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Research Article

Generalization and Properties of κ-Intuitionistic Fuzzy Metric Spaces with Applications to Fixed-Point Theorems

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Abstract: This study introduces κ-intuitionistic fuzzy metric spaces, significantly broadening the scope of intuitionistic fuzzy metric spaces. This framework offers greater flexibility and applicability by incorporating multiple parameters (κ) into an intuitionistic fuzzy set. The study explores the properties of κ-intuitionistic fuzzy metric spaces, demonstrating that their topology is first-countable and that the corresponding metric space is Hausdorff. We establish a fixed-point theorem that generalizes and extends existing results for intuitionistic fuzzy metric spaces.

***Keywords*:** κ-intuitionistic fuzzy metric spaces, intuitionistic fuzzy metric space, intuitionistic Banach fixed-point theorem, Hausdorff spaces

MSC: 47H10, 03E72, 46S40, 54H25, 54E35

Abbreviation

FS	Fuzzy Set
IFS	Intuitionistic Fuzzy Set
MS	Metric Space
PMS	Probabilistic Metric Space
FMS	Fuzzy Metric Space
IFM	Intuitionistic Fuzzy Metric
IFMS	Intuitionistic Fuzzy Metric Space
IFCM	Intuitionistic Fuzzy Contraction Mapping

1. Introduction

The mathematical representation of uncertainty and imprecision has evolved significantly since Zadeh [1] introduced the fuzzy set (FS) theory in 1965. This groundbreaking framework revolutionized traditional mathematical concepts by introducing gradual membership assessment, particularly benefiting applications in engineering and natural sciences. The evolution continued with Menger's [2] development of probabilistic metric spaces (PMS), which transformed conventional

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metric spaces by incorporating probabilistic distance measures. In this framework, the distance between elements x and y is represented by a distribution function $\mathscr{M}_{x,\, y}$, where $\mathscr{M}_{x,\, y}(t)$ denotes the probability that their distance is less than t. These distribution functions are characterized as left-continuous, non-decreasing functions from $\mathbb{R}$ to $[0,\ 1]$, satisfying $\inf_{t\in\mathbb{R}} \mathscr{M}(t) = 0$ and $\sup_{t\in\mathbb{R}} \mathscr{M}(t) = 1$. A significant advancement came with Atanassov's [3] introduction of Intuitionistic Fuzzy Sets (IFS), which extended the capabilities of fuzzy set theory. This development led to Park's [4] formulation of Intuitionistic Fuzzy Metric Spaces (IFMS), incorporating Atanassov's concepts. The theoretical foundation was further strengthened by Alaca et al.'s [5] fixed-point theorems and subsequent investigations of Cauchy sequences by various researchers [6–8]. The field continued to expand through contributions to both fuzzy metric spaces [9–11] and their intuitionistic counterparts [12, 13].

Recent developments have significantly broadened the scope of these mathematical structures. Gopal et al. [14] introduced k-FMS, offering enhanced flexibility compared to the classical fuzzy metric spaces proposed by George and Veeramani. Nazeem et al. [15] further advanced this framework by establishing fixed-point theorems for Kannan-type contractions and demonstrating their application to fractional differential equations. The parameter t in fuzzy metrics has proven particularly valuable across diverse fields, including color image filtering [16], perceptual color difference assessment [17], self-similarity measurement [18], and dynamic system equilibrium modeling [19–23]. The evolution of fixed-point theory has paralleled these developments, with significant contributions emerging in various mathematical frameworks. Younis et al. [24] advanced the field by analyzing fixed-point computations in graphical spaces, particularly in elastic beam deformation studies. Ahmad et al. [25] extended this work to bipolar b-metric spaces, while subsequent research by Younis et al. [26] explored Ćirić contractions in bipolar metric spaces. Additional innovations came from Ahmad et al. [27] with their work on double-controlled partial metric spaces and Ahmad's [28] practical applications of fixed points in controlled metric spaces.

Our research introduces κ-Intuitionistic Fuzzy Metric Spaces (κ-IFMS), representing a significant advancement in the field. This framework is motivated by the observation that real-world distance measurements frequently involve multiple parameters and varying degrees of uncertainty. While traditional IFMS effectively models fuzzy distances through single-parameter closeness measures, many practical applications require a more comprehensive approach. Consider international trade relationships: the economic "distance" between nations such as the United States (x) and China (y) encompasses multiple factors, including shipping duration, transportation costs, tariff structures, and regulatory requirements. The κ-IFMS framework, where $\kappa \in \{1,\ 2,\ 3,\ \ldots\}$ represents the parameter count, enables more sophisticated analysis of such multi-dimensional relationships. It simultaneously evaluates multiple criteria while incorporating closeness and non-closeness degrees across various dimensions. This approach provides a more nuanced and realistic representation of complex relationships between elements in the space.

Our work establishes the theoretical foundations of κ-IFMS and investigates contractive mappings within these spaces. Through rigorous analysis, we extend classical fixed-point theorems to this generalized setting, offering new tools for modeling complex systems where multiple parameters and uncertainty play crucial roles. These results have significant implications for both pure mathematics and applied sciences.

This paper structure is as follows: Section 2 presents fundamental concepts of t-norms, t-conorms, and IFMS; Section 3 introduces and develops κ-IFMS and their properties; Section 4 establishes fixed-point theorems for κ-IFMS; and Section 5 provides concluding remarks and future directions.

2. Preliminaries

This section provides the fundamental definition related to IFMS.

Definition 1 Triangular Norms (t-norms) [29] A triangular norm, often denoted as a t-norm, is a binary operation $\bigstar : [0,\ 1] \times [0,\ 1] \to [0,\ 1]$ that fulfills the following properties:

(a) $\bigstar$ satisfies commutativity,

(b) $\bigstar$ is associative,

(c) $\bigstar$ has an identity, i.e., $\acute{p} \bigstar 1 = \acute{p}$ for all $\acute{p} \in [0,\ 1]$,

(d) $\bigstar$ is monotonic, that is, if $\acute{p} \leq \acute{r}$ and $\acute{r} \leq \acute{s}$, then $\acute{p}\bigstar\acute{q} \leq \acute{q}\bigstar\acute{s}$ for all $\acute{p},\ \acute{q},\ \acute{r},\ \acute{s} \in [0,\ 1]$.

If $\bigstar$ is continuous, it is called a continuous t-norm.

Definition 2 Triangular Conorms [29]

A triangular conorm, often denoted as a t-conorm, is a binary operation $\bigstar : [0,\ 1] \times [0,\ 1] \to [0,\ 1]$ that fulfills the following properties:

(a) $\blacklozenge$ satisfies commutativity,

(b) $\blacklozenge$ is associative,

(c) $\blacklozenge$ has an identity, i.e., $\acute{p}\blacklozenge 0 = \acute{p} \forall \acute{p} \in [0,\ 1]$,

(d) $\blacklozenge$ is monotonicity, ie., $\acute{p} \leq \acute{r}$ and $\acute{q} \leq \acute{s}$, then $\acute{p}\blacklozenge\acute{q} \leq \acute{r}\blacklozenge\acute{s}\ \ \forall \acute{p},\ \acute{q},\ \acute{r},\ \acute{s} \in [0,\ 1]$.

If $\blacklozenge$ is continuous, it is called a continuous t-conorm.

Remark 1 (i) For any $\varepsilon_1,\ \varepsilon_2 \in (0,\ 1)$ with $\varepsilon_1 > \varepsilon_2$, there exist $\varepsilon_3,\ \varepsilon_4 \in (0,\ 1)$ such that $\varepsilon_1\bigstar\varepsilon_3 \geq \varepsilon_2$ and $\varepsilon_1 \geq \varepsilon_4\blacklozenge\varepsilon_2$.

(ii) For any $\varepsilon_5 \in (0,\ 1)$, there exist $\varepsilon_6,\ \varepsilon_7 \in (0,\ 1)$ such that $\varepsilon_6\bigstar\varepsilon_6 \geq \varepsilon_5$ and $\varepsilon_6\blacklozenge\varepsilon_7 \leq \varepsilon_5$.

Definition 3 IFMS [14] Let Δ be a non-empty set where $\bigstar$ represents a continuous t-norm, $\blacklozenge$ represents a continuous t-conorm, and $\mathscr{M}$ and $\mathscr{N}$ are FS defined on $\Delta^2 \times (0,\ \infty)$. An ordered 5-tuple IFMS $(\Delta,\ \mathscr{M},\ \mathscr{N},\ \bigstar,\ \blacklozenge)$ is called an IFMS if the ensuing conditions are satisfied $\forall x,\ y \in \Delta$, and $t,\ s > 0$.

(i) $\mathscr{M}(x,\ y,\ t) + \mathscr{N}(x,\ y,\ t) \leq 1$,

(ii) $\mathscr{M}(x,\ y,\ t) > 0$,

(iii) $\mathscr{M}(x,\ y,\ t) = 1 \iff x = y$,

(iv) $\mathscr{M}(x,\ y,\ t) = \mathscr{M}(y,\ x,\ t)$,

(v) $\mathscr{M}(x,\ z,\ s+t) \geq \mathscr{M}(x,\ y,\ s)\bigstar\mathscr{M}(y,\ z,\ t)$,

(vi) $\mathscr{M}(x,\ y,\ .) : (0,\ \infty) \to [0,\ 1]$ is continuous,

(vii) $\mathscr{N}(x,\ y,\ t) < 1$,

(viii) $\mathscr{N}(x,\ y,\ t) = 0 \iff x = y$,

(ix) $\mathscr{N}(x,\ y,\ t) = \mathscr{N}(y,\ x,\ t)$,

(x) $\mathscr{N}(x,\ z,\ t+s) \leq \mathscr{N}(x,\ y,\ t)\blacklozenge\mathscr{N}(y,\ z,\ s)$,

(xi) $\mathscr{N}(x,\ y,\ .) : (0,\ \infty) \to [0,\ 1]$ is continuous.

In this context, an IFMS $(\Delta,\ \mathscr{M},\ \mathscr{N},\ \bigstar,\ \blacklozenge)$, where $\mathscr{M}$ is associated with $\bigstar$ and $\mathscr{N}$ is associated with $\blacklozenge$, is referred to as an IFM on Δ. For any $x,\ y \in \Delta$ and $t > 0$, $\mathscr{M}(x,\ y,\ t)$ and $\mathscr{N}(x,\ y,\ t)$ represent the degree of closeness and degree of non-closeness between x and y with respect to t. The pair $(\mathscr{M},\ \mathscr{N})$ constitutes an IFM on Δ.

Definition 4 κ-FMS [14] Let Δ be a non-empty set with a continuous t-norm $\bigstar$. Let $\mathscr{M}$ be FS defined on $\Delta^2 \times (0,\ \infty)^\kappa$. An ordered triple $(\Delta,\ \mathscr{M},\ \bigstar)$ is said to be a κ-FMS if the following conditions are satisfied: $\forall\ t,\ s > 0$ with $t_1,\ t_2,\ \ldots,\ t_\kappa > 0$,

(κ1) $\mathscr{M}(x,\ y,\ t_1,\ t_2,\ \cdots,\ t_\kappa) + \mathscr{N}(x,\ y,\ t_1,\ t_2,\ \cdots,\ t_\kappa) \leq 1$,

(κ2) $\mathscr{M}(x,\ y,\ t_1,\ t_2,\ \cdots,\ t_\kappa) > 0$,

(κ3) $\mathscr{M}(x,\ y,\ t_1,\ t_2,\ \cdots,\ t_\kappa) = 1 \iff x = y$,

(κ4) $\mathscr{M}(x,\ y,\ t_1,\ t_2,\ \cdots,\ t_\kappa) = \mathscr{M}(y,\ x,\ t_1,\ t_2,\ \cdots,\ t_\kappa)$,

(κ5) for any $\ell \in \{1,\ 2,\ \cdots,\ \kappa\}$, we have

$$\mathscr{M}(x,\ z,\ t_1,\ t_2,\ \cdots,\ t_{\ell-1},\ t+s,\ t_{\ell+1},\ \cdots,\ t_\kappa)$$

$$\geq \mathscr{M}(x,\ y,\ t_1,\ t_2,\ \cdots,\ t_{\ell-1},\ t,\ t_{\ell+1},\ \cdots,\ t_\kappa)\bigstar\mathscr{M}(y,\ z,\ t_1,\ t_2,\ \cdots,\ t_{\ell-1},\ s,\ t_{\ell+1},\ \cdots,\ t_\kappa), \tag{1}$$

(κ6) for every fixed $x,\ y \in \Delta$ the function $\mathscr{M}(x,\ y,\ .) : (0,\ \infty)^\kappa \to [0,\ 1]$ is continuous.

