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Research Article

Some New Fractional Interval-Valued Inequalities for Set-Valued H(α, $1 - \alpha$)-Godunova-Levin Mappings with Applications

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Abstract: Convex integral inequalities are an area of substantial interest in mathematical analysis because of their applications to a variety of fields such as optimization, probability theory, and functional analysis. This study derives general forms of convex integral inequalities and presents several new results in the context of $\mathbf{H}(\alpha,\ 1-\alpha)$-Godunova-Levin mappings via **AB** fractional integral operators, including Jensen's, Pachapatte, Ostrowski, and Hermite-Hadamard. Additionally, we developed a new type of Jensen-type inequality in sequential form as well as a new Ostrowki-type inequality using a Moore-metric Hausdorff distance approach, which is really an innovative approach to such inequality. Our analysis of convex integral inequalities introduces novel bounds and constraints that characterize the behavior of generalized convex functions. We have further developed several remarks to demonstrate the accuracy of our results that lead to several other generalized convex mappings that have never been introduced so far for this type of generalized mapping, as well as several interesting non-trivial examples. Additionally, we show that our results also correlated with special means as an application configured appropriately.

***Keywords*:** Hermite-Hadamard, Pachpatte, Ostrowski-type, Atangana-Baleanu, Jensen-type inequality, $\mathbf{H}(\alpha,\ 1-\alpha)$-Godunova-Levin

MSC: 26D15, 26A51, 26A33

1. Introduction

Fractional calculus represents a significant extension of classical calculus, enabling the analysis of phenomena that cannot be adequately described by integer-order derivatives and integrals. Over the centuries, mathematicians like Euler, Laplace, Riemann, and Liouville contributed to its development. However, it wasn't until the 20th century that fractional calculus started gaining significant attention and applications. Fractional calculus often provides more accurate and flexible models for real-world phenomena. It can capture memory effects, non-locality, and complex dynamics that traditional integer-order calculus might miss. It has numerous applications in various fields of science and engineering,

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including: It is applied in the analysis of electromagnetic fields and waves, where fractional derivatives help model complex behaviors. Fractional calculus is utilized in control systems to model and analyze dynamical systems, enhancing the design of controllers for systems with memory effects. In materials science, fractional calculus models the behavior of viscoelastic materials, which exhibit both viscous and elastic characteristics, allowing for a more accurate representation of their stress-strain relationships. For some other applications in various domains, check [1–5] and the references therein.

Convex analysis provides a powerful mathematical framework for analyzing problems in various fields, especially due to the well-behaved characteristics of convex functions and sets. Its uses span a variety of fields, including control theory, economics, machine learning, and optimization. In control theory, systems are often formulated as convex problems, where the system needs to minimize energy or error subject to dynamic constraints; in signal processing, it aids in the design of codes that minimize transmission errors, enhancing communication reliability. Convex analysis is intimately related to economic theory, notably the study of utility functions, which depict rational consumer preferences in which utility grows with consumption but at a decreasing pace. For more current uses in several disciplines of applied sciences, we refer to [6–10] and the references therein.

Interval analysis is a mathematical methodology that allows numerical algorithms to address uncertainty more rigorously. It has applications in a variety of domains, including numerical computation, global optimization, control systems, engineering, and computer graphics. Borwein et al. [11] initially defined convex interval-valued functions (IVFs) in 1981, and since then, several researchers have extended and promoted different types of convexity by using IVFs. For example, include preinvex [12], harmonic convex [13], Godunova-Levin [14], (h_1, h_2)-convex [15], log-convex [16], coordinated convex [17], and various others [18–20] and the references therein. It's important to remember that the partial order relation defines these convex IVFs, meaning that any two intervals may not be comparable. This indicates that the maximum-minimum problem cannot be solved since it is impossible to determine which of them is the greatest or smallest interval using these orderings. Hu and Wang [21] introduced the *cr*-order, which takes into account the midpoint and radius of two intervals to address this limitation. This order is total, meaning that any two interval numbers are comparable. In 2020, the authors in [22] provided the appropriate constrained conditions for the objective function and provided a novel definition of convex IVFs using *cr*-order.

Convex inequalities and fractional calculus are intimately related, especially in optimization, stability analysis, and complex system modeling. Convexity simplifies fractional-order system analysis, allowing tools such as Jensen's and Hermite-Hadamard (H-H) inequalities to be used to tackle real-world issues in engineering, economics, and applied sciences. The inequality is defined as follows:

Suppose $\Psi : \Omega \subseteq \mathsf{R} \to \mathsf{R}$ be a mapping defined on the convex set over the interval Ω with $\sigma_1, \sigma_2 \in \Omega$. Then, the inequality stated below is true:

$$\Psi\left(\frac{\sigma_1+\sigma_2}{2}\right) \le \frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\Psi(\Theta)\, d\Theta \le \frac{\Psi(\sigma_1)+\Psi(\sigma_2)}{2}. \tag{1}$$

In recent years, academics have proposed numerous versions of the Hermite-Hadamard inequality, taking into account other convexity classes or applying the inequality to new types of functions. The unifying idea behind these generalizations is to extend the concept of convexity and apply it to new types of functions, leading to more refined versions of the classical inequality. Each generalization involves adjusting the bounds or introducing new terms that capture the behavior of the specific class of functions under consideration. In 2007, Sanja [23] developed the idea of *h*-convex mappings and refined the conventional Hermite-Hadamard inequality in a new way with several interesting applications. Subsequently, several authors expand and broaden this concept by employing more generalized forms of convexities.

In 2018, Awan et al. [24] proposed a notion of $(\mathsf{h}_1, \mathsf{h}_2)$-convex mappings and created several product forms of Hermite-Hadamard type inequalities, generalizing Sanja's idea of *h*-convex mappings.

Theorem 1.1 Let $\Psi : \Omega \subseteq \mathsf{R} \to \mathsf{R}$. If Ψ is $(\mathsf{h}_1, \mathsf{h}_2)$-convex, and $\mathsf{h}_1\left(\frac{1}{2}\right)\mathsf{h}_2\left(\frac{1}{2}\right) \neq 0$. Then

$$\frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\Psi\left(\frac{\sigma_1+\sigma_2}{2}\right) \le \frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\Psi(\Theta)\,d\Theta \le [\Psi(\sigma_1)+\Psi(\sigma_2)]\int_0^1 h_1(\eth)h_2(1-\eth)d\eth.$$

Later in 2019, An et al. [25] further generalized the concept of (h_1, h_2)-convex mappings by introducing interval-valued mappings utilizing inclusion relations.

Theorem 1.2 Let $\Psi, \Phi : [\sigma_1, \sigma_2] \to R_I^+$, $h_1, h_2 : (0, 1) \to R^+$ such that $H\left(\frac{1}{2}, \frac{1}{2}\right) \neq 0$. If $\Psi, \Phi \in SX((h_1, h_2), [\sigma_1, \sigma_2], R_I^+)$ and $\Psi\Phi \in IR_{([\sigma_1, \sigma_2])}$, then

$$\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\Psi(\Theta)\Phi(\Theta)\,d\Theta \supseteq M(\sigma_1, \sigma_2)\int_0^1 H^2(\eth, 1-\eth)d\eth$$

$$+N(\sigma_1, \sigma_2)\int_0^1 H(\eth, \eth)H(1-\eth, 1-\eth)d\eth.$$

Following these results, numerous authors used different forms of generalized convex mappings and created various sorts of relevant inequalities connected to the presented results in this note. For example, Afzal et al. [26, 27] used (h_1, h_2)-Godunova-Levin convex and harmonic convex functions to build numerous integral inequalities, including H-H and its different variants, as well as Jensen-type inequalities by using classical the Riemann integral operator. Khan et al. [28] used (h_1, h_2)-convex fuzzy valued mappings to establish several integral inequalities, including H-H and its different product and symmetric forms. Bai et al. [29] constructed H-H and Jensen-type inclusions by combining interval (h_1, h_2)-non-convex functions with p-convex mappings. Ahmadini et al. [30] utilized interval preinvex (h_1, h_2)-Godunova-Levin functions and produced Trapezium, weighted Fejer, and H-H type inclusions with applications to means and special functions. Sahoo et al. [31] used interval-valued (m, h_1, h_2)-Godunova-Levin functions defined on the harmonic set to develop distinct product forms of Hermite-Hadamard type inclusions. Yasin et al. [32] proposed a notion of (h_1, h_2, s)-convex functions defined on an s-convex set and created several product forms of H-H and Fejer's type results. Jesus Medina-Viloria [33] presented a notion of (m, h_1, h_2)-convex functions defined on harmonic-convex set and discussed different features and developed product forms of H-H and Fejer's type results. For some other relevant results connected to the outcomes established in this note using standard and inclusion order relations, see [34–39] and the references therein.

In 2021, the authors [40] show that the inclusion ($\subseteq$) relation has problems since they lack the property of comparability between intervals, which they show in example 3 that was constructed for central Milne type inequality.

Theorem 1.3 Suppose that $\Psi, \Phi : [0, 1] \to R$ are two positive and Lebesgue integrable functions. Additionally, if Ψ, Φ are comonotone functions, the following inequality is not true:

$$\int_0^1 \frac{\Psi\Phi}{\Psi+\Phi}\,d\eth \int_0^1 (\Psi+\Phi)d\eth \not\le \int_0^1 \Psi\,d\eth \int_0^1 \Phi\,d\eth. \tag{2}$$

To address this problem in 2022, Liu et al. [41] employed a novel kind of order relation known as the cr-total order relation, which includes a number of additional features such as integral preservation and interval-order comparability that are absent in inclusion relations in multiple outcomes. Taking motivation from these findings Saeed et al. [42] used two different forms of generalized (h_1, h_2)-type mappings defined on convex and harmonic Godunova-Levin convex sets and developed several outputs linked to these results.

Theorem 1.4 Let $\Psi, \Phi : [\sigma_1, \sigma_2] \to \mathsf{R}^+_\mathrm{I}, \mathsf{h}_1, \mathsf{h}_2 : (0, 1) \to \mathsf{R}^+$ such that $\mathsf{H}\left(\frac{1}{2}, \frac{1}{2}\right) \neq 0$. If $\Psi, \Phi \in \mathsf{SGHX}((cr\text{-}\mathsf{h}_1, \mathsf{h}_1), [\sigma_1, \sigma_2], \mathsf{R}^+_\mathrm{I})$ and $\Psi\Phi \in \mathsf{IR}_{([\sigma_1, \sigma_2])}$, then

$$\frac{\left[\mathsf{H}\left(\frac{1}{2}, \frac{1}{2}\right)\right]}{2}\Psi\left(\frac{2\sigma_1\sigma_2}{\sigma_1+\sigma_2}\right) \leq_{\mathbf{CR}} \frac{\sigma_1\sigma_2}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\frac{\Psi(\eth)}{\eth^2}d\eth \leq_{\mathbf{CR}} [\Psi(\sigma_1)+\Psi(\sigma_2)]\int_0^1 \frac{d\eth}{\mathsf{H}(\eth, 1-\eth)}.$$

Other relevant results obtained by using different classes of convex mappings under cr-total order relation are given in [43–45] and its references.

1.1 *Originality and importance*

This study is considered important and original because we developed a new Jensen-type inequality in sequential form and a new Ostrowki type inequality using a Moore metric Hausdorff distance approach by using a generalized $\mathbf{H}(\alpha, 1-\alpha)$-Godunova-Levin mappings. Furthermore, we produced Hermite-Hadamard and its different products by utilizing fractional integral operators, while recently the result obtained by using classical standard integral operators. We have added a few remarks to show the precision of our findings, which result in a number of additional generalized convex mappings that have never been shown before, along with a number of intriguing non-trivial cases.

The format of this article is as follows. In Section 2, we should review some crucial ideas connected to fractional and interval calculus, as well as certain fundamental definitions and results required to proceed with the article. In Section 3, we discuss our main findings, including Jensen, Ostrowski, and distinct product forms of Hermite-Hadamard type inequalities, with several noteworthy cases and remarks. In Section 4, we relate some of our main findings to show some applications for special means. In Section 5, we will discuss the results and draw conclusions. Finally, in Section 6, we forecast some potential future work for interested readers.

2. Preliminaries

This part begins by reviewing several key definitions and results from fractional calculus, interval analysis, and other relevant results used in creating primary outcomes. The real closed and bounded interval $[\underline{\sigma}, \overline{\sigma}]$ is defined as

$$[\underline{\sigma}, \overline{\sigma}] = \{\mathrm{x} \in \mathsf{R} : \underline{\sigma} \leq \mathrm{x} \leq \overline{\sigma}\},$$

where $\underline{\sigma} \leq \overline{\sigma}$ and $\underline{\sigma}, \overline{\sigma} \in \mathsf{R}$. We designate by $\mathbf{L}(\alpha)$ the length of the interval $[\underline{\sigma}, \overline{\sigma}]$. If $\mathbf{L}(\alpha) = 0$, then $[\underline{\sigma}, \overline{\sigma}]$ is degenerated. If $\underline{\sigma} > 0$, then $[\underline{\sigma}, \overline{\sigma}]$ is described as a positive interval. Analogously, if $\overline{\sigma} < 0$, then $[\underline{\sigma}, \overline{\sigma}]$ denotes a negative interval. All real intervals of R and all positive intervals are indicated by R_I and R^+_I, respectively.

If $[\underline{\sigma}, \overline{\sigma}], [\underline{\nu}, \overline{\nu}] \in \mathsf{R}_\mathsf{I}$ and $\Gamma \in \mathsf{R}$, then the interval $\Gamma[\underline{\sigma}, \overline{\sigma}]$ is represented as follows:

$$\Gamma[\underline{\sigma}, \overline{\sigma}] = \begin{cases} [\Gamma\underline{\sigma}, \Gamma\overline{\sigma}] & \text{if } \Gamma > 0, \\ \{0\} & \text{if } \Gamma = 0, \\ [\Gamma\overline{\sigma}, \Gamma\underline{\sigma}] & \text{if } \Gamma < 0 \end{cases}$$

and the four operations of arithmetic are given as

$$[\underline{\sigma}, \overline{\sigma}] + [\underline{\nu}, \overline{\nu}] = [\underline{\sigma} + \underline{\nu}, \overline{\sigma} + \overline{\nu}],$$

$$[\underline{\sigma}, \overline{\sigma}] - [\underline{\nu}, \overline{\nu}] = [\underline{\sigma} - \underline{\nu}, \overline{\sigma} - \overline{\nu}],$$

$$[\underline{\sigma}, \overline{\sigma}] \cdot [\underline{\nu}, \overline{\nu}] = [\min\{\underline{\sigma\nu}, \underline{\sigma}\overline{\nu}, \overline{\sigma}\underline{\nu}, \overline{\sigma\nu}\}, \max\{\underline{\sigma\nu}, \underline{\sigma}\overline{\nu}, \overline{\sigma}\underline{\nu}, \overline{\sigma\nu}\}],$$

$$[\underline{\sigma}, \overline{\sigma}]/[\underline{\nu}, \overline{\nu}] = [\min\{\underline{\sigma}/\underline{\nu}, \underline{\sigma}/\overline{\nu}, \overline{\sigma}/\underline{\nu}, \overline{\sigma}/\overline{\nu}\}, \max\{\underline{\sigma}/\underline{\sigma}, \underline{\sigma}/\overline{\nu}, \overline{\sigma}/\underline{\nu}, \overline{\sigma}/\overline{\nu}\}],$$

where $0 \notin [\underline{\nu}, \overline{\nu}]$.

The Hausdorff metric on R_I^+ is defined by

$$\mathbf{M}(\mathtt{C}, \mathtt{D}) = \max\{\mathbf{d}(\mathtt{C}, \mathtt{D}), \mathbf{d}(\mathtt{D}, \mathtt{C})\}, \tag{3}$$

where $\mathtt{d}(\mathtt{C}, \mathtt{D}) = \max_{\mathtt{c}\in\mathtt{C}} \mathtt{d}(\mathtt{c}, \mathtt{D})$ and $\mathtt{d}(\mathtt{c}, \mathtt{D}) = \min_{\mathtt{d}\in\mathtt{D}} \mathtt{d}(\mathtt{c}, \mathtt{d}) = \min_{\mathtt{d}\in\mathtt{D}} |\mathtt{c} - \mathtt{d}|$.

Remark 2.1 An similar form of the Hausdorff metric found in (3) is as follows:

$$\mathbf{M}([\underline{\sigma}, \overline{\sigma}], [\underline{\nu}, \overline{\nu}]) = \max\{|\underline{\sigma} - \underline{\nu}|, |\overline{\sigma} - \overline{\nu}|\}. \tag{4}$$

For $[\underline{\sigma}, \overline{\sigma}], [\underline{\nu}, \overline{\nu}] \in \mathsf{R}_\mathsf{I}$, the center and radius order "$\preceq_{\mathbf{CR}}$" is defined as below.

