i\rho(z)} + X_H(z) = ze^{i\rho(z)} + X_{H(2)}(z) + \ldots, \tag{36}$$

where $\rho(z)$ denotes the characteristic exponent associated with the periodic orbit passing from $z \in U \subset \mathbb{R}$. Now we note that the terms starting from the second one in the expansion (36) have degrees at least two, and hence are $O(|z|^4)$. The analysis of the function $\rho(z)$ is based on the bifurcation theory of the equation (19). More specifically, to have some estimate of $\rho(z)$, one writes the restriction of the periodic matrix A in the equation (19) to the open set U as

$$A|_U = A_0 + A_1 z + A_2 z^2 + \ldots, \tag{37}$$

as a bifurcation near zero. Again by using the normal form of Hamiltonian H, we have $A_1 = 0$. We know that $DF(0)$ has eigenvalues $e^{i\rho}$. The flow passing through z can be written as

$$\rho(z) = \rho(0) + \eta(0).z + O(z^{1+1/m(\rho(0))}),\ |z| < \delta \tag{38}$$

for δ small enough and m is the multiplicity of the eigenvalue $\rho(0)$ and in our case is equal to one, see equation (17). The factor $\eta(0)$ is an eigenvalue of the matrix $[\langle A_1^{m_i - m_j} v_i,\ v_j^* \rangle]$ where v_i and v_j^* are eigenvectors of A_0 and A_0^* (star denotes the transpose conjugate). In our case because $A_1 = 0$ we have $\eta(0) = 0$. This proves the first part of the theorem. The second part follows easily because the symmetry $\mathscr{R}$ is a symplectic reflection.

As in the case of real multipliers, we formulate the analog of Theorem 8 for the case of non-real multipliers. We need the following lemma, which is the analog of the exchange lemma 2, the proof of which is similar [30].

Lemma 3 (Modified exchange lemma-nonreal multipliers) Consider the following system of differential equations satisfying the Assumptions 1-4 of the previous section,

$$
\begin{aligned}
&v^c_x = \delta(\phi_0 + h(v^c,\ w)w^1w^2),\\
&w^1_x = -h^1(v^c,\ w^1,\ w^2)w^1,\\
&w^2_x = h^2(v^c,\ w^1,\ w^2)w^2,
\end{aligned}
\tag{39}
$$

where $v = (v^c,\ w) \in \mathbb{R}\times\mathbb{R}^2$, $w = (w^1,\ w^2)$, and $\phi_0 = [1,\ 0,\ 0]$ such that the multipliers are conjugate non-real numbers. We assume that the functions h and the entries in the matrix h^1 and h^2 are smooth and uniformly bounded and bounded away from 0. Then for small fixed r, sufficiently large x, and any sufficiently small δ and any z there exists a unique solution $v(x)$ such that

$$
\begin{aligned}
&v(0) = (z,\ r,\ a_1e^{-i\rho x}) + O(e^{-\rho_c x}(0,\ e^{-i\rho x},\ 0)),\\
&v(x) = (z + rx\phi_0,\ a_2e^{i\rho x},\ r) + O(e^{-\rho_c x}(e^{i\rho x},\ 0,\ r)).
\end{aligned}
\tag{40}
$$

Proof. The proof is a consequence of discussion and results in [30]. The existence and uniqueness follow, for instance, from [30], Theorem 4 and Section 5. The expansions follow from [30] Section 5 and [31] Theorem 2.1.

The difference between Lemma 3 and the analogous one in [30] is that in the above lemma ρ is complex. However the formal form of the solution does not change. In this case the integral curve lies on a torus. By applying the Lemma 3 we obtain the following.