3. Main results

This section developed κ-IFMS and examined several properties substantiating the framework.

Definition 5 κ-IFMS : Let Δ be a non-empty set equipped with a continuous t-norm $\bigstar$ and a continuous t-conorm $\blacklozenge$. Let $\mathscr{M}$ and $\mathscr{N}$ be FS defined on $\Delta^2 \times (0, \infty)^\kappa$. An ordered 5-tuple $(\Delta, \mathscr{M}, \mathscr{N}, \bigstar, \blacklozenge)$ is said to be a κ-IFMS if the following conditions are satisfied: $\forall\, t, s > 0$ with $t_1, t_2, \ldots, t_\kappa > 0$,

(κ1) $\mathscr{M}(x, y, t_1, t_2, \cdots, t_\kappa) + \mathscr{N}(x, y, t_1, t_2, \cdots, t_\kappa) \leq 1$,

(κ2) $\mathscr{M}(x, y, t_1, t_2, \cdots, t_\kappa) > 0$,

(κ3) $\mathscr{M}(x, y, t_1, t_2, \cdots, t_\kappa) = 1 \iff x = y$,

(κ4) $\mathscr{M}(x, y, t_1, t_2, \cdots, t_\kappa) = \mathscr{M}(y, x, t_1, t_2, \cdots, t_\kappa)$,

(κ5) for any $\ell \in \{1, 2, \cdots, \kappa\}$, we have

$$\begin{aligned}&\mathscr{M}(x, z, t_1, t_2, \cdots, t_{\ell-1}, t+s, t_{\ell+1}, \cdots, t_\kappa)\\ &\geq \mathscr{M}(x, y, t_1, t_2, \cdots, t_{\ell-1}, t, t_{\ell+1}, \cdots, t_\kappa) \bigstar \mathscr{M}(y, z, t_1, t_2, \cdots, t_{\ell-1}, s, t_{\ell+1}, \cdots, t_\kappa), \end{aligned} \tag{2}$$

(κ6) for every fixed $x, y \in \Delta$ the function $\mathscr{M}(x, y, .) : (0, \infty)^\kappa \to [0, 1]$ is continuous,

(κ7) $\mathscr{N}(x, y, t_1, t_2, \cdots, t_\kappa) < 1$,

(κ8) $\mathscr{N}(x, y, t_1, t_2, \cdots, t_\kappa) = 0 \iff x = y$,

(κ9) $\mathscr{N}(x, y, t_1, t_2, \cdots, t_\kappa) = \mathscr{N}(y, x, t_1, t_2, \cdots, t_\kappa)$,

(κ10) for any $\ell \in \{1, 2, \cdots, \kappa\}$, we have

$$\begin{aligned}&\mathscr{N}(x, z, t_1, t_2, \cdots, t_{\ell-1}, t+s, t_{\ell+1}, \cdots, t_\kappa)\\ &\leq \mathscr{N}(x, y, t_1, t_2, \cdots, t_{\ell-1}, t, t_{\ell+1}, \cdots, t_\kappa) \blacklozenge \mathscr{N}(y, z, t_1, t_2, \cdots, t_{\ell-1}, s, t_{\ell+1}, \cdots, t_\kappa), \end{aligned} \tag{3}$$

(κ11) for every fixed $x, y \in \Delta$ the function $\mathscr{N}(x, y, .) : (0, \infty)^\kappa \to [0, 1]$ is continuous.

Remark 2 For the special case where $\kappa = 1$, the κ-IFMS simplifies to the IFMS, as defined by Jin Han Park [4].

Example 1 (Induced κ-IFMS) Let (Δ, d) be a metric space, where $\bigstar$ represents the product t-norm, and $\blacklozenge$ stands for the Lukasiewicz t-conorm. Let $\mathscr{M}_d$ and $\mathscr{N}_d$ be FS defined on $\Delta^2 \times (0, \infty)^\kappa$, where $\kappa \in \mathbb{N}$, and $\omega > 0$ according to the expressions:

$$\mathscr{M}(x, y, t_1, t_2, t_3, \cdots, t_\kappa) = \frac{\omega t_1 t_2 t_3, \cdots, t_\kappa}{\omega t_1 t_2 t_3, \cdots, t_\kappa + d(x, y)},$$

$$\mathscr{N}(x, y, t_1, t_2, t_3, \cdots, t_\kappa) = \frac{d(x, y)}{\omega t_1 t_2 t_3, \cdots, t_\kappa + d(x, y)},$$

$\forall x, y \in \Delta$ and $t_1, t_2, \cdots, ., t_\kappa > 0$. Then, the ordered 5-tuple $(\Delta, \mathscr{M}, \mathscr{N}, \bigstar, \blacklozenge)$ constitutes a κ-IFMS.

Remark 3 Example 1 remains valid even when using minimum t-norm and maximum t-conorm.

Example 2 Let Δ be a positive real number, $\bigstar$ defined on the Minimum t-norm and $\blacklozenge$ defined on the maximum t-conorm. Let $\mathscr{M}$ and $\mathscr{N}$ be FS on $\Delta^2 \times (0, \infty)^\kappa$, $\kappa \in \mathbb{N}$ by

$$\mathscr{M}(x, y, t_1, t_2, t_3) = e^{-\frac{|x-y|}{t_1 t_2 t_3}}, \quad \mathscr{N}(x, y, t_1, t_2, t_3) = 1 - e^{-\frac{|x-y|}{t_1 t_2 t_3}},$$

$\forall x, y \in \Delta$ and $t_1, t_2, t_3 > 0$. Then $(\Delta, \mathscr{M}, \mathscr{N}, \bigstar, \blacklozenge)$ is a κ-IFMS. Figure 1 presents the graphical representation of $\mathscr{M}$ and $\mathscr{N}$.

Example 3 Let $\Delta = \mathbb{R}^n$ be the n-dimensional plane, with Euclidean metric d on Δ. assume that 't_1' represents the duration it takes for goods to be transported and 't_2' represents the economic interaction cost required for goods to be transported from point x to point y. then the degree of the closeness of x and y with respect to t_1 and t_2 can be measured by the 2-IFM $\mathscr{M}$ and $\mathscr{N}$, which is FS defined on $X^2 \times (0, \infty)^2$, where $\kappa = 2$, $\bigstar$ represents the product t-norm, and $\blacklozenge$ represents the maximum t-conorm.

$$\mathscr{M}(x, y, t_1, t_2) = \frac{1}{e^{\left(\frac{d(x, y)}{t_1+t_2}\right)}}, \quad \mathscr{N}(x, y, t_1, t_2) = 1 - \frac{1}{e^{\left(\frac{d(x, y)}{t_1+t_2}\right)}}, \qquad \forall x, y \in \Delta, \text{ and } t_1, t_2 > 0.$$

Definition 6 Let $(\Delta, \mathscr{M}, \mathscr{N}, \bigstar, \blacklozenge)$ be a κ-IFMS is said to be ℓ-natural κ-IFMS if there exists a $\ell \in \{1, 2, \cdots, \kappa\}$ then

$$\lim_{t_\ell \to \infty} \mathscr{M}(x, y, t_1, t_2, \cdots, t_\ell, \cdots, t_\kappa) = 1 \text{ and } \lim_{t_\ell \to \infty} \mathscr{N}(x, y, t_1, t_2, \cdots, t_\ell, \cdots, t_\kappa) = 0,$$

$\forall x, y \in \Delta$ and $t_1, t_2, t_3, \cdots, t_\kappa > 0$.

We use the notations $\mathscr{M}(x, y, t_1^\kappa)$ instead of $\mathscr{M}(x, y, t_1, t_2, \cdots, t_\kappa)$ and $\mathscr{N}(x, y, t_1^\kappa)$ instead of $\mathscr{N}(x, y, t_1, t_2, \cdots, t_\kappa)$ for simplicity.

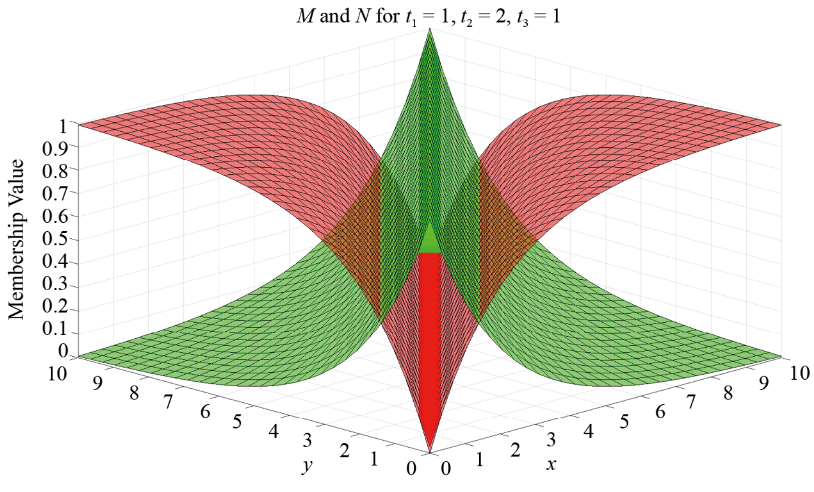

Figure 1. The visual representation illustrates the graphical patterns of $\mathscr{M}$ and $\mathscr{N}$, where t_1, t_2 and t_3 are respectively 1, 2 and 1 and $x, y \in (0, 10]$. The green colour portrays the behaviour of the function $\mathscr{M}$, while the red colour portrays the behaviour of the function $\mathscr{N}$

Lemma 1 Let $(\Delta, \mathscr{M}, \mathscr{N}, \bigstar, \blacklozenge)$ be a κ-IFMS, $t, t_1, t_2, \cdots, t_\kappa > 0$. If $t_\ell < t$ for some $\ell \in \{1, 2, 3, \cdots, \kappa\}$, then the following inequalities hold:

$$\mathscr{M}(x,\ y,\ t_1^\kappa) \le \mathscr{M}(x,\ y,\ t_1,\ t_2,\ \cdots,\ t_{\ell-1},\ t,\ t_{\ell+1},\ \cdots,\ t_\kappa),$$

$$\mathscr{N}(x,\ y,\ t_1^\kappa) \ge \mathscr{N}(x,\ y,\ t_1,\ t_2,\ \cdots,\ t_{\ell-1},\ t,\ t_{\ell+1},\ \cdots,\ t_\kappa).$$

Proof. Utilizing the characteristic of t-norm and $(\kappa 4)$ for every pair of elements x and y in the set Δ, we derive

$$\begin{aligned}\mathscr{M}(x,\ y,\ t_1^\kappa) &= \mathscr{M}(x,\ y,\ t_1^\kappa)\bigstar 1\\ &= \mathscr{M}(x,\ y,\ t_1^\kappa)\bigstar\mathscr{M}(y,\ y,\ t_1,\ t_2,\ \cdots,\ t_{\ell-1},\ t-t_\ell,\ t_{\ell+1},\ \cdots,\ t_\kappa)\\ &\le \mathscr{M}(x,\ y,\ t_1,\ \cdots,\ t_{\ell-1},\ t,\ t_{\ell+1},\ \cdots,\ t_\kappa).\end{aligned}$$