Definition 2.1 (see [43]) The centre and radius form is another way to express intervals. This form of relation is defined as for some intervals $\mathtt{A}_1 = [\underline{\sigma_1}, \overline{\sigma_2}] = \langle \sigma_\mathbf{C}, \sigma_\mathbf{R} \rangle = \left\langle \frac{\underline{\sigma_1} + \overline{\sigma_2}}{2}, \frac{\overline{\sigma_2} - \underline{\sigma_1}}{2} \right\rangle$, $\mathtt{A}_2 = [\underline{\nu_1}, \overline{\nu_2}] = \langle \nu_\mathbf{C}, \nu_\mathbf{R} \rangle = \left\langle \frac{\underline{\nu_1} + \overline{\nu_2}}{2}, \frac{\overline{\nu_2} - \underline{\nu_1}}{2} \right\rangle$ are represented as:

$$\mathtt{A}_1 \preceq_{\mathbf{CR}} \mathtt{A}_2 \Longleftrightarrow \begin{cases} \sigma_\mathbf{C} < \nu_\mathbf{C}, & \text{if } \sigma_\mathbf{C} \neq \nu_\mathbf{C}; \\ \sigma_\mathbf{R} \leq \nu_\mathbf{R}, & \text{if } \sigma_\mathbf{R} = \nu_\mathbf{R}. \end{cases}$$

The relation $\preceq_{\mathbf{CR}}$ has the following relational features for any three intervals $\mathtt{A}_1 = [\underline{\sigma_1}, \overline{\sigma_2}] = \langle \sigma_\mathbf{C}, \sigma_\mathbf{R} \rangle$, $\mathtt{A}_2 = [\underline{\nu_1}, \overline{\nu_2}] = \langle \nu_\mathbf{C}, \nu_\mathbf{R} \rangle$ and $\mathtt{H}_3 = [\underline{\eta_1}, \overline{\eta_2}] = \langle \eta_\mathbf{C}, \eta_\mathbf{R} \rangle$: **Reflexivity:** $\mathtt{A}_1 \preceq_{\mathbf{CR}} \mathtt{A}_1$. **Anti-symmetry:** $\mathtt{A}_1 \preceq_{\mathbf{CR}} \mathtt{A}_2$ and $\mathtt{A}_2 \preceq_{\mathbf{CR}} \mathtt{A}_1$. **Transitivity:** $\mathtt{A}_1 \preceq_{\mathbf{CR}} \mathtt{A}_2$ and $\mathtt{A}_2 \preceq_{\mathbf{CR}} \mathtt{A}_3$ then $\mathtt{A}_1 \preceq_{\mathbf{CR}} \mathtt{A}_3$. **Comparability:** $\mathtt{A}_2 \preceq_{\mathbf{CR}} \mathtt{H}_3$ or $\mathtt{A}_3 \preceq_{\mathbf{CR}} \mathtt{A}_2$.

Remark 2.2 We note that if $[\sigma_1, \sigma_2]$, $[\nu_1, \nu_2]$ and $[\eta_1, \eta_2]$ are intervals with positive endpoints, then

$$[\sigma_1, \sigma_2] \geq [\eta_1, \eta_2] \Leftrightarrow \frac{[\sigma_1, \sigma_2]}{[\nu_1, \nu_2]} \geq \frac{[\eta_1, \eta_2]}{[\nu_1, \nu_2]},$$

$$[\nu_1, \nu_2] \leq [\eta_1, \eta_2] \Leftrightarrow \frac{[\sigma_1, \sigma_2]}{[\nu_1, \nu_2]} \geq \frac{[\sigma_1, \sigma_2]}{[\eta_1, \eta_2]}.$$

Given a monotone and continuous function $\Psi(\Omega)$ over the interval $\Omega = [\sigma_1, \sigma_2]$, we may specify as

$$\Psi([\sigma_1, \sigma_2]) = [\min\{\Psi(\sigma_1), \Psi(\sigma_2)\}, \max\{\Psi(\sigma_1), \Psi(\sigma_2)\}].$$

2.1 *Integral of set-valued functions*

If $\Psi : \Omega \to \mathsf{R}_\mathsf{I}$ is an IVF defined over closed interval $\Omega = [\sigma_1, \sigma_2]$, then it is defined as follows:

$$\Psi(\sigma) = [\underline{\Phi}(\sigma), \overline{\Phi}(\sigma)],$$

where $\underline{\Phi}(\sigma) \le \overline{\Phi}(\sigma)$, $\forall \sigma \in \Omega$. The functions $\underline{\Phi}(\sigma)$ and $\overline{\Phi}(\sigma)$ are called the lower and the upper endpoint functions of Ψ, respectively. For IVF, it is clear that $\Psi : \Omega \to \mathsf{R}_\mathsf{I}$ is continuous at $\sigma_0 \in \Omega$ if

$$\lim_{\sigma \to \sigma_0} \Psi(\sigma) = \Psi(\sigma_0),$$

consequently, Ψ is continuous at $\sigma_0 \in \Omega$ if and only if its endpoint functions $\underline{\Phi}$ and $\overline{\Phi}$ are continuous functions at $\sigma_0 \in \Omega$.

Theorem 2.1 (see [43]) Let $\Psi : [\sigma_1, \sigma_2] \to \mathsf{R}_\mathtt{I}$ be an interval-valued mapping represented as $\Psi(\Omega) = [\underline{\Psi}(\sigma), \overline{\Psi}(\sigma)]$. $\Psi \in \mathsf{IR}_{([\sigma_1, \sigma_2])}$, iff $\underline{\Psi}(\sigma), \overline{\Psi}(\sigma) \in \mathsf{R}_{([\sigma_1, \sigma_2])}$ and

$$(\mathrm{IR}) \int_{\sigma_1}^{\sigma_2} \Psi(\sigma)\,\mathrm{d}\sigma = \left[(\mathsf{R}) \int_{\sigma_1}^{\sigma_2} \underline{\Psi}(\sigma)\,\mathrm{d}\sigma, (\mathsf{R}) \int_{\sigma_1}^{\sigma_2} \overline{\Psi}(\sigma)\,\mathrm{d}\sigma\right].$$

Theorem 2.2 (see [43]) Let $\Psi, \Phi : [\sigma_1, \sigma_2] \to \mathsf{R}_\mathtt{I}$ be an interval-valued mapping represented as $\Phi = [\underline{\Phi}, \overline{\Phi}]$, $\Psi = [\underline{\Psi}, \overline{\Psi}]$. If $\Psi(\sigma) \preceq_{\mathbf{CR}} \Phi(\sigma)$ for all $\sigma \in [\sigma_1, \sigma_2]$, then

$$\int_{\sigma_1}^{\sigma_2} \Psi(\sigma)\mathrm{d}\sigma \preceq_{\mathbf{CR}} \int_{\sigma_1}^{\sigma_2} \Phi(\sigma)\mathrm{d}\sigma.$$

Example 2.1 Taking into account the assumptions of Theorem 2.2 and let $\Psi = [\sigma, 2\sigma]$ and $\Phi = [\sigma^2, \sigma^2 + 2]$, then for $\sigma \in [0, 1]$, we have

$$\Psi_{\mathbf{C}} = \frac{3\sigma}{2}, \Psi_{\mathbf{R}} = \frac{\sigma}{2}, \Phi_{\mathbf{C}} = \sigma^2 + 1 \text{ and } \Phi_{\mathbf{R}} = 1.$$

From Definition 2.1, it follows $\Psi(\sigma) \preceq_{\mathbf{CR}} \Phi(\sigma)$ for $\sigma \in [0, 1]$.

Since,

$$\int_0^1 [\sigma, 2\sigma]\mathrm{d}\sigma = \left[\frac{1}{2}, 1\right]$$

and

$$\int_0^1 [\sigma^2, \sigma^2+2]\mathrm{d}\sigma = \left[\frac{1}{3}, \frac{7}{3}\right].$$

Also, from Theorem 2.2, we have

$$\int_0^1 \Psi(\sigma)\mathrm{d}\sigma \preceq_{\mathbf{CR}} \int_0^1 \Phi(\sigma)\mathrm{d}\sigma.$$

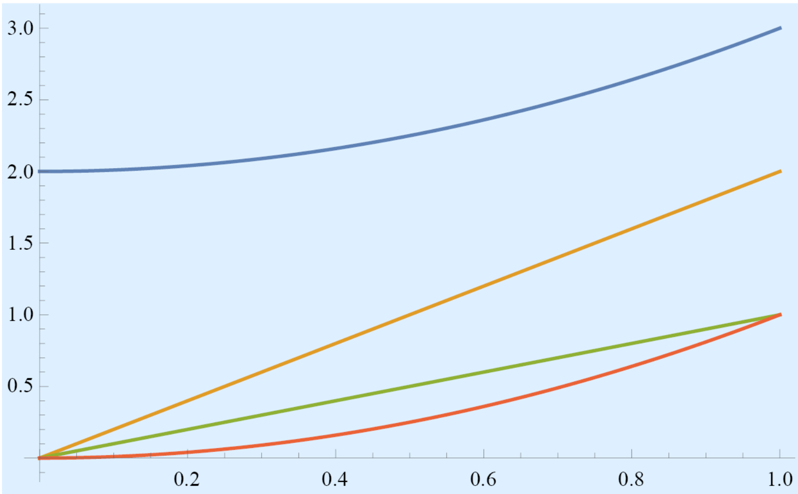

Figure 1. A blue color indicates σ^2+2; an orange color indicates 2σ; a green color indicates σ; and a red color indicates σ^2

The Figure 1 above clearly illustrates the validity of the $\preceq_{\mathbf{CR}}$-order relationship.

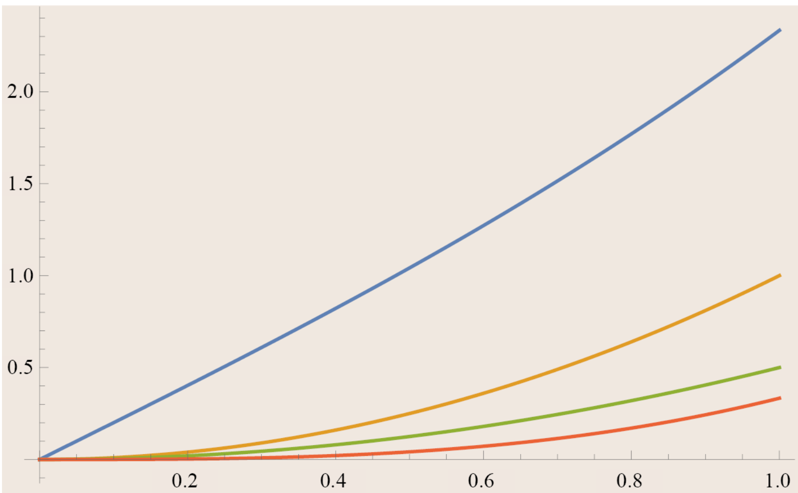

Figure 2. A blue color indicates $2\sigma+\frac{\sigma^3}{3}$; an orange color indicates σ^2; a green color indicates $\frac{\sigma^2}{2}$; and a red color indicates $\frac{\sigma^3}{3}$

The validity of Theorem 2.2 is evident from the Figure 2.

Definition 2.2 (see [46]) Let $\Psi : \Omega \subset \mathsf{R} \to \mathsf{R}$ be defined over convex set Ω; then, Ψ is called to be convex if

$$\Psi(\eth\sigma_1 + (1-\eth\sigma_2) \leq \eth\Psi(\sigma_1) + (1-\eth)\Psi(\sigma_2),$$

holds for all $\sigma_1,\ \sigma_2 \in \Omega \subset \mathsf{R}$ and $\eth \in [0,\ 1]$.

Definition 2.3 (see [46]) A mapping $\Psi : \Omega \subset \mathsf{R} \to \mathsf{R}$ is called to be h-convex if

$$\Psi(\eth\sigma_1 + (1-\eth\sigma_2) \leq \mathsf{h}(\eth)\Psi(\sigma_1) + \mathsf{h}(1-\eth)\Psi(\sigma_2),$$

holds for all $\sigma_1,\ \sigma_2 \in \Omega \subset \mathsf{R}$ and $\eth \in (0,\ 1)$, where $\mathsf{h} : [0,\ 1] \to (0,\ \infty)$ such that $\mathsf{h} \neq 0$.

Definition 2.4 (see [46]) A mapping $\Psi : \Omega \subset \mathsf{R} \to \mathsf{R}$ is called to be h-Godunova-Levin if and only if

$$\Psi(\eth\sigma_1 + (1-\eth\sigma_2) \leq \frac{\Psi(\sigma_1)}{\mathsf{h}(\eth)} + \frac{\Psi(\sigma_2)}{\mathsf{h}(1-\eth)},$$

holds for all $\sigma_1,\ \sigma_2 \in \Omega \subset \mathsf{R}$ and $\eth \in (0,\ 1)$, where $\mathsf{h} : (0,\ 1) \to (0,\ \infty)$ such that $\mathsf{h} \neq 0$.

Definition 2.5 (see [47]) A mapping $\Psi : \Omega \subseteq \mathsf{R} \to \mathsf{R}$ is called to be $(\mathsf{h}_1,\ \mathsf{h}_2)$-Godunova-Levin, if and only if

$$\Psi(\eth\sigma_1 + (1-\eth\sigma_2) \leq \frac{\Psi(\sigma_1)}{\mathsf{h}_1(\eth)\mathsf{h}_2(1-\eth)} + \frac{\Psi(\sigma_2)}{\mathsf{h}_1(1-\eth)\mathsf{h}_2(\eth)},$$

holds for all $\sigma_1,\ \sigma_2 \in \Omega \subset \mathsf{R}$ and $\eth \in (0,\ 1)$, where $\mathsf{h}_1,\ \mathsf{h}_2 : (0,\ 1) \to (0,\ \infty)$ such that $\mathsf{h}_1,\ \mathsf{h}_2 \neq 0$.

Definition 2.6 (see [47]) A mapping $\Psi : \Omega \subseteq \mathsf{R} \to \mathsf{R}$ is called to be harmonical $(\mathsf{h}_1,\ \mathsf{h}_2)$-Godunova-Levin, if and only if

$$\Psi\left(\frac{\sigma_1\sigma_2}{\eth\sigma_1 + (1-\eth\sigma_2}\right) \leq \frac{\Psi(\sigma_1)}{\mathsf{h}_1(\eth)\mathsf{h}_2(1-\eth)} + \frac{\Psi(\sigma_2)}{\mathsf{h}_1(1-\eth)\mathsf{h}_2(\eth)},$$

holds for all $\sigma_1,\ \sigma_2 \in \Omega \subset \mathsf{R}$ and $\eth \in (0,\ 1)$, where $\mathsf{h}_1,\ \mathsf{h}_2 : (0,\ 1) \to (0,\ \infty)$ such that $\mathsf{h}_1,\ \mathsf{h}_2 \neq 0$.

Note: In our main results, we used interval-valued mappings under center-radius total order relations, so we extend the above definitions that have intervals as their codomains.

Definition 2.7 (see [47]) A mapping $\Psi = [\underline{\Psi},\ \overline{\Psi}] : \Omega \subseteq \mathsf{R} \to \mathsf{R}_{\mathtt{I}}^{+}$ is called to be interval-valued **CR**-$(\mathsf{h}_1,\ \mathsf{h}_2)$-Godunova-Levin, if and only if

$$\Psi(\eth\sigma_1 + (1-\eth\sigma_2) \preceq_{\mathbf{CR}} \frac{\Psi(\sigma_1)}{\mathsf{h}_1(\eth)\mathsf{h}_2(1-\eth)} + \frac{\Psi(\sigma_2)}{\mathsf{h}_1(1-\eth)\mathsf{h}_2(\eth)}, \tag{5}$$

holds for all $\sigma_1,\ \sigma_2 \in \Omega \subset \mathsf{R}$ and $\eth \in (0,\ 1)$, where $\mathsf{h}_1,\ \mathsf{h}_2 : (0,\ 1) \to (0,\ \infty)$ such that $\mathsf{h}_1,\ \mathsf{h}_2 \neq 0$. If the relation (5) is reversed, then Ψ is said to be **CR**-$(\mathsf{h}_1,\ \mathsf{h}_2)$-Godunova-Levin concave. The class of all **CR**-$(\mathsf{h}_1,\ \mathsf{h}_2)$-Godunova-Levin convex mappings are denoted by $\mathtt{SGX}(\mathbf{CR}\text{-}(\mathsf{h}_1,\ \mathsf{h}_2),\ [\sigma_1,\ \sigma_2],\ \mathsf{R}_{\mathtt{I}}^{+})$.

Remark 2.3

• If the mapping fulfill the condition $\underline{\Psi} = \overline{\Psi}$ with $h_1(\eth) = \frac{1}{\eth}$, $h_2(\eth) = 1$, then Definition 2.7 transforms into classical convex function [26].

• If the mapping fulfill the condition $\underline{\Psi} = \overline{\Psi}$ with $h_1(\eth) = h(\eth)$, $h_2(\eth) = 1$, then Definition 2.7 transforms into h-Godunova-Levin function [48].

• If $h_1(\eth) = h_2(\eth) = 1$, then Definition 2.7 transforms into **CR**-p-convex function [49].

• If the mapping fulfill the condition $\underline{\Psi} = \overline{\Psi}$ with $h_1(\eth) = \frac{1}{\eth^s}$, $h_2(\eth) = 1$, then Definition 2.7 transforms into s-convex function [50].

• If the mapping fulfill the condition $\underline{\Psi} = \overline{\Psi}$, then Definition 2.7 transforms into Definition 2.5.

• If the mapping fulfill the condition $\underline{\Psi} = \overline{\Psi}$ with $h_1(\eth) = \eth^s$, $h_2(\eth) = 1$, then Definition 2.7 transforms into s-Godunova-Levin function [50].

Definition 2.8 (see [47]) A mapping $\Psi = [\underline{\Psi}, \overline{\Psi}] : \Omega \subseteq \mathsf{R} \to \mathsf{R}_{\mathtt{I}}^+$ is called to be interval-valued harmonical **CR**-(h_1, h_2)-Godunova-Levin, if and only if

$$\Psi\left(\frac{\sigma_1\sigma_2}{\eth\sigma_1 + (1-\eth\sigma_2}\right) \preceq_{\mathbf{CR}} \frac{\Psi(\sigma_1)}{h_1(\eth)h_2(1-\eth)} + \frac{\Psi(\sigma_2)}{h_1(1-\eth)h_2(\eth)}, \tag{6}$$

holds for all $\sigma_1, \sigma_2 \in \Omega \subset \mathsf{R}$ and $\eth \in (0, 1)$, where $h_1, h_2 : (0, 1) \to \mathsf{R}^+$ such that $h_1, h_2 \neq 0$. If the relation (6) is reversed, then Ψ is said to be harmonical **CR**-(h_1, h_2)-Godunova-Levin concave. The class of all harmonical **CR**-(h_1, h_2)-Godunova-Levin convex mappings are denoted by $\mathtt{SGHX}(\mathbf{CR}\text{-}(h_1, h_2), [\sigma_1, \sigma_2], \mathsf{R}_{\mathtt{I}}^+)$.