Theorem 11 There exists positive constants L_0 and η such that for $L > L_0$ and $\phi_0 \in S^1$ there is a unique solution $v(x)$, also denoted $v(x,\ \phi_0)$, defined for $x \in [-L,\ L]$ so that $v^c(0) = \phi_0$. Furthermore, we have

$$
\begin{aligned}
&v(x) = \Big(\phi_0 + O(e^{-\eta x}),\ re^{-i\rho(\mu)x}(1 + O(e^{-\eta x})),\ re^{+i\rho(\mu)x}(1 + O(e^{-\eta x}))\Big),\\
&v(0) = \Big(\phi_0,\ re^{-i\rho(\mu)x}(1 + O(e^{-\eta x})),\ re^{+i\rho(\mu)x}(1 + O(e^{-\eta x}))\Big).
\end{aligned}
\tag{41}
$$

Proof. The theorem follows from the Lemma 3. The estimates for the solutions in Theorem 11 are written from those in the Lemma 3. The formulas for solutions are written from their formal form obtained by Floquét theory (see also [30] and [31]).

5. Application to control theory-main result

In this section, we design two kinds of observers (in the context of control theory) for some bilinear equation derived from the equation (6). In control theory, we deal with differential equations including some input factors, i.e., an extra term that contains a function, namely, input function. The strategy is we can affect the solution of the equation by changing the input. In this way, one is naturally dealing with an output function depending on the solution too. Another kind of analysis is to associate with the original ODE a variational equation called its observer. An observer is an equation that is naturally constructed from our original ODE and its output. The purpose of the observer is to measure the difference between the output and the real solution of the ODE. Thus in the structure of the observer naturally, some difference between the output and the real solution will appear formally. The choice of the observer equation is not unique, and usually, one can design different kinds of observers for a given control ODE.

5.1 *Design of a high-gain observer-linear case*

In the first example, we explain a special kind of observer that is called a high-gain observer. A high-gain observer is associated with some systems that depend on a parameter namely the "high-gain parameter". It means that when this parameter is sufficiently big enough it will affect all other ingredients of the system to be convergent, see [18, 20, 24, 32]. In the following example, we design a high gain observer for the system (8).

Question 1 Design a high-gain observer for the equation

$$\begin{aligned} \dot{\phi} &= A(0).\phi + uB\phi, \\ \psi &= x_1, \\ A &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ c & 0 & -\mu & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_1 & 0 & a_3 & 0 \end{bmatrix}, \ \phi = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}. \end{aligned} \tag{42}$$

The entries a_i are constants and affect the terms of the equation (6). The function $u = u(t)$ is the input control, and we interpret it as a way we can affect the specific terms in the Kawahara equation. The bilinear systems of the form (42) have been studied in [20], where a high gain observer is designed for this kind of equation. Thus, a high gain observer for system (42) can be given by

$$\dot{\hat{\phi}} = A(0)\hat{\phi} + uB\hat{\phi} + K(\psi - C\hat{\phi}), \tag{43}$$

where $C = [1, 0, 0, 0]$, $K = [k_1, \ldots, k_4]^{\mathrm{T}}$ is a column block matrix of feedback coefficients, and $u(t)$ is bounded [20, 24]. The error $e = \phi - \hat{\phi}$ satisfies

$$\dot{e} = A_0 e + uBe, \tag{44}$$

where

$$A_0 = A(0) - KC = \begin{pmatrix} -k_1 & 1 & 0 & 0 \\ -k_2 & 0 & 1 & 0 \\ -k_3 & 0 & 0 & 1 \\ -k_4 + c & 0 & \mu & 0 \end{pmatrix}. \tag{45}$$

In [20] a standard hierarchy of high-gain observers is stated such that the eigenvalues of the matrix A_0 can be proportional to a high-gain factor (see [20] lemma 1). Specifically, the hierarchy states that if the feedback coefficients k_i are big enough, then the eigenvalues of the matrix A_0 are all real and negative (A_0 is Hurwitz). In this case, the contribution of the coefficients c and μ in the characteristic polynomial of A_0 can be neglected. The hierarchy is to select the gain factors as a function of the parameter θ in the form $k_i(\theta) = \overline{k_i}\theta^i + o(i)$ such that the characteristic polynomial of $A_0 = A(0) - KC$ is given by

$$\rho(s) = |s^4 + k_1(\theta)s^3 + ... + k_4(\theta)|, \tag{46}$$

Then, the spectrum of the matrix A_0 is

$$\begin{aligned} &\mathrm{Sp}(A_0) = \{\theta\alpha_1(\theta), \ \ldots, \ \theta\alpha_4(\theta)\}, \\ &\lim_{\theta\to\infty} \alpha_i(\theta) = \overline{\alpha_i}. \end{aligned} \tag{47}$$