By applying the t-conorm and $(\kappa 11)$ properties to each element $x,\ y$ in Δ, we derive

$$\begin{aligned}\mathscr{N}(x,\ y,\ t_1^\kappa) &= \mathscr{N}(x,\ y,\ t_1^\kappa)\blacklozenge 0\\ &= \mathscr{N}(x,\ y,\ t_1^\kappa)\blacklozenge\mathscr{N}(y,\ y,\ t_1,\ t_2,\ \cdots,\ t_{\ell-1},\ t-t_\ell,\ t_{\ell+1},\ \cdots,\ t_\kappa)\\ &\ge \mathscr{N}(x,\ y,\ t_1,\ \cdots,\ t_{\ell-1},\ t,\ t_{\ell+1},\ \cdots,\ t_\kappa).\end{aligned}$$

□

Remark 4 Let $(\Delta,\ \mathscr{M},\ \mathscr{N},\ \bigstar,\ \blacklozenge)$ be a κ-IFMS, $(\Delta,\ \mathscr{M},\ \mathscr{N},\ \bigstar,\ \blacklozenge)$, if $\mathscr{M}(x,\ y,\ t_1^\kappa) > 1-\varepsilon$, and $\mathscr{N}(x,\ y,\ t_1^\kappa) < \varepsilon$ for all $x,\ y \in \Delta,\ t_1,\ t_2,\ t_3,\ \cdots,\ t_\kappa > 0$, and $\varepsilon \in (0,\ 1)$, then for each $\ell \in \{1,\ 2,\ 3,\ \cdots,\ \kappa\}$, there exists $t \in (0,\ t_\ell)$ such that

$$\mathscr{M}(x,\ y,\ t_1,\ t_2,\ \cdots,\ t_{\ell-1},\ t,\ t_{\ell+1},\ \cdots,\ t_\kappa) > 1-\varepsilon,\ \text{and}\ \mathscr{N}(x,\ y,\ t_1,\ t_2,\ \cdots,\ t_{\ell-1},\ t,\ t_{\ell+1},\ \cdots,\ t_\kappa) < \varepsilon.$$

Definition 7 Let x be a point in κ-IFMS $(\Delta,\ \mathscr{M},\ \mathscr{N},\ \bigstar,\ \blacklozenge)$. For any real number $\varepsilon \in (0,\ 1)$. $\mathfrak{B}(x,\ \varepsilon,\ t_1,\ t_2,\ t_3,\ \cdots,\ t_\kappa) = \{y \in \Delta : \mathscr{M}(x,\ y,\ t_1^\kappa) > 1-\varepsilon$ and $\mathscr{N}(x,\ y,\ t_1^\kappa) < \varepsilon\}$ is a subset of Δ is said to be open ball with centered at $x \in \Delta$ and a radius $\varepsilon \in (0,\ 1)$ with respect to the parameters $t_1,\ t_2,\ t_3,\ \cdots,\ t_\kappa > 0$.

Definition 8 Let $(\Delta,\ \mathscr{M},\ \mathscr{N},\ \bigstar,\ \blacklozenge)$ be a κ-IFMS. Let $\mathfrak{C}$ is an open set of $\Delta \iff$ there is an open ball $\mathfrak{D}$ then $\mathfrak{D} \subset \mathfrak{C}$.

Definition 9 Let x be a point in κ-IFMS $(\Delta,\ \mathscr{M},\ \mathscr{N},\ \bigstar,\ \blacklozenge)$. For any real number $\varepsilon \in (0,\ 1)$. $\mathfrak{B}(x,\ \varepsilon,\ t_1,\ t_2,\ t_3,\ \cdots,\ t_\kappa) = \{y \in \Delta : \mathscr{M}(x,\ y,\ t_1^\kappa) \ge 1-\varepsilon$ and $\mathscr{N}(x,\ y,\ t_1^\kappa) \le \varepsilon\}$ is a subset of Δ is said to be closed ball with centered at $x \in \Delta$ and a radius $\varepsilon \in (0,\ 1)$ with respect to the parameters $t_1,\ t_2,\ t_3,\ \cdots,\ t_\kappa > 0$.

Theorem 1 Let $(\Delta,\ \mathscr{M},\ \mathscr{N},\ \bigstar,\ \blacklozenge)$ be a κ-IFMS. Every open ball is an open set.

Proof. Let $(\Delta,\ \mathscr{M},\ \mathscr{N},\ \bigstar,\ \blacklozenge)$ be a κ-IFMS. Let $x \in \Delta,\ t_1,\ t_2,\ t_3,\ \cdots,\ t_\kappa > 0$, and $\varepsilon \in (0,\ 1)$. Suppose $y \in \mathfrak{B}(x,\ \varepsilon,\ t_1,\ t_2,\ t_3,\ \cdots,\ t_\kappa)$. Therefore $\mathscr{M}(x,\ y,\ t_1^\kappa) > 1-\varepsilon$ and $\mathscr{N}(x,\ y,\ t_1^\kappa) < \varepsilon$.

Consequently, there exist $\ell \in \{1,\ 2,\ 3,\ \cdots,\ \kappa\}$ and $t \in (0,\ t_\ell)$ such that $\varepsilon_0 = \mathscr{M}(x,\ y,\ t_1,\ t_2,\ \cdots,\ t_{\ell-1},\ t_\ell,\ t_{\ell+1},\ \cdots,\ t_\kappa)$. as $\varepsilon_0 > 1-\varepsilon$, by Remark 1, there exists $\varepsilon' \in (0,\ 1)$ such that $\varepsilon_0 > 1-\varepsilon' > 1-\varepsilon$.

Now, given ε_0 and ε' with $\varepsilon_0 > 1-\varepsilon'$, there exist $\varepsilon_1,\ \varepsilon_2 \in (0,\ 1)$ satisfying $\varepsilon_0 \cdot \varepsilon_1 > 1-\varepsilon'$ and $(1-\varepsilon_0)\blacklozenge(1-\varepsilon_2) \leq \varepsilon'$. Where $\varepsilon_3 = max\{\varepsilon_1,\ \varepsilon_2\}$ then the open ball is $\mathfrak{B}(y,\ 1-\varepsilon_3,\ t_1,\ t_2,\ \cdots,\ t_{\ell-1},\ t-t_\ell,\ t_{\ell+1},\ \cdots,\ t_\kappa)$.

We assert that $\mathfrak{B}(y,\ 1-\varepsilon_3,\ t_1,\ t_2,\ \cdots,\ t_{\ell-1},\ t-t_\ell,\ t_{\ell+1},\ \cdots,\ t_\kappa) \subset \mathfrak{B}(x,\ \varepsilon,\ t_1^\kappa)$. To verify this claim, Suppose $z \in \mathfrak{B}(y,\ 1-\varepsilon_3,\ t_1,\ t_2,\ \cdots,\ t_{\ell-1},\ t-t_\ell,\ t_{\ell+1},\ \cdots,\ t_\kappa)$. Then, $\mathscr{M}(y,\ z,\ t_1,\ t_2,\ \cdots,\ t_{\ell-1},\ t-t_\ell,\ t_{\ell+1},\ \cdots,\ t_\kappa) > \varepsilon_3$ and $\mathscr{N}(y,\ z,\ t_1,\ t_2,\ \cdots,\ t_{\ell-1},\ t-t_\ell,\ t_{\ell+1},\ \cdots,\ t_\kappa) < \varepsilon_3$.

$$\begin{aligned}
\mathscr{M}(x,\ z,\ t_1^\kappa) &\geq \mathscr{M}(x,\ y,\ t_1,\ t_2,\ \cdots,\ t_{\ell-1},\ t,\ t_{\ell+1},\ \cdots,\ t_\kappa)\bigstar\mathscr{M}(y,\ z,\ t_1,\ t_2,\ \cdots,\ t_{\ell-1},\ t-t_\ell,\ t_{\ell+1},\ \cdots,\ t_\kappa)\\
&\geq \varepsilon_0\bigstar\varepsilon_3\\
&\geq \varepsilon_0\bigstar\varepsilon_1\\
&\geq 1-\varepsilon'\\
&> 1-\varepsilon.\\
\mathscr{N}(x,\ z,\ t_1^\kappa) &\leq \mathscr{N}(x,\ y,\ t_1,\ t_2,\ \cdots,\ t_{\ell-1},\ t,\ t_{\ell+1},\ \cdots,\ t_\kappa)\blacklozenge\mathscr{N}(y,\ z,\ t_1,\ t_2,\ \cdots,\ t_{\ell-1},\ t-t_\ell,\ t_{\ell+1},\ \cdots,\ t_\kappa)\\
&\leq (1-\varepsilon_0)\blacklozenge(1-\varepsilon_3)\\
&\leq (1-\varepsilon_0)\blacklozenge(1-\varepsilon_2)\\
&\leq \varepsilon'\\
&< \varepsilon.
\end{aligned}$$

$$\therefore z \in \mathfrak{B}(x,\ y,\ t_1^\kappa) \text{ then } \mathfrak{B}(y,\ 1-\varepsilon_3,\ t_1^\kappa - t_0) \subseteq \mathfrak{B}(x,\ \varepsilon,\ t_1^\kappa).$$

□

Based on the aforementioned theorem, we can derive the following corollary:

Corollary 1 Let $(\Delta,\ \mathscr{M},\ \mathscr{N},\ \bigstar,\ \blacklozenge)$ be a κ-IFMS. Let $\tau_{(\mathscr{M},\ \mathscr{N})} = \{A \subseteq \Delta : \forall x \in \Delta$, there exist $t_1,\ t_2,\ t_3,\ \cdots,\ t_\kappa > 0$ and $\varepsilon \in (0,\ 1)$ such that $\mathfrak{B}(x,\ \varepsilon; t_1,\ t_2,\ t_3,\ \cdots,\ t_\kappa) \subseteq A\}$. Then $\tau_{(\mathscr{M},\ \mathscr{N})}$ forms a topology on Δ.

Remark 5 (a) From Theorem 1, and Corollary 1, any κ-IFMS $(\mathscr{M},\ \mathscr{N})$ on Δ where $\tau_{(\mathscr{M},\ \mathscr{N})}$ is an induced topology on Δ. This topology is consisting of open sets, $\{\mathfrak{B}(x,\ \varepsilon : t_1,\ t_2,\ t_3,\ \cdots,\ t_\kappa) :\ x \in \Delta,\ \ \varepsilon \in (0,\ 1),\ \ t > 0\}$.

(b) For $B_x = \left\{\mathfrak{B}(x,\ \frac{1}{n} : t_1,\ t_2,\ t_3,\ \cdots,\ t_\kappa) : n \in \mathbb{N}\right\}$, where $t_1 = t_2 = t_3 =,\ \cdots,\ = t_\kappa = \frac{1}{n}$, forms a local base at a point x. The topology $\tau_{(\mathscr{M},\ \mathscr{N})}$ is a first countable.

Theorem 2 Every κ-IFMS is Hausdorff.

Proof. Let $(\Delta,\ \mathscr{M},\ \mathscr{N},\ \bigstar,\ \blacklozenge)$ be a κ-IFMS.

Let x and y represent two different points in Δ. For any given $t_1,\ t_2,\ t_3,\ \cdots,\ t_\kappa > 0$. We observe that $0 < \mathscr{M}(x,\ y,\ t_1^\kappa) < 1$ and $0 < \mathscr{N}(x,\ y,\ t_1^\kappa) < 1$. Let $\varepsilon_1 = \mathscr{M}(x,\ y,\ t_1^\kappa) \in (0,\ 1)$, $\ \ \varepsilon_2 = \mathscr{N}(x,\ y,\ t_1^\kappa) \in (0,\ 1)$ and $\varepsilon = \max\{\varepsilon_1,\ 1-\varepsilon_2\}$.