Remark 2.4

• If $h_1(\eth) = \frac{1}{\eth}$, $h_2(\eth) = 1$, then Definition 2.8 reduces to harmonical **CR**-convex function [51].

• If $h_1(\eth) = h(\eth)$, $h_2(\eth) = 1$, then Definition 2.8 transforms into harmonical **CR**-h-Godunova-Levin function [52].

• If $h_1(\eth) = h_2(\eth) = 1$, then Definition 2.7 reduces to harmonical **CR**-p-convex function [41].

• If $h_1(\eth) = \frac{1}{\eth^s}$, $h_2(\eth) = 1$, then Definition 2.7 transforms into harmonical **CR**-s-convex function [41].

• If the mapping fulfill the condition $\underline{\Psi} = \overline{\Psi}$, then Definition 2.7 transforms into Definition 2.6.

• If the mapping fulfill the condition $\underline{\Psi} = \overline{\Psi}$ with $h_1(\eth) = \frac{1}{\eth^s}$, $h_2(\eth) = 1$, then Definition 2.7 transforms into harmonical s-convex function [53].

We have now defined an interval-valued fractional integral operator which we used in our main results.

Definition 2.9 (see [54]) Let $\Psi : [\sigma_1, \sigma_2] \to \mathsf{R}_{\mathtt{I}}^+$ be an IVF represented as $\Psi = [\underline{\Psi}, \overline{\Psi}]$. The interval-valued left and right sided Atangana-Baleanu integrals of function Ψ and order $\eta > 0$ are defined as follows

$${}^{\mathtt{AB}}_{\sigma_1}\mathtt{I}^{\eta}_{\mathtt{t}}\{\Psi(\mathtt{t})\} = \left[{}^{\mathtt{AB}}_{\sigma_1}\mathtt{I}^{\eta}_{\mathtt{t}}\{\underline{\Psi}(\mathtt{t})\}, {}^{\mathtt{AB}}_{\sigma_1}\mathtt{I}^{\eta}_{\mathtt{t}}\{\overline{\Psi}(\mathtt{t})\}\right]$$

and

$${}^{\mathtt{AB}}\mathtt{I}^{\eta}_{\sigma_2}\{\Psi(\mathtt{t})\} = \left[{}^{\mathtt{AB}}\mathtt{I}^{\eta}_{\sigma_2}\{\underline{\Psi}(\mathtt{t})\}, {}^{\mathtt{AB}}\mathtt{I}^{\eta}_{\sigma_2}\{\overline{\Psi}(\mathtt{t})\}\right],$$

where

$$
{}^{AB}_{\sigma_1}I^{\eta}_{t}\{\Psi(t)\} = \frac{1-\eta}{B(\eta)}\Psi(t) + \frac{\eta}{B(\eta)\Gamma(\eta)}\int_{\sigma_1}^{t}\Psi(\eth)(t-\eth)^{\eta-1}\,d\eth,
$$

$$
{}^{AB}I^{\eta}_{\sigma_2}\{\Psi(t)\} = \frac{1-\eta}{B(\eta)}\Psi(t) + \frac{\eta}{B(\eta)\Gamma(\eta)}\int_{t}^{\sigma_2}\Psi(\eth)(\eth-t)^{\eta-1}\,d\eth,
$$

also $\sigma_1 < \sigma_2$, $\eta \in (0, 1]$, $\Gamma(\eth) = \int_0^\infty t^{\eth-1}e^{-t}\,dt$ is the Gamma function, $B(\eta) > 0$ such that $B(0) = B(1) = 1$, $\|B(\eta)\| = 1$, and $\beta_a = \beta_a(p, q) = \int_0^a \eth^{p-1}(1-\eth)^{q-1}\,d\eth$ is the beta integral in incomplete sense.

As part of our main results, we also use the following two inequalities: Hölder's inequality and its generalized case at several points.

Theorem 2.3 (see [46]) (Hölder inequality) Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Consider two real-valued functions Ψ and Φ on $[\sigma_1, \sigma_2]$ with $|\Psi|^p$, $|\Phi|^q$ are also integrable on $[\sigma_1, \sigma_2]$, then one has

$$
\int_{\sigma_1}^{\sigma_2}|\Psi(\eth)\Phi(\eth)|d\eth \le \left(\int_{\sigma_1}^{\sigma_2}|\Psi(\eth)|^p d\eth\right)^{\frac{1}{p}}\left(\int_{\sigma_1}^{\sigma_2}|\Phi(\eth)|^q d\eth\right)^{\frac{1}{q}}
$$

Theorem 2.4 (see [55]) Let $q \ge 1$ and two real-valued functions Ψ and Φ defined on $[\sigma_1, \sigma_2]$ with $|\Psi|$, $|\Psi||\Phi|^q$ are integrable on $[\sigma_1, \sigma_2]$, then we have

$$
\int_{\sigma_1}^{\sigma_2}|\Psi(\eth)\Phi(\eth)|d\eth \le \left(\int_{\sigma_1}^{\sigma_2}|\Psi(\eth)|d\eth\right)^{1-\frac{1}{q}}\left(\int_{\sigma_1}^{\sigma_2}|\Psi(\eth)||\Phi(\eth)|^q d\eth\right)^{\frac{1}{q}}.
$$

Theorem 2.5 (see [56]) (Young's inequality) Let p, q be two positive real numbers satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Then if Ψ, Φ are two nonnegative real-valued functions, then one has

$$
\Psi\Phi \le \frac{\Psi^p}{p} + \frac{\Phi^q}{q},
$$

and equality holds iff $\Psi^p = \Phi^q$.

In their paper, Iscan et al. [57] developed new Hermite-Hadamard and Bullen inequalities that were applied to trapezoidal formulae and special means, and they also introduced the following interesting results that are also essential to developing the application section of this article.

Lemma 2.1 (see [57]) Let $\Psi : \Omega^\circ \subset R \to R$ is a differentiable mapping on Ω°, where $\sigma_1, \sigma_2 \in \Omega^\circ$, with $\sigma_1 < \sigma_2$. If $\Psi' \in L[\sigma_1, \sigma_2]$ (the class of all Lebesgue measurable functions), then we have

$$\Omega_{k}(\Psi, \sigma_1, \sigma_2) = \sum_{\zeta=0}^{k-1} \frac{1}{2k}\left[\Psi\left(\frac{(k-\zeta)\sigma_1+\zeta\sigma_2}{k}\right)+\Psi\left(\frac{(k-\zeta-1)\sigma_1+(\zeta+1)\sigma_2}{k}\right)\right]$$

$$-\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\Psi(\eta)d\eta$$

$$=\sum_{\zeta=0}^{k-1}\frac{\sigma_2-\sigma_1}{2k^2}\left[\int_0^1(1-2\eth)\Psi'\left(\eth\frac{(k-\zeta)\sigma_1+\zeta\sigma_2}{k}\right.\right.$$

$$\left.\left.+(1-\eth)\frac{(k-\zeta-1)\sigma_1+(\zeta+1)\sigma_2}{k}\right)d\eth\right].$$

holds.

Using the standard order relation, Fernandez and Mohammed [58] derived the following H-H inequality based on the Atangana-Baleanu integral operators.

Theorem 2.6 (see [58]) If $\Psi : [\sigma_1, \sigma_2] \to R$ is convex over the interval $[\sigma_1, \sigma_2]$ and $\Psi \in L_1([\sigma_1, \sigma_2])$, then the inequality stated below

$$\Psi\left(\frac{\sigma_1+\sigma_2}{2}\right)\leq\frac{B(\eta)\Gamma(\eta)}{2\left[(\sigma_2-\sigma_1)^{\eta}+(1-\eta)\Gamma(\eta)\right]}\left[{}^{AB}I^{\eta}_{\sigma_1^+}\Psi(\sigma_2)+{}^{AB}I^{\eta}_{\sigma_2}\Psi(\sigma_1)\right]\leq\frac{\Psi(\sigma_1)+\Psi(\sigma_2)}{2}$$

hold for $\eta \in (0, 1)$.

Onalan et al. [59] developed the following H-H type inequality by using fractional integral operators with Mittag-Leffler kernels.

Theorem 2.7 (see [59]) If $\Psi : [\sigma_1, \sigma_2] \to R$ is s-convex on $[\sigma_1, \sigma_2]$ and $\Psi \in L([\sigma_1, \sigma_2])$, then the inequality stated below

$$2^{s}\frac{\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)}{B(\eta)\Gamma(\eta)}+\frac{1-\eta}{(\sigma_2-\sigma_1)^{\eta}}\left[\frac{\Psi(\sigma_1)+\Psi(\sigma_2)}{B(\eta)}\right]$$

$$\leq\frac{1}{(\sigma_2-\sigma_1)^{\eta}}\left[{}^{AB}_{\sigma_1}I^{\eta}_{\sigma_2}\{\Psi(\sigma_2)\}+{}^{AB}I^{\eta}_{\sigma_2}\{\Psi(\sigma_1)\}\right]$$

$$\leq\left[\frac{\Psi(\sigma_1)+\Psi(\sigma_2)}{B(\eta)\Gamma(\eta)}\right]\left[\frac{\eta}{\Gamma(\eta)(\eta+s)}+\frac{1-\eta}{(\sigma_2-\sigma_1)^{\eta}}+\frac{\eta\beta(\eta, s+1)}{\Gamma(\eta)}\right],$$

hold for $\eta \in (0, 1)$.

In Yildiz et al. [60], s-convex mappings were used to develop the following fascinating results that are also used in our primary findings.

Theorem 2.8 (see [60]) Let $\Psi : \Omega \subset R \to R$ is a differentiable mapping on Ω°, where $\sigma_1, \sigma_2 \in \Omega^{\circ}$, with $\sigma_1 < \sigma_2$. If $|\Psi'|^{q}$ is s-convex on $[\sigma_1, \sigma_2]$ for some $q > 1$, then one has

$$
\begin{aligned}
&|\Omega_{k}(\Psi, \sigma_1, \sigma_2)| \\
&\leq \sum_{\zeta=0}^{k-1} \frac{\sigma_2-\sigma_1}{2k^2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}} \\
&\times\left[\left|\Psi'\left(\frac{(k-\zeta)\sigma_1+\zeta\sigma_2}{k}\right)\right|^{q}+\left|\Psi'\left(\frac{(k-\zeta-1)\sigma_1+(\zeta+1)\sigma_2}{k}\right)\right|^{q}\right]^{\frac{1}{q}}
\end{aligned}
$$

holds, where $\frac{1}{p}+\frac{1}{q}=1$.

Here is the lemma that will be used in several main results:

Lemma 2.2 (see [59]) Let $\sigma_1<\sigma_2$, $\sigma_1, \sigma_2 \in \mathsf{R}^+$, $\Psi: \mathsf{R}^+ \to \mathsf{R}^+$ is a differentiable mapping. If $\Psi'' \in \mathsf{L}[\sigma_1, \sigma_2]$, for each $\eta \in (0, 1]$, then one has

$$
\begin{aligned}
&\frac{1}{\sigma_2-\sigma_1}\left[{}^{AB}I^{\eta}_{\frac{\sigma_2+\sigma_1}{2}}\{\Psi(\sigma_1)\}+{}^{AB}_{\frac{\sigma_2+\sigma_1}{2}}I^{\eta}_{\sigma_2}\{\Psi(\sigma_2)\}\right] \\
&-\frac{1}{(\sigma_2-\sigma_1)B(\eta)}[\Psi(\sigma_1)+\Psi(\sigma_2)]-\frac{(\sigma_2-\sigma_1)^{\eta-1}}{2^{\eta-1}B(\eta)\Gamma(\eta)}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right) \\
&=\frac{(\sigma_2-\sigma_1)^{\eta-1}}{2(\eta+1)B(\eta)\Gamma(\eta)}\int_0^1 w^{\eta}(\eth)\left[\Psi''(\eth\sigma_1+(1-\eth)\sigma_2)+\Psi''(\eth\sigma_2+(1-\eth)\sigma_1)\right]d\eth,
\end{aligned}
$$

where

$$
w^{\eta}(\eth)=\begin{cases}\eth^{\eta+1},\ \eth\in\left[0, \frac{1}{2}\right), \\ (1-\eth)^{\eta+1},\ \eth\in\left[\frac{1}{2}, 1\right].\end{cases}
$$

3. The main results

This section develops our main findings, specifically H-H, Jensen and Ostrowski-type inequalities for the $\mathbf{H}(\alpha, 1-\alpha)$-Godunova-Levin mappings via **AB** integral Operators.

Proposition 3.1 Let $f_i, \sigma_i \in \mathsf{R}^+$ and $h_1, h_2: (0, 1] \to (0, \infty)$ be a supermutiplicative mapping such that $\sum_{i=1}^{d} h_1\left(\frac{f_i}{F_d}\right) h_2\left(\frac{F_{i-1}}{F_d}\right) \preceq_{\mathbf{CR}} (0, 1]$. If $\Psi \in SGHX(\mathbf{CR}\text{-}(h_1, h_2), [\sigma_1, \sigma_2], \mathsf{R}_I^+)$, then we have

$$\Psi\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_d}-\frac{1}{F_d}\sum_{i=1}^{d}\frac{f_i}{\sigma_i}}\right) \preceq_{\mathbf{CR}} \Psi(\sigma_1)+\Psi(\sigma_d)-\sum_{i=1}^{d}\left[\frac{\Psi(\sigma_i)}{H\left(\frac{f_i}{F_d},\frac{F_{i-1}}{F_d}\right)}\right]. \tag{7}$$

Proof. Since $F_d=\sum_{i=1}^{d} f_i$ and $\Psi \in \mathtt{SGHX}(\mathbf{CR}\text{-}(h_1, h_2), [\sigma_1, \sigma_2], R_I^+)$, and taking into account [61], Theorem 2.5, we obtain

$$\Psi\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_d}-\frac{1}{F_d}\sum_{i=1}^{d}\frac{f_i}{\sigma_i}}\right)=\Psi\left(\frac{1}{\sum_{i=1}^{d}\frac{f_i}{F_d}\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_d}-\frac{1}{\sigma_i}\right)}\right)\preceq_{\mathbf{CR}}\sum_{i=1}^{d}\left[\frac{\Psi\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_d}-\frac{1}{\sigma_k}}\right)}{h_1\left(\frac{f_i}{F_d}\right)h_2\left(\frac{F_{i-1}}{F_d}\right)}\right].$$

Since, $\sum_{i=1}^{d} h_1\left(\frac{f_i}{F_d}\right)h_2\left(\frac{F_{i-1}}{F_d}\right)\preceq_{\mathbf{CR}}(0,1]$, then we have

$$\Psi\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_d}-\frac{1}{F_d}\sum_{i=1}^{d}\frac{f_i}{\sigma_i}}\right)\preceq_{\mathbf{CR}}\sum_{i=1}^{d}\left[\frac{\Psi\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_d}-\frac{1}{\sigma_k}}\right)}{h_1\left(\frac{f_i}{F_d}\right)h_2\left(\frac{F_{i-1}}{F_d}\right)}\right]\preceq_{\mathbf{CR}}\sum_{i=1}^{d}\left[\frac{\Psi(\sigma_1)+\Psi(\sigma_d)-\Psi(\sigma_i)}{h_1\left(\frac{f_i}{F_d}\right)h_2\left(\frac{F_{i-1}}{F_d}\right)}\right]$$

$$\preceq_{\mathbf{CR}}\sum_{i=1}^{d}\left[\frac{\Psi(\sigma_1)+\Psi(\sigma_d)}{h_1\left(\frac{f_i}{F_d}\right)h_2\left(\frac{F_{i-1}}{F_d}\right)}-\frac{\Psi(\sigma_i)}{h_1\left(\frac{f_i}{F_d}\right)h_2\left(\frac{F_{i-1}}{F_d}\right)}\right]$$

$$\preceq_{\mathbf{CR}}\Psi(\sigma_1)+\Psi(\sigma_d)-\sum_{i=1}^{d}\left[\frac{\Psi(\sigma_i)}{h_1\left(\frac{f_i}{F_d}\right)h_2\left(\frac{F_{i-1}}{F_d}\right)}\right]$$

$$=\Psi(\sigma_1)+\Psi(\sigma_d)-\sum_{i=1}^{d}\left[\frac{\Psi(\sigma_i)}{H\left(\frac{f_i}{F_d},\frac{F_{i-1}}{F_d}\right)}\right].$$

Next, we demonstrate that the mappings $\Psi_{\mathbf{C}}$ and $\Psi_{\mathbf{R}}$ are harmonic $(\mathtt{h}_1, \mathtt{h}_2)$-Godunova-Levin convex in the Jensen sense.

$$\Psi_{\mathbf{C}}\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_d}-\frac{1}{F_d}\sum_{i=1}^{d}\frac{f_i}{\sigma_i}}\right) \preceq_{\mathbf{CR}} \Psi_{\mathbf{C}}(\sigma_1)+\Psi_{\mathbf{C}}(\sigma_d)-\sum_{i=1}^{d}\left[\frac{\Psi_{\mathbf{C}}(\sigma_i)}{H\left(\frac{f_i}{F_d}, \frac{F_{i-1}}{F_d}\right)}\right],$$

and

$$\Psi_{\mathbf{R}}\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_d}-\frac{1}{F_d}\sum_{i=1}^{d}\frac{f_i}{\sigma_i}}\right) \preceq_{\mathbf{CR}} \Psi_{\mathbf{R}}(\sigma_1)+\Psi_{\mathbf{R}}(\sigma_d)-\sum_{i=1}^{d}\left[\frac{\Psi_{\mathbf{R}}(\sigma_i)}{H\left(\frac{f_i}{F_d}, \frac{F_{i-1}}{F_d}\right)}\right].$$