Now, the point is that in this method, one can start from arbitrary real numbers $\overline{\alpha_i}$ and define all the parameters backward. That is, when θ grows, we are able to obtain negative real numbers as the eigenvalues of A_0 proportional to $\overline{\alpha_i}$ by the gain parameter $\theta \gg 0$. The goal here, is to find some bounds on the error dynamics, i.e. the difference between the real solution ϕ and its observer $\hat{\phi}$. Goncharov [20] proves that in general the error $e(t)$ in (44) is exponentially bounded, that is, there exist constants $M, \ a > 0$ and a polynomial $P(\theta)$ such that,

$$\| e \| \leqslant P(\theta)\exp((-(\theta - a)t)), \ (\theta > M). \tag{48}$$

and θ is *the high-gain factor*, see [20, 21, 23, 24].

Example 1 As a specific example, we design a high-gain observer for (42) with $c = 10, \mu = \pm 1$ and $a_1 = a_2 = 1$. We treat the case $\mu = 1$. Notice that we assumed the only available state is $\phi(1) = x_1$. Also, in the equation of the observer (43), we assume $\theta > 0$. Thus we have the following observer [21, 24].

$$\dot{\hat{\phi}} = A(0)\hat{\phi} + B\hat{\phi}u - KC^{\mathsf{T}}(C\hat{\phi} - y), \tag{49}$$

where $y = C\phi$ with $C = (1, \ 0, \ 0, \ 0)$ and

$$K = \begin{pmatrix} 4\theta & 6\theta^2 & 4\theta^3 & \theta^4 \\ 6\theta^2 & 14\theta^3 & 11\theta^4 & 3\theta^5 \\ 4\theta^3 & 11\theta^4 & 10\theta^5 & 3\theta^6 \\ \theta^4 & 3\theta^5 & 3\theta^6 & \theta^7 \end{pmatrix}. \tag{50}$$

To maintain the states bounded, we choose the output-feedback control given by,

$$u = -0.1\hat{\phi}(4) - 20, \tag{51}$$

where $\hat{\phi}(4) = \hat{x}_4$. We have simulated the above systems with the response depicted in Figure 1. The initial condition for the system is $\phi(0) = (1, -1, -2, 1)^{\mathsf{T}}$, with zero initial conditions for the observer, and gain $\theta = 30$. Notice the convergence of errors and the oscillating response in the control input.

