UNIVERSAL WISER
PUBLISHER

On Random Population Growth Punctuated by Geometric Catastrophic Events

Thierry E. Huillet
Laboratory of Physics; Theory and Models, CY Cergy Paris University, CNRS UMR-8089, 2 avenue Adolphe-Chauvin, 95302 Cergy-Pontoise, France
Email: Thierry.Huillet@cyu.fr

Abstract: Catastrophe Markov chain population models have received a lot of attention in the recent past. Besides systematic random immigration events promoting growth, we study a particular case of populations simultaneously subject to the effect of geometric catastrophes that causes recurrent mass removal. We describe the subtle balance between the two such contradictory effects.
Keywords: Markov chain, random population growth, geometric catastrophic events, recurrence/transience transition, height and length of excursions, extinction events, time to extinction, large deviations

1. Introduction

Besides systematic immigration events, populations are often subject to the effect of catastrophic events that cause recurrent mass removal. This results is in a subtle balance between the two contradictory effects. Many Markov chains models have been designed in an attempt to explain the transient and large-time behavior of the size of such populations. Some results concern the evaluation of the risk of extinction and the distribution of the population size in the case of total disasters where all individuals in the population are removed simultaneously[1-2]. Other works deal with the basic immigration process subject to either binomial or geometric catastrophes; that is when the population size is iteratively reduced according either to a binomial or a geometric law[3-4]. Such Markov chains are random walks on the nonnegative integers (as a semigroup) which differ from standard random walks on the integers (as a group) in that a one-step move down from some positive integer cannot take the walker to a negative state, resulting in transition probabilities being state-dependent.

The binomial effect is appropriate when, on a catastrophic event, the individuals of the current population each die or survive in an independent and even way, resulting in a drastic depletion of individuals at each step[4-6].

The geometric effect corresponds to softer deletion cases where the sequential decline in the population keeps going on but is stopped as soon as the current population size is exhausted. In more precise words, if a geometric catastrophic event occurs, given the chain is in state x, the population shrinks by a random (geometrically distributed) amount so long as this amount does not exceed x; if it does, the population size is set to 0 (a disaster event). In [4], this model was called for this reason the 'truncated geometric' catastrophe model. This may be appropriate for some forms of catastrophic epidemics or when the catastrophe has a sequential propagation effect like in predator-prey models: the predator kills preys until it becomes satisfied so long as resources are available. More examples and motivations can be found in Artalejo et al.[7] and in Cairns and Pollett[8], essentially in continuous-times.

In this work[4], we revisited the latter geometric catastrophe model in discrete-time, as a Markov chain on the non-negative integers.

Using a generating function approach, we first discussed the condition, under which this process is recurrent (either positive or null) or transient. The recurrence/transience transition is sharp.

In the positive-recurrent case, we described the shape and features of the invariant probability measure. We emphasized that, in the null-recurrent case, no nontrivial invariant measure exists. Focus is then put on the time spent by the walker in the ground state $\{0\}$. We then showed how to compute the double (space/time) generating functional of the process. Using this representation, we derived the generating functions of the first return time to zero (the length of the excursions) and of the first extinction time when the process is started at $x_0 > 0$. In the recurrent case, extinction occurs

DOI: https://doi.org/10.37256/cm.152020600

with probability 1. The harmonic (or scale) function of a version of the process forced to be absorbed in state {0} is used to give an expression of the law of the height of the excursions (the maximum population size reached between any two consecutive extinctions).

In the transient case, after a finite number of visits to {0}, the chain drifts to ∞. We first emphasized that no nontrivial invariant measure exists either. Using the expression of the double generating functional of the process, we gain access to a precise large deviation result. The harmonic (or scale) function of a version of the process forced to be absorbed in state, {0} is then used to give an expression of the probability of a first extinction. The law of the height of the excursions is also derived, with an atom at ∞.

More complex growth/collapse or decay/surge models in the same vein, although in the continuum, were considered in [9-11], where physical applications were developed, such as queueing processes arising in the physics of dams or stress release issues arising in the physics of earthquakes.

2. Geometric catastrophes

Let us introduce the process under concern, as a Markov chain[3-4].

The model.

Birth (growth): Let $(\beta_n)_{n \geq 1}$ be an independent identically distributed (iid) sequence taking values in $\mathbb{N} := \{1, 2, ...\}$, with $b_x := \mathbf{P}(\beta_1 = x)$, $x \geq 1$. We shall let $B(z) := \mathbf{E}(z^{\beta})$ be the common probability generating function of the βs.

Death (depletion): Let $(\delta_n)_{n \geq 1}$ be an iid shifted geometric(α)-distributed sequence [A geometric(α) rv with success probability α takes values in $\mathbb{N}$. A shifted geometric (α) rv with success probability α takes values in $\mathbb{N}_0$. It is obtained while shifting the former one by one unit], with success parameter $\alpha \in (0,1)$. With $\overline{\alpha} := 1 - \alpha$, δ_1 then takes values in $\mathbb{N}_0 :=$ (0, 1, 2, ...} with $\mathbf{P}(\delta_1 = x) = d_x = \alpha\overline{\alpha}^x$, $x \geq 0$.

Consider the Markov chain with temporal evolution[4]:

$$X_{n+1} = X_n + \beta_{n+1} \text{ with probability } p$$

$$= (X_n - \delta_{n+1})_+ \text{ with probability } q = 1 - p. \quad (1)$$

At each step n, the walker moves up with probability p and the amplitude of the upward move is β_{n+1}. The number of step-wise removed individuals (whenever available) is δ_{n+1}, with probability q : the distribution of the sizes of the catastrophes δ_{n+1} does not depend on the current size of the population, so long as there is enough prey. Stated differently, given the population size is x at n, the magnitude of the downward jump is $x \wedge \delta_{n+1}$.

Note that if $X_n = 0$, then $X_{n+1} = \beta_{n+1}$ with probability p (reflection at 0) and $X_{n+1} = 0$ with probability q (absorption at 0).