For each $\varepsilon_0 \in (\varepsilon,\ 1)$ there exists ε_3 and ε_4 such that $\varepsilon_3\bigstar\varepsilon_3 \geq \varepsilon_0$ and $(1-\varepsilon_4)\blacklozenge(1-\varepsilon_4) \leq 1-\varepsilon_0$.

Put $\varepsilon_5 = \max\{\varepsilon_3,\ \varepsilon_4\}$ and consider the open balls, $\mathfrak{B}(x,\ 1-\varepsilon_5 : t_1,\ t_2,\ t_3,\ \cdots,\ t_{\frac{\ell}{2}},\ \cdots,\ t_\kappa)$ and $\mathfrak{B}(y,\ 1-\varepsilon_5 : t_1,\ t_2,\ t_3,\ \cdots,\ t_{\frac{\ell}{2}},\ \cdots,\ t_\kappa)$. Then clearly $B_{xy} = \mathfrak{B}(x,\ 1-\varepsilon_5 : t_1,\ t_2,\ t_3,\ \cdots,\ t_{\frac{\ell}{2}},\ \cdots,\ t_\kappa) \cap \mathfrak{B}(y,\ 1-\varepsilon_5 : t_1,\ t_2,\ t_3,\ \cdots,\ t_{\frac{\ell}{2}},\ \cdots,\ t_\kappa) = \emptyset$. Assume that $B_{xy} \neq \emptyset$, i.e., there exists $z \in B_{xy}$ then we have

$$\varepsilon_1 = \mathscr{M}(x,\ y,\ t_1^\kappa) \geq \mathscr{M}(x,\ z,\ t_1,\ t_2,\ t_3,\ \cdots,\ t_{\frac{\ell}{2}},\ \cdots,\ t_\kappa) \bigstar \mathscr{M}(z,\ y,\ t_1,\ t_2,\ t_3,\ \cdots,\ t_{\frac{\ell}{2}},\ \cdots,\ t_\kappa)$$

$$\geq \varepsilon_5 \bigstar \varepsilon_5 \geq \varepsilon_3 \bigstar \varepsilon_3 \geq \varepsilon_0 > \varepsilon_1.$$

$$\varepsilon_2 = \mathscr{N}(x,\ y,\ t_1^\kappa) \leq \mathscr{N}(x,\ z,\ t_1,\ t_2,\ t_3,\ \cdots,\ t_{\frac{\ell}{2}},\ \cdots,\ t_\kappa) \blacklozenge \mathscr{N}(z,\ y,\ t_1,\ t_2,\ t_3,\ \cdots,\ t_{\frac{\ell}{2}},\ \cdots,\ t_\kappa)$$

$$\leq (1-\varepsilon_5) \blacklozenge (1-\varepsilon_5) \leq (1-\varepsilon_3) \blacklozenge (1-\varepsilon_3) \leq 1-\varepsilon_0 < \varepsilon_2.$$

Hence it is contradiction, therefore $(\Delta,\ \mathscr{M},\ \mathscr{N},\ \bigstar,\ \blacklozenge)$ is a Hausdorff space. □

Remark 6 A metric space $(\Delta,\ d)$. Define the κ-IFMS $(\mathscr{M},\ \mathscr{N})$ on Δ as follows:

$$\mathscr{M}(x,\ y,\ t_1^\kappa) = \frac{t_1^\kappa}{t_1^\kappa + d(x,\ y)},$$

$$\mathscr{N}(x,\ y,\ t_1^\kappa) = \frac{d(x,\ y)}{h t_1^\kappa + d(x,\ y)} \quad h \in \mathbb{R}^+.$$

Let τ_d be the topology induced by a metric d, which is similar to $\tau_{(\mathscr{M},\ \mathscr{N})}$ is a topology induced by the κ-IFM $(\mathscr{M},\ \mathscr{N})$.

Definition 10 Let $(\Delta,\ \mathscr{M},\ \mathscr{N},\ \bigstar,\ \blacklozenge)$ be a κ-IFMS, if there exist $t_1,\ t_2,\ t_3,\ \cdots,\ t_\kappa > 0$ and $0 < \varepsilon < 1$ such that $\mathscr{M}(x,\ y,\ t_1^\kappa) > 1-\varepsilon$ and $\mathscr{N}(x,\ y,\ t_1^\kappa) < \varepsilon$, $\forall x,\ y \in \Delta$. where $\mathfrak{C}$ is a subset of Δ is called (Intuitionistic Fuzzy-bounded) IF-bounded.

Remark 7 Let a κ-IFMS $(\Delta,\ \mathscr{M},\ \mathscr{N},\ \bigstar,\ \blacklozenge)$ induced by a metric space d on Δ. The subset $A \subset \Delta$ is IF-bounded if and only if it is bounded.

Definition 11 A subset $A \subseteq \Delta$ of a κ-IFMS $(\Delta,\ \mathscr{M},\ \mathscr{N},\ \bigstar,\ \blacklozenge)$ is compact, if every sequence in A has a subsequence that converges to a point within A.

Theorem 3 Let $(\Delta,\ \mathscr{M},\ \mathscr{N},\ \bigstar,\ \blacklozenge)$ be a κ-IFMS. Every compact subset A is IF-bounded.

Proof. Let A be a compact subset of a κ-IFMS. Fix positive values $t_1,\ t_2,\ t_3,\ \cdots,\ t_\kappa$ and $\varepsilon \in (0,\ 1)$. Consider an open cover $\{\mathfrak{B}(x,\ r : t_1,\ t_2,\ t_3,\ \cdots,\ t_\ell,\ \cdots,\ t_\kappa) : x \in A\}$ of A. Since A is compact, there exist $x_1,\ x_2,\ x_3,\ \cdots,\ x_n \in A$ such that $A \subseteq \bigcup_{i=1}^{n} \mathfrak{B}(x_i,\ \varepsilon,\ t)$. For any $x,\ y \in A$, it follows that $x \in \mathfrak{B}(x_i,\ \varepsilon : t_1,\ t_2,\ t_3,\ \cdots,\ t_\ell,\ \cdots,\ t_\kappa)$ for some $i,\ j$.

Thus we have $\mathscr{M}(x,\ x_i,\ t_1^\kappa) > 1-\varepsilon,\ \mathscr{N}(x,\ x_i,\ t_1^\kappa) < \varepsilon,\ \mathscr{M}(y,\ x_j,\ t_1^\kappa) > 1-\varepsilon,\ \mathscr{N}(y,\ x_j,\ t_1^\kappa) < \varepsilon$. let $\alpha = \min\{\mathscr{M}(x_i,\ x_j,\ t_1^\kappa) : 1 \leq i,\ j \leq n\}$, $\beta = \max\{\mathscr{N}(x_i,\ x_j,\ t_1^\kappa) : 1 \leq i,\ j \leq n\}$ then $\alpha > 0,\ \beta > 0$. Now we have

$$\mathscr{M}(x,\ y,\ 3t_1^\kappa) \geq \mathscr{M}(x,\ x_i,\ t_1^\kappa) \bigstar \mathscr{M}(x_i,\ x_j,\ t_1^\kappa) \bigstar \mathscr{M}(x_j,\ y,\ t_1^\kappa)$$

$$\geq (1-\varepsilon) \bigstar (1-\varepsilon) \bigstar \alpha$$

$$> 1-\varepsilon_1^{\cdot},\ \text{ for some } 0 < \varepsilon_1^{\cdot} < 1.$$

$$\mathscr{N}(x,\ y,\ 3t_1^{\kappa}) \leq \mathscr{N}(x,\ x_i,\ t_1^{\kappa}) \blacklozenge \mathscr{N}(x_i,\ x_j,\ t_1^{\kappa}) \blacklozenge \mathscr{N}(x_j,\ y,\ t_1^{\kappa})$$

$$\leq \varepsilon \blacklozenge \varepsilon \blacklozenge \beta$$

$$< \varepsilon_2^{`}, \quad \text{for some } 0 < \varepsilon_2^{`} < 1.$$

Taking $\varepsilon^{`} = \max\{\varepsilon_1^{`},\ \varepsilon_2^{`}\}$ and $t' = 3t_1^{\kappa}$.

We have $\mathscr{M}(x,\ y,\ t') > 1 - \varepsilon^{`}$ and $\mathscr{N}(x,\ y,\ t') < \varepsilon^{`}$ $\forall x,\ y \in A$. Hence, A is IF-bounded. □

Remark 8 From the above Theorem 3, and Remark 7, in a κ-IFMS, every compact set is closed and bounded.

Theorem 4 Let $(\Delta,\ \mathscr{M},\ \mathscr{N},\ \bigstar,\ \blacklozenge)$ be a κ-IFM. Let $\tau_{(M,\ N)}$ be the topology on Δ induced by the κ-IFMS then for a $\{x_n\} \in \Delta$, $x_n \to x \iff \mathscr{M}(x_n,\ x,\ t_1^{\kappa}) \to 1$ and $\mathscr{N}(x_n,\ x,\ t_1^{\kappa}) \to 0$ as $n \to \infty$.

Proof. Fix $t_1,\ t_2,\ t_3,\ \cdots,\ t_\ell,\ \cdots,\ t_\kappa > 0$. Let (x_n) be a real sequence. We say that $x_n \to x$ if for any given $1 > \varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $x_n \in \mathfrak{B}(x,\ r : t_1,\ t_2,\ t_3,\ \cdots,\ t_\ell,\ \cdots,\ t_\kappa)$ $\forall n \geq n_0$, $1 - \mathscr{M}(x_n,\ x,\ t_1^{\kappa}) < \varepsilon$ and $\mathscr{N}(x_n,\ x,\ t_1^{\kappa}) < \varepsilon$ and $\mathscr{M}(x_n,\ x,\ t_1^{\kappa}) \to 1$ and $\mathscr{N}(x_n,\ x,\ t_1^{\kappa}) \to 0$ as $n \to \infty$. Conversely, if for every $t_1,\ t_2,\ t_3,\ \cdots,\ t_\ell,\ \cdots,\ t_\kappa > 0$, $\mathscr{M}(x_n,\ x,\ t_1^{\kappa}) \to 1$ and $\mathscr{N}(x_n,\ x,\ t_1^{\kappa}) \to 0$ as $n \to \infty$. For any $1 > \varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $1 - \mathscr{M}(x_n,\ x,\ t_1^{\kappa}) < \varepsilon$ and $\mathscr{N}(x_n,\ x,\ t_1^{\kappa}) < \varepsilon$, $\forall\ n \geq n_0$. This implies that $\mathscr{M}(x_n,\ x,\ t_1^{\kappa}) > 1 - \varepsilon$ and $\mathscr{N}(x_n,\ x,\ t_1^{\kappa}) < \varepsilon$, $\forall\ n \geq n_0$.

Thus $x \in \mathfrak{B}(x,\ r : t_1,\ t_2,\ t_3,\ \cdots,\ t_\ell,\ \cdots,\ t_\kappa)$, for all $n \geq n_0$, and $x_n \to x$. □

Definition 12 Consider $(\Delta,\ \mathscr{M},\ \mathscr{N},\ \bigstar,\ \blacklozenge)$ as a κ-IFMS. Let x_n in Δ is said convergent, and converging to $x \in \Delta$, iff, for every real number $\varepsilon \in (0,\ 1)$, there exists a natural number n_0 such that $\mathscr{M}(x_n,\ x,\ {t_1}^{\kappa}) > 1 - \varepsilon$, $\mathscr{N}(x_n,\ x,\ {t_1}^{\kappa}) < \varepsilon$, $\forall n > n_0$, and $t_1,\ t_2,\ t_3,\ \cdots,\ t_\kappa > 0$.