Now, if

$$\Psi_{\mathbf{C}}\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_d}-\frac{1}{F_d}\sum_{i=1}^{d}\frac{f_i}{\sigma_i}}\right) \neq \Psi_{\mathbf{C}}(\sigma_1)+\Psi_{\mathbf{C}}(\sigma_d)-\sum_{i=1}^{d}\left[\frac{\Psi_{\mathbf{C}}(\sigma_i)}{H\left(\frac{f_i}{F_d}, \frac{F_{i-1}}{F_d}\right)}\right],$$

then for each $f_i, F_d \in (0, 1)$ and for all $\sigma_i \in [\sigma_1, \sigma_2]$,

$$\Psi_{\mathbf{C}}\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_d}-\frac{1}{F_d}\sum_{i=1}^{d}\frac{f_i}{\sigma_i}}\right) < \Psi_{\mathbf{C}}(\sigma_1)+\Psi_{\mathbf{C}}(\sigma_d)-\sum_{i=1}^{d}\left[\frac{\Psi_{\mathbf{C}}(\sigma_i)}{H\left(\frac{f_i}{F_d}, \frac{F_{i-1}}{F_d}\right)}\right],$$

then

$$\Psi\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_d}-\frac{1}{F_d}\sum_{i=1}^{d}\frac{f_i}{\sigma_i}}\right) \preceq_{\mathbf{CR}} \sum_{i=1}^{d}\left[\frac{\Psi\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_d}-\frac{1}{\sigma_k}}\right)}{h_1\left(\frac{f_i}{F_d}\right)h_2\left(\frac{F_{i-1}}{F_d}\right)}\right].$$

Otherwise, for each $f_i, F_d \in (0, 1)$ and for all $\sigma_i \in [\sigma_1, \sigma_2]$,

$$\Psi_{\mathbf{R}}\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_d}-\frac{1}{F_d}\sum_{i=1}^{d}\frac{f_i}{\sigma_i}}\right)\preceq_{\mathbf{CR}}\Psi_{\mathbf{R}}(\sigma_1)+\Psi_{\mathbf{R}}(\sigma_d)-\sum_{i=1}^{d}\left[\frac{\Psi_{\mathbf{R}}(\sigma_i)}{H\left(\frac{f_i}{F_d},\frac{F_{i-1}}{F_d}\right)}\right],$$

Combining all of the aforementioned and from Definition 2.7 it can be written as:

$$\Psi\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_d}-\frac{1}{F_d}\sum_{i=1}^{d}\frac{f_i}{\sigma_i}}\right)\preceq_{\mathbf{CR}}\sum_{i=1}^{d}\left[\frac{\Psi\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_d}-\frac{1}{\sigma_k}}\right)}{h_1\left(\frac{f_i}{F_d}\right)h_2\left(\frac{F_{i-1}}{F_d}\right)}\right].$$

The proof is now complete. □

Corollary 3.1 If $h_1(\eth)=h(\eth)$, $h_2(\eth)=1$, then Theorem 3.1 reduces to harmonical **CR**-h-Godunova-Levin function which is also new.

$$\Psi\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_d}-\frac{1}{F_d}\sum_{i=1}^{d}\frac{f_i}{\sigma_i}}\right)\preceq_{\mathbf{CR}}\Psi(\sigma_1)+\Psi(\sigma_d)-\sum_{i=1}^{d}\left[\frac{\Psi(\sigma_i)}{h\left(\frac{f_i}{F_d}\right)}\right]. \tag{8}$$

Remark 3.1 If the mapping fulfill the condition $\underline{\Psi}=\overline{\Psi}$ with $h_1(\eth)=h_2(\eth)=1$, then Theorem 3.1 reduces to Lemma 2.1 for harmonical p-convex function which is obtained by the authors in [61].

$$\Psi\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_d}-\frac{1}{F_d}\sum_{i=1}^{d}\frac{f_i}{\sigma_i}}\right)\preceq_{\mathbf{CR}}\Psi(\sigma_1)+\Psi(\sigma_d)-\Psi(\sigma_i). \tag{9}$$

Remark 3.2 If the mapping fulfill the condition $\underline{\Psi}=\overline{\Psi}$ with $h_1(\eth)=\frac{1}{h(\eth)}$, $h_2(\eth)=1$, then Theorem 3.1 reduces to Theorem 2.8 for harmonical h-convex function which is obtained by the authors in [61].

$$\Psi\left(\frac{1}{\frac{1}{\sigma_1}+\frac{1}{\sigma_d}-\frac{1}{F_d}\sum_{i=1}^{d}\frac{f_i}{\sigma_i}}\right)\preceq_{\mathbf{CR}}\Psi(\sigma_1)+\Psi(\sigma_d)-\sum_{i=1}^{d}\left[h\left(\frac{f_i}{F_d}\right)\right]\Psi(\sigma_i). \tag{10}$$

3.1 *New Ostrowski-type inequality via interval CR-ordering relation for H(α, 1 − α)-Godunova-Levin mappings*

In this section, we used Definition 2.8 to develop a new type of generalized Ostrowski-type inequality via **CR**-ordering relation. To further develop this new result, we can use the authors' lemma for *h*-convex mappings in classical standard order relation [62].

Lemma 3.1 (see [62]) Let $\Psi : \Omega \subseteq (0, \infty) \to \mathsf{R}$ be a differentiable mapping on Ω^o. If Ψ' is integrable on interval $[\sigma_1, \sigma_2]$, then one has:

$$\Psi(\nu) - \frac{1}{\sigma_2 - \sigma_1} \int_{\sigma_1}^{\sigma_2} \Psi(\eth) d\eth$$

$$= \frac{(\nu - \sigma_1)^2}{\sigma_2 - \sigma_1} \int_0^1 \eth\Psi'(\eth\nu + (1 - \eth)\sigma_1) d\eth - \frac{(\sigma_2 - \nu)^2}{\sigma_2 - \sigma_1} \int_0^1 \eth\Psi'(\eth\nu + (1 - \eth)\sigma_2) d\eth, \ \forall \nu \in [\sigma_1, \sigma_2].$$

Theorem 3.1 Let three super-multiplicative mappings $\mathtt{h}, \mathtt{h}_1, \mathtt{h}_2 : (0, 1) \to \mathsf{R}$ with $\eth \preceq_{\mathbf{CR}} \frac{1}{\mathtt{H}(\eth, 1 - \eth)}$ for each $\eth \in (0, 1)$. Let $\Psi = [\underline{\Psi}, \overline{\Psi}] : \mathsf{R} \to \mathsf{R}_{\mathtt{I}}^{+}$ be an $\mathtt{IVF}$ such that $\underline{\Psi}, \overline{\Psi}$ are continuously differentiable functions on Ω^o. If $|\Psi'|$ be an interval-valued $(\mathtt{h}_1, \mathtt{h}_2)$-Godunova-Levin function and satisfying $|\Psi'(\nu)| = \left|[\underline{\Psi}'(\nu), \overline{\Psi}'(\nu)]\right| \preceq_{\mathbf{CR}} [\underline{\Sigma}, \overline{\Sigma}]$ for each ν, then we have

$$\mathbf{M}\left([\underline{\Psi}(\nu), \overline{\Psi}(\nu)], \left[\frac{1}{\sigma_2 - \sigma_1} \int_{\sigma_1}^{\sigma_2} \underline{\Psi}(\eth) d\eth, \frac{1}{\sigma_2 - \sigma_1} \int_{\sigma_1}^{\sigma_2} \overline{\Psi}(\eth) d\eth\right]\right)$$

$$= \max\left\{\left|\underline{\Psi}(\nu) - \frac{1}{\sigma_2 - \sigma_1} \int_{\sigma_1}^{\sigma_2} \underline{\Psi}(\eth) d\eth\right|, \left|\overline{\Psi}(\nu) - \frac{1}{\sigma_2 - \sigma_1} \int_{\sigma_1}^{\sigma_2} \overline{\Psi}(\eth) d\eth\right|\right\}$$

$$\preceq_{\mathbf{CR}} \frac{\Sigma\left[(\nu - \sigma_1)^2 + (\sigma_2 - \nu)^2\right]}{\sigma_2 - \sigma_1} \int_0^1 \left[\frac{1}{\mathbb{H}(\eth^2, \eth - \eth^2)} + \frac{1}{\mathbb{H}(\eth - \eth^2, \eth^2)}\right] d\eth.$$

Proof. Taking into account Lemma 3.1 and the Moore metric described in equation (4) in interval space, further we know that if $|\Psi'| \in \mathtt{SGX}(\mathbf{CR}\text{-}(\mathtt{h}_1, \mathtt{h}_2), [\sigma_1, \sigma_2], \mathsf{R}_{\mathtt{I}}^{+})$, then we have

$$\mathbf{M}\left([\underline{\Psi}(\nu), \overline{\Psi}(\nu)], \left[\frac{1}{\sigma_2 - \sigma_1} \int_{\sigma_1}^{\sigma_2} \underline{\Psi}(\eth) d\eth, \frac{1}{\sigma_2 - \sigma_1} \int_{\sigma_1}^{\sigma_2} \overline{\Psi}(\eth) d\eth\right]\right)$$

$$= \max\left\{\left|\underline{\Psi}(\nu) - \frac{1}{\sigma_2 - \sigma_1} \int_{\sigma_1}^{\sigma_2} \underline{\Psi}(\eth) d\eth\right|, \left|\overline{\Psi}(\nu) - \frac{1}{\sigma_2 - \sigma_1} \int_{\sigma_1}^{\sigma_2} \overline{\Psi}(\eth) d\eth\right|\right\}$$

$$\preceq_{\mathbf{CR}} \max\left\{\frac{(\nu - \sigma_1)^2}{\sigma_2 - \sigma_1} \int_0^1 \eth\left|\underline{\Psi}'(\eth\nu + (1 - \eth)\sigma_1)\right| d\eth + \frac{(\sigma_2 - \nu)^2}{\sigma_2 - \sigma_1} \int_0^1 \eth\left|\underline{\Psi}'(\eth\nu + (1 - \eth)\sigma_2)\right| d\eth,\right.$$

$$\frac{(\nu-\sigma_1)^2}{\sigma_2-\sigma_1}\int_0^1 \eth\left|\overline{\Psi}'(\eth\nu+(1-\eth)\sigma_1)\right|d\eth+\frac{(\sigma_2-\nu)^2}{\sigma_2-\sigma_1}\int_0^1 \eth\left|\overline{\Psi}'(\eth\nu+(1-\eth)\sigma_2)\right|d\eth\Bigg\}$$

$$\preceq_{CR}\max\left\{\frac{(\nu-\sigma_1)^2}{\sigma_2-\sigma_1}\int_0^1 \eth\left[\frac{|\underline{\Psi}'(\nu)|}{\mathtt{H}(\eth,1-\eth)}+\frac{|\underline{\Psi}'(\sigma_1)|}{\mathtt{H}(1-\eth,\eth)}\right]d\eth+\frac{(\sigma_2-\nu)^2}{\sigma_2-\sigma_1}\int_0^1 \eth\left[\frac{|\underline{\Psi}'(\nu)|}{\mathtt{H}(\eth,1-\eth)}+\frac{|\underline{\Psi}'(\sigma_2)|}{\mathtt{H}(1-\eth,\eth)}\right]d\eth,\right.$$

$$\left.\frac{(\nu-\sigma_1)^2}{\sigma_2-\sigma_1}\int_0^1 \eth\left[\frac{|\overline{\Psi}'(\nu)|}{\mathtt{H}(\eth,1-\eth)}+\frac{|\overline{\Psi}'(\sigma_1)|}{\mathtt{H}(1-\eth,\eth)}\right]d\eth+\frac{(\sigma_2-\nu)^2}{\sigma_2-\sigma_1}\int_0^1 \eth\left[\frac{|\overline{\Psi}'(\nu)|}{\mathtt{H}(\eth,1-\eth)}+\frac{|\overline{\Psi}'(\sigma_2)|}{\mathtt{H}(1-\eth,\eth)}\right]d\eth\right\}$$

$$\preceq_{CR}\max\left\{\frac{\underline{\Sigma}(\nu-\sigma_1)^2}{\sigma_2-\sigma_1}\int_0^1\left[\frac{1}{\mathtt{H}(\eth^2,\eth-\eth^2)}+\frac{1}{\mathtt{H}(\eth-\eth^2,\eth^2)}\right]d\eth+\frac{(\sigma_2-\nu)^2}{\sigma_2-\sigma_1}\int_0^1\left[\frac{1}{\mathtt{H}(\eth^2,\eth-\eth^2)}+\frac{1}{\mathtt{H}(\eth-\eth^2,\eth^2)}\right]d\eth,\right.$$

$$\left.\frac{\overline{\Sigma}(\nu-\sigma_1)^2}{\sigma_2-\sigma_1}\int_0^1\left[\frac{1}{\mathtt{H}(\eth^2,\eth-\eth^2)}+\frac{1}{\mathtt{H}(\eth-\eth^2,\eth^2)}\right]d\eth+\frac{(\sigma_2-\nu)^2}{\sigma_2-\sigma_1}\int_0^1\left[\frac{1}{\mathtt{H}(\eth^2,\eth-\eth^2)}+\frac{1}{\mathtt{H}(\eth-\eth^2,\eth^2)}\right]d\eth\right\}$$

$$=\frac{\Sigma(\nu-\sigma_1)^2}{\sigma_2-\sigma_1}\int_0^1\left[\frac{1}{\mathtt{H}(\eth^2,\eth-\eth^2)}+\frac{1}{\mathtt{H}(\eth-\eth^2,\eth^2)}\right]d\eth+\frac{(\sigma_2-\nu)^2}{\sigma_2-\sigma_1}\int_0^1\left[\frac{1}{\mathtt{H}(\eth^2,\eth-\eth^2)}+\frac{1}{\mathtt{H}(\eth-\eth^2,\eth^2)}\right]d\eth.$$

The proof is finished. □

Remark 3.3

• If the mapping fulfill the condition $\underline{\Psi}=\overline{\Psi}$ with $\mathtt{h}(\eth)=\frac{1}{\eth}$, $\mathtt{h}_1(\eth)=\frac{1}{\mathtt{h}(\eth)}$, $\mathtt{h}_2(\eth)=1$ with $\underline{\Psi}=\overline{\Psi}$, then Theorem 3.1 reduces to Theorem 2 for $\mathtt{h}$-convex-mapping, as developed by the author [62].

• If the mapping fulfill the condition $\underline{\Psi}=\overline{\Psi}$ with $\mathtt{h}(\eth)=\frac{1}{\eth^s}$, $\mathtt{h}_1(\eth)=1$, $\mathtt{h}_2(\eth)=1$, then Theorem 3.1 transforms into Theorem 2 for s-convex-mapping, as developed by the author [63].

3.2 *New fractional Hermite-Hadamard and Pachpatte-type type inequalities via interval CR-ordering relation*

In this part, we used Definition 2.7 to build a new form of generalized Hermite-Hadamard and Pachpatte-type integral inequalities using the center-radius ordering relation.

Theorem 3.2 Let $\mathsf{h}_1,\mathsf{h}_2:(0,1)\to\mathsf{R}^+$ such that $\mathsf{H}\left(\frac{1}{2},\frac{1}{2}\right)\neq 0$, and $\Phi:[\sigma_1,\sigma_2]\to\mathsf{R}^+$ is symmetric function about $\frac{\sigma_1+\sigma_2}{2}$. If $\Psi\in\mathtt{SGX}(\mathbf{CR}\text{-}(\mathsf{h}_1,\mathsf{h}_2),[\sigma_1,\sigma_2],\mathsf{R}_\mathtt{I}^+)$, then we have

$$\frac{\mathsf{H}\left(\frac{1}{2}, \frac{1}{2}\right)}{2}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)\left[{}^{\mathtt{AB}}_{\sigma_1}\mathtt{I}^{\eta}_{\sigma_2}\{\Phi(\sigma_2)\}+{}^{\mathtt{AB}}\mathtt{I}^{\eta}_{\sigma_2}\{\Phi(\sigma_1)\}\right]-\frac{\mathsf{H}\left(\frac{1}{2}, \frac{1}{2}\right)}{2}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)\frac{1-\eta}{\mathtt{B}(\eta)}\left[\Phi(\sigma_1)+\Phi(\sigma_2)\right]$$

$$+\frac{1-\eta}{\mathtt{B}(\eta)}\left[\Psi(\sigma_1)\Phi(\sigma_1)+\Psi(\sigma_2)\Phi(\sigma_2)\right]\preceq_{\mathbf{CR}}{}^{\mathtt{AB}}_{\sigma_1}\mathtt{I}^{\eta}_{\sigma_2}\{(\Psi\Phi(\sigma_2)\}+{}^{\mathtt{AB}}\mathtt{I}^{\eta}_{\sigma_2}\{(\Psi\Phi(\sigma_1)\}$$

$$\preceq_{\mathbf{CR}}\frac{\eta(\sigma_2-\sigma_1)^{\eta}}{\mathtt{B}(\eta)\Gamma(\eta)}\left[\Psi(\sigma_1)+\Psi(\sigma_2)\right]\times\int_0^1\eth^{\eta-1}\left[\frac{1}{\mathsf{H}(\eth, 1-\eth)}+\frac{1}{\mathsf{H}(1-\eth, \eth)}\right]\Phi(\eth\sigma_2+(1-\eth)\sigma_1)d\eth$$

$$+\frac{1-\eta}{\mathtt{B}(\eta)}\left[\Psi(\sigma_1)\Phi(\sigma_1)+\Psi(\sigma_2)\Phi(\sigma_2)\right], \tag{11}$$

where $\eta\in(0, 1]$.