The one-step stochastic transition matrix P (obeying $P\mathbf{1} = \mathbf{1}$, where $\mathbf{1}$ is a column vector of ones) of the Markov chain $\{X_n\}$ is:

$$P(0,0) = q, P(0,y) = pb_y, y \geq 1$$

$$P(x,y) = qd_{x-y} = q\alpha\overline{\alpha}^{x-y}, x \geq 1\% \text{ and } 1 \leq y \leq x$$

$$P(x,0) = q\sum_{k \geq x} d_k = q\overline{\alpha}^x, x \geq 1 \text{ and } 0 = y < x$$

$$P(x,y) = pb_{y-x}, x \geq 1 \text{ and } y > x. \quad (2)$$

With $\boldsymbol{\pi}_n := (\mathbf{P}_{x_0}(X_n = 0), \mathbf{P}_{x_0}(X_n = 1), ...)'$ the column vector [In the sequel, a boldface variable, say $\mathbf{x}$, will represent a column-vector so that its transpose, say $\mathbf{x}'$, will be a row-vector.] of the states' occupation probabilities at time n, $\boldsymbol{\pi}'_{n+1} = \boldsymbol{\pi}'_n P$, $X_0 \overset{d}{\sim} \delta_{x_0}$, is the master equation of its temporal evolution. An equivalent recurrence for the probability generating function of $\{X_n\}$ started at $X_0 = x_0$ is:

Lemma 1 With $\Phi_n(z) := \mathbf{E}_{x_0}(z^{X_n}) = \sum_{x\geq 0} z^x \mathbf{P}_{x_0}(X_n = x)$, such that $\Phi_0(z) = z^{x_0}$, it holds,

$$\Phi_{n+1}(z) = \left(pB(z) - \frac{q\alpha z}{\overline{\alpha} - z}\right)\Phi_n(z) + q\left(1 + \frac{\alpha z}{\overline{\alpha} - z}\right)\Phi_n(\overline{\alpha}). \tag{3}$$

Proof. From (2),

$$\Phi_{n+1}(z) = \pi_{n+1}(0) + pB(z)\Phi_n(z) + q\sum_{y\geq 1} z^y \sum_{x\geq y} \pi_n(x)\alpha\overline{\alpha}^{x-y}$$

$$= \pi_{n+1}(0) + pB(z)\Phi_n(z) + \frac{q\alpha z}{\overline{\alpha} - z}\left(\Phi_n(\overline{\alpha}) - \Phi_n(z)\right)$$

$$\Phi_{n+1}(0) = \pi_{n+1}(0) = q\Phi_n(\overline{\alpha}).$$

Remark: The geometric Markov chain is time-homogeneous, irreducible and aperiodic. As a result, all states are either recurrent or transient.

3. The recurrent case

The geometric Markov chain being either transient or recurrent, we shall first exhibit when this transition occurs. We shall then deal first with the recurrent case before switching to the transient regime.

3.1 Existence and shape of an invariant probability measure

Theorem 2[4] The chain $\{X_n\}$ with geometric catastrophe is ergodic if and only if $\rho := \mathbf{E}(\beta_1) < \rho_c := qp^{-1}\frac{\overline{\alpha}}{\alpha} < \infty$. The probability generating function of is then given by (4).

Proof. If a limiting random variable X_∞ exists, from (3), it has probability generating function $\Phi_\infty(z) := \mathbf{E}(z^{X_\infty})$ obeying

$$\Phi_\infty(z) = \frac{\overline{\alpha}(1-z)q\Phi_\infty(\overline{\alpha})}{(\overline{\alpha} - z)(1 - pB(z)) + q\alpha z} =: \frac{N(z)}{D(z)}.$$

Suppose $\rho := B'(1) = \mathbf{E}(\beta_1) < \infty$. Only in such a case does the numerator N and denominator D both tend to 0 as $z \to 1$ while the ratio Φ_∞ tends to 1, as required for $\Phi_\infty(z)$ to be a probability generating function. By L'Hospital rule $\frac{N'(z)}{D'(z)} \to 1$ as $z \to 1$, leading to

$$\Phi_\infty(\overline{\alpha}) = \frac{q\overline{\alpha} - p\alpha\rho}{q\overline{\alpha}}.$$

Thus

$$\pi(0) := \mathbf{P}(X_\infty = 0) = \Phi_\infty(0) = q\Phi_\infty(\overline{\alpha}) = q - p\alpha\rho / \overline{\alpha},$$

and, with $\Phi_Y(z) := q/(1 - pB(z))$ the probability generating function of a random variable Y obtained as a shifted-geometric(q) convolution of the βs

$$\Phi_\infty(z) = \frac{(1-z)(q\overline{\alpha} - p\alpha\rho)}{(\overline{\alpha} - z)(1 - pB(z)) + q\alpha z} = \frac{\Phi_Y(z)(1-z)(q\overline{\alpha} - p\alpha\rho)}{(\overline{\alpha} - z)q + q\alpha z\Phi_Y(z)}$$

$$\pi(x) := \pi_\infty(x) = \left[z^x\right]\Phi_\infty(z) = \Phi_\infty(0)\left[z^x\right]\frac{1-z}{(\overline{\alpha}-z)(1-pB(z))+q\alpha z},\ x \geq 1.$$

If and only if $\rho = \mathbf{E}(\beta_1) < \infty$ and $\zeta := \alpha(1 + q^{-1}p\rho) < 1$ ($\rho < \rho c$), then $\pi(0) = \frac{q}{\overline{\alpha}}(1 - \zeta) \in (0, 1)$ and the Markov chain is ergodic (positive recurrent and aperiodic). The probability generating function of the invariant probability measure then takes the alternative form

$$\Phi_\infty(z) = (1-\zeta)\Phi_Y(z)\left[1-\alpha\frac{1-z\Phi_Y(z)}{1-z}\right]^{-1}.$$

Note that, with $\Phi_Z(z) := \frac{1-z\Phi_Y(z)}{(1+m)(1-z)} = \frac{1-\Phi_{Y+1}(z)}{\mathbf{E}(Y+1)(1-z)}$ the probability generating function of some random variable Z, this is also

$$\Phi_\infty(z) = \Phi_Y(z)\frac{1-\zeta}{1-\zeta\Phi_Z(z)}, \tag{4}$$

which is the product of the probability generating function of Y times the one of a shifted compound geometric($1 - \zeta$) random variable with compounding probability generating function $\Phi_Z(z)$ (the delay probability generating function of Y + 1).

Note

$$\zeta < 1 \Leftrightarrow p\mathbf{E}(\beta_1) = p\rho < q\frac{\overline{\alpha}}{\alpha} = q\mathbf{E}(\delta_1),$$

observing $\overline{\alpha}/\alpha = \mathbf{E}(\delta_1)$. This is a sub-criticality condition stating that the one-step unlimited average move down $q\mathbf{E}(\delta_1)$ must exceed the average move up $p\rho$ of the geometric chain.

Corollary 3 If $\rho < \rho_c$, the random variable X_∞ exists and is infinitely divisible (compound Poisson).