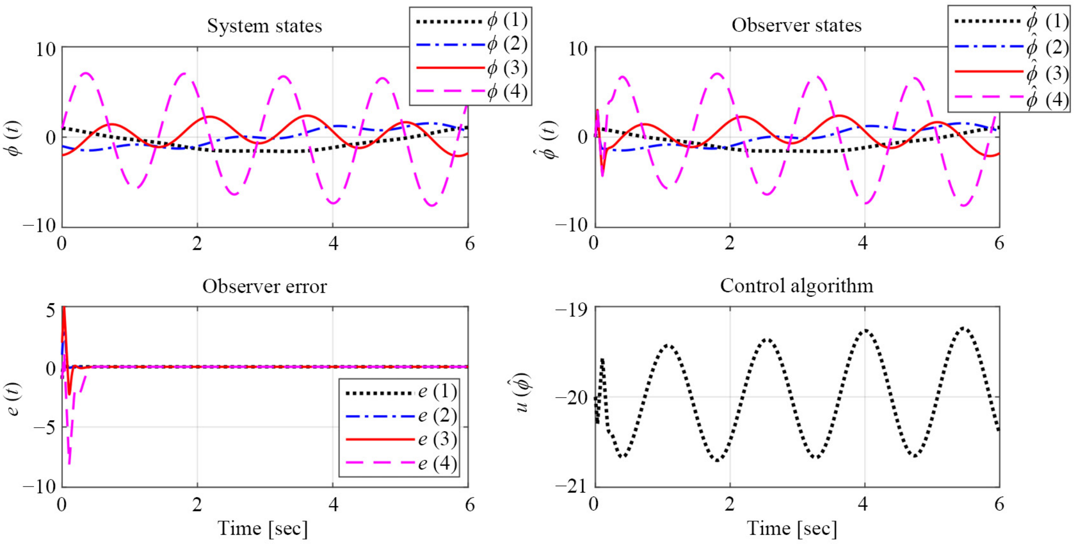

Figure 1. The simulation experiment explores the behavior of the system (42) with observer (49) and an output feedback control (51). Moving from the upper left corner to clockwise, we observe the following distinct components: The response of the system states, characterized by oscillations that remain within bounded limits. The observer states that this enables a comprehensive understanding of the system's operation and allows for extracting valuable information about unobservable quantities. The control algorithm, the ultimate driving force behind the system, leverages the observer's estimations and feedback information to generate control signals. Finally, the error observer, a critical element in the control loop, continuously evaluates the disparity between the estimated states and the actual states of the system

5.2 *Design of observer-nonlinear case*

As another example we design a high gain observer for the nonlinear equation (19), i.e. we consider

$$\begin{aligned} \dot{\phi} &= A.\phi + uB\phi, \\ \psi &= x_1, \end{aligned} \tag{52}$$

where A is as in (19). An observer for this equation is (cf. [21, 24, 33])

$$\dot{\hat{\phi}} = A(\hat{\phi})\hat{\phi} + Bu - \theta\Delta_\theta P(t)C^{\mathrm{T}}(C\hat{\phi} - y), \tag{53}$$

where $\hat{\phi} = [\hat{x_1}, \hat{x_2}, \hat{x_3}, \hat{x_4}]^{\mathrm{T}}$ and $\Delta_\theta = \mathrm{diag}[1, \theta^{-1}, \theta^{-2}, \theta^{-3}]$, and $P(t)$ is a symmetric positive definite solution of the Ricatti equation

$$\dot{P}(t) = \theta[A(\hat{X})P(t) + P(t)A(\hat{X})^{\mathrm{T}} - P(t)C^{\mathrm{T}}CP(t) + P(t)], \tag{54}$$

where $P(0) = P(0)^{\mathrm{T}} > 0$. The value

$$e = C\hat{x} - y = \hat{x} - x, \tag{55}$$

is called the observation error. The variable $\hat{X}$ is called the system estimate. The observer in (53) is of a special type. In the equation (53) the parameter θ is called the gain parameter. One denotes $K(t) := P(t)C^{\mathrm{T}} = [K_1(t),\ K_2(t),\ K_3(t),\ K_3(t)]^{\mathrm{T}}$ called the feedback law. In [21] we show that the error dynamics for the observer (53) also satisfies (48). The primary difference between them lies in the type of system they are applied to and how they handle the dynamics of the system.

5.3 *Closed loop system with periodic input-linear case*

In control dynamics by a suitable change of coordinates the analysis of equation (42) can be equivalently translated to that of the transfer function

$$T(s) = C(sI - A(0))^{-l}B. \tag{56}$$

Here we can assume $B \in M_{4\times m},\ C = [1,\ 0,\ 0,\ 0]$ but $u = u(t) \in \mathbb{R}^m,\ m \leq 4$. The transfer function relates the input vector $u \in U = \mathbb{R}^m$ to the corresponding output $\psi \in Y$, by $\hat{\psi}(x) = T(s)\hat{u}(s)$, without explicit mention of the state $x \in \mathbb{R}^4$, [17, 18, 23, 34]. Now, a feedback law in the linear context is just a linear map $K : Y \to U = \mathbb{R}^m$ and the corresponding closed loop feedback has dynamics given by

$$\dot{\phi} = (A(0) - BKC)\phi + Bu. \tag{57}$$

The instability, or rather stability, question is thus whether

$$\chi(K) = \det(sI - A(0) + BKC) \tag{58}$$

has its roots in the left half plane. The inverse problem is naturally deeper and applicable.