Lemma 2 Let $(\Delta,\ \mathscr{M},\ \mathscr{N},\ \bigstar,\ \blacklozenge)$ be a κ-IFMS. A sequence $x_n \in \Delta$ is said to be convergent and converges to $x \in \Delta$ if and only if $lim_{n\to\infty}\mathscr{M}(x_n,\ x,\ {t_1}^{\kappa}) = 1$, and $lim_{n\to\infty}\mathscr{N}(x_n,\ x,\ {t_1}^{\kappa}) = 0$, $\forall\ t_1,\ t_2,\ t_3,\ \cdots,\ t_\kappa > 0$, and $x,\ y \in \Delta$.

Definition 13 Let $(\Delta,\ \mathscr{M},\ \mathscr{N},\ \bigstar,\ \blacklozenge)$ be a κ-IFMS, then a sequence $x_n \in \Delta$ is said to be Cauchy if for $\varepsilon > 0$ and each $t_1,\ t_2,\ t_3,\ \cdots,\ t_n > 0$ and $\exists\ n_0 \in \mathbb{N}$ such that $\mathscr{M}(x_n,\ x_m,\ {t_1}^{\kappa}) > 1 - \varepsilon$ and $\mathscr{N}(x_n,\ x_m,\ {t_1}^{\kappa}) < \varepsilon\ \forall,\ n,\ \mathfrak{m} \geq n_0$.

Definition 14 Let $(\Delta,\ \mathscr{M},\ \mathscr{N},\ \bigstar,\ \blacklozenge)$ be a κ-IFMS, then every Cauchy sequence is convergent with respect to $\tau_{(\mathscr{M},\ \mathscr{N})}$. Then $(\Delta,\ \mathscr{M},\ \mathscr{N},\ \bigstar,\ \blacklozenge)$ is complete κ-IFMS.

Example 4 Let $\Delta = \left\{\dfrac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$ and $\bigstar$ be the continuous t-norm and $\blacklozenge$ be the continuous t-conorm defined by $a \bigstar b = ab$, $a \blacklozenge b = \min\{1,\ a+b\}$ $\forall a,\ b \in [0,\ 1]$ respectively. For any $t_1^{\kappa} \in (0,\ 1)^{\kappa}$ and for any $x,\ y \in \Delta$, Define a FS $\mathscr{M}$ and $\mathscr{N}$ on $\Delta^2 \times (0,\ \infty)^{\kappa}$ by

$$\mathscr{M}(x,\ y,\ t_1^{\kappa}) = \left\{\frac{t_1^{\kappa}}{t_1^{\kappa} + d(x,\ y)},\quad t_1^{\kappa} > 0\right\} \quad \text{and} \quad \mathscr{N}(x,\ y,\ t_1^{\kappa}) = \left\{\frac{d(x,\ y)}{t_1^{\kappa} + d(x,\ y)},\quad t_1^{\kappa} > 0\right\},$$

then $(\Delta,\ \mathscr{M},\ \mathscr{N},\ \bigstar,\ \blacklozenge)$ is a complete κ-IFMS.

Definition 15 Let $(\Delta,\ \mathscr{M},\ \mathscr{N},\ \bigstar,\ \blacklozenge)$ is said to be a complete κ-IFMS if every Cauchy sequence in Δ has convergent subsequence in κ-IFMS.

Theorem 5 Let $(\Delta,\ \mathscr{M},\ \mathscr{N},\ \bigstar,\ \blacklozenge)$ be a κ-IFMS and let $\mathscr{H}$ be a subset of Δ with subspace κ-IFM $\left(\mathscr{M}_{\mathscr{H}},\ \mathscr{N}_{\mathscr{H}}\right) = \left(\mathscr{M}|_{\mathscr{H}^2\times(0,\ \infty)^{\kappa}},\ \mathscr{N}|_{\mathscr{H}^2\times(0,\ \infty)^{\kappa}}\right)$. Then $(\mathscr{H},\ \mathscr{M}_{\mathscr{H}},\ \mathscr{N}_{\mathscr{H}},\ \bigstar,\ \blacklozenge)$ is complete if and only if $\mathscr{H}$ is a closed subset of Δ.

Proof. Let $\mathscr{H}$ be a closed subset of Δ. Let $\{x_n\}$ be a Cauchy sequence in $(\mathscr{H},\ \mathscr{M}_{\mathscr{H}},\ \mathscr{N}_{\mathscr{H}},\ \bigstar,\ \blacklozenge)$. Then $\{x_n\}$ is also a Cauchy sequence in Δ. Since Δ is complete, there exists $x \in \Delta$ such that $x_n \to x$. As $\bar{\mathscr{H}} = \mathscr{H}$ is closed, $x \in \mathscr{H}$. Therefore, $\{x_n\}$ converges in $\mathscr{H}$, proving that $(\mathscr{H},\ \mathscr{M}_{\mathscr{H}},\ \mathscr{N}_{\mathscr{H}},\ \bigstar,\ \blacklozenge)$ is complete.

Suppose $(\mathscr{H},\ \mathscr{M}_{\mathscr{H}},\ \mathscr{N}_{\mathscr{H}},\ \bigstar,\ \blacklozenge)$ is complete. We proceed by contradiction. Assume $\mathscr{H}$ is not closed in Δ. Then there exists $x \in \bar{\mathscr{H}} \setminus \mathscr{H}$. By the definition of closure, there exists a sequence $\{x_n\}$ in $\mathscr{H}$ converging to x. This sequence is Cauchy in $\mathscr{H}$, so for every $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that:

$$\mathscr{M}(x_n,\ x_m,\ t_1^{\kappa}) > 1 - \varepsilon \quad \text{and} \quad \mathscr{N}(x_n,\ x_m,\ t_1^{\kappa}) < \varepsilon$$

for all $n,\ m \geq n_0$. By the completeness of $(\mathscr{H},\ \mathscr{M}_{\mathscr{H}},\ \mathscr{N}_{\mathscr{H}},\ \bigstar,\ \blacklozenge)$, there exists $y \in \mathscr{H}$ such that $x_n \to y$, satisfying:

$$\mathscr{M}_{\mathscr{H}}(y,\ x_n,\ t_1^{\kappa}) > 1 - \varepsilon \quad \text{and} \quad \mathscr{N}_{\mathscr{H}}(y,\ x_n,\ t_1^{\kappa}) < \varepsilon$$

for all $n \geq n_0$. Since $\{x_n\} \subset \mathscr{H}$ and $y \in \mathscr{H}$, we have:

$$\mathscr{M}(y,\ x_n,\ t_1^{\kappa}) = \mathscr{M}_{\mathscr{H}}(y,\ x_n,\ t_1^{\kappa}) \quad \text{and} \quad \mathscr{N}(y,\ x_n,\ t_1^{\kappa}) = \mathscr{N}_{\mathscr{H}}(y,\ x_n,\ t_1^{\kappa}).$$

Thus, $\{x_n\}$ converges to both x and y in $(\Delta,\ \mathscr{M},\ \mathscr{N},\ \bigstar,\ \blacklozenge)$. Since $x \neq y$, this contradicts the uniqueness of limits in a κ-IFMS. Therefore, $\mathscr{H}$ must be a closed subset of Δ. □

4. Fixed point theorems on κ-IFMS

We establish numerous fixed point outcomes within κ-IFMS. To simplify, for any κ-IFMS $(\Delta,\ \mathscr{M},\ \mathscr{N},\ \bigstar,\ \blacklozenge)$, where $\ell \in \{1,\ 2,\ \cdots,\ \kappa\}$, $\alpha > 0$, $x,\ y \in \Delta$, and $t_1,\ t_2,\ \cdots,\ t_\kappa > 0$.

The expressions $\mathscr{M}_{\ell}^{\alpha}(x,\ y,\ t_1^{\kappa})$ and $\mathscr{N}_{\ell}^{\alpha}(x,\ y,\ t_1^{\kappa})$ are used as alternative notations for the more detailed forms $\mathscr{M}(x,\ y,\ t_\ell,\ \cdots,\ t_{\ell-1},\ t_\ell/\alpha,\ t_{\ell+1},\ \cdots,\ t_\kappa)$ and $\mathscr{N}(x,\ y,\ t_\ell,\ \cdots,\ t_{\ell-1},\ t_\ell/\alpha,\ t_{\ell+1},\ \cdots,\ t_\kappa)$, respectively.

Theorem 6 Let $(\Delta,\ \mathscr{M},\ \mathscr{N},\ \bigstar,\ \blacklozenge)$ be a complete κ-IFMS and $\Omega : \Delta \to \Delta$ be a mapping satisfying:

$$\mathscr{M}_{\ell}^{1/\mu}(\Omega x,\ \Omega y,\ t_1^{\kappa}) \geq \mathscr{M}(x,\ y,\ t_1^{\kappa}), \tag{4}$$

$$\mathscr{N}_{\ell}^{1/\mu}(\Omega x,\ \Omega y,\ t_1^{\kappa}) \leq \mathscr{N}(x,\ y,\ t_1^{\kappa}), \tag{5}$$

$\forall x,\ y \in \Delta$, and $t_1,\ t_2,\ \cdots,\ t_\kappa > 0$, $\ell \in \{1,\ 2,\ \cdots,\ \kappa\}$ and $\mu \in (0,\ 1)$ is a constant. Assuming $(\Delta,\ \mathscr{M},\ \mathscr{N},\ \bigstar,\ \blacklozenge)$ is a ℓ-natural κ-IFMS, then Ω has a unique fixed point.

Proof. Assume there exists a unique fixed point for Ω. Suppose $\mathfrak{a}$ and $\mathfrak{b}$ are fixed points of Ω. From (4) and (5)

$$\begin{aligned}
\mathscr{M}(\mathfrak{a},\ \mathfrak{b},\ t_1^{\kappa}) &= \mathscr{M}(\Omega\mathfrak{a},\ \Omega\mathfrak{b},\ t_1^{\kappa}) \\
&\geq \mathscr{M}(\mathfrak{a},\ \mathfrak{b},\ t_\ell,\ \cdots,\ t_{\ell-1},\ t_\ell/\mu,\ t_{\ell+1},\ \cdots,\ t_\kappa) \\
&= \mathscr{M}_{\ell}^{\mu}(\mathfrak{a},\ \mathfrak{b},\ t_1^{\kappa}),
\end{aligned}$$

$$\mathscr{N}(\mathfrak{a},\ \mathfrak{b},\ t_1^{\kappa}) = \mathscr{N}(\Omega\mathfrak{a},\ \Omega\mathfrak{b},\ t_1^{\kappa})$$

$$\leq \mathscr{N}(\mathfrak{a},\ \mathfrak{b},\ t_\ell,\ \cdots,\ t_{\ell-1},\ t_\ell/\mu,\ t_{\ell+1},\ \cdots,\ t_\kappa)$$

$$= \mathscr{N}_\ell^{\mu}(\mathfrak{a},\ \mathfrak{b},\ t_1^{\kappa}).$$

By repeating, we get:

$$\mathscr{M}(\mathfrak{a},\ \mathfrak{b},\ t_1^{\kappa}) \geq \mathscr{M}_\ell^{\mu^n}(\mathfrak{a},\ \mathfrak{b},\ t_1^{\kappa}) \text{ and } \mathscr{N}(\mathfrak{a},\ \mathfrak{b},\ t_1^{\kappa}) \leq \mathscr{N}_\ell^{\mu^n}(\mathfrak{a},\ \mathfrak{b},\ t_1^{\kappa}),\ \forall n \in \mathbb{N}. \tag{6}$$

If $\{c_n\}$ is any sequence with $c_n > 0$ and $\lim_{n\to\infty} c_n = 0$, due to the ℓ-natural property of $(\Delta,\ \mathscr{M},\ \mathscr{N},\ \bigstar,\ \blacklozenge)$:

$$\lim_{n\to\infty} \mathscr{M}_\ell^{c_n}(x,\ y,\ t_1^{\kappa}) = 1 \text{ and } \lim_{n\to\infty} \mathscr{N}_\ell^{c_n}(x,\ y,\ t_1^{\kappa}) = 0, \quad \forall\ t_1,\ t_2,\ \cdots,\ t_\kappa > 0.$$

Using this in (6), we obtain $\mathscr{M}(\mathfrak{a},\ \mathfrak{b},\ t_1^{\kappa}) = 1$ and $\mathscr{N}(\mathfrak{a},\ \mathfrak{b},\ t_1^{\kappa}) = 0, \forall\, t_1,\ t_2,\ \cdots,\ t_\kappa > 0$, i.e., $\mathfrak{a} = \mathfrak{b}$.