Proof. Consider first the random variable Y with probability generating function $\Phi_Y(z) := \mathbf{E}(z^Y) = \frac{q}{1-pB(z)}$. It is a compound shifted-geometric random variable with compounding random variable β. It is infinitely divisible because $\Phi_Y(z) = \exp - \lambda(1 - \psi(z))$ for some $\lambda > 0$ and some probability generating function $\psi(z)$ with $\psi(0) = 0$. Indeed, with $q =: e^{-\lambda}$, there exists a probability generating function ψ solving $\Phi_Y(z) := q/(1 - pB(z)) = \exp - \lambda(1 - \psi(z))$. It is

$$\psi(z) = \frac{-\log(1-pB(z))}{-\log q},$$

which is recognized as the probability generating function of a Fisherlog-series random variable. Shifted-geometric convolution random variables are infinitely divisible. For the same reason, the random variable with probability generating function $\frac{1-\zeta}{1-\zeta\Phi_Z(z)}$ is infinitely divisible as a shifted-compound geometric random variable with compounding random variable Z. From (4), $\Phi_\infty(z)$ is the product of these two probability generating functions and so is itself the probability generating function of an infinitely divisible random variable X_∞.

Miscellaneous:

Neuts[4] observed that if β is itself geometrically distributed (more generally of phase-type), then so are Y, Z, together with X_∞, so with an explicit expression of $\pi(x)$. The random variable X_∞ in particular admits geometric tails. Discrete phase-type random variables are the ones obtained as first hitting times of the absorbing state of a terminating Markov chain with finitely many states[12].

In the general case for the law of β, an expression of $\pi(x) := [z^x]\Phi_\infty(z)$ can be obtained from (4), in principle, using the Faa di Bruno formula[13], observing $\Phi_\infty(z) = H(B(z))$ for some generating function H, as a composition of generating functions.

We also have: $\mu := \mathbf{E}(X_\infty) < \infty \Leftrightarrow \mathbf{E}(Y^2) < \infty \Leftrightarrow \mathbf{E}(\beta^2) < \infty$, owing to: $\mathbf{E}(Y^2) = 2\mathbf{E}(Y)^2 + \mathbf{E}(G)\mathbf{E}(\beta^2)$, $\mathbf{E}(G) = p/q$.

Specifically, if $\mathbf{E}(\beta^2) < \infty$

$$\mu = m + \alpha \frac{\mathbf{E}\left(Y^2\right) - \mathbf{E}\left(Y\right)^2}{2(1-\zeta)} = m\left(1 + \frac{\alpha}{\mathbf{E}(\beta)} \frac{\mathbf{E}(\beta)m + \mathbf{E}\left(\beta^2\right)}{2(1-\zeta)}\right).$$

More generally, it can also be checked that if β only has moments of order $q < 1 + a (a > 0)$, then X_∞ only has moments of order $q < a$.

Finally, with $m = \Phi_Y'(1) = \mathbf{E}(Y) = q^{-1}p\rho = q^{-1}p\mathbf{E}(\beta)$, as $z \to 1$

$$\frac{1 - z\Phi_Y(z)}{1-z} \sim \frac{1 - z\left(1 - m(1-z)\right)}{1-z} \sim 1 + m,$$

so that

$$\Phi_\infty(1) \sim (1-\zeta)\left[1 - \alpha(1+m)\right]^{-1} = 1.$$

In the positive recurrent case, the geometric process is not time-reversible with respect to the invariant probability measure $\boldsymbol{\pi}$: there is no detailed balance.

Considerable simplifications are expected when $\beta \overset{d}{\sim} \delta_1$: in this case, the transition matrix P in (2) is of Hessenberg type, that is lower triangular with a non-zero upper diagonal. The corresponding process is skip-free to the right.

3.2 Time spent in the ground state {0}

Whenever the process $\{X_n\}$ is ergodic, it visits infinitely often all the states, in particular the state $\{0\}$, and a sample path of it is made of iid successive non-negative excursions (arches) through that state. By the ergodic theorem, the asymptotic fraction of time spent by $\{X_n\}$ in this state is

$$\frac{1}{N}\sum_{n=1}^{N} \mathbf{1}\left(X_n = 0\right) \to \pi(0) = \mathbf{P}\left(X_\infty = 0\right), \text{ almost surely, as } N \to \infty. \tag{5}$$

Let $\tau_{0,0} := \inf(n \geq 1 : X_n = 0 \mid X_0 = 0)$ be the first return time to state $\{0\}$. By Kac's theorem[14], its expected value is $\mathbf{E}(\tau_{0,0}) = 1/\pi(0)$ where $\pi(0) = a\,(1 + q^{-1}p\rho)\,\frac{q}{\alpha}(1 - \zeta)$.

Suppose $\{X_n\}$ enters state $\{0\}$ from above at some time n_1. The first return time to state $\{0\}$, $\tau_{0,0} := \inf(n \geq n_1 : X_n = 0 \mid X_{n_1} = 0)$, is:

either 1 if X_{n_1} stays there with probability $P(0, 0) = q$ in the next step; this corresponds to a trivial excursion of length 1 and height 0.

or, with probability $p = 1 - q$, $\{X_n\}$ starts a 'true' excursion with positive height and length $\tau_{0,0}^+ \geq 2$. Thus,

$$\mathbf{E}\left(\tau_{0,0}\right) = \frac{1}{\pi(0)} = 1 + p\mathbf{E}\left(\tau_{0,0}^+\right) \text{ and } \mathbf{E}\left(\tau_{0,0}^+\right) = \frac{1}{p}\left(\frac{1}{\pi(0)} - 1\right) \geq 2, \tag{6}$$

entailing $\pi(0) < 1/(1 + 2p)$. Given $\{X_n\}$ enters state $\{0\}$ from above at some time n_1, it stays there with probability $P(0, 0) = q$ in the next step, so $\{X_n\}$ will quit state $\{0\}$ at time $n_1 + G$ where G is a geometric random time with success probability $p = 1 - q$. After $n_1 + G$, the chain moves up before returning to state $\{0\}$ again and the time it takes is $\tau_{0,0}^+$. Considering two consecutive instants where $\{X_n\}$ enters state $\{0\}$ from above (defining an alternating renewal process), the mean fraction of time θ spent in state $\{0\}$ is:

$$\theta := \frac{\mathbf{E}(G)}{\mathbf{E}(G)+\mathbf{E}\left(\tau_{0,0}^{+}\right)}.$$

From the expression $\mathbf{E}(G) = 1/p$ and the value of $\mathbf{E}(\tau_{0,0}^{+})$, consistently with (5), we get:
Proposition 4 In the positive recurrent case, the mean fraction of time θ spent in state $\{0\}$ is:

$$\theta = \frac{1}{1+p\mathbf{E}\left(\tau_{0,0}^{+}\right)} = \pi(0). \tag{7}$$

3.3 No nontrivial ($\neq$ 0) invariant measure in the null-recurrent case

If $\zeta = 1$ (or $\rho = \rho_c$), the critical chain is null-recurrent with

$$\pi(0) = \frac{q\overline{\alpha}-p\alpha\rho}{\overline{\alpha}} = \frac{q}{\overline{\alpha}}(1-\zeta) = 0 \Rightarrow \pi(x) = 0 \text{ for all } x \geq 1.$$

The chain has no nontrivial ($\neq 0$) invariant positive measure.
It is not Harris-recurrent[15-17].