Question 2 Can one find K so that (58) has its roots in the left half plane?

The question 2 is a special case of the classical pole placement problem in control theory. The problem has been deeply studied in classical control theory, see [34, 35] and the references therein. Below we consider a special case applicable to our equation. We sketch the proof for the sake of completeness.

Proposition 5 The above problem is solvable (has an affirmative answer) when the vectors

$$CB,\ CA(0)B,\ \ldots,\ CA(0)^3B$$

are independent in $\mathbb{R}^4$.

Proof. (see [34]), One tries to compute the rank of the derivative of the function $\chi(K) = \det(sI - A(0) + BKC)$ at $0 \in \mathbb{R}^4$. We denote the map χ as a map $\mathbb{R}^m \to \mathbb{R}^4$ by identifying the coordinates in the image with the coefficients of the characteristic polynomial (58). In control theory, it is sufficient to change the coordinates by use of the frequency domain and transmit the above question to a question on the transfer function. First, we write a co-prime factorization for the transfer function (56) as $T(s) = P(s)D(s)^{-1}$. Then, the transfer function of the observer (57) is written as

$$T_K(s) = T(s)(I - KT)^{-1}.$$

It follows that to solve an equation $\chi(K) = p(s)$ is equivalent to solve an equation

$$\det(I - KT(s)) = p(s)/\det D(s), \tag{59}$$

for rational functions. The coefficients in the polynomial $\det(I - KT(s))$ are the characteristic coefficients of the matrix $KT(s)$. Thus, to compute the derivative of the map $\chi(K)$, one should look for the first order approximation of χ at $0 \in \mathbb{R}^4$. By what we explained χ is given to the first order as

$$\chi(K) \sim \mathrm{trace}(KT(s)) =: \langle K,\, T(s)\rangle.$$

Since $T(s)$ is rational, the Jacobian of χ at $0 \in \mathbb{R}^4$ is given by

$$d\chi(K) = [\langle K,\, CA(0)^i B\rangle]_{0 \le i \le 3}. \tag{60}$$

The last formula proves the claim of the proposition.

We make the following assumption.

Assumption 5 The input function $u = u(t) \in \mathbb{R}^m$ is periodic with period T the same as the original matrix A.

Now we are able to prove the following lemma.

Corollary 2 Under the Assumption 5, there exists a constant $C > 0$ independent of "the function $u(t)$ and the choice of the matrix B" such that a general solution of the equation (42) satisfies

$$|\phi(t)| \le C\left(\int_0^T |Bu|^2 ds\right)^{1/2},\ t \in [0,\ T]. \tag{61}$$

A similar bound exists for the equation (57).

Proof. When the assumption 5 is satisfied, the equation (42) falls into the class of a periodic perturbation of (19) and the inequality (61) is a special case of (14), with $f(t) = Bu(t)$.

It follows that under assumption 5 the solutions of (42) are asymptotically bounded near the periodic orbit.

6. Discussion

The Kawahara equation is a nonlinear partial differential equation that models the dynamics of long waves in dispersive systems with higher-order dispersive effects. The interaction between third-order (u_{xxx}) and fifth-order (u_{xxxxx}) dispersion plays a critical role in determining the stability. The relative sign and magnitude of these terms affect the wave shape and the balance between dispersive effects. When the signs of dispersion terms oppose, more complex dynamics and instabilities can emerge. Traveling wave solutions are typically of the form $u(x,\ t) = f(kx - \omega t)$, where f describes the wave shape, k is the wavenumber, and ω is the frequency.

In this work, we have used several analytical and numerical techniques to analyze the stability of periodic solutions, focusing on the growth rate of small perturbations around traveling wave solutions. Floquét multipliers are employed to measure the growth or decay of perturbations, predicting whether solutions tend to quasiperiodic orbits or diverge. In systems with small perturbations, stable orbits correspond to quasiperiodic motion, while unstable orbits may lead to chaotic trajectories. In nearly integrable Hamiltonian systems, Floquét multipliers identify stable periodic orbits, which are the building blocks of invariant tori described by Kolmogorov-Arnold-Moser theory. Our study confirms that the stability of traveling waves in the Kawahara equation is influenced by several factors:

1. **Relative strength of third-and fifth-order dispersion:** Stability depends on the balance between third-order (u_{xxx}) and fifth-order (u_{xxxxx}) dispersive terms. The relative strength determines whether a wave evolves into stable solutions or destabilizes into chaotic structures.