Thus, there is only one fixed point of Ω. Choose $x_0 \in \Delta$ and define the iterative sequence $\{x_n\}$ by setting $x_n = \Omega x_{n-1}$, $\forall n \in \mathbb{N}$. Consequently, x_n represents the fixed point of Ω. Suppose $x_n \neq x_{n-1}$, $\forall n \in \mathbb{N}$. Given any $n \in \mathbb{N}$ and $t_1,\ t_2,\ \cdots,\ t_\kappa > 0$

$$\mathscr{M}(x_n,\ x_{n+1},\ t_1^{\kappa}) = \mathscr{M}(\Omega x_{n-1},\ \Omega x_n,\ t_1^{\kappa}) \geq \mathscr{M}(x_{n-1},\ x_n,\ t_\ell,\ \cdots,\ t_{\ell-1},\ t_{\frac{\ell}{\mu}},\ t_{\ell+1},\ \cdots,\ t_\kappa) = \mathscr{M}_\ell^{\mu}(x_{n-1},\ x_n,\ t_1^{\kappa}),$$

$$\mathscr{N}(x_n,\ x_{n+1},\ t_1^{\kappa}) = \mathscr{N}(\Omega x_{n-1},\ \Omega x_n,\ t_1^{\kappa}) \leq \mathscr{N}(x_{n-1},\ x_n,\ t_\ell,\ \cdots,\ t_{\ell-1},\ t_{\frac{\ell}{\mu}},\ t_{\ell+1},\ \cdots,\ t_\kappa) = \mathscr{N}_\ell^{\mu}(x_{n-1},\ x_n,\ t_1^{\kappa}).$$

By repeating this technique, we get $\mathscr{M}(x_n,\ x_{n+1},\ t_1^{\kappa}) \geq \mathscr{M}_\ell^{\mu^n}(x_0,\ x_1,\ t_1^{\kappa})\ \forall n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, $t_1,\ t_2,\ \cdots,\ t_\kappa > 0$ and $p > 0$, we have

$$\mathscr{M}(x_n,\ x_{n+p},\ t_1^{\kappa}) \geq \mathscr{M}(x_n,\ x_{n+1},\ t_\ell,\ \cdots,\ t_{\ell-1},\ t_{\frac{\ell}{2}},\ t_{\ell-1},\ \cdots,\ t_\kappa) \bigstar \mathscr{M}(x_{n+1},\ x_{n+p},\ t_\ell,\ \cdots,\ t_{\ell-1},\ t_{\frac{\ell}{2}},\ t_\ell,\ \cdots,\ t_\kappa)$$

$$\geq \mathscr{M}_\ell^{2}(x_n,\ x_{n+1},\ t_1^{\kappa}) \bigstar \mathscr{M}(x_{n+1},\ x_{n+2},\ t_\ell,\ \cdots,\ t_{\ell-1},\ t_{\frac{\ell}{2^2}},\ t_{\ell-1},\ \cdots,\ t_\kappa)$$

$$\bigstar \mathscr{M}(x_{n+2},\ x_{n+p},\ t_\ell,\ \cdots,\ t_{\ell-1},\ t_{\frac{\ell}{2^2}},\ t_\ell,\ \cdots,\ t_\kappa)$$

$$\geq \mathscr{M}_\ell^{2}(x_n,\ x_{n+1},\ t_1^{\kappa}) \bigstar \mathscr{M}_\ell^{2^2}(x_{n+1},\ x_{n+2},\ t_1^{\kappa}) \bigstar,\ \cdots,\ \bigstar \mathscr{M}_\ell^{2^{p-1}}(x_{n+p-2},\ x_{n+p-1},\ t_1^{\kappa})$$

$$\bigstar \mathscr{M}_\ell^{2^{p-1}}(x_{n+p-1},\ x_{n+p},\ t_1^{\kappa}).$$

$$\mathscr{N}(x_n, x_{n+p}, t_1^\kappa) \leq \mathscr{N}(x_n, x_{n+1}, t_\ell, \cdots, t_{\ell-1}, t_{\frac{\ell}{2}}, t_{\ell-1}, \cdots, t_\kappa) \blacklozenge \mathscr{N}(x_{n+1}, x_{n+p}, t_\ell, \cdots, t_{\ell-1}, t_{\frac{\ell}{2}}, t_\ell, \cdots, t_\kappa)$$

$$\leq \mathscr{N}_\ell^2(x_n, x_{n+1}, t_1^\kappa) \blacklozenge \mathscr{N}(x_{n+1}, x_{n+2}, t_\ell, \cdots, t_{\ell-1}, t_{\frac{\ell}{2^2}}, t_{\ell-1}, \cdots, t_\kappa)$$

$$\blacklozenge \mathscr{N}(x_{n+2}, x_{n+p}, t_\ell, \cdots, t_{\ell-1}, t_{\frac{\ell}{2^2}}, t_\ell, \cdots, t_\kappa)$$

$$\leq \mathscr{N}_\ell^2(x_n, x_{n+1}, t_1^\kappa) \blacklozenge \mathscr{N}_\ell^{2^2}(x_{n+1}, x_{n+2}, t_1^\kappa) \blacklozenge, \cdots,$$

$$\blacklozenge \mathscr{N}_\ell^{2^{p-1}}(x_{n+p-2}, x_{n+p-1}, t_1^\kappa) \blacklozenge \mathscr{N}_\ell^{2^{p-1}}(x_{n+p-1}, x_{n+p}, t_1^\kappa).$$

By using (6), we obtain

$$\mathscr{M}(x_n, x_{n+p}, t_1^\kappa) \geq \mathscr{M}^{2\mu^n}(x_0, x_1, t_1^\kappa) \bigstar, \cdots, \bigstar \mathscr{M}^{2^2\mu^{n+1}}(x_0, x_1, t_1^\kappa) \bigstar \mathscr{M}^{2^{p+1}\mu^{n+p-1}}(x_0, x_1, t_1^\kappa),$$

$$\mathscr{N}(x_n, x_{n+p}, t_1^\kappa) \leq \mathscr{N}_\ell^{2\mu^n}(x_0, x_1, t_1^\kappa) \blacklozenge, \cdots, \blacklozenge \mathscr{N}_\ell^{2^2\mu^{n+1}}(x_0, x_1, t_1^\kappa) \blacklozenge \mathscr{N}_\ell^{2^{p+1}\mu^{n+p-1}}(x_0, x_1, t_1^\kappa).$$

Since $(\Delta, \mathscr{M}, \mathscr{N}, \bigstar, \blacklozenge)$ is ℓ-natural, it follows from the above inequality that

$$\lim_{n\to\infty} \mathscr{M}(x_n, x_{n+p}, t_1^\kappa) = 1 \text{ and } \lim_{n\to\infty} \mathscr{N}(x_n, x_{n+p}, t_1^\kappa) = 0, \qquad \forall \ t_1, t_2, \cdots, t_\kappa > 0.$$

Therefore $\{x_n\}$ is a cauchy sequence.
By the completeness of $(\Delta, \mathscr{M}, \mathscr{N}, \bigstar, \blacklozenge)$ there exist $\mathfrak{a} \in \Delta$ such that

$$\lim_{n\to\infty} \mathscr{M}(x_n, \mathfrak{a}, t_1^\kappa) = 1 \quad \text{and} \quad \lim_{n\to\infty} \mathscr{N}(x_n, \mathfrak{a}, t_1^\kappa) \ \forall \ t_1, t_2, \cdots, t_\kappa > 0, \tag{7}$$

Then $\mathfrak{a}$ is a fixed point of Ω. For each $t_1, t_2, \cdots, t_\kappa > 0$, we have

$$\mathscr{M}(\mathfrak{a}, \Omega\mathfrak{a}, t_1^\kappa) \geq \mathscr{M}_\ell^2(\mathfrak{a}, x_n, t_1^\kappa) \bigstar \mathscr{M}_\ell^2(x_n, \Omega\mathfrak{a}, t_1^\kappa)$$

$$= \mathscr{M}_\ell^2(\mathfrak{a}, x_n, t_1^\kappa) \bigstar \mathscr{M}_\ell^2(\Omega x_{n-1}, \Omega\mathfrak{a}, t_1^\kappa)$$

$$\geq \mathscr{M}_\ell^2(\mathfrak{a}, x_n, t_1^\kappa) \bigstar \mathscr{M}_\ell^{2\mu}(x_{n-1}, \mathfrak{a}, t_1^\kappa),$$

$$\mathscr{N}(\mathfrak{a},\ \Omega\mathfrak{a},\ t_1^{\kappa}) \leq \mathscr{N}_\ell^2(\mathfrak{a},\ x_n,\ t_1^{\kappa}) \blacklozenge \mathscr{N}_\ell^2(x_n,\ \Omega\mathfrak{a},\ t_1^{\kappa})$$

$$= \mathscr{N}_\ell^2(\mathfrak{a},\ x_n,\ t_1^{\kappa}) \blacklozenge \mathscr{N}_\ell^2(\Omega x_{n-1},\ \Omega\mathfrak{a},\ t_1^{\kappa})$$

$$\leq \mathscr{N}_\ell^2(\mathfrak{a},\ x_n,\ t_1^{\kappa}) \blacklozenge \mathscr{N}_\ell^{2\mu}(x_{n-1},\ \mathfrak{a},\ t_1^{\kappa}).$$

By Using (7) in the above inequality, we obtain

$$\mathscr{M}(\mathfrak{a},\ \Omega\mathfrak{a},\ t_1^{\kappa}) = 1 \text{ and } \mathscr{N}(\mathfrak{a},\ \Omega\mathfrak{a},\ t_1^{\kappa}) = 0\ \ \forall\, t_1,\ t_2,\ \cdots,\ t_\kappa > 0.$$

i.e., $\Omega\mathfrak{a} = \mathfrak{a}$. There is only one fixed point of Ω. □

Corollary 2 When $\kappa = 1$, the theorem mentioned above simplifies to the following outcome of the Intuitionistic Fuzzy Banach Contraction Theorem (Alaca et al. [5]). Let $(\Delta,\ \mathscr{M},\ \mathscr{N},\ \bigstar,\ \blacklozenge)$ be a complete IFMS. A function $\Omega : \Delta \to \Delta$ if there exists $\mu \in (0,\ 1)$ such that

$$\mathscr{M}(\Omega x,\ \Omega y,\ \mu t) \geq \mathscr{M}(x,\ y,\ t) \quad \text{and} \quad \mathscr{N}(\Omega x,\ \Omega y,\ \mu t) \leq \mathscr{N}(x,\ y,\ t), \quad \forall\ x,\ y \in \Delta. \tag{8}$$

Then Ω has only one solution.