3.4 The generating functional of the geometric model

With $x_0 \geq 1$, defining the double generating function

$$\Phi_{x_0}(u,z) = \sum_{n\geq 0} u^n \Phi_n(z) = \sum_{n\geq 0} u^n \mathbf{E}_{x_0}\left(z^{X_n}\right),$$

from (3), we get

$$\frac{1}{u}\left(\Phi_{x_0}(u,z) - z^{x_0}\right) = \left(pB(z) - \frac{q\alpha z}{\overline{\alpha}-z}\right)\Phi_{x_0}(u,z) + q\left(1+\frac{\alpha z}{\overline{\alpha}-z}\right)\Phi_{x_0}(u,\overline{\alpha}),$$

together with

$$\Phi_{x_0}(u,0) = qu\Phi_{x_0}(u,\overline{\alpha}).$$

We obtain

$$\Phi_{x_0}(u,z) = \frac{z^{x_0}(\overline{\alpha}-z) + q\overline{\alpha}u\Phi_{x_0}(u,\overline{\alpha})(1-z)}{(\overline{\alpha}-z)\left[1-puB(z)\right]+q\alpha zu} = \frac{z^{x_0}(\overline{\alpha}-z)+\overline{\alpha}(1-z)\Phi_{x_0}(u,0)}{(\overline{\alpha}-z)\left[1-puB(z)\right]+q\alpha zu}. \tag{8}$$

So far, $\Phi_{x_0}(u, z)$ is unknown since it requires the knowledge of $\Phi_{x_0}(u, \overline{\alpha})$ or $\Phi_{x_0}(u, 0)$.
Letting $\Phi_{x_0}(u, z) =: N(u, z)/D(u, z)$, the denominator $D(u, z)$ vanishes at

$$u = u(z) = \frac{z-\overline{\alpha}}{q\alpha z + p(z-\overline{\alpha})B(z)}. \tag{9}$$

Note

$$u(1)=1 \text{ and } u^{'}(1)=\frac{1}{\alpha}(q\overline{\alpha}-\alpha p\rho)=\frac{q}{\alpha}(1-\zeta). \tag{10}$$

The generating function $\Phi_{x_0}(u, z)$ is well-defined when $u < u(z)$ and possibly when $u = u(z)$ as in the recurrent case.

In the recurrent case ($\zeta \le 1$), $u(z)$ is concave and monotone increasing on the interval $[\overline{\alpha}, 1]$, owing to $u'(1) \ge 0$. The function $u(z)$ has a well-defined inverse $z(u)$ which maps [0, 1] to $[\overline{\alpha}, 1]$; this inverse is monotone increasing and convex on this interval. Because in the recurrent case, $[u^n]\Phi_{x_0}(u, z) \to \Phi_\infty(z)$ as $n \to \infty$, $\Phi_{x_0}(u, z)$ also converges as $z \to z(u)$. So both the numerator N and the denominator D of $\Phi_{x_0}(u, z)$ must tend to 0 as $z \to z(u)$, meaning (by L'Hospital rule) that

$$\lim_{z\to z(u)}\frac{N(u,z)}{D(u,z)}=\lim_{z\to z(u)}\frac{N^{'}(u,z)}{D^{'}(u,z)}.$$

Near $z = z(u)$,

$$N(u,z)=A(z)+B(z)\Phi_{x_0}(u,0)$$

$$\sim A(z(u))+B(z(u))\Phi_{x_0}(u,0)+(z-z(u))\left[A^{'}(z(u))+B^{'}(z(u))\Phi_{x_0}(u,0)\right]$$

$$D(u,z)=C(z)-D(z)u\sim(z-z(u))\left[C^{'}(z(u))-D^{'}(z(u))u\right],$$

imposing $A(z(u)) + B(z(u))\Phi_{x_0}(u, 0) = 0$, thereby fixing

$$\Phi_{x_0}(u,0)=\frac{z(u)^{x_0}(z(u)-\overline{\alpha})}{\overline{\alpha}(1-z(u))}=:G_{x_0,0}(u)\ge 0. \tag{11}$$

Note $G_{x_0,0}(1) = \infty$ and $G_{x_0,0}(0) = \mathbf{P}_{x_0}(X_0 = 0) = 0$. The function

$$G_{x_0,0}(u)=\sum_{n\ge 1}u^n\mathbf{P}_{x_0}(X_n=0)=\sum_{n\ge 1}u^nP^n(x_0,0),$$

is the Green kernel of the chain at the endpoints $(x_0, 0)$[18]. The matrix element $P^n(x_0, 0)$ is the contact probability at 0 at time n, starting from x_0.

From (8) and (11), we thus get a closed form expression of $\Phi_{x_0}(u, z)$ when $x_0 \ge 1$, as ($u \in [0, 1]$)

Proposition 5 In the recurrent case ($\zeta \le 1$), with $z(u)$ the inverse of $u(z)$ defined in (9),

$$\Phi_{x_0}(u,z)=\frac{z(u)^{x_0}(z(u)-\overline{\alpha})(1-z)-z^{x_0}(z-\overline{\alpha})(1-z(u))}{(1-z(u))\left[(\overline{\alpha}-z)(1-puB(z))+q\alpha zu\right]}. \tag{12}$$

Remark: If $z = 1$, $\Phi_{x_0}(u, 1) = 1/(1 - u)$ and $\Phi_{x_0}(0, z) = z^{x_0}$, as required.

3.5 First return time to 0

When $x_0 = 0$, observing now $G_{0,0}(0) = \mathbf{P}_0(X_0 = 0) = 1$,

$$G_{0,0}(u)=1+\Phi_0(u,0)=1+\frac{z(u)-\overline{\alpha}}{\overline{\alpha}(1-z(u))}=\frac{\alpha z(u)}{\overline{\alpha}(1-z(u))}, \text{ with } G_{0,0}(1)=\infty\ (z(1)=1). \tag{13}$$

This function is the Green kernel at the endpoints (0, 0).