2. **Initial and boundary conditions:** Stability can be highly sensitive to the imposed initial profile and boundary conditions, which may amplify or dampen perturbations in the traveling wave.

3. **External perturbations and dissipation:** Perturbations or damping terms added to the Kawahara equation can either stabilize or destabilize wave dynamics depending on their interaction with the natural modes. These external effects can be studied through control feedback systems, which can modify and regulate the dynamics of traveling waves.

This study also highlights the application of Floquét multipliers in predicting the long-term behavior of solutions. Floquét theory provides insights into the stability properties of periodic orbits and their transitions to instability through bifurcations. These tools are particularly effective for analyzing Hamiltonian systems with periodic coefficients and complex stability properties.

A comparative analysis between the Kawahara and Korteweg-de Vries (KdV) equations further emphasizes their differences. The KdV equation, being integrable, supports soliton solutions that retain their shape during propagation and interaction due to invariants of motion. In contrast, the Kawahara equation, with its fifth-order dispersive term, introduces oscillatory tails in solutions and greater complexity in their stability characteristics. While the KdV equation is ideal for modeling systems with well-defined stability, the Kawahara equation is better suited for phenomena with higher-order dispersion or breaking events. This distinction makes the Kawahara equation a valuable extension for modeling wave phenomena in scenarios sensitive to dispersive instabilities.

From a control perspective, this paper advances the study of stability by developing feedback stabilization strategies for differential equations. High Gain Observers (HGO) and Riccati-based designs were employed to stabilize traveling waves and ensure convergence to desired states. The methods presented here are broadly applicable and extendable to similar systems in other areas, such as optics and fiber laser systems.

Future research directions include leveraging machine learning techniques to identify reduced-order models for the Kawahara equation that retain the essence of periodic orbit dynamics. For example, neural networks can predict the stability of periodic solutions based on input parameters and initial conditions. Additionally, advanced numerical schemes, such as adaptive time-stepping or spectral methods, can improve the efficiency and accuracy of studying the long-term dynamics of solutions. These efforts aim to enhance the theoretical understanding and expand the applicability of the Kawahara equation to complex physical systems.

7. Conclusions

In this study, we analyzed the stability of periodic solutions of the Kawahara equation, emphasizing cases with complex, non-real Floquét multipliers in the associated ODE. Using a combination of analytical and numerical methods, we established the orbital stability of these solutions, providing asymptotic bounds derived from the Poincaré map. Additionally, we extended the analysis to include control stabilization techniques, where high-gain observers and Riccati-based control methods were employed to stabilize solutions effectively. These contributions offer a significant step forward in understanding the interplay between stability, nonlinear dynamics, and control in higher-order dispersive systems. Our findings not only enhance the theoretical framework of the Kawahara equation but also provide practical insights that can be extended to other nonlinear systems, such as those encountered in optics and fiber lasers.

Future research should delve deeper into numerical approaches, particularly adaptive methods that can address the stiffness introduced by the fifth-order term while maintaining accuracy over long simulations. Additionally, machine learning techniques could be explored to develop reduced-order models that retain the essential dynamics of periodic orbits, enabling faster predictions of stability under varying conditions. Another promising direction lies in the study of parameter sensitivity and bifurcations, where external perturbations, boundary conditions, and dissipation could significantly alter the system's behavior. Finally, advanced control strategies, including the design of boundary feedback and external forcing mechanisms, could be developed to manipulate periodic solutions and further broaden the applicability of the Kawahara equation to complex physical systems. These endeavors will enrich the theoretical understanding and practical relevance of nonlinear dispersive wave models.