Lemma 3 for any $\ell \in \{1,\ 2,\ \cdots,\ \kappa\}$, for each $t_1,\ t_2,\ \cdots,\ t_\kappa > 0$ and $0 < \varepsilon < t$, if $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then

$$\mathscr{M}(x,\ y,\ t_1,\ t_2,\ \cdots,\ t_{\ell-1},\ t+\varepsilon,\ t_{\ell+1},\ \cdots,\ t_\kappa) \leq \liminf_{n\to\infty} \mathscr{M}(x_n,\ y_n,\ t_1,\ t_2,\ \cdots,\ t_{\ell-1},\ t+\varepsilon,\ t_{\ell+1},\ \cdots,\ t_\kappa),$$

$$\mathscr{N}(x,\ y,\ t_1,\ t_2,\ \cdots,\ t_{\ell-1},\ t+\varepsilon,\ t_{\ell+1},\ \cdots,\ t_\kappa) \geq \limsup_{n\to\infty} \mathscr{N}(x_n,\ y_n,\ t_1,\ t_2,\ \cdots,\ t_{\ell-1},\ t+\varepsilon,\ t_{\ell+1},\ \cdots,\ t_\kappa),$$

$$\mathscr{M}(x,\ y,\ t_1,\ t_2,\ \cdots,\ t_{\ell-1},\ t+\varepsilon,\ t_{\ell+1},\ \cdots,\ t_\kappa) \leq \limsup_{n\to\infty} \mathscr{M}(x_n,\ y_n,\ t_1,\ t_2,\ \cdots,\ t_{\ell-1},\ t+\varepsilon,\ t_{\ell+1},\ \cdots,\ t_\kappa),$$

$$\mathscr{N}(x,\ y,\ t_1,\ t_2,\ \cdots,\ t_{\ell-1},\ t+\varepsilon,\ t_{\ell+1},\ \cdots,\ t_\kappa) \geq \liminf_{n\to\infty} \mathscr{N}(x_n,\ y_n,\ t_1,\ t_2,\ \cdots,\ t_{\ell-1},\ t+\varepsilon,\ t_{\ell+1},\ \cdots,\ t_\kappa).$$

Remark 9 Consider $(\Delta,\ \mathscr{M},\ \mathscr{N},\ \bigstar,\ \blacklozenge)$ as an IFMS. Let $\Omega : \Delta \to \Delta$ be a function. The equation (8) indicating contractivity implies the function Ω contracts the space concerning the parameter t, ensuring that the degree of closeness and degree of non-closeness of the distance between the images of any two points under Ω is at least as large as the corresponding degree of closeness and degree of non-closeness between the original points.

The theorem (6) asserts that the mapping contracts only concerning the parameter t_ℓ for some $\ell \in \{1,\ 2,\ 3,\ \cdots,\ \kappa\}$, and it may not exhibit contractive behavior concerning other parameters. Similarly, it is assumed that $(\Delta,\ \mathscr{M},\ \mathscr{N},\ \bigstar,\ \blacklozenge)$ is an ℓ-natural κ-IFMS for a minimum on one $\ell \in \{1,\ 2,\ 3,\ \cdots,\ \kappa\}$ only. The subsequent examples verify this remark.

Example 5 Consider $\Delta = [0,\ 1] \times [0,\ 1] \times [0,\ 1]$, where $\bigstar$ denotes the product t-norm, $\blacklozenge$ represents the Lukasiewicz t-conorm and FS $\mathscr{M}$ and $\mathscr{N}$ on $\Delta^2 \times (0,\ \infty)^\kappa$ as follows:

$$\mathscr{M}(x, y, t_1, t_2) = \frac{t_1}{t_1 + |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|},$$

$$\mathscr{N}(x, y, t_1, t_2) = \frac{|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|}{t_1 + |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|},$$

$\forall x = (x_1, x_2), y = (y_1, y_2) \in \Delta$ and $t_1, t_2 > 0$. Then $(\Delta, \mathscr{M}, \mathscr{N}, \bigstar, \blacklozenge)$ is a complete 2-IFMS. Moreover. $\lim_{n\to\infty} \mathscr{M}(x, y, t_1, t_2) = 1$, and $\lim_{n\to\infty} \mathscr{N}(x, y, t_1, t_2) = 0$, $\forall x, y \in \Delta$, $t_2 > 0$. i.e., $(\Delta, \mathscr{M}, \mathscr{N}, \bigstar, \blacklozenge)$ is a 1-natural 2-IFMS. Define a mapping $\Omega : \Delta \to \Delta$ by $\Omega(x_1, x_2, x_3) = (\frac{x_1}{3}, \frac{x_2}{3}, \frac{x_3}{3})$, $\forall (x_1, x_2, x_3) \in \Delta$. For $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in \Delta$ $t_1, t_2 > 0$. We have

$$\begin{aligned}
\mathscr{M}(\Omega x, \psi y, \mu t_1, t_2) &= \frac{\mu t_1}{\mu t_1 + \frac{|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|}{3}} \\
&= \frac{3\mu t_1}{3\mu t_1 + |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|} \\
&\geq \frac{t_1}{t_1 + |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|} \\
&= \mathscr{M}(x, y, t_1, t_2),
\end{aligned}$$

$$\begin{aligned}
\mathscr{N}(\Omega x, \Omega y, \mu t_1, t_2) &= \frac{\frac{|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|}{3}}{\mu t_1 + \frac{|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|}{3}} \\
&= \frac{|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|}{3\mu t_1 + |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|} \\
&\leq \frac{t_1}{t_1 + |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|} \\
&= \mathscr{N}(x, y, t_1, t_2).
\end{aligned}$$

For $\mu \in \left[\frac{1}{3}, 1\right)$, theorem (6) guarantees the existence of a unique fixed point for the mapping Ω. In this particular instance, the point $(0, 0, 0) \in \Delta$ serves as a fixed point for Ω.

Remark 10 In theorem (6), under equations (4) and (5), we presume that the space $(\Delta, \mathscr{M}, \mathscr{N}, \bigstar, \blacklozenge)$ is ℓ-natural. It is important to observe that when considering the existence of a fixed point, the condition of ℓ-naturalness cannot be substituted by $\mathfrak{m}$-naturalness, where $\mathfrak{m}$ is not equal to ℓ. The subsequent example validates this assertion.

Example 6 Consider $\Delta = [0, 1] \times [0, 1] \times [0, 1]$, where $\bigstar$ denotes the product t-norm, $\blacklozenge$ represents the Lukasiewicz t-conorm and FS $\mathscr{M}$ and $\mathscr{N}$ on $\Delta^2 \times (0, \infty)^\kappa$ as follows:

$$\mathscr{M}(x,\ y,\ t_1,\ t_2) = \frac{t_2}{t_2 + |\ x_1 - y_1\ | + |\ x_2 - y_2\ | + |\ x_3 - y_3\ |},$$

$$\mathscr{N}(x,\ y,\ t_1,\ t_2) = \frac{|\ x_1 - y_1\ | + |\ x_2 - y_2\ | + |\ x_3 - y_3\ |}{t_2 + |\ x_1 - y_1\ | + |\ x_2 - y_2\ | + |\ x_3 - y_3\ |},$$

$$\forall x = (x_1,\ x_2),\ y = (y_1,\ y_2) \in \Delta \text{ and } t_1,\ t_2 > 0.$$

Then $(\Delta,\ \mathscr{M},\ \mathscr{N},\ \bigstar,\ \blacklozenge)$ is a complete 2-IFMS. Furthermore

$$\lim_{n\to\infty} \mathscr{M}(x,\ y,\ t_1,\ t_2) = 1, \quad \forall\ x,\ y \in \Delta,\ t_1 > 0, \tag{9}$$

$$\lim_{n\to\infty} \mathscr{N}(x,\ y,\ t_1,\ t_2) = 0, \quad \forall\ x,\ y \in \Delta,\ t_1 > 0. \tag{10}$$

i.e., $(\Delta,\ \mathscr{M},\ \mathscr{N},\ \bigstar,\ \blacklozenge)$ is a 1-natural 2-IFMS. Define a mapping $\Omega : \Delta \to \Delta$ by $\Omega(x_1,\ x_2,\ x_3) = \big(x_1,\ x_2,\ x_3\big)$, $\forall (x_1,\ x_2,\ x_3) \in \Delta$.

For $x = (x_1,\ x_2,\ x_3),\ y = (y_1,\ y_2,\ y_3) \in \Delta,\ t_1,\ t_2 > 0$.

We have $\mathscr{M}(\Omega x,\ \Omega y,\ \mu t_1,\ t_2) \geq \mathscr{M}(x,\ y,\ t_1,\ t_2)$, $\mathscr{N}(\Omega x,\ \Omega y,\ \mu t_1,\ t_2) \leq \mathscr{N}(x,\ y,\ t_1,\ t_2)$. For $\mu \in (0,\ 1)$, as per theorem (9), However, the fixed point of Ω is not unique. In fact, every point $(x_1,\ x_2,\ x_3) \in \Delta$ is a fixed point of Ω. Ultimately, the definition of a κ-IFCM is as follows.

Definition 16 Let $(\Delta,\ \mathscr{M},\ \mathscr{N},\ \bigstar,\ \blacklozenge)$ denote a κ-IFMS. A mapping $\Omega : \Delta \to \Delta$ is called a κ-IFCM if $0 \leq \mu < 1$ such that, for all $x,\ y \in \Delta$ and $t_1,\ t_2,\ t_3,\ \cdots,\ t_\kappa > 0$, the following conditions hold:

$$\frac{1}{\mathscr{M}(\Omega x,\ \Omega y,\ t_1^\kappa)} - 1 \leq \mu \left[\frac{1}{\mathscr{M}(x,\ y,\ t_1^\kappa)} - 1\right] \quad \text{and} \quad \mathscr{N}(\Omega x,\ \Omega y,\ t_1^\kappa) \leq \mu \mathscr{N}(x,\ y,\ t_1^\kappa),$$

where μ is the contractive factor of Ω.

Theorem 7 Let $(\Delta,\ \mathscr{M},\ \mathscr{N},\ \bigstar,\ \blacklozenge)$ be a κ-IFMS and $\Omega : \Delta \to \Delta$ be a $\kappa-$IFCM. Then, Ω has a unique fixed point.