If $n \geq 1$, from the recurrence

$$\mathbf{P}_0\left(X_n=0\right)=: P^n\left(0,0\right)=\sum_{m=0}^{n}\mathbf{P}\left(\tau_{0,0}=m\right)P^{n-m}\left(0,0\right),$$

we see, from taking the generating function of both sides and observing the right-hand side is a convolution, that the pgf $\phi_{0,0}(u)=\mathbf{E}(u^{\tau_{0,0}})$ of the first return time to 0, $\tau_{0,0}$ and $G_{0,0}(u)$ are related by the Feller relation[19-20]: $G_{0,0}(u)=1+G_{0,0}(u)\phi_{0,0}(u)$. Hence, with $\phi_{0,0}(0)=0$,

$$\phi_{0,0}\left(u\right)=\mathbf{E}\left(u^{\tau_{0,0}}\right)=1-\frac{1}{G_{0,0}\left(u\right)}=\frac{z\left(u\right)-\bar{\alpha}}{\alpha z\left(u\right)},\ \phi_{0,0}\left(1\right)=1. \tag{14}$$

In particular, observing $z'(1)=1/u'(1)=\frac{\alpha}{q(1-\zeta)}$,we get

$$\mathbf{E}\left(\tau_{0,0}\right)=\bar{\alpha}z'\left(1\right)/\alpha=\frac{\bar{\alpha}}{q\left(1-\zeta\right)} \text{ if } \zeta<1 \text{ (positive recurrence),}$$

$$=\infty \text{ if } \zeta=1 \text{ (null recurrence).}$$

Note as required from Kac's theorem: $\mathbf{E}(\tau_{0,0})=1/\pi(0)$.

3.6 Contact probability at 0 and first time to extinction

Also, with $x_0 \geq 1$ ($\phi_{x_0,0}(0)=0$), using (11) and (13),

$$\mathbf{E}\left(u^{\tau_{x_0,0}}\right)=\phi_{x_0,0}\left(u\right)=\frac{G_{x_0,0}\left(u\right)}{G_{0,0}\left(u\right)}=\frac{1}{\alpha}z\left(u\right)^{x_0-1}\left(z\left(u\right)-\bar{\alpha}\right)=\phi_{0,0}\left(u\right)z\left(u\right)^{x_0}, \tag{15}$$

gives the probability generating function of the first hitting time of 0, starting from $x_0 \geq 1$ (the first extinction time of the chain). We also get

$$\mathbf{E}\left(\tau_{x_0,0}\right)=\frac{z'(1)}{\alpha}\left(\alpha x_0+\bar{\alpha}\right)=\frac{\alpha x_0+\bar{\alpha}}{q\left(1-\zeta\right)} \text{ if } \zeta<1 \text{ (positive recurrence),}$$

$$=\infty \text{ if } \zeta=1 \text{ (null recurrence).}$$

If $z''(1)<\infty$ ($\sigma^2(\beta)<\infty$), the variance of $\tau_{x_0,0}$ in the positive-recurrent case is finite and found to be

$$\sigma^2\left(\tau_{x_0,0}\right)=\frac{z''\left(1\right)+z'\left(1\right)}{\alpha}\left(\alpha x_0+\bar{\alpha}\right)-\frac{z'\left(1\right)^2}{\alpha^2}\left(\alpha^2 x_0+1-\alpha^2\right)$$

$$=x_0\left(z''\left(1\right)+z'\left(1\right)-z'\left(1\right)^2\right)+\frac{\bar{\alpha}}{\alpha^2}\left(\alpha\left(z''\left(1\right)+z'\left(1\right)\right)-\left(1+\alpha\right)z'\left(1\right)^2\right).$$

If $\sigma^2(\beta)=\infty$, $\sigma^2(\tau_{x_0,0})=\infty$.
We observe from (14)

$$\phi_{0,0}\left(u\right)=\mathbf{E}\left(u^{\tau_{0,0}}\right)=\frac{z\left(u\right)-\bar{\alpha}}{\alpha z\left(u\right)},\ \text{entailing } z\left(u\right)=\frac{\bar{\alpha}}{1-\alpha\phi_{0,0}\left(u\right)}.$$

The function $\phi_{0,0}(u)$ being a probability generating function, $z(u)$ is the probability generating function of some random variable $\tau \geq 0$; namely, the one of a compound shifted geometric($\overline{\alpha}$) probability generating function with compounding probability generating function $\phi_{0,0}(u)$. And $z(u)^{x_0}$ is the probability generating function of the random variable

$$\tau_{x_0} = \sum_{k=1}^{x_0} \tau_k, \tag{16}$$

where τ_k is an iid sequence with $\tau_k \sim \tau$. We have thus proven

Theorem 6 In the recurrent case, the function $z(u)$ is the probability generating function of some random variable $\tau \geq 0$. With τ_{x_0} defined in (16), the first time to extinction random variable $\tau_{x_0,0}$ is decomposable as

$$\tau_{x_0,0} \overset{d}{=} \tau_{x_0} + \tau_{0,0}, \tag{17}$$

where $\tau_{x_0} \geq 0$ and $\tau_{0,0} \geq 1$ are independent. With $\phi_{0,0}(u)$ given by (14), the probability generating function of $\tau_{x_0,0}$ is given in (15) with $z(u) = \overline{\alpha} / (1 - \alpha\phi_{0,0}(u))$.

The function $u(z)$ is explicitly defined in (9) and $z(u)$ can be obtained from the Lagrange inversion formula as follows: develop $u(z)$ as a power series in $\hat{z} := (z - \overline{\alpha}) / \alpha$, where

$$u(\hat{z}) = \frac{\hat{z}}{q\overline{\alpha} + \hat{z}\left(q\alpha + pB\left(\overline{\alpha} + \alpha\hat{z}\right)\right)}$$

maps [0, 1] to [0, 1]. It can in principle be obtained by Faa di Bruno formula for the composition of analytic functions[13], observing $\hat{z}^{-1}u(\hat{z}) = u_1(u_2(\hat{z}))$ where $u_1(z) = 1/(q\overline{\alpha} + z)$ and $u_2(\hat{z}) = \hat{z}(q\alpha + pB(\overline{\alpha} + \alpha\hat{z}))$ both with well-defined series expansion near 0. Then, by using Lagrange inversion formula[13], we can have $\hat{z}(u)$ satisfied $u(\hat{z}(u)) = u$ and then also the output

$$\mathbf{E}\left(u^{\tau_{x_0,0}}\right) = \frac{1}{\alpha} z(u)^{x_0 - 1}\left(z(u) - \overline{\alpha}\right) = \hat{z}(u)\left(\alpha\hat{z}(u) + \overline{\alpha}\right)^{x_0 - 1}.$$

The probability generating function $z(u)$ is well-defined in the range $u \in [0, 1]$ in the recurrent case and $u \in [0, 1] \to z(u) \in [\overline{\alpha}, 1]$ is its range.