Proof. As $\Psi\in\mathtt{SGX}(\mathbf{CR}\text{-}(\mathsf{h}_1, \mathsf{h}_2), [\sigma_1, \sigma_2], \mathsf{R}^+_{\mathtt{I}})$, we have

$$\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)\leq\frac{1}{\mathsf{H}\left(\frac{1}{2}, \frac{1}{2}\right)}\left[\Psi(\eth\sigma_1+(1-\eth)\sigma_2)+\Psi(\eth\sigma_2+(1-\eth)\sigma_1)\right]. \tag{12}$$

Multiplying the equation (12) with $\mathsf{H}\left(\frac{1}{2}, \frac{1}{2}\right)\eth^{\eta-1}\Phi(\eth\sigma_2+(1-\eth)\sigma_1)$, and integrating over (0, 1), we have

$$\mathsf{H}\left(\frac{1}{2}, \frac{1}{2}\right)\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)\int_0^1\eth^{\eta-1}\Phi(\eth\sigma_2+(1-\eth)\sigma_1)d\eth$$

$$\preceq_{\mathbf{CR}}\int_0^1\eth^{\eta-1}\left[\Psi(\eth\sigma_1+(1-\eth)\sigma_2)+\Psi(\eth\sigma_2+(1-\eth)\sigma_1)\right]\Phi(\eth\sigma_2+(1-\eth)\sigma_1)d\eth.$$

Assuming $\mathrm{u}=\eth\sigma_2+(1-\eth)\sigma_1$, the above relation becomes

$$\mathsf{H}\left(\frac{1}{2}, \frac{1}{2}\right)\frac{1}{(\sigma_2-\sigma_1)^{\eta}}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)\int_{\sigma_1}^{\sigma_2}(\mathrm{u}-\sigma_1)^{\eta-1}\Phi(\mathrm{u})d\mathrm{u}$$

$$\preceq_{\mathbf{CR}}\frac{1}{(\sigma_2-\sigma_1)^{\eta}}\left[\int_{\sigma_1}^{\sigma_2}(\mathrm{u}-\sigma_1)^{\eta-1}\Psi(\sigma_2+\sigma_1-\mathrm{u})\Phi(\mathrm{u})d\mathrm{u}\right.$$

$$\left.+\int_{\sigma_1}^{\sigma_2}(\mathrm{u}-\sigma_1)^{\eta-1}\Psi(\mathrm{u})\Phi(\mathrm{u})d\mathrm{u}\right].$$

Making a modification in the foregoing relation, $\Psi=\sigma_2+\sigma_1-\mathrm{u}$, then $\Phi(\sigma_2+\sigma_1-\Psi)=\Phi(\Psi)$, becomes

$$\mathsf{H}\left(\frac{1}{2},\frac{1}{2}\right)\frac{1}{(\sigma_2-\sigma_1)^{\eta}}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)\int_{\sigma_1}^{\sigma_2}(\mathrm{u}-\sigma_1)^{\eta-1}\Phi(\mathrm{u})d\mathrm{u}$$

$$\preceq_{\mathbf{CR}}\frac{1}{(\sigma_2-\sigma_1)^{\eta}}\left[\int_{\sigma_1}^{\sigma_2}(\sigma_2-\Psi)^{\eta-1}\Psi(\Psi)\Phi(\Psi)d\Psi+\int_{\sigma_1}^{\sigma_2}(\mathrm{u}-\sigma_1)^{\eta-1}\Psi(\mathrm{u})\Phi(\mathrm{u})d\mathrm{u}\right].$$

Multiplying above relation with $\frac{\eta(\sigma_2-\sigma_1)^{\eta}}{\mathtt{B}(\eta)\Gamma(\eta)}$ and adding the expression $\frac{1-\eta}{\mathtt{B}(\eta)}\left[\Psi(\sigma_1)+\Psi(\sigma_2)\right]$ to the both sides, we get

$$\mathsf{H}\left(\frac{1}{2},\frac{1}{2}\right)\frac{\eta}{\mathtt{B}(\eta)\Gamma(\eta)}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)\int_{\sigma_1}^{\sigma_2}(\mathrm{u}-\sigma_1)^{\eta-1}\Phi(\mathrm{u})d\mathrm{u}+\frac{1-\eta}{\mathtt{B}(\eta)}\left[\Psi(\sigma_1)\Phi(\sigma_1)+\Psi(\sigma_2)\Phi(\sigma_2)\right]$$

$$\preceq_{\mathbf{CR}}\frac{\eta}{\mathtt{B}(\eta)\Gamma(\eta)}\left[\int_{\sigma_1}^{\sigma_2}(\sigma_2-\Psi)^{\eta-1}\Psi(\Psi)\Phi(\Psi)d\Psi+\int_{\sigma_1}^{\sigma_2}(\mathrm{u}-\sigma_1)^{\eta-1}\Psi(\mathrm{u})\Phi(\mathrm{u})d\mathrm{u}\right]$$

$$+\frac{1-\eta}{\mathtt{B}(\eta)}\left[\Psi(\sigma_1)\Phi(\sigma_1)+\Psi(\sigma_2)\Phi(\sigma_2)\right].$$

From this, it can be follows as

$$\mathsf{H}\left(\frac{1}{2},\frac{1}{2}\right)\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right){}^{\mathtt{AB}}\mathtt{I}_{\sigma_2}^{\eta}\{\Phi(\sigma_1)\}$$

$$-\mathsf{H}\left(\frac{1}{2},\frac{1}{2}\right)\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)\frac{1-\eta}{\mathtt{B}(\eta)}\Phi(\sigma_1)+\frac{1-\eta}{\mathtt{B}(\eta)}\left[\Psi(\sigma_1)\Phi(\sigma_1)+\Psi(\sigma_2)\Phi(\sigma_2)\right]$$

$$\preceq_{\mathbf{CR}}{}^{\mathtt{AB}}_{\sigma_1}\mathtt{I}_{\sigma_2}^{\eta}\{(\Psi\Phi(\sigma_2)\}+{}^{\mathtt{AB}}\mathtt{I}_{\sigma_2}^{\eta}\{(\Psi\Phi(\sigma_1)\}. \tag{13}$$

Similarly, multiplying $\mathsf{H}\left(\frac{1}{2},\frac{1}{2}\right)\eth^{\eta-1}\Phi(\eth\sigma_1+(1-\eth)\sigma_2)$ on both sides of (12) and integrating, we have

$$\mathsf{H}\left(\frac{1}{2},\frac{1}{2}\right)\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)\int_0^1\eth^{\eta-1}\Phi(\eth\sigma_1+(1-\eth)\sigma_2)d\eth$$

$$\preceq_{\mathbf{CR}}\int_0^1\eth^{\eta-1}\left[\Psi(\eth\sigma_1+(1-\eth)\sigma_2)+\Psi(\eth\sigma_2+(1-\eth)\sigma_1)\right]\Phi(\eth\sigma_1+(1-\eth)\sigma_2)d\eth.$$

Let $\mathrm{u}=\eth\sigma_1+(1-\eth)\sigma_2$, then the above relation becomes

$$\mathsf{H}\left(\frac{1}{2},\frac{1}{2}\right)\frac{1}{(\sigma_2-\sigma_1)^{\eta}}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)\int_{\sigma_1}^{\sigma_2}(\sigma_2-\mathrm{u})^{\eta-1}\Phi(\mathrm{u})\mathrm{du}$$

$$\preceq_{\mathbf{CR}}\frac{1}{(\sigma_2-\sigma_1)^{\eta}}\left[\int_{\sigma_1}^{\sigma_2}(\sigma_2-\mathrm{u})^{\eta-1}\Psi(\mathrm{u})\Phi(\mathrm{u})\mathrm{du}\right.$$

$$\left.+\int_{\sigma_1}^{\sigma_2}(\sigma_2-\mathrm{u})^{\eta-1}\Psi(\sigma_2+\sigma_1-\mathrm{u})\Phi(\mathrm{u})\mathrm{du}\right].$$

Same as earlier, making a modification in the foregoing relation, $\Psi=\sigma_2+\sigma_1-\mathrm{u}$, then $\Phi(\sigma_2+\sigma_1-\Psi)=\Phi(\Psi)$, becomes

$$\mathsf{H}\left(\frac{1}{2},\frac{1}{2}\right)\frac{1}{(\sigma_2-\sigma_1)^{\eta}}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)\int_{\sigma_1}^{\sigma_2}(\sigma_2-\mathrm{u})^{\eta-1}\Phi(\mathrm{u})\mathrm{du}$$

$$\preceq_{\mathbf{CR}}\frac{1}{(\sigma_2-\sigma_1)^{\eta}}\left[\int_{\sigma_1}^{\sigma_2}(\sigma_2-\mathrm{u})^{\eta-1}\Psi(\mathrm{u})\Phi(\mathrm{u})\mathrm{du}+\int_{\sigma_1}^{\sigma_2}(\Psi-\sigma_1)^{\eta-1}\Psi(\Psi)\Phi(\Psi)\mathrm{d}\Psi\right].$$

As before, multiplying the aforementioned relation with $\frac{\eta(\sigma_2-\sigma_1)^{\eta}}{\mathsf{B}(\eta)\Gamma(\eta)}$ and adding the expression $\frac{1-\eta}{\mathsf{B}(\eta)}\left[\Psi(\sigma_1)+\Psi(\sigma_2)\right]$ to the both sides, we get

$$\mathsf{H}\left(\frac{1}{2},\frac{1}{2}\right)\frac{\eta}{\mathsf{B}(\eta)\Gamma(\eta)}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)\int_{\sigma_1}^{\sigma_2}(\sigma_2-\mathrm{u})^{\eta-1}\Phi(\mathrm{u})\mathrm{du}$$

$$+\frac{1-\eta}{\mathsf{B}(\eta)}\left[\Psi(\sigma_1)\Phi(\sigma_1)+\Psi(\sigma_2)\Phi(\sigma_2)\right]$$

$$\preceq_{\mathbf{CR}}\frac{\eta}{\mathsf{B}(\eta)\Gamma(\eta)}\left[\int_{\sigma_1}^{\sigma_2}(\sigma_2-\mathrm{u})^{\eta-1}\Psi(\mathrm{u})\Phi(\mathrm{u})\mathrm{du}+\int_{\sigma_1}^{\sigma_2}(\Psi-\sigma_1)^{\eta-1}\Psi(\Psi)\Phi(\Psi)\mathrm{d}\Psi\right]$$

$$+\frac{1-\eta}{\mathsf{B}(\eta)}\left[\Psi(\sigma_1)\Phi(\sigma_1)+\Psi(\sigma_2)\Phi(\sigma_2)\right].$$

From this, it can be follows that

$$\mathsf{H}\left(\frac{1}{2},\frac{1}{2}\right)\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right){}^{\mathtt{AB}}_{\sigma_1}\mathtt{I}^{\eta}_{\sigma_2}\{\Phi(\sigma_2)\}$$

$$-\mathsf{H}\left(\frac{1}{2},\frac{1}{2}\right)\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)\frac{1-\eta}{\mathtt{B}(\eta)}\Phi(\sigma_2)+\frac{1-\eta}{\mathtt{B}(\eta)}\left[\Psi(\sigma_1)\Phi(\sigma_1)+\Psi(\sigma_2)\Phi(\sigma_2)\right]$$

$$\preceq_{\mathbf{CR}}{}^{\mathtt{AB}}_{\sigma_1}\mathtt{I}^{\eta}_{\sigma_2}\{(\Psi\Phi(\sigma_2)\}+{}^{\mathtt{AB}}\mathtt{I}^{\eta}_{\sigma_2}\{(\Psi\Phi(\sigma_1)\}. \tag{14}$$

Adding (13) and (14), we can get the first relation in (11) that is

$$\frac{\mathsf{H}\left(\frac{1}{2},\frac{1}{2}\right)}{2}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)\left[{}^{\mathtt{AB}}_{\sigma_1}\mathtt{I}^{\eta}_{\sigma_2}\{\Phi(\sigma_2)\}+{}^{\mathtt{AB}}\mathtt{I}^{\eta}_{\sigma_2}\{\Phi(\sigma_1)\}\right]$$

$$-\frac{\mathsf{H}\left(\frac{1}{2},\frac{1}{2}\right)}{2}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)\frac{1-\eta}{\mathtt{B}(\eta)}\left[\Phi(\sigma_1)+\Phi(\sigma_2)\right]$$

$$+\frac{1-\eta}{\mathtt{B}(\eta)}\left[\Psi(\sigma_1)\Phi(\sigma_1)+\Psi(\sigma_2)\Phi(\sigma_2)\right]. \tag{15}$$

Once more considering Definition 2.7 for the second relation, we have

$$\Psi(\eth\sigma_1+(1-\eth)\sigma_2)\preceq_{\mathbf{CR}}\frac{\Psi(\sigma_1)}{\mathsf{H}(\eth,1-\eth)}+\frac{\Psi(\sigma_2)}{\mathsf{H}(1-\eth,\eth)},$$

$$\Psi(\eth\sigma_2+(1-\eth)\sigma_1)\preceq_{\mathbf{CR}}\frac{\Psi(\sigma_2)}{\mathsf{H}(\eth,1-\eth)}+\frac{\Psi(\sigma_1)}{\mathsf{H}(1-\eth,\eth)}.$$

Adding the above two relations we have

$$\Psi(\eth\sigma_1+(1-\eth)\sigma_2)+\Psi(\eth\sigma_2+(1-\eth)\sigma_1)\preceq_{\mathbf{CR}}\left[\frac{1}{\mathsf{H}(\eth,1-\eth)}+\frac{1}{\mathsf{H}(1-\eth,\eth)}\right]\left[\Psi(\sigma_1)+\Psi(\sigma_2)\right]. \tag{16}$$

Multiplying relation (16) with $\eth^{\eta-1}\Phi(\eth\sigma_2+(1-\eth)\sigma_1)$ and integrating, we have

$$\int_0^1\eth^{\eta-1}\left[\Psi(\eth\sigma_1+(1-\eth)\sigma_2)+\Psi(\eth\sigma_2+(1-\eth)\sigma_1)\right]\Phi(\eth\sigma_2+(1-\eth)\sigma_1)d\eth$$

$$\preceq_{\mathbf{CR}}\left[\Psi(\sigma_1)+\Psi(\sigma_2)\right]\int_0^1\eth^{\eta-1}\left[\frac{1}{\mathsf{H}(\eth,1-\eth)}+\frac{1}{\mathsf{H}(1-\eth,\eth)}\right]\Phi(\eth\sigma_2+(1-\eth)\sigma_1)d\eth.$$

Making a change to the previously mentioned relationship, $\Psi = \sigma_2 + \sigma_1 - \mathrm{u}$, then $\Phi(\sigma_2 + \sigma_1 - \Psi) = \Phi(\Psi)$, becomes

$$\frac{1}{(\sigma_2-\sigma_1)^\eta}\left[\int_{\sigma_1}^{\sigma_2}(\sigma_2-\Psi)^{\eta-1}\Psi(\Psi)\Phi(\Psi)d\Psi+\int_{\sigma_1}^{\sigma_2}(\mathrm{u}-\sigma_1)^{\eta-1}\Psi(\mathrm{u})\Phi(\mathrm{u})d\mathrm{u}\right]$$

$$\preceq_{\mathbf{CR}}\left[\Psi(\sigma_1)+\Psi(\sigma_2)\right]\int_0^1 \eth^{\eta-1}\left[\frac{1}{\mathsf{H}(\eth,1-\eth)}+\frac{1}{\mathsf{H}(1-\eth,\eth)}\right]\Phi(\eth\sigma_2+(1-\eth)\sigma_1)d\eth.$$

Multiplying above relation with $\frac{\eta(\sigma_2-\sigma_1)^\eta}{\mathsf{B}(\eta)\Gamma(\eta)}$ and adding the expression $\frac{1-\eta}{\mathsf{B}(\eta)}\left[\Psi(\sigma_1)+\Psi(\sigma_2)\right]$, we have

$$\frac{\eta}{\mathsf{B}(\eta)\Gamma(\eta)}\left[\int_{\sigma_1}^{\sigma_2}(\sigma_2-\Psi)^{\eta-1}\Psi(\Psi)\Phi(\Psi)d\Psi+\int_{\sigma_1}^{\sigma_2}(\mathrm{u}-\sigma_1)^{\eta-1}\Psi(\mathrm{u})\Phi(\mathrm{u})d\mathrm{u}\right]$$

$$+\frac{1-\eta}{\mathsf{B}(\eta)}\left[\Psi(\sigma_1)\Phi(\sigma_1)+\Psi(\sigma_2)\Phi(\sigma_2)\right]$$

$$\preceq_{\mathbf{CR}}\frac{\eta(\sigma_2-\sigma_1)^\eta}{\mathsf{B}(\eta)\Gamma(\eta)}\left[\Psi(\sigma_1)+\Psi(\sigma_2)\right]$$

$$\times\int_0^1 \eth^{\eta-1}\left[\frac{1}{\mathsf{H}(\eth,1-\eth)}+\frac{1}{\mathsf{H}(1-\eth,\eth)}\right]\Phi(\eth\sigma_2+(1-\eth)\sigma_1)d\eth$$

$$+\frac{1-\eta}{\mathsf{B}(\eta)}\left[\Psi(\sigma_1)\Phi(\sigma_1)+\Psi(\sigma_2)\Phi(\sigma_2)\right],$$

it follows that

$${}^{\mathtt{AB}}_{\sigma_1}\mathtt{I}^{\eta}_{\sigma_2}\{(\Psi\Phi(\sigma_2)\}+{}^{\mathtt{AB}}\mathtt{I}^{\eta}_{\sigma_2}\{(\Psi\Phi(\sigma_1)\}$$

$$\preceq_{\mathbf{CR}}\frac{\eta(\sigma_2-\sigma_1)^\eta}{\mathsf{B}(\eta)\Gamma(\eta)}\left[\Psi(\sigma_1)+\Psi(\sigma_2)\right]$$

$$\times\int_0^1 \eth^{\eta-1}\left[\frac{1}{\mathsf{H}(\eth,1-\eth)}+\frac{1}{\mathsf{H}(1-\eth,\eth)}\right]\Phi(\eth\sigma_2+(1-\eth)\sigma_1)d\eth$$

$$+\frac{1-\eta}{\mathsf{B}(\eta)}\left[\Psi(\sigma_1)\Phi(\sigma_1)+\Psi(\sigma_2)\Phi(\sigma_2)\right]. \tag{17}$$

Taking into account relation (15) and (17), we obtain the needed relation (11). □

Example 3.1 Taking into account the assumptions of Theorem 3.2.