Proof. Consider $x_0 \in \Delta$. Let $\{x_n\}$ is defined as $x_n = \Omega x_{n-1} \forall n \in \mathbb{N}$. For every $n \in \mathbb{N}$,

$$\begin{aligned}
\frac{1}{\mathscr{M}(x_n,\ x_{n+1},\ t_1^\kappa)} - 1 &= \frac{1}{\mathscr{M}(\Omega x_{n-1},\ \Omega x_n,\ t_1^\kappa)} - 1 \\
&\leq \mu \left[\frac{1}{\mathscr{M}(x_{n-1},\ x_n,\ t_1^\kappa)} - 1\right] \\
&= \mu \left[\frac{1}{\mathscr{M}(\Omega x_{n-2},\ \Omega x_{n-1},\ t_1^\kappa)} - 1\right] \\
&\leq \mu^2 \left[\frac{1}{\mathscr{M}(x_{n-2},\ x_{n-1},\ t_1^\kappa)} - 1\right]
\end{aligned}$$

$$\vdots$$

$$\leq \mu^n \left[\frac{1}{\mathscr{M}(x_0,\, x_1,\, t_1^{\kappa})} - 1 \right].$$

Finally, we get

$$\frac{1}{\mathscr{M}(x_n,\, x_{n+1},\, t_1^{\kappa})} - 1 \leq \mu^n \left[\frac{1}{\mathscr{M}(x_0,\, x_1,\, t_1^{\kappa})} - 1 \right]. \tag{11}$$

For every natural number n, given that $0 \leq \mu < 1$, we infer from equation (11) that

$$\lim_{n\to\infty} \left[\frac{1}{\mathscr{M}(x_n,\, x_{n+1},\, t_1^{\kappa})} - 1 \right] \leq 0,$$

$$\text{i.e.} \lim_{n\to\infty} \frac{1}{\mathscr{M}(x_n,\, x_{n+1},\, t_1^{\kappa})} = 1,\ \forall t_1,\, t_2,\, \cdots,\, t_\kappa > 0. \tag{12}$$

For each $n \in \mathbb{N},\ p > 0$ and $t_1,\, t_2,\, \cdots,\, t_\kappa > 0$, we have

$$\mathscr{M}(x_n,\, x_{n+p},\, t_1^{\kappa}) \geq \mathscr{M}_\ell^2(x_n,\, x_{n+1},\, t_1^{\kappa}) \bigstar \mathscr{M}_\ell^2(x_{n+1},\, x_{n+p},\, t_1^{\kappa})$$

$$\geq \mathscr{M}_\ell^2(x_n,\, x_{n+1},\, t_1^{\kappa}) \bigstar \mathscr{M}_\ell^{2^2}(x_{n+1},\, x_{n+2},\, t_1^{\kappa}) \bigstar,\ \cdots,\ .$$

$$\bigstar \mathscr{M}_\ell^{2^{p-1}}(x_{n+p-2},\, x_{n+p-1},\, t_1^{\kappa}) \bigstar \mathscr{M}_\ell^{2^{p-1}}(x_{n+p-1},\, x_{n+p},\, t_1^{\kappa}). \tag{13}$$

From (12), we have $\lim_{n\to\infty} \mathscr{M}_\ell^{\alpha}(x_n,\, x_{n+1},\, t_1^{\kappa}) = 1$, for all $t_1,\, t_2,\, \cdots,\, t_\kappa > 0$ and $\alpha > 0$, which together with inequality (13) yields $\lim_{n\to\infty} \mathscr{M}(x_n,\, x_{n+1},\, t_1^{\kappa}) \geq 1 \bigstar 1 \bigstar,\ \cdots,\ \bigstar 1 = 1$.

For any positive real numbers $t_1,\, t_2,\, \cdots,\, t_\kappa$ and p. Let $\{x_n\}$ be a Cauchy sequence in Δ then converge to itself is said to be complete, $\exists\ \mathfrak{a} \in \Delta$ such that the sequence $\{x_n\} \to \mathfrak{a}$.

In other words,

$$\lim_{n\to\infty} \mathscr{M}(x_n,\, \mathfrak{a},\, t_1^{\kappa}) = 1,\ \forall\ \ t_1,\, t_2,\, \cdots,\, t_\kappa > 0. \tag{14}$$

Similarly, we obtain the above definition that

$$
\begin{aligned}
\mathscr{N}(x_n,\, x_{n+1},\, t_1^{\kappa}) &= \mathscr{N}(\Omega x_{n-1},\, \Omega x_n,\, t_1^{\kappa})\\
&\leq \mu \mathscr{N}(x_{n-1},\, x_n,\, t_1^{\kappa})\\
&= \mu \mathscr{N}(\Omega x_{n-2},\, \Omega x_{n-1},\, t_1^{\kappa})\\
&\leq \mu^2 \mathscr{N}(x_{n-2},\, x_{n-1},\, t_1^{\kappa})\\
&\quad \vdots\\
&= \mu^{n-1} \mathscr{N}(\Omega x_1,\, \Omega x_2,\, t_1^{\kappa})\\
&\leq \mu^n \mathscr{N}(x_0,\, x_1,\, t_1^{\kappa}), \qquad (15)
\end{aligned}
$$

since $0 \leq \mu < 1$, we conclude from (15) that

$$\lim_{n\to\infty} \mathscr{N}(x_n,\, x_{n+1},\, t_1^{\kappa}) = 0,\, \forall\, t_1,\, t_2,\, \cdots,\, t_\kappa > 0. \qquad (16)$$

For $n \in \mathbb{N},\ p > 0,\ t_1,\, t_2,\, \cdots,\, t_\kappa > 0$, we have

$$
\begin{aligned}
\mathscr{N}(x_n,\, x_{n+p},\, t_1^{\kappa}) &\leq \mathscr{N}_\ell^2(x_n,\, x_{n+1},\, t_1^{\kappa}) \blacklozenge \mathscr{N}_\ell^2(x_{n+1},\, x_{n+p},\, t_1^{\kappa})\\
&\leq \mathscr{N}_\ell^2(x_n,\, x_{n+1},\, t_1^{\kappa}) \blacklozenge \mathscr{N}_\ell^{2^2}(x_{n+1},\, x_{n+2},\, t_1^{\kappa})\\
&\quad \blacklozenge \cdots \blacklozenge \mathscr{N}_\ell^{2^{p-1}}(x_{n+p-2},\, x_{n+p-1},\, t_1^{\kappa})\\
&\quad \blacklozenge \mathscr{N}_\ell^{2^{p-1}}(x_{n+p-1},\, x_{n+p},\, t_1^{\kappa}). \qquad (17)
\end{aligned}
$$

From (16), we have $\lim_{n\to\infty} \mathscr{N}_\alpha^\ell(x_n,\, x_{n+1},\, t_1^{\kappa}) = 0$, for all $t_1,\, t_2,\, \cdots,\, t_\kappa > 0$ and $\alpha > 0$. Which together with inequality (17) yeilds

$$\lim_{n\to\infty} \mathscr{N}(x_n,\, x_{n+1},\, t_1^{\kappa}) \leq 0 \blacklozenge 0 \blacklozenge,\, \cdots,\, \blacklozenge 0 = 0.$$

For any positive real numbers $t_1,\ t_2,\ \cdots,\ t_\kappa$ and p. Let $\{x_n\}$ be a Cauchy sequence in $\varDelta$ then converge to itself is said to be complete, $\exists\ \mathfrak{a} \in \varDelta$ such that the sequence $\{x_n\} \to \mathfrak{a}$.

In other words,

$$\lim_{n\to\infty} \mathscr{N}(x_n,\ \mathfrak{a},\ t_1^\kappa) = 0,\ \forall\, t_1,\ t_2,\ \cdots,\ t_\kappa > 0. \tag{18}$$

We will demonstrate that $\mathfrak{a}$ constitutes a fixed point for Ω. For every $n \in \mathbb{N}$ and $t_1,\ t_2,\ \cdots,\ t_\kappa > 0$, we observe that $\frac{1}{\mathscr{M}(x_{n+1},\ \Omega\mathfrak{a},\ t_1^\kappa)} - 1 = \frac{1}{\mathscr{M}(\Omega x_n,\ \Omega\mathfrak{a},\ t_1^\kappa)} - 1 \leq \mu\left[\frac{1}{\mathscr{M}(x_n,\ \mathfrak{a},\ t_1^\kappa)} - 1\right]$.

$$\lim_{n\to\infty}\left[\frac{1}{\mathscr{M}(x_{n+1},\ \Omega\mathfrak{a},\ t_1^\kappa)} - 1\right] = 0,\ \text{using (14)}$$

$$\text{i.e., } \lim_{n\to\infty} \mathscr{M}(x_{n+1},\ \Omega\mathfrak{a},\ t_1^\kappa) = 1,$$

$$\mathscr{M}(\mathfrak{a},\ \Omega\mathfrak{a},\ t_1^\kappa) \geq \mathscr{M}_\ell^2(\mathfrak{a},\ x_{n+1},\ t_1^\kappa) \bigstar \mathscr{M}_\ell^2(x_{n+1},\ \Omega\mathfrak{a},\ t_1^\kappa), \tag{19}$$

For any $n \in \mathbb{N}$ and for every $t_1,\ t_2,\ \cdots,\ t_\kappa > 0$, we have which together with (13) and (19) yields: $\mathscr{M}(\mathfrak{a},\ \Omega\mathfrak{a},\ t_1^\kappa) = 1,\ \forall t_1,\ t_2,\ \cdots,\ t_\kappa > 0$,

$$\mathscr{N}(x_n,\ \Omega\mathfrak{a},\ t_1^\kappa) = \mathscr{N}(\Omega x_n,\ \Omega\mathfrak{a},\ t_1^\kappa) \leq \mathscr{N}(x_n,\ \mathfrak{a},\ t_1^\kappa).$$

$$\lim_{n\to\infty} \mathscr{N}(x_{n+1},\ \Omega\mathfrak{a},\ t_1^\kappa) = 0,\ \text{using (18)},$$

$$\mathscr{N}(\mathfrak{a},\ \Omega\mathfrak{a},\ t_1^\kappa) \leq \mathscr{N}_\ell^2(\mathfrak{a},\ x_{n+1},\ t_1^\kappa) \blacklozenge \mathscr{N}_\ell^2(x_{n+1},\ \Omega\mathfrak{a},\ t_1^\kappa), \tag{20}$$

for all $t_1,\ t_2,\ \cdots,\ t_\kappa > 0$. For any $n \in \mathbb{N}$, we have which together with (16) and (20) yields $\mathscr{N}(\mathfrak{a},\ \Omega\mathfrak{a},\ t_1^\kappa) = 0,\ \forall t_1,\ t_2,\ \cdots,\ t_\kappa > 0$.

Signifying that $\mathfrak{a}$ functions as a fixed point for Ω, $\mathfrak{b}$ is an alternative fixed point of Ω, different from $\mathfrak{a}$. Consequently, there exist positive values $r_1,\ r_2,\ \cdots,\ r_\kappa$. $\mathscr{M}(\mathfrak{a},\ \mathfrak{b},\ r_1^\kappa) < 1,\ \mathscr{N}(\mathfrak{a},\ \mathfrak{b},\ r_1^\kappa) > 0$.

Now, we have

$$\frac{1}{\mathscr{M}(\mathfrak{a},\ \mathfrak{b},\ r_1^\kappa)} - 1 = \frac{1}{\mathscr{M}(\Omega\mathfrak{a},\ \Omega\mathfrak{b},\ r_1^\kappa)} - 1 \leq \mu\left[\frac{1}{\mathscr{M}(\mathfrak{a},\ \mathfrak{b},\ r_1^\kappa)} - 1\right],$$

$$\mathscr{N}(\mathfrak{a},\ \mathfrak{b},\ r_1^\kappa) = \mathscr{N}(\Omega\mathfrak{a},\ \Omega\mathfrak{b},\ r_1^\kappa) \leq \mu\ \mathscr{N}(\mathfrak{a},\ \mathfrak{b},\ r_1^\kappa) < \mathscr{N}(\mathfrak{a},\ \mathfrak{b},\ r_1^\kappa).$$

Given that μ is less than 1, the abovementioned inequality leads to a contradiction. Consequently, it is necessary for $\mathfrak{a}$ to equal $\mathfrak{b}$. Consequently, a single point of Ω is established as unique. □

Remark 11 In this situation where $\kappa = 1$, the theorem simplifies to the following outcome presented by Rafi and Noorani [13].

5. Conclusions

This research has established and analyzed the theoretical foundations of κ-intuitionistic fuzzy metric spaces (κ-IFMS), making substantial contributions to fixed-point theory. Our work extends the traditional framework of intuitionistic fuzzy metric spaces by introducing multiple parameters, enabling more sophisticated approaches to uncertainty modeling in mathematical structures. The theoretical framework developed herein offers several key advantages. First, it provides a more comprehensive approach to modeling uncertainty and imprecision in metric spaces, addressing the limitations of existing frameworks. Second, our fixed-point theorems extend classical results to this new setting, offering powerful tools for analyzing complex systems. Third, the framework's flexibility allows applications across diverse mathematical domains, from pure theory to practical implementations. Our results demonstrate that κ-IFMS provides a natural and powerful generalization of traditional fuzzy metric spaces while maintaining essential mathematical properties that enable practical applications. The fixed-point theorems established in this work advance theoretical understanding and provide foundational tools for solving concrete problems in various domains.

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Conflict of interest

The authors declare no competing financial interest.

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