3.7 Harmonic (scale) function and the height of an excursion (recurrent case)

Suppose $\{X_n\}$, with transition matrix P, is recurrent. Let then $P \to \overline{P}$ with $\overline{P}(0, y) = \delta_{0,y}$, forcing state 0 to be absorbing, corresponding to the substitution of processes: $X_n \to Y_n := X_{n \wedge \tau_{x,0}}$ (the process X_n started at x_0 and stopped when it first hits 0). The matrix $\overline{P}$ is stochastic but non-irreducible, having {0} as an absorbing class. For the process Y_n with transition matrix $\overline{P}$, eventually $Y_\infty = 0$ with probability 1. The harmonic (or scale) sequence $\boldsymbol{\phi}$ solves:

$$\overline{P}\boldsymbol{\phi} = \boldsymbol{\phi}, \tag{18}$$

where $\boldsymbol{\phi} = (\phi(0), \phi(1), ...)'$ is the right column eigenvector of $\overline{P}$ associated to the eigenvalue 1. With $h \gg 1$ and $x \in \{1, ..., h - 1\}$, let

$$\tau_{x,h} = \inf(n \geq 0 : X_n \geq h \mid X_0 = x) \text{ if such an } n \text{ exists,}$$

$$= +\infty \text{ if not.}$$

By induction, with $\phi(0) := 0$ and $\tau_x^{(h)} := \tau_{x,0} \wedge \tau_{x,h}$

$$\forall x \in \{1,...,h-1\},\ \forall n \geq 0:\ \phi(x) = \mathbf{E}\phi\left(Y_{n \wedge \tau_{x,h}}\right) = \mathbf{E}\phi\left(X_{n \wedge \tau_x^{(h)}}\right).$$

The harmonic function ϕ on $\{1, ..., h\}$, makes $\phi(X_{n \wedge \tau_x^{(h)}})$ a martingale. Therefore, as $n \to \infty$, $\forall x \in \{1, ..., h-1\}$:

$$\phi(x) = \mathbf{E}\phi\left(X_{\tau_x^{(h)}}\right) = \phi(0)\mathbf{P}\left(\tau_{x,0} < \tau_{x,h}\right) + \mathbf{E}\phi\left(X_{\tau_{x,h}}\right)\mathbf{P}\left(\tau_{x,h} < \tau_{x,0}\right). \tag{19}$$

As $h \to \infty$, both $\tau_{x,h}$ and the over shoot $X_{\tau_{x,h}} \to \infty$. Assuming there is a solution $\phi(x) \geq 0$ such that $\phi(0) = 0$ and $\phi(x) \to \infty$ as $x \to \infty$, owing to $X_{\tau_{x,h}} \to \infty$ yields

$$\mathbf{P}\left(\tau_{x,h} < \tau_{x,0}\right) = \frac{\phi(x)}{\mathbf{E}\phi\left(X_{\tau_{x,h}}\right)} \underset{h \to \infty}{\to} 0 = \mathbf{P}\left(\tau_{x,\infty} < \tau_{x,0}\right), \tag{20}$$

indeed consistent with the guess: Y_n does not escape, it goes extinct with probability 1, whatever the initial condition x is.

Let us now consider the problem of evaluating the law of the height H of an excursion in the recurrent case. Firstly, $H = 0$ with probability q (corresponding to a trivial excursion). Now, for nontrivial excursions, the event $H \geq h \geq 1$ is realized whenever a first birth event occurs with size $\beta_1 \geq h$ or, if $0 < \beta_1 < h$, when the time (starting from β_1) needed to first hit $(h, h + 1, ...\}$ is less than the time needed to first hit 0, namely the event $\tau_{\beta_1,h} < \tau_{\beta_1,0}$. Hence,

$$\mathbf{P}(H \geq h) = p\mathbf{P}(\beta_1 \geq h) + p\sum_{x=1}^{h-1} \mathbf{P}_0(\beta_1 = x)\mathbf{P}\left(\tau_{x,h} < \tau_{x,0}\right),$$

where $\mathbf{P}(\beta_1 = x) = bx$, $\mathbf{P}(\beta_1 \geq h) = \Sigma_{y \geq h} b_y$ and $\mathbf{P}(\tau_{x,h} < \tau_{x,0})$ is given in (20). Note $\mathbf{P}(H \geq 1) = p$ and, as required since $\mathbf{P}(\tau_{x,\infty} < \tau_{x,0}) = 0$, $H < \infty$ with probability 1.

4. The transient case

In the transient case ($\zeta > 1$ or $\rho > \rho_c$), the denominator of $\Phi_{x_0}(u(z), z)$ tends to 0 as $u \to u(z)$ and $u(z)$ does not cancel the numerator: $u(z)$ is a true pole of $\Phi_{x_0}(u, z)$. When $z \in [\overline{\alpha}, 1]$, $u(z)$ is concave but it has a maximum $u(z_*)$ strictly larger than 1, attained at some z_* inside $(\overline{\alpha}, 1)$, owing to $u(\overline{\alpha}) = 0$, $u(1) = 1$ and $u'(1) < 0$. We anticipate that for some constant $C > 0$ depending on x_0,

$$\mathbf{P}_{x_0}(X_n = 0) \sim C \cdot u(z_*)^{-n} \text{ as } n \to \infty,$$

stating that $\{X_n\}$ only visits 0 a finite number of times before drifting to ∞.

4.1 No nontrivial ($\neq 0$) invariant measure in the transient case

Before turning to this question, let us observe the following: suppose $\zeta > 1$ ($\rho > \rho_c$), so that the super-critical geometric chain is transient. For the same reason as for the null-recurrent case, the chain has no nontrivial ($\neq 0$) invariant positive measure either. It is not Harris-transient[16].

4.2 Large deviations

Consider $v(z) := 1/u(z)$ where $u(z)$, as a pole, cancels the denominator of $\Phi_{x_0}(u, z)$ without cancelling its numerator, so

$$v(z) = \frac{q\alpha z + p(z - \overline{\alpha})B(z)}{z - \overline{\alpha}}.$$

Over the domain $1 \geq z > \overline{\alpha}$, $v(z)$ is convex with

$$v'(z) = pB'(z) - q\alpha\bar{\alpha}(z-\bar{\alpha})^{-2} \text{ and } v''(z) = pB''(z) + 2q\alpha\bar{\alpha}(z-\bar{\alpha})^{-3} > 0.$$

Thus

$$v(1) = 1 \text{ and } v'(1) = \frac{p\alpha\rho - q\bar{\alpha}}{\alpha} = p\rho - q\bar{\alpha}/\alpha.$$

We have $v'(1) > 0$ if and only if the chain is transient. In this transient case,

$$\Phi_n(z)^{1/n} \to v(z) \text{ as } n \to \infty.$$

Define

$$F(\lambda) := -\log v\left(e^{-\lambda}\right) = \log u\left(e^{-\lambda}\right).$$

The function $F(\lambda)$ is concave on its definition domain $\lambda \in [0, -\log\bar{\alpha})$. It first increases, attains a maximum and then decreases to $-\infty$ while crossing zero in between. There exists $\lambda^* \in (0, -\log\bar{\alpha})$ such that $\omega_* = F'(\lambda^*) = 0$.