Let $\Psi(\sigma)=\left[2e^{\eth}+1,\, 3e^{\eth}+\frac{\sqrt{\eth}}{3}\right]$ defined over $[\sigma_1,\, \sigma_2]=[1,\, 2]$ with $\mathsf{h}_1(\eth)=\frac{1}{\sigma}$, $\mathsf{h}_2(\eth)=1$, $\mathtt{B}(\eta)=1$, $\eta=\frac{1}{2}$ and a symmetric functions $\Phi(\eth)=\eth-1$ for $\eth\in\left[1,\, \frac{5}{2}\right]$ and $\Phi(\eth)=-\eth+4$ for $\eth\in\left[\frac{5}{2},\, 4\right]$, then we have

$$\frac{\mathsf{H}\left(\frac{1}{2},\frac{1}{2}\right)}{2}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)\left[{}^{\mathtt{AB}}_{\sigma_1}\mathtt{I}^{\eta}_{\sigma_2}\{\Phi(\sigma_2)\}+{}^{\mathtt{AB}}\mathtt{I}^{\eta}_{\sigma_2}\{\Phi(\sigma_1)\}\right]$$

$$-\frac{\mathsf{H}\left(\frac{1}{2},\frac{1}{2}\right)}{2}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)\frac{1-\eta}{\mathtt{B}(\eta)}\left[\Phi(\sigma_1)+\Phi(\sigma_2)\right]$$

$$+\frac{1-\eta}{\mathtt{B}(\eta)}\left[\Psi(\sigma_1)\Phi(\sigma_1)+\Psi(\sigma_2)\Phi(\sigma_2)\right]$$

$$=\Psi\left(\frac{5}{2}\right)\left[\frac{1}{2}\Phi\left(\frac{5}{2}\right)+\frac{1}{2\sqrt{\Pi}}\int_1^{\frac{5}{2}}(\eth-1)\left(\frac{5}{2}-\eth\right)^{\frac{-1}{2}}+\frac{1}{2}\Phi\left(\frac{5}{2}\right)+\frac{1}{2\sqrt{\Pi}}\int_{\frac{5}{2}}^{4}(-\eth+4)\left(\eth-\frac{5}{2}\right)^{\frac{-1}{2}}\right]\mathsf{d}\eth$$

$$=\left[3e^{\frac{5}{2}}+\frac{3}{2}+2e^{\frac{5}{2}}\left(\frac{6}{\Pi}\right)^{\frac{1}{2}}+\left(\frac{6}{\Pi}\right)^{\frac{1}{2}},\, \frac{9e^{\frac{5}{2}}}{2}+\frac{5^{\frac{1}{2}}}{2\cdot 2^{\frac{1}{2}}}+\frac{5^{\frac{1}{2}}}{3^{\frac{1}{2}}\Pi^{\frac{1}{2}}}+3e^{\frac{5}{2}}\left(\frac{6}{\Pi}\right)^{\frac{1}{2}}\right]$$

$$\approx[73.10130,\, 106.84792].$$

And

$${}^{\mathtt{AB}}_{\sigma_1}\mathtt{I}^{\eta}_{\sigma_2}\{(\Psi\Phi(\sigma_2))\}+{}^{\mathtt{AB}}\mathtt{I}^{\eta}_{\sigma_2}\{(\Psi\Phi(\sigma_1))\}$$

$$=\left[\frac{1}{2}\Psi\left(\frac{3}{2}\right)+\frac{1}{2\sqrt{\Pi}}\int_1^{\frac{5}{2}}\left[2e^{\eth}+1,\, 3e^{\eth}+\frac{\sqrt{\eth}}{3}\right](\eth-1)\left(\frac{5}{2}-\eth\right)^{\frac{-1}{2}}+\frac{1}{2}\Psi\left(\frac{3}{2}\right)\right.$$

$$\left.+\frac{1}{2\sqrt{\Pi}}\int_{\frac{5}{2}}^{4}\left[2e^{\eth}+1,\, 3e^{\eth}+\frac{\sqrt{\eth}}{3}\right](-\eth+4)\left(\eth-\frac{5}{2}\right)^{\frac{-1}{2}}\right]\mathsf{d}\eth$$

$$\approx[81.16120,\, 111.35182].$$

Also, we have

$$\frac{\eta(\sigma_2-\sigma_1)^\eta}{\mathrm{B}(\eta)\Gamma(\eta)}\left[\Psi(\sigma_1)+\Psi(\sigma_2)\right]\times\int_0^1 \eth^{\eta-1}\left[\frac{1}{\mathrm{H}(\eth,\,1-\eth)}+\frac{1}{\mathrm{H}(1-\eth,\,\eth)}\right]\Phi(\eth\sigma_2+(1-\eth)\sigma_1)d\eth$$

$$+\frac{1-\eta}{\mathrm{B}(\eta)}\left[\Psi(\sigma_1)\Phi(\sigma_1)+\Psi(\sigma_2)\Phi(\sigma_2)\right]$$

$$\approx[85.14621,\ 115.36241].$$

Consequently, Theorem 3.2 is correct.

$$[73.10130,\ 106.84792]\preceq_{\mathbf{CR}}[81.16120,\ 111.35182]\preceq_{\mathbf{CR}}[85.14621,\ 115.36241].$$

Corollary 3.2 If $\mathrm{h}_1(\eth)=\frac{1}{\eth^s}$, $\mathrm{h}_2(\eth)=1$, then Theorem 3.2 yields an outcome for the s-convex function for Atangana-Baleanu integral operators:

$$2^{s-1}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)\left[{}^{\mathrm{AB}}_{\sigma_1}\mathrm{I}^{\eta}_{\sigma_2}\{\Phi(\sigma_2)\}+{}^{\mathrm{AB}}\mathrm{I}^{\eta}_{\sigma_2}\{\Phi(\sigma_1)\}\right]$$

$$-2^{s-1}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)\frac{1-\eta}{\mathrm{B}(\eta)}\left[\Phi(\sigma_1)+\Phi(\sigma_2)\right]$$

$$+\frac{1-\eta}{\mathrm{B}(\eta)}\left[\Psi(\sigma_1)\Phi(\sigma_1)+\Psi(\sigma_2)\Phi(\sigma_2)\right]$$

$$\preceq_{\mathbf{CR}}{}^{\mathrm{AB}}_{\sigma_1}\mathrm{I}^{\eta}_{\sigma_2}\{(\Psi\Phi(\sigma_2)\}+{}^{\mathrm{AB}}\mathrm{I}^{\eta}_{\sigma_2}\{(\Psi\Phi(\sigma_1)\}$$

$$\preceq_{\mathbf{CR}}\frac{\eta(\sigma_2-\sigma_1)^\eta}{\mathrm{B}(\eta)\Gamma(\eta)}\left[\Psi(\sigma_1)+\Psi(\sigma_2)\right]$$

$$\times\int_0^1\eth^{\eta-1}\left[\eth^s+(1-\eth)^s\right]\Phi(\eth\sigma_2+(1-\eth)\sigma_1)d\eth$$

$$+\frac{1-\eta}{\mathrm{B}(\eta)}\left[\Psi(\sigma_1)\Phi(\sigma_1)+\Psi(\sigma_2)\Phi(\sigma_2)\right]. \tag{18}$$

Corollary 3.3 If $\mathrm{h}_1(\eth)=\mathrm{h}(\eth)$, $\mathrm{h}_2(\eth)=1$, then Theorem 3.2 yields an outcome for the h-Godunova-Levin function for Atangana-Baleanu integral operators:

$$\frac{h\left(\frac{1}{2}\right)}{2}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)\left[{}^{\mathtt{AB}}_{\sigma_1}\mathtt{I}^{\eta}_{\sigma_2}\{\Phi(\sigma_2)\}+{}^{\mathtt{AB}}\mathtt{I}^{\eta}_{\sigma_2}\{\Phi(\sigma_1)\}\right]$$

$$-\frac{1}{2h\left(\frac{1}{2}\right)}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)\frac{1-\eta}{\mathtt{B}(\eta)}\left[\Phi(\sigma_1)+\Phi(\sigma_2)\right]$$

$$+\frac{1-\eta}{\mathtt{B}(\eta)}\left[\Psi(\sigma_1)\Phi(\sigma_1)+\Psi(\sigma_2)\Phi(\sigma_2)\right]$$

$$\preceq_{\mathbf{CR}}{}^{\mathtt{AB}}_{\sigma_1}\mathtt{I}^{\eta}_{\sigma_2}\{(\Psi\Phi(\sigma_2)\}+{}^{\mathtt{AB}}\mathtt{I}^{\eta}_{\sigma_2}\{(\Psi\Phi(\sigma_1)\}$$

$$\preceq_{\mathbf{CR}}\frac{\eta(\sigma_2-\sigma_1)^{\eta}}{\mathtt{B}(\eta)\Gamma(\eta)}\left[\Psi(\sigma_1)+\Psi(\sigma_2)\right]$$

$$\times\int_0^1\eth^{\eta-1}\left[\frac{1}{\mathsf{h}(\eth)}+\frac{1}{\mathsf{h}(1-\eth)}\right]\Phi(\eth\sigma_2+(1-\eth)\sigma_1)\mathrm{d}\eth$$

$$+\frac{1-\eta}{\mathtt{B}(\eta)}\left[\Psi(\sigma_1)\Phi(\sigma_1)+\Psi(\sigma_2)\Phi(\sigma_2)\right].$$

Corollary 3.4 If $\mathsf{h}_1(\eth)=\mathsf{h}_2(\eth)=1$, then Theorem 3.2 yields an outcome for the p-convex function for Atangana-Baleanu integral operators:

$$\frac{1}{2}f\left(\frac{\sigma_2+\sigma_1}{2}\right)\left[{}^{\mathtt{AB}}_{\sigma_1}\mathtt{I}^{\eta}_{\sigma_2}\{\Phi(\sigma_2)\}+{}^{\mathtt{AB}}\mathtt{I}^{\eta}_{\sigma_2}\{\Phi(\sigma_1)\}\right]-\frac{1}{2}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)\frac{1-\eta}{\mathtt{B}(\eta)}\left[\Phi(\sigma_1)+\Phi(\sigma_2)\right]$$

$$+\frac{1-\eta}{\mathtt{B}(\eta)}\left[\Psi(\sigma_1)\Phi(\sigma_1)+\Psi(\sigma_2)\Phi(\sigma_2)\right]$$

$$\preceq_{\mathbf{CR}}{}^{\mathtt{AB}}_{\sigma_1}\mathtt{I}^{\eta}_{\sigma_2}\{(\Psi\Phi(\sigma_2)\}+{}^{\mathtt{AB}}\mathtt{I}^{\eta}_{\sigma_2}\{(\Psi\Phi(\sigma_1)\}$$

$$\preceq_{\mathbf{CR}}\frac{2\eta(\sigma_2-\sigma_1)^{\eta}}{\mathtt{B}(\eta)\Gamma(\eta)}\left[\Psi(\sigma_1)+\Psi(\sigma_2)\right]$$

$$\times\int_0^1\eth^{\eta-1}\Phi(\eth\sigma_2+(1-\eth)\sigma_1)\mathrm{d}\eth$$

$$+\frac{1-\eta}{\mathtt{B}(\eta)}\left[\Psi(\sigma_1)\Phi(\sigma_1)+\Psi(\sigma_2)\Phi(\sigma_2)\right].$$

Theorem 3.3 Let $\mathsf{h}_1, \mathsf{h}_2 : (0, 1) \to \mathsf{R}^+$ such that $\mathsf{H}\left(\frac{1}{2}, \frac{1}{2}\right) \neq 0$, and $\Psi : [\sigma_1, \sigma_2] \to \mathsf{R}_{\mathtt{I}}^+$. If $\Psi \in \mathtt{SGX}(\mathbf{CR}\text{-}(\mathsf{h}_1, \mathsf{h}_2), [\sigma_1, \sigma_2], \mathsf{R}_{\mathtt{I}}^+)$, then we have

$$\mathsf{H}\left(\frac{1}{2}, \frac{1}{2}\right)\frac{(\sigma_2-\sigma_1)^\eta}{\mathtt{B}(\eta)\Gamma(\eta)}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)+\frac{1-\eta}{\mathtt{B}(\eta)}\left[\Psi(\sigma_1)+\Psi(\sigma_2)\right]$$

$$\preceq_{\mathbf{CR}} {}^{\mathtt{AB}}_{\sigma_1}\mathtt{I}^{\eta}_{\sigma_2}\{\Psi(\sigma_2)\}+{}^{\mathtt{AB}}\mathtt{I}^{\eta}_{\sigma_2}\{\Psi(\sigma_1)\}$$

$$\preceq_{\mathbf{CR}}\left[\frac{\Psi(\sigma_1)+\Psi(\sigma_2)}{\mathtt{B}(\eta)}\right]\left[1-\eta+\frac{\eta(\sigma_2-\sigma_1)^\eta}{\Gamma(\eta)}\right.$$

$$\left.\times\int_0^1 \eth^{\eta-1}\left(\frac{1}{\mathsf{H}(\eth, 1-\eth)}+\frac{1}{\mathsf{H}(1-\eth, \eth)}\right)d\eth\right], \tag{19}$$

where $\eta \in (0, 1)$.

Proof. As $\Psi \in \mathtt{SGX}(\mathbf{CR}\text{-}(\mathsf{h}_1, \mathsf{h}_2), [\sigma_1, \sigma_2], \mathsf{R}_{\mathtt{I}}^+)$, we have

$$\left[\mathsf{H}\left(\frac{1}{2}, \frac{1}{2}\right)\right]\Psi\left(\frac{\nu_1+\nu_2}{2}\right)\preceq_{\mathbf{CR}}\Psi(\nu_1)+\Psi(\nu_2),$$

let $\nu_1 = \eth\sigma_1+(1-\eth)\sigma_2$, $\nu_2 = \eth\sigma_2+(1-\eth)\sigma_1$, the above relation becomes

$$\mathsf{H}\left(\frac{1}{2}, \frac{1}{2}\right)\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)\preceq_{\mathbf{CR}}\left[\Psi(\eth\sigma_1+(1-\eth)\sigma_2)+\Psi(\eth\sigma_2+(1-\eth)\sigma_1)\right]. \tag{20}$$

Multiplying by $\eth^{\eta-1}$ in (20) and integrating, we have

$$\frac{1}{\eta}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)\preceq_{\mathbf{CR}}\frac{1}{\mathsf{H}\left(\frac{1}{2}, \frac{1}{2}\right)}\left[\int_0^1\eth^{\eta-1}\Psi(\eth\sigma_1+(1-\eth)\sigma_2)d\eth+\int_0^1\eth^{\eta-1}\Psi(\eth\sigma_2+(1-\eth)\sigma_1)d\eth\right],$$

it follows that

$$\frac{\mathsf{H}\left(\frac{1}{2}, \frac{1}{2}\right)}{\eta}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)\preceq_{\mathbf{CR}}\int_0^1\eth^{\eta-1}\Psi(\eth\sigma_1+(1-\eth)\sigma_2)d\eth$$

$$+\int_0^1\eth^{\eta-1}\Psi(\eth\sigma_2+(1-\eth)\sigma_1)d\eth.$$

Multiplying the above relation with $\frac{\eta(\sigma_2-\sigma_1)^\eta}{\mathtt{B}(\eta)\Gamma(\eta)}$ and adding the expression $\frac{1-\eta}{\mathtt{B}(\eta)}\left[\Psi(\sigma_1)+\Psi(\sigma_2)\right]$, we get

$$\mathsf{H}\left(\frac{1}{2},\frac{1}{2}\right)\frac{(\sigma_2-\sigma_1)^{\eta}}{\mathtt{B}(\eta)\Gamma(\eta)}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)+\frac{1-\eta}{\mathtt{B}(\eta)}\left[\Psi(\sigma_1)+\Psi(\sigma_2)\right]$$

$$\preceq_{\mathbf{CR}}\frac{\eta(\sigma_2-\sigma_1)^{\eta}}{\mathtt{B}(\eta)\Gamma(\eta)}\int_0^1\eth^{\eta-1}\Psi(\eth\sigma_1+(1-\eth)\sigma_2)\mathrm{d}\eth$$

$$+\frac{\eta(\sigma_2-\sigma_1)^{\eta}}{\mathtt{B}(\eta)\Gamma(\eta)}\int_0^1\eth^{\eta-1}\Psi(\eth\sigma_2+(1-\eth)\sigma_1)\mathrm{d}\eth+\frac{1-\eta}{\mathtt{B}(\eta)}\left[\Psi(\sigma_1)+\Psi(\sigma_2)\right].$$

It follows that

$$\mathsf{H}\left(\frac{1}{2},\frac{1}{2}\right)\frac{(\sigma_2-\sigma_1)^{\eta}}{\mathtt{B}(\eta)\Gamma(\eta)}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)+\frac{1-\eta}{\mathtt{B}(\eta)}\left[\Psi(\sigma_1)+\Psi(\sigma_2)\right]\preceq_{\mathbf{CR}}{}^{\mathtt{AB}}_{\sigma_1}\mathtt{I}^{\eta}_{\sigma_2}\{\Psi(\sigma_2)\}+{}^{\mathtt{AB}}\mathtt{I}^{\eta}_{\sigma_2}\{\Psi(\sigma_1)\},$$

so the first relation of (19) holds.