Author contributions

Mohammad Reza Rahmati: Conceptualization,Methodology, Formal analysis,Writing-Original Draft.
L. A. Rodriguez-Morales: Validation, Writing-Review & Editing.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] Haragus M, Lombardi E, Scheel A. Spectral stability of wave trains in the Kawahara equation. *Journal of Mathematical Fluid Mechanics*. 2006; 8(4): 482-509. Available from: https://doi.org/10.1007/s00021-005-0185-3.
[2] Sprenger P, Bridges TJ, Shearer M. The Kawahara equation: Traveling wave solutions joining periodic waves. *arXiv:2203.01906*. 2022. Available from: https://doi.org/10.48550/arXiv.2203.01906.
[3] Kawahara T. Oscillatory solitary waves in dispersive media. *Journal of the Physical Society of Japan*. 1972; 33(1): 260-264. Available from: https://doi.org/10.1143/JPSJ.33.260.
[4] Amick CJ, Toland JF. Homoclinic orbits in the dynamic phase-space analogy of an elastic strut. *European Journal of Applied Mathematics*. 1992; 3(2): 97-114. Available from: https://doi.org/10.1017/S0956792500000735.
[5] Champneys A. Homoclinic orbits in reversible systems and their applications in mechanics, fluids and optics. *Physica D: Nonlinear Phenomena*. 1998; 112(1-2): 158-186.
[6] Benilov ES, Grimshaw R, Kuznetsova EP. The generation of radiating waves in a singularly-perturbed Korteweg-de Vries equation. *Physica D: Nonlinear Phenomena*. 1993; 69(3-4): 270-278.
[7] Hunter JK, Scheurle J. Existence of perturbed solitary wave solutions to a model equation for water waves. *Physica D: Nonlinear Phenomena*. 1988; 32(2): 253-268. Available from: https://doi.org/10.1016/0167-2789(88)90054-1.
[8] Ablowitz MJ, Segur H. *Solitons and Inverse Scattering Transform*. Philadelphia: SIAM; 1981.

[9] Alyousef HA, Salas AH, Alharthi MR, El-Tantawy SA. New periodic and localized traveling wave solutions to a Kawahara-type equation: Applications to plasma physics. *Complexity*. 2022; 2022(2): 1-15. Available from: https://doi.org/10.1155/2022/9942267.

[10] Natali F. A note on the stability for Kawahara-KdV type equations. *Applied Mathematics Letters*. 2007; 23(5): 591-596. Available from: https://doi.org/10.1016/j.aml.2010.01.017.

[11] Mancas SC. Traveling wave solutions to Kawahara and related equations. *Differential Equations and Dynamical Systems*. 2016; 27(2): 19-37. Available from: https://doi.org/10.1007/s12591-017-0367-5.

[12] Aougab T, Beck M, Carter P, Desai S, Sandstede B, Stadt M, et al. Isolas versus snaking of localized rolls. *Journal of Dynamics and Differential Equations*. 2019; 31(3): 1199-1222.

[13] Beck M, Knobloch J, Lloyd DJB, Sandstede B, Wagenknecht T. Snakes, ladders, and isolas of localized patterns. *SIAM Journal on Mathematical Analysis*. 2009; 41(3): 936-972. Available from: https://doi.org/10.1137/080713306.

[14] Abraham R, Marsden JE. *Foundations of Mechanics*. 2nd ed. Redwood City, CA: Addison-Wesley Publishing Company, Inc; 1987.

[15] Ye F, Tian J, Zhang X, Jiang C, Ouyang T, Gu Y. All traveling wave exact solutions of the Kawahara equation using the complex method. *Axioms*. 2022; 11(7): 330. Available from: https://doi.org/10.3390/axioms11070330.

[16] Trichtchenko O, Deconinck B, Kollár R. Stability of periodic traveling wave solutions to the Kawahara equation. *SIAM Journal on Applied Dynamical Systems*. 2018; 17(4): 2761-2783. Available from: https://doi.org/10.1137/18M1196121.

[17] Khalil HK. *High-Gain Observers in Nonlinear Feedback Control*. Philadelphia, PA: Society for Industrial and Applied Mathematics; 2017.