With $\omega \in (F'(-\log\bar{\alpha}), F'(0)]$, define

$$f(\omega) = \inf_{0 \le \lambda < -\log\bar{\alpha}} \left(\omega\lambda - F(\lambda)\right) \le 0, \tag{21}$$

the Legendre conjugate of $F(\lambda)$. The variable ω is Legendre conjugate to λ with $\omega = F'(\lambda)$ and $\lambda = f'(\omega)$. Note $F'(-\log\bar{\alpha}) = -\infty$ and $F'(0) = v'(1) > 0$. On its definition domain, $f(\omega) \le 0$ is increasing and concave, starting from $f(-\infty) = -\infty$ and ending with $f(F'(0)) = 0$ where $f'(F'(0)) = 0$.

From [21], we get:

Proposition 7 For those ω in the range $[\omega_* = 0, F'(0)]$ and for any $x_0 > 0$,

$$\frac{1}{n}\log \mathbf{P}_{x_0}\left(\frac{1}{n}X_n \le \omega\right) \underset{n\to\infty}{\to} f(\omega). \tag{22}$$

In particular, at $\omega = F'(0) = v'(1) > 0$ where $f(F'(0)) = -F(0) = 0$, we get

$$\frac{1}{n}X_n \overset{a.s.}{\to} v'(1) \text{ as } n \to \infty.$$

To keep $\omega = F'(\lambda)$ in the nonnegative range $[\omega_* = 0, F'(0)]$, the range of λ should then equivalently be restricted to $[0, \lambda^*]$. We clearly have $f(\omega_*) = -F(\lambda^*) < 0$ and, from (22),

$$-\frac{1}{n}\log \mathbf{P}_{x_0}\left(\frac{1}{n}X_n \le 0\right) = -\frac{1}{n}\log \mathbf{P}_{x_0}\left(X_n \le 0\right) \to F\left(\lambda^*\right) \text{ as } n \to \infty. \tag{23}$$

This shows the rate at which $\mathbf{P}_{x_0}(X_n = 0)$ decays exponentially with n. Equivalently, with $z_* = e^{-\lambda}$, $\mathbf{P}(X_n = 0) \sim C \cdot u(z_*)^{-n}$ as guessed and the Green series $\Phi_{x_0}(1, 0) = G_{x_0,0}(1) = \Sigma_{n\ge1}\mathbf{P}_{x_0}(X_n = 0)$ now is summable for all $x_0 \ge 0$ (translating that $\{X_n\}$ visits 0 only a finite number of times). In the transient case,

$$\mathbf{P}\left(\tau_{x_0,0} < \infty\right) = \phi_{x_0,0}(1) = \frac{G_{x_0,0}(1)}{G_{0,0}(1)} < 1.$$

4.3 The scale function and the extinction probability

In the transient case, the chain $\{X_n\}$ started at $x > 0$ can drift to ∞ before it first hits 0. There is thus only a probability smaller than 1 that $\{X_n\}$ gets extinct for the first time.

In the transient case, let $P \to \overline{P}$ with $\overline{P}\ (0, y) = \delta_{0,\, y}$ forcing state 0 to be absorbing, corresponding to: $X_n \to Y_n := X_{n\wedge\tau_{x,0}}$. The matrix $\overline{P}$ is stochastic but non irreducible, having an absorbing class. The harmonic (or scale) sequence $\boldsymbol{\phi}$ solves:

$$\overline{P}\boldsymbol{\phi} = \boldsymbol{\phi}, \tag{24}$$

where $\boldsymbol{\phi} = (\phi(0), \phi(1), \ldots)'$ is a column vector. With $h \gg 1$ and $x \in (1, \ldots, h-1\}$, let

$$\tau_{x,h} = \inf\left(n \geq 0 : X_n \geq h \mid X_0 = x\right) \text{ if such an } n \text{ exists,}$$

$$= +\infty \text{ if not.}$$

By induction, with $\phi(0) = 1$ and $\tau_x^{(h)} := \tau_{x,0} \wedge \tau_{x,h}$

$$\forall x \in \{1, \ldots, h-1\},\ \forall n \geq 0\text{: } \phi(x) = \mathbf{E}\phi\left(Y_{n\wedge\tau_{x,h}}\right) = \mathbf{E}\phi\left(X_{n\wedge\tau_x^{(h)}}\right).$$

The harmonic function on $\{1, \ldots, h\}$, makes $\boldsymbol{\phi}(X_{n\wedge\tau_x^{(h)}})$ a martingale[22]. As $n \to \infty$, $\forall x \in \{1, \ldots, h-1\}$:

$$\phi(x) = \mathbf{E}\phi\left(X_{\tau_x}\right) = \phi(0)\mathbf{P}\left(\tau_{x,0} < \tau_{x,h}\right) + \mathbf{E}\phi\left(X_{\tau_{x,h}}\right)\mathbf{P}\left(\tau_{x,h} < \tau_{x,0}\right). \tag{25}$$

As $h \to \infty$, both $\tau_{x,h}$ and the overshoot $X_{\tau_{x,h}} \to \infty$. Assuming there is a solution $\phi(x) > 0$ such that $\phi(x) \to 0$ as $x \to \infty$, yields

$$\mathbf{P}\left(\tau_{x,0} < \tau_{x,\infty}\right) = \phi(x), \tag{26}$$

indeed consistent with the guess: $\phi > 0$ and vanishing at ∞. This expression also shows that $\phi(x)$ must be decreasing with x. Note $\mathbf{P}(\tau_{x,\,0} < \tau_{x,\,\infty}) = \mathbf{P}(X_{\tau_x} = 0)$ where $\tau_x := \tau_{x,\,0} \wedge \tau_{x,\,\infty}$. We obtained:

Proposition 8 In the transient case, with $\boldsymbol{\phi}$ defined in (24) obeying $\phi(0) = 1$,

$$\phi(x) = \mathbf{P}(\tau_{x,\,0} < \tau_{x,\,\infty}),$$

is the probability of extinction starting from state x.

Remark: The function $\overline{\boldsymbol{\phi}} := 1 - \boldsymbol{\phi}$ clearly also is a (increasing) harmonic function so that so is any convex combination of $\boldsymbol{\phi}$ and $\overline{\boldsymbol{\phi}}$ [18]. Clearly, $\overline{\phi}(x) = \mathbf{P}(\tau_{x,\,\infty} < \tau_{x,\,0}) = \mathbf{P}(X_{\tau_x} = \infty)$.