Considering Definition 2.7 for the second relation, we have

$$\Psi(\eth\sigma_1+(1-\eth)\sigma_2)\preceq_{\mathbf{CR}}\frac{\Psi(\sigma_1)}{\mathsf{h}_1(\eth)\mathsf{h}_2(1-\eth)}+\frac{\Psi(\sigma_2)}{\mathsf{h}_1(1-\eth)\mathsf{h}_2(\eth)}.$$

Multiplying aforementioned relation with $\eth^{\eta-1}$, and integrating, we have

$$\int_0^1\eth^{\eta-1}\Psi(\eth\sigma_1+(1-\eth)\sigma_2)\mathrm{d}\eth$$

$$\preceq_{\mathbf{CR}}\Psi(\sigma_1)\int_0^1\frac{\eth^{\eta-1}\mathrm{d}\eth}{\mathsf{h}_1(\eth)\mathsf{h}_2(1-\eth)}+\Psi(\sigma_2)\int_0^1\frac{\eth^{\eta-1}\mathrm{d}\eth}{\mathsf{h}_1(1-\eth)\mathsf{h}_2(\eth)}. \tag{21}$$

Multiplying both sides of (21) by $\frac{\eta(\sigma_2-\sigma_1)^{\eta}}{\mathtt{B}(\eta)\Gamma(\eta)}$ and adding the expression $\frac{1-\eta}{\mathtt{B}(\eta)}\Psi(\sigma_2)$ to both sides, we get

$$\frac{\eta(\sigma_2-\sigma_1)^{\eta}}{\mathtt{B}(\eta)\Gamma(\eta)}\int_0^1\eth^{\eta-1}\Psi(\eth\sigma_1+(1-\eth)\sigma_2)\mathrm{d}\eth+\frac{1-\eta}{\mathtt{B}(\eta)}\Psi(\sigma_2)$$

$$\preceq_{\mathbf{CR}}\frac{\eta(\sigma_2-\sigma_1)^{\eta}}{\mathtt{B}(\eta)\Gamma(\eta)}\left[\Psi(\sigma_1)\int_0^1\frac{\eth^{\eta-1}\mathrm{d}\eth}{\mathsf{h}_1(\eth)\mathsf{h}_2(1-\eth)}\right.$$

$$\left.+\Psi(\sigma_2)\int_0^1\frac{\eth^{\eta-1}\mathrm{d}\eth}{\mathsf{h}_1(1-\eth)\mathsf{h}_2(\eth)}\right]+\frac{1-\eta}{\mathtt{B}(\eta)}\Psi(\sigma_2). \tag{22}$$

Making a change to the previously mentioned relationship, $\Psi=\sigma_2+\sigma_1-\mathtt{u}$, then $\Phi(\sigma_2+\sigma_1-\Psi)=\Phi(\Psi)$, becomes

$$
{}_{\sigma_1}^{\mathtt{AB}}\mathtt{I}_{\sigma_2}^{\eta}\{\Psi(\sigma_2)\} \preceq_{\mathbf{CR}} \frac{\eta(\sigma_2-\sigma_1)^{\eta}}{\mathtt{B}(\eta)\Gamma(\eta)}\left[\Psi(\sigma_1)\int_0^1 \frac{\eth^{\eta-1}\mathrm{d}\eth}{\mathsf{H}(\eth,\, 1-\eth)}\right.
$$

$$
\left.+\Psi(\sigma_2)\int_0^1 \frac{\eth^{\eta-1}\mathrm{d}\eth}{\mathsf{H}(1-\eth,\, \eth)}\right] + \frac{1-\eta}{\mathtt{B}(\eta)}\Psi(\sigma_2). \tag{23}
$$

Again by Definition 2.7, we have

$$
\Psi(\eth\sigma_1 + (1-\eth)\sigma_2) \preceq_{\mathbf{CR}} \frac{\Psi(\sigma_1)}{\mathsf{h}_1(\eth)\mathsf{h}_2(1-\eth)} + \frac{\Psi(\sigma_2)}{\mathsf{h}_1(1-\eth)\mathsf{h}_2(\eth)}.
$$

Multiplying aforementioned relation with $\eth^{\eta-1}$, and integrating, we have

$$
\frac{\eta(\sigma_2-\sigma_1)^{\eta}}{\mathtt{B}(\eta)\Gamma(\eta)}\int_0^1 \eth^{\eta-1}\Psi(\eth\sigma_2+(1-\eth)\sigma_1)\mathrm{d}\eth + \frac{1-\eta}{\mathtt{B}(\eta)}\Psi(\sigma_1)
$$

$$
\preceq_{\mathbf{CR}} \frac{\eta(\sigma_2-\sigma_1)^{\eta}}{\mathtt{B}(\eta)\Gamma(\eta)}\left[\Psi(\sigma_2)\int_0^1 \frac{\eth^{\eta-1}\mathrm{d}\eth}{\mathsf{h}_1(\eth)\mathsf{h}_2(1-\eth)}\right.
$$

$$
\left.+\Psi(\sigma_1)\int_0^1 \frac{\eth^{\eta-1}\mathrm{d}\eth}{\mathsf{h}_1(1-\eth)\mathsf{h}_2(\eth)}\right] + \frac{1-\eta}{\mathtt{B}(\eta)}\Psi(\sigma_1).
$$

Same as earlier, making a modification in the foregoing relation, $\Psi = \sigma_2 + \sigma_1 - \mathrm{u}$, then $\Phi(\sigma_2+\sigma_1-\Psi) = \Phi(\Psi)$, becomes

$$
{}^{\mathtt{AB}}\mathtt{I}_{\sigma_2}^{\eta}\{\Psi(\sigma_1)\} \preceq_{\mathbf{CR}} \frac{\eta(\sigma_2-\sigma_1)^{\eta}}{\mathtt{B}(\eta)\Gamma(\eta)}\left[\Psi(\sigma_2)\int_0^1 \frac{\eth^{\eta-1}\mathrm{d}\eth}{\mathsf{H}(\eth,\, 1-\eth)}\right.
$$

$$
\left.+\Psi(\sigma_1)\int_0^1 \frac{\eth^{\eta-1}\mathrm{d}\eth}{\mathsf{H}(1-\eth,\, \eth)}\right] + \frac{1-\eta}{\mathtt{B}(\eta)}\Psi(\sigma_1). \tag{24}
$$

Adding (23) and (24), we can get that the second relation of (19). This completes the evidence. □

Example 3.2 Taking into account the assumptions of Theorem 3.3. Let $\Psi(\sigma) = \left[2e^{\eth}+1,\, 3e^{\eth}+\frac{\sqrt{\eth}}{3}\right]$ defined over $[\sigma_1,\, \sigma_2] = [1,\, 4]$ with $\mathsf{h}_1(\eth) = \frac{1}{\sigma}$, $\mathsf{h}_2(\eth) = 1$, $\mathtt{B}(\eta) = 1$, $\eta = \frac{1}{2}$, then we have

$$\mathsf{H}\left(\frac{1}{2},\frac{1}{2}\right)\frac{(\sigma_2-\sigma_1)^{\eta}}{\mathsf{B}(\eta)\Gamma(\eta)}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)+\frac{1-\eta}{\mathsf{B}(\eta)}\left[\Psi(\sigma_1)+\Psi(\sigma_2)\right]$$

$$=\frac{\sqrt{3}}{\sqrt{\Pi}}\left[\mathrm{e}^4+\mathrm{e}+2\mathrm{e}^{\frac{5}{2}}+2,\ \frac{3\mathrm{e}^4+3\mathrm{e}+1}{2}+\mathrm{e}^{\frac{5}{2}}\sqrt{\frac{5}{2}}\right]$$

$$\approx[81.77390,\ 103.32659].$$

And

$${}^{\mathtt{AB}}_{\sigma_1}\mathtt{I}^{\eta}_{\sigma_2}\{\Psi(\sigma_2)\}+{}^{\mathtt{AB}}\mathtt{I}^{\eta}_{\sigma_2}\{\Psi(\sigma_1)\}$$

$$=\left[\frac{1}{2}+\Phi\left(\frac{5}{2}\right)+\frac{1}{2\sqrt{\Pi}}\right]\left(\int_1^{\frac{5}{2}}\left[2\mathrm{e}^{\eth}+1,\ 3\mathrm{e}^{\eth}+\frac{\sqrt{\eth}}{3}\right]\left(-\eth+\frac{5}{2}\right)^{\frac{-1}{2}}\right.$$

$$\left.+\int_{\frac{5}{2}}^{4}\left[2\mathrm{e}^{\eth}+1,\ 3\mathrm{e}^{\eth}+\frac{\sqrt{\eth}}{3}\right]\left(\eth-\frac{5}{2}\right)^{\frac{-1}{2}}\right)\mathrm{d}\eth$$

$$=\left[2\mathrm{e}^{\frac{5}{2}}+\frac{3}{2}+\frac{1}{2\sqrt{\Pi}},\ 3\mathrm{e}^{\frac{5}{2}}+\frac{\sqrt{\frac{5}{2}}}{3}+\frac{1}{\sqrt{\Pi}}\right]$$

$$\times\left(\left[2\mathrm{e}^{\frac{5}{2}}\sqrt{\Pi}\,\mathrm{erf}\left(\frac{\sqrt{3}}{\sqrt{2}}\right)+\sqrt{2}\sqrt{3},\ \frac{-27\sqrt{2}\mathrm{e}+27\sqrt{2}\mathrm{e}^{\frac{5}{2}}+5^{\frac{3}{2}}-2^{\frac{3}{2}}}{9\sqrt{2}}\right]\right.$$

$$\left.+\left[\sqrt{2}\sqrt{3}-2\mathrm{e}^{\frac{5}{2}}\sqrt{\Pi}\,\mathrm{erf}\left(\frac{\sqrt{3}}{\sqrt{2}}\right),\ \frac{27\sqrt{2}\mathrm{e}^4-27\sqrt{2}\mathrm{e}^{\frac{5}{2}}-5^{\frac{3}{2}}+2^{\frac{9}{2}}}{9\sqrt{2}}\right]\right)$$

$$\approx[90.33565,\ 131.54364].$$

Also, we have

$$\left[\frac{\Psi(\sigma_1)+\Psi(\sigma_2)}{\mathrm{B}(\eta)}\right]\left[1-\eta+\frac{\eta(\sigma_2-\sigma_1)^\eta}{\Gamma(\eta)}\right.$$

$$\left.\times\int_0^1 \eth^{\eta-1}\left(\frac{1}{\mathrm{H}(\eth,\, 1-\eth)}+\frac{1}{\mathrm{H}(1-\eth,\, \eth)}\right)\mathrm{d}\eth\right]$$

$$=\left[e^4+e+\frac{\sqrt{3}\sqrt{\Pi}\left(2e^4+2e+2\right)}{3\Pi}+1,\ \left(3e^4+3e+1\right)\left(\frac{1}{2}+\frac{\sqrt{3}\sqrt{\Pi}}{3\Pi}\right)\right]$$

$$\approx [96.30783,\ 142.81028].$$

Consequently, Theorem 3.3 is correct.

$$[81.77390,\ 103.32659] \preceq_{\mathbf{CR}} [90.33565,\ 131.54364] \preceq_{\mathbf{CR}} [96.30783,\ 142.81028].$$

Corollary 3.5 If $\mathrm{h}_1(\eth)=\frac{1}{\eth^s}$, $\mathrm{h}_2(\eth)=1$, then Theorem 3.3 yields an outcome for the s-convex function for Atangana-Baleanu integral operators:

$$2^s\frac{(\sigma_2-\sigma_1)^\eta}{\mathrm{B}(\eta)\Gamma(\eta)}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)+\frac{1-\eta}{\mathrm{B}(\eta)}\left[\Psi(\sigma_1)+\Psi(\sigma_2)\right]$$

$$\preceq_{\mathbf{CR}} {}^{\mathrm{AB}}_{\sigma_1}\mathrm{I}^{\eta}_{\sigma_2}\{\Psi(\sigma_2)\}+{}^{\mathrm{AB}}\mathrm{I}^{\eta}_{\sigma_2}\{\Psi(\sigma_1)\}$$

$$\preceq_{\mathbf{CR}}\left[\frac{\Psi(\sigma_1)+\Psi(\sigma_2)}{\mathrm{B}(\eta)}\right]\left[1-\eta+\frac{\eta(\sigma_2-\sigma_1)^\eta}{\Gamma(\eta)(\mathrm{s}+\eta)}+\frac{\eta(\sigma_2-\sigma_1)^\eta}{\Gamma(\eta)}\frac{\Gamma(\eta)\Gamma(\mathrm{s}+1)}{\Gamma(\eta+\mathrm{s}+2)}\right].$$

Corollary 3.6 If $\mathrm{h}_1(\eth)=\mathrm{h}(\eth)$, $\mathrm{h}_2(\eth)=1$, then Theorem 3.3 yields an outcome for the h-Godunova-Levin function for Atangana-Baleanu integral operators:

$$h\left(\frac{1}{2}\right)\frac{(\sigma_2-\sigma_1)^\eta}{\mathrm{B}(\eta)\Gamma(\eta)}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)+\frac{1-\eta}{\mathrm{B}(\eta)}\left[\Psi(\sigma_1)+\Psi(\sigma_2)\right]$$

$$\preceq_{\mathbf{CR}} {}^{\mathrm{AB}}_{\sigma_1}\mathrm{I}^{\eta}_{\sigma_2}\{\Psi(\sigma_2)\}+{}^{\mathrm{AB}}\mathrm{I}^{\eta}_{\sigma_2}\{\Psi(\sigma_1)\}$$

$$\preceq_{\mathbf{CR}}\left[\frac{\Psi(\sigma_1)+\Psi(\sigma_2)}{\mathrm{B}(\eta)}\right]\left[1-\eta+\frac{\eta(\sigma_2-\sigma_1)^\eta}{\Gamma(\eta)}\int_0^1\eth^{\eta-1}\left(\frac{1}{\mathrm{h}(\eth)}+\frac{1}{\mathrm{h}(1-\eth)}\right)\mathrm{d}\eth\right].$$

Corollary 3.7 If $\mathrm{h}_1(\eth)=\mathrm{h}_2(\eth)=1$, then Theorem 3.3 yields an outcome for the p-convex function for Atangana-Baleanu integral operators:

$$\frac{(\sigma_2-\sigma_1)^\eta}{\mathtt{B}(\eta)\Gamma(\eta)}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)+\frac{1-\eta}{\mathtt{B}(\eta)}\left[\Psi(\sigma_1)+\Psi(\sigma_2)\right]$$

$$\preceq_{\mathbf{CR}}{}^{\mathtt{AB}}_{\sigma_1}\mathtt{I}^\eta_{\sigma_2}\{\Psi(\sigma_2)\}+{}^{\mathtt{AB}}\mathtt{I}^\eta_{\sigma_2}\{\Psi(\sigma_1)\}$$

$$\preceq_{\mathbf{CR}}\left[\frac{\Psi(\sigma_1)+\Psi(\sigma_2)}{\mathtt{B}(\eta)}\right]\left[1-\eta+\frac{2(\sigma_2-\sigma_1)^\eta}{\Gamma(\eta)}\right].$$

3.3 *Upper bounds for Hermite-Hadamard type inequality for twice differentiable mappings*

Based on the identity in Lemma 2.2, we present a novel refinement of H-H type fractional integral inequalities when the function Ψ is twice differentiable.

Theorem 3.4 Let $\mathsf{h}_1, \mathsf{h}_2 : (0, 1) \to \mathsf{R}^+$ such that $\mathsf{H}\left(\frac{1}{2}, \frac{1}{2}\right) \neq 0$, and $\Psi : [\sigma_1, \sigma_2] \to \mathsf{R}^+_{\mathtt{I}}$. If $\Psi'' \in \mathtt{SGX}(\mathbf{CR}\text{-}(\mathsf{h}_1, \mathsf{h}_2), [\sigma_1, \sigma_2], \mathsf{R}^+_{\mathtt{I}})$, then we have

$$\frac{1}{\sigma_2-\sigma_1}\left[{}^{\mathtt{AB}}\mathtt{I}^\eta_{\frac{\sigma_2+\sigma_1}{2}}\{\Psi(\sigma_1)\}+{}^{\mathtt{AB}}_{\frac{\sigma_2+\sigma_1}{2}}\mathtt{I}^\eta_{\sigma_2}\{\Psi(\sigma_2)\}\right]$$

$$-\frac{1}{(\sigma_2-\sigma_1)\mathtt{B}(\eta)}\left[\Psi(\sigma_1)+\Psi(\sigma_2)\right]-\frac{(\sigma_2-\sigma_1)^{\eta-1}}{2^{\eta-1}\mathtt{B}(\eta)\Gamma(\eta)}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)$$

$$\preceq_{\mathbf{CR}}\frac{(\sigma_2-\sigma_1)^{\eta-1}}{(\eta+1)\mathtt{B}(\eta)\Gamma(\eta)}\left[|\Psi''(\sigma_1)|+|\Psi''(\sigma_2)|\right]$$

$$\times\int_0^{\frac{1}{2}}\eth^{\eta+1}\left[\frac{1}{\mathsf{H}(\eth,1-\eth)}+\frac{1}{\mathsf{H}(1-\eth,\eth)}\right]d\eth, \tag{25}$$

where $\eta \in (0, 1]$.