[18] Isidori A. *Nonlinear Control Systems*. London: Springer-Verlag; 1995.

[19] Astolfi D, Isidori A, Marconi L, Praly L. Nonlinear output regulation by post-processing internal model for multi-input multi-output systems. *IFAC Proceedings Volumes*. 2013; 46(23): 295-300. Available from: https://doi.org/10.3182/20130904-3-FR-2041.00118.

[20] Goncharov OI. Observer design for bilinear systems of a special form. *Differential Equations*. 2012; 48(12): 1596-1606. Available from: https://doi.org/10.1134/S0012266112120063.

[21] Reza-Rahmati M, Flores G. A nonlinear observer for bilinear systems in block form. *European Journal of Control*. 2023; 70: 100780. Available from: https://doi.org/10.1016/j.ejcon.2023.100780.

[22] Dufour P, Flila S, Hammouri H. Observer design for MIMO non-uniformly observable systems. *IEEE Transactions on Automatic Control*. 2012; 57(2): 511-516. Available from: https://doi.org/10.1109/TAC.2011.2166667.

[23] Astolfi D. *Observers and Robust Output Regulation for Nonlinear Systems*. Ph.D. Thesis. de l'Université de recherche Paris Sciences et Lettres PSL Research University [Paris Sciences et Lettres Research University]; 2016.

[24] Farza M, Ménard T, Ltaief A, Bouraoui I, M'Saad M, Maatoug T. Extended high gain observer design for a class of MIMO non-uniformly observable systems. *Automatica*. 2017; 86: 138-146. Available from: https://doi.org/10.1016/j.automatica.2017.08.002.

[25] Astolfi D, Jungers M, Zaccarian L. Output injection filtering redesign in high-gain observers. In: *2018 European Control Conference*. Limassol, Cyprus: IEEE; 2018. p.1957-1962.

[26] Yakubovich VA, Starzhinskii VM. *Linear Differential Equations with Periodic Coefficients*. USA: John Wiley and Sons, Inc; 1975.

[27] Erugin NP. *Linear Systems of Ordinary Differential Equations With Periodic and Quasi-Periodic Coefficients*. San Diego: Academic Press; 1966.

[28] Shtokalo IZ. *Linear Differential Equations with Variable Coefficients: Criteria of Stability and Unstability of Their Solutions*. India: Hindustan Publishing Corporation; 1961.

[29] Grébert B. Birkhoff normal form and hamiltonian PDEs. *arXiv:math/0604132*. 2006. Available from: https://doi.org/10.48550/arXiv.math/0604132.

[30] Krupa M, Sandstede B, Szmolyan P. Fast and slow waves in the FitzHugh-Nagumo equation. *Journal of Differential Equations*. 1997; 133(1): 49-97. Available from: https://doi.org/10.1006/jdeq.1996.3198.

[31] Schecter S. Exchange lemmas 1: Deng's lemma. *Journal of Differential Equations*. 2008; 245(2): 392-410. Available from: https://doi.org/10.1016/j.jde.2007.08.011.

[32] Gauthier JP, Hammouri H, Othman S. A simple observer for nonlinear systems applications to bioreactors. *IEEE Transactions on Automatic Control*. 1992; 37(6): 875-880. Available from: https://doi.org/10.1109/9.256352.

[33] Farza M, Rodriguez-Mata AE, Robles-Magdaleno J, M'Saad M. A new filtered high gain observer design for the estimation of the components concentrations in a photobioreactor in microalgae culture. *IFAC-PapersOnLine*. 2019; 52(1): 904-909. Available from: https://doi.org/10.1016/j.ifacol.2019.06.177.

[34] Byrnes CI. Algebraic and geometric aspects of the analysis of feedback systems. In: Byrnes CI, Martin CF. (eds). *Geometrical Methods for the Theory of Linear Systems*. Dordrecht: Springer Netherlands; 1980. p.85-124.

[35] Kimura H. Pole assignment by gain output feedback. *IEEE Transactions on Automatic Control*. 1975; 20(4): 509-516. Available from: https://doi.org/10.1109/TAC.1975.1101028.