4.4 The height of an excursion (transient case)

The law of the height H of an excursion is given by

$$\mathbf{P}\left(H \geq h\right) = \mathbf{P}_0\left(X_1 \geq h\right) + \sum_{x=1}^{h-1}\mathbf{P}_0\left(X_1 = x\right)\mathbf{P}\left(X_{\tau_x} \geq h\right)$$

$$= \mathbf{P}_0\left(X_1 \geq h\right) + \sum_{x=1}^{h-1}\mathbf{P}_0\left(X_1 = x\right)\mathbf{P}\left(\tau_{x,h} < \tau_{x,0}\right),$$

where $\mathbf{P}_0(X_1 = x) = P(0, x) = pb_x$, $\mathbf{P}_0(X_1 \geq h) = p\Sigma_{y \geq h} b_y$ and, from (25),

$$\mathbf{P}\left(\tau_{x,h} < \tau_{x,0}\right) = \frac{1-\phi(x)}{1-\mathbf{E}\phi\left(X_{\tau_{x,h}}\right)}.$$

Note, as required, that $H = 0$ with probability q and

$$H = \infty \text{ with probability } \sum_{x \geq 1} \mathbf{P}_0\left(X_1 = x\right)\overline{\phi}\left(x\right).$$

4.5 Doob transform: conditioning on non-extinction

In the transient case when the chain can either drift to infinity or go extinct, the harmonic sequence ϕ plays some additional role in a conditioning. With D_ϕ := diag($\phi(0)$, $\phi(1)$, ...), the matrix

$$P_\phi := D_\phi^{-1} \overline{P} D_\phi$$

is a stochastic matrix. We have $P_\phi(m, 0) = \delta_{m, 0}$ so that state 0 is inaccessible from any other state than 0 itself: state 0 is isolated and disconnected and so may be removed from the state space. We have[22-23]:

Proposition 9 In the transient case, P_ϕ is the one-step transition matrix corresponding to the process $X_{n \wedge \tau_x}$ conditioned not to be absorbed at 0.

Remark: this is a selection of paths procedure allowing to focus only on those paths of the transient chain that do not go extinct.

5. Concluding remarks

We revisited the geometric catastrophe model in discrete-time, as a Markovian population dynamics on the non-negative integers. For this process, a collapse move from some state is geometrically distributed so long as it does not exhaust the current value of this state. It is balanced by random growth moves with arbitrary distribution. This process may be viewed as a generalized birth and death chain. Using generating function techniques, conditions under which the birth and death competing events yield a process that is stable (recurrent) have been highlighted. When recurrent, the shape of the invariant probability measure was described. When unstable (transient), the chain either drifts to infinity or goes extinct, a feature similar to supercritical branching processes[24]. The height and length of excursions, extinction probability, time to extinction, have been studied both in the recurrent and transient setups. The harmonic (scale) function was shown to play an important role in the analysis.

Other interesting random walks in the same spirit, but with different collapse rules, were introduced in [4]. We plan, as we do here for the truncated geometric case, to lift the veil on some of their intrinsic statistical properties which are expected to be of a completely different nature.

Acknowledgments

T.H. acknowledges partial support from the labex MME-DII (Dynamic mathematics and economic model, Uncertainty and interaction), ANR11-LBX-0023-01. This work also benefited from the support of the Chair “Mathematical modeling and biodiversity” of Veolia-Ecole Polytechnique-MNHN-Fondation X.

References

[1] Swift R. J. Transient probabilities for a simple birth-death-immigration process under the influence of total catastrophes. *Internat. J. Math. Math. Sci.* 2001; 25: 689-692.

[2] Goncalves B., Huillet T. Scaling features of two special Markov chains involving total disasters. *J. Stat. Phys.* 2020; 178: 499-531.

[3] Brockwell P. J., Gani J., Resnick S. I. Birth, immigration and catastrophe processes. *Adv. in Appl. Probab.* 1982;

14(4): 709-731.
[4] Neuts M. F. An interesting random walk on the non-negative integers. *J. Appl. Probab.* 1994; 31: 48-58.
[5] Ben-Ari I., Roitershtein A., Schinazi R. B. A random walk with catastrophes. *Electron. J. Probab.* 2019; 24: 28.
[6] Fontes L. R., Schinazi R. B. Metastability of a random walk with catastrophes. *Electron. J. Probab.* 2019; 24: 1-10.
[7] Artalejo J. R., Economou A., Lopez-Herrero M. J. Evaluating growth measures in an immigration process subject to binomial and geometric catastrophes. *Math. Biosci. Eng.* 2007; 4(4): 573-594.
[8] Cairns B., Pollett P. K. Extinction times for a general birth, death and catastrophe process. *J. Appl. Probab.* 2004; 41(4): 1211-1218.
[9] Eliazar I., Klafter J. Growth-collapse and decay-surge evolutions, and geometric Langevin equations. *Physica A* 2006; 387: 106-128.
[10] Eliazar I., Klafter J. Nonlinear Shot Noise: From aggregate dynamics to maximal dynamics. *Euro-Phys. Lett.* 2007; 78: 40001.
[11] Eliazar I., Klafter J. The maximal process of nonlinear shot noise. *Physica A.* 2009; 388: 1755-1779.
[12] Neuts M. F. *Matrix-Geometric-Solutions in Stochastic Models: an algorithmic approach*. Chapter 2: Probability Distributions of Phase-Type. Dover Publications Inc.; 1981.
[13] Comtet L. *Analyse combinatoire*. (p.146, 159) Tome 1. Paris: French University Press; 1970.
[14] Kac M. Random walk and the theory of Brownian motion. *Amer. Math. Monthly.* 1947; 54(7): 369-391.
[15] Harris T. E. The existence of stationary measures for certain Markov processes. *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability*. Berkeley and Los Angeles: University of California Press; 1956; 2: 113-124.
[16] Harris T. E. Transient Markov chains with stationary measures. *Proc. Amer. Math. Soc.* 1957; 8: 937-942.
[17] Jamison B., Orey S. Markov chains recurrent in the sense of Harris. *Z. Wahrsch. Theo. verw. Geb.* 1967; 8: 41-48.
[18] Neveu J. Chaines de Markov et theorie du potentiel. *Ann. Fac. Sci. Univ. Clermont-Ferrand.* 1964; 24: 37-89.
[19] Feller W. *An introduction to probability theory and its applications.* New York: Wiley; 1971.
[20] Bingham N. H. Random walk and fluctuation theory. (p.3-4) *Handbook of Statistics.* 2001: 19: 171-213. Available from: https://doi.org/10.1016/S0169-7161(01)19009-7.
[21] Varadhan S. R. S. Large deviations (Special invited paper). *The Ann. of Prob.* 2008; 36(2): 397-419.
[22] Norris J. R. *Markov chains.* Cambridge University Press; 1998.
[23] Rogers L. C., Williams D. *Diffusions, Markov Processes and Martingales.*(p.327) Chichester; 1994.
[24] Harris T. E. *The theory of branching processes.* Berlin: Springer-Verlag; 1963.