Proof. By considering Lemma 1 in [59], we have

$$\frac{1}{\sigma_2-\sigma_1}\left[{}^{\mathtt{AB}}\mathtt{I}^\eta_{\frac{\sigma_2+\sigma_1}{2}}\{\Psi(\sigma_1)\}+{}^{\mathtt{AB}}_{\frac{\sigma_2+\sigma_1}{2}}\mathtt{I}^\eta_{\sigma_2}\{\Psi(\sigma_2)\}\right]$$

$$-\frac{1}{(\sigma_2-\sigma_1)\mathtt{B}(\eta)}\left[\Psi(\sigma_1)+\Psi(\sigma_2)\right]-\frac{(\sigma_2-\sigma_1)^{\eta-1}}{2^{\eta-1}\mathtt{B}(\eta)\Gamma(\eta)}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)$$

$$\preceq_{\mathbf{CR}}\frac{(\sigma_2-\sigma_1)^{\eta-1}}{2(\eta+1)\mathtt{B}(\eta)\Gamma(\eta)}\times\int_0^1|\mathtt{w}^\eta(\eth)|\left[|\Psi''(\eth\sigma_1+(1-\eth)\sigma_2)|+|\Psi''(\eth\sigma_2+(1-\eth)\sigma_1)|\right]d\eth. \tag{26}$$

Since $\Psi'' \in \mathtt{SGX}(\mathbf{CR}\text{-}(\mathsf{h}_1, \mathsf{h}_2), [\sigma_1, \sigma_2], \mathsf{R}^+_{\mathtt{I}})$, then we have

$$\frac{1}{\sigma_2-\sigma_1}\left[{}^{\mathtt{AB}}\mathtt{I}^{\eta}_{\frac{\sigma_2+\sigma_1}{2}}\{\Psi(\sigma_1)\}+{}^{\mathtt{AB}}_{\frac{\sigma_2+\sigma_1}{2}}\mathtt{I}^{\eta}_{\sigma_2}\{\Psi(\sigma_2)\}\right]$$

$$-\frac{1}{(\sigma_2-\sigma_1)\mathtt{B}(\eta)}\left[\Psi(\sigma_1)+\Psi(\sigma_2)\right]-\frac{(\sigma_2-\sigma_1)^{\eta-1}}{2^{\eta-1}\mathtt{B}(\eta)\Gamma(\eta)}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)$$

$$\leq\frac{(\sigma_2-\sigma_1)^{\eta-1}}{2(\eta+1)\mathtt{B}(\eta)\Gamma(\eta)}\times\left\{\int_0^{\frac{1}{2}}\eth^{\eta+1}\left[\frac{|\Psi''(\sigma_1)|}{\mathsf{h}_1(\eth)\mathsf{h}_2(1-\eth)}+\frac{|\Psi''(\sigma_2)|}{\mathsf{h}_1(1-\eth)\mathsf{h}_2(\eth)}\right]d\eth\right.$$

$$+\int_{\frac{1}{2}}^{1}(1-\eth)^{\eta+1}\left[\frac{|\Psi''(\sigma_1)|}{\mathsf{h}_1(\eth)\mathsf{h}_2(1-\eth)}+\frac{|\Psi''(\sigma_2)|}{\mathsf{h}_1(1-\eth)\mathsf{h}_2(\eth)}\right]d\eth$$

$$+\int_0^{\frac{1}{2}}\eth^{\eta+1}\left[\frac{|\Psi''(\sigma_2)|}{\mathsf{h}_1(\eth)\mathsf{h}_2(1-\eth)}+\frac{|\Psi''(\sigma_1)|}{\mathsf{h}_1(1-\eth)\mathsf{h}_2(\eth)}\right]d\eth$$

$$\left.+\int_{\frac{1}{2}}^{1}(1-\eth)^{\eta+1}\left[\frac{|\Psi''(\sigma_2)|}{\mathsf{h}_1(\eth)\mathsf{h}_2(1-\eth)}+\frac{|\Psi''(\sigma_1)|}{\mathsf{h}_1(1-\eth)\mathsf{h}_2(\eth)}\right]d\eth\right\}$$

$$=\frac{(\sigma_2-\sigma_1)^{\eta-1}}{(\eta+1)\mathtt{B}(\eta)\Gamma(\eta)}$$

$$\times\left\{\int_0^{\frac{1}{2}}\eth^{\eta+1}\left[\frac{|\Psi''(\sigma_1)|}{\mathsf{h}_1(\eth)\mathsf{h}_2(1-\eth)}+\frac{|\Psi''(\sigma_2)|}{\mathsf{h}_1(1-\eth)\mathsf{h}_2(\eth)}\right]d\eth\right.$$

$$\left.+\int_0^{\frac{1}{2}}\eth^{\eta+1}\left[\frac{|\Psi''(\sigma_2)|}{\mathsf{h}_1(\eth)\mathsf{h}_2(1-\eth)}+\frac{|\Psi''(\sigma_1)|}{\mathsf{h}_1(1-\eth)\mathsf{h}_2(\eth)}\right]d\eth\right\}$$

$$=\frac{(\sigma_2-\sigma_1)^{\eta-1}}{(\eta+1)\mathtt{B}(\eta)\Gamma(\eta)}\left[|\Psi''(\sigma_1)|+|\Psi''(\sigma_2)|\right]\int_0^{\frac{1}{2}}\eth^{\eta+1}\left[\frac{1}{\mathsf{H}(\eth,1-\eth)}+\frac{1}{\mathsf{H}(1-\eth,\eth)}\right]d\eth.$$

□

Example 3.3 Taking into account the assumptions of Theorem 3.4. Let $\Psi(\sigma)=\left[2e^{\eth}+1,\,3e^{\eth}+\frac{\sqrt{\eth}}{3}\right]$ defined over $[\sigma_1,\sigma_2]=[1,4]$ with $\mathsf{h}_1(\eth)=\frac{1}{\sigma}$, $\mathsf{h}_2(\eth)=1$, $\mathtt{B}(\eta)=1$, $\eta=\frac{1}{2}$.

Then, all the hypothesis of Theorem 3.4 are satisfied.

Corollary 3.8 If $\mathsf{h}_1(\eth)=\frac{1}{\eth^s}$, $\mathsf{h}_2(\eth)=1$, then Theorem 3.4 yields an outcome for the s-convex function for Atangana-Baleanu integral operators:

$$\frac{1}{\sigma_2-\sigma_1}\left[{}^{AB}I^{\eta}_{\frac{\sigma_2+\sigma_1}{2}}\{\Psi(\sigma_1)\}+{}^{AB}_{\frac{\sigma_2+\sigma_1}{2}}I^{\eta}_{\sigma_2}\{\Psi(\sigma_2)\}\right]$$

$$-\frac{1}{(\sigma_2-\sigma_1)B(\eta)}\left[\Psi(\sigma_1)+\Psi(\sigma_2)\right]-\frac{(\sigma_2-\sigma_1)^{\eta-1}}{2^{\eta-1}B(\eta)\Gamma(\eta)}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)$$

$$\leq\frac{(\sigma_2-\sigma_1)^{\eta-1}}{(\eta+1)B(\eta)\Gamma(\eta)}\left[|\Psi''(\sigma_1)|+|\Psi''(\sigma_2)|\right]\left[\frac{\left(\frac{1}{2}\right)^{\eta+s+2}}{\eta+s+2}+\beta_{\frac{1}{2}}(\eta+2,\, s+1)\right].$$

Corollary 3.9 If $h_1(\eth)=h(\eth)$, $h_2(\eth)=1$, then Theorem 3.4 yields an outcome for the h-Godunova-Levin function for Atangana-Baleanu integral operators:

$$\frac{1}{\sigma_2-\sigma_1}\left[{}^{AB}I^{\eta}_{\frac{\sigma_2+\sigma_1}{2}}\{\Psi(\sigma_1)\}+{}^{AB}_{\frac{\sigma_2+\sigma_1}{2}}I^{\eta}_{\sigma_2}\{\Psi(\sigma_2)\}\right]$$

$$-\frac{1}{(\sigma_2-\sigma_1)B(\eta)}\left[\Psi(\sigma_1)+\Psi(\sigma_2)\right]-\frac{(\sigma_2-\sigma_1)^{\eta-1}}{2^{\eta-1}B(\eta)\Gamma(\eta)}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)$$

$$\leq\frac{(\sigma_2-\sigma_1)^{\eta-1}}{(\eta+1)B(\eta)\Gamma(\eta)}\left[|\Psi''(\sigma_1)|+|\Psi''(\sigma_2)|\right]\int_0^{\frac{1}{2}}\eth^{\eta+1}\left[\frac{1}{h(1-\eth)}+\frac{1}{h(\eth)}\right]d\eth.$$

Corollary 3.10 If $h_1(\eth)=h_2(\eth)=1$, then Theorem 3.4 yields an outcome for the p-convex function for Atangana-Baleanu integral operators:

$$\frac{1}{\sigma_2-\sigma_1}\left[{}^{AB}I^{\eta}_{\frac{\sigma_2+\sigma_1}{2}}\{\Psi(\sigma_1)\}+{}^{AB}_{\frac{\sigma_2+\sigma_1}{2}}I^{\eta}_{\sigma_2}\{\Psi(\sigma_2)\}\right]$$

$$-\frac{1}{(\sigma_2-\sigma_1)B(\eta)}\left[\Psi(\sigma_1)+\Psi(\sigma_2)\right]-\frac{(\sigma_2-\sigma_1)^{\eta-1}}{2^{\eta-1}B(\eta)\Gamma(\eta)}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)$$

$$\leq\frac{\left(\frac{1}{2}\right)^{\eta+1}}{(\eta+1)(\eta+2)}\frac{(\sigma_2-\sigma_1)^{\eta-1}}{B(\eta)\Gamma(\eta)}\left[|\Psi''(\sigma_1)|+|\Psi''(\sigma_2)|\right].$$

Theorem 3.5 Let h_1, $h_2:(0,1)\to R^+$ such that $H\left(\frac{1}{2},\frac{1}{2}\right)\neq 0$, and $\Psi:[\sigma_1,\sigma_2]\to R_I^+$. If $\Psi''\in SGX(\mathbf{CR}\text{-}(h_1,h_2),[\sigma_1,\sigma_2],R_I^+)$, then we have

$$\frac{1}{\sigma_2-\sigma_1}\left[{}^{\mathtt{AB}}\mathtt{I}^{\eta}_{\frac{\sigma_2+\sigma_1}{2}}\{\Psi(\sigma_1)\}+{}^{\mathtt{AB}}_{\frac{\sigma_2+\sigma_1}{2}}\mathtt{I}^{\eta}_{\sigma_2}\{\Psi(\sigma_2)\}\right]$$

$$-\frac{1}{(\sigma_2-\sigma_1)\mathtt{B}(\eta)}\left[\Psi(\sigma_1)+\Psi(\sigma_2)\right]-\frac{(\sigma_2-\sigma_1)^{\eta-1}}{2^{\eta-1}\mathtt{B}(\eta)\Gamma(\eta)}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)$$

$$\preceq_{\mathbf{CR}}\frac{(\sigma_2-\sigma_1)^{\eta-1}}{(\eta+1)\mathtt{B}(\eta)\Gamma(\eta)}\left(\frac{\left(\frac{1}{2}\right)^{\eta\mathtt{p}+2\mathtt{p}}}{\eta\mathtt{p}+\mathtt{p}+1}\right)^{\frac{1}{\mathtt{p}}}$$

$$\left[|\Psi''(\sigma_1)|+|\Psi''(\sigma_2)|\right]\times\left[\left(\int_0^1\frac{d\eth}{\mathsf{H}(\eth,1-\eth)}\right)^{\frac{1}{\mathtt{q}}}+\left(\int_0^1\frac{d\eth}{\mathsf{H}(1-\eth,\eth)}\right)^{\frac{1}{\mathtt{q}}}\right], \tag{27}$$

where $\eta\in(0,1]$, $\frac{1}{\mathtt{p}}+\frac{1}{\mathtt{q}}=1$.

Proof. According to Lemma 2.2 and taking into account Hölder's inequality and result (26), we have

$$\frac{1}{\sigma_2-\sigma_1}\left[{}^{\mathtt{AB}}\mathtt{I}^{\eta}_{\frac{\sigma_2+\sigma_1}{2}}\{\Psi(\sigma_1)\}+{}^{\mathtt{AB}}_{\frac{\sigma_2+\sigma_1}{2}}\mathtt{I}^{\eta}_{\sigma_2}\{\Psi(\sigma_2)\}\right]$$

$$-\frac{1}{(\sigma_2-\sigma_1)\mathtt{B}(\eta)}\left[\Psi(\sigma_1)+\Psi(\sigma_2)\right]-\frac{(\sigma_2-\sigma_1)^{\eta-1}}{2^{\eta-1}\mathtt{B}(\eta)\Gamma(\eta)}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)$$

$$\preceq_{\mathbf{CR}}\frac{(\sigma_2-\sigma_1)^{\eta-1}}{2(\eta+1)\mathtt{B}(\eta)\Gamma(\eta)}\left(\int_0^1|\mathtt{w}^{\eta}(\eth)|^{\mathtt{p}}d\eth\right)^{\frac{1}{\mathtt{p}}}\left[\left(\int_0^1|\Psi''(\eth\sigma_1+(1-\eth)\sigma_2)|^{\mathtt{q}}d\eth\right)^{\frac{1}{\mathtt{q}}}\right. \tag{28}$$

$$\left.+\left(\int_0^1|\Psi''(\eth\sigma_2+(1-\eth)\sigma_1)|^{\mathtt{q}}d\eth\right)^{\frac{1}{\mathtt{q}}}\right]. \tag{29}$$

As $\Psi''\in\mathtt{SGX}(\mathbf{CR}\text{-}(\mathsf{h}_1,\mathsf{h}_2),[\sigma_1,\sigma_2],\mathsf{R}^+_{\mathtt{I}})$, then we have

$$\frac{1}{\sigma_2-\sigma_1}\left[{}^{AB}I^{\eta}_{\frac{\sigma_2+\sigma_1}{2}}\{\Psi(\sigma_1)\}+{}^{AB}_{\frac{\sigma_2+\sigma_1}{2}}I^{\eta}_{\sigma_2}\{\Psi(\sigma_2)\}\right]$$

$$-\frac{1}{(\sigma_2-\sigma_1)B(\eta)}\left[\Psi(\sigma_1)+\Psi(\sigma_2)\right]-\frac{(\sigma_2-\sigma_1)^{\eta-1}}{2^{\eta-1}B(\eta)\Gamma(\eta)}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)$$

$$\preceq_{CR}\frac{(\sigma_2-\sigma_1)^{\eta-1}}{(\eta+1)B(\eta)\Gamma(\eta)}\left(\frac{\left(\frac{1}{2}\right)^{\eta p+2p}}{\eta p+p+1}\right)^{\frac{1}{p}}$$

$$\times\left\{\left[\int_0^1\left(\frac{|\Psi''(\sigma_1)|^q}{h_1(\eth)h_2(1-\eth)}+\frac{|\Psi''(\sigma_2)|^q}{h_1(1-\eth)h_2(\eth)}\right)d\eth\right]^{\frac{1}{q}}\right.$$

$$\left.+\left[\int_0^1\left(\frac{|\Psi''(\sigma_2)|^q}{h_1(\eth)h_2(1-\eth)}+\frac{|\Psi''(\sigma_1)|^q}{h_1(1-\eth)h_2(\eth)}\right)d\eth\right]^{\frac{1}{q}}\right\}.$$

Then, we apply the fact that

$$\sum_{k=1}^{\sigma_2}(u_k+v_k)^{\sigma_1}\leq\sum_{k=1}^{\sigma_2}{u_k}^{\sigma_1}+\sum_{k=1}^{\sigma_2}{v_k}^{\sigma_1},$$

for $0<\sigma_1<1$, u_1, u_2s, $u_{\sigma_2}\geq 0$, Ψ_1, Ψ_2s, $\Psi_{\sigma_2}\geq 0$. So, we have the following inequality:

$$\frac{1}{\sigma_2-\sigma_1}\left[{}^{AB}I^{\eta}_{\frac{\sigma_2+\sigma_1}{2}}\{\Psi(\sigma_1)\}+{}^{AB}_{\frac{\sigma_2+\sigma_1}{2}}I^{\eta}_{\sigma_2}\{\Psi(\sigma_2)\}\right]$$

$$-\frac{1}{(\sigma_2-\sigma_1)B(\eta)}\left[\Psi(\sigma_1)+\Psi(\sigma_2)\right]-\frac{(\sigma_2-\sigma_1)^{\eta-1}}{2^{\eta-1}B(\eta)\Gamma(\eta)}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)$$

$$\preceq_{CR}\frac{(\sigma_2-\sigma_1)^{\eta-1}}{(\eta+1)B(\eta)\Gamma(\eta)}\left(\frac{\left(\frac{1}{2}\right)^{\eta p+2p}}{\eta p+p+1}\right)^{\frac{1}{p}}$$

$$\times\left[\left(\int_0^1\frac{|\Psi''(\sigma_1)|^q}{h_1(\eth)h_2(1-\eth)}d\eth\right)^{\frac{1}{q}}+\left(\int_0^1\frac{|\Psi''(\sigma_2)|^q}{h_1(1-\eth)h_2(\eth)}d\eth\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^1 \frac{|\Psi''(\sigma_2)|^{q}}{h_1(\eth)h_2(1-\eth)}d\eth\right)^{\frac{1}{q}}+\left(\int_0^1 \frac{|\Psi''(\sigma_1)|^{q}}{h_1(1-\eth)h_2(\eth)}d\eth\right)^{\frac{1}{q}}\Bigg]$$

$$=\frac{(\sigma_2-\sigma_1)^{\eta-1}}{(\eta+1)B(\eta)\Gamma(\eta)}\left(\frac{\left(\frac{1}{2}\right)^{\eta p+2p}}{\eta p+p+1}\right)^{\frac{1}{p}}\left[|\Psi''(\sigma_1)|+|\Psi''(\sigma_2)|\right]$$

$$\times\left[\left(\int_0^1 \frac{d\eth}{H(\eth,\, 1-\eth)}\right)^{\frac{1}{q}}+\left(\int_0^1 \frac{d\eth}{H(1-\eth,\, \eth)}\right)^{\frac{1}{q}}\right].$$

The proof is completed. □

Example 3.4 Taking into account the assumptions of Theorem 3.5. Let $\Psi(\sigma)=\left[4e^{\eth}+7+\eth,\ \frac{\sqrt{2+\eth}}{5}\right]$ defined over $[\sigma_1,\, \sigma_2]=[1,\, 3]$ with $h_1(\eth)=\frac{1}{\sigma}$, $h_2(\eth)=1$, $B(\eta)=1$, $\eta=\frac{1}{2}$, $p=q=2$.

Then, all the hypothesis of Theorem 3.5 are satisfied.

Corollary 3.11 If $h_1(\eth)=\frac{1}{\eth^s}$, $h_2(\eth)=1$, then Theorem 3.5 yields an outcome for the s-convex function for Atangana-Baleanu integral operators:

$$\frac{1}{\sigma_2-\sigma_1}\left[{}^{AB}I^{\eta}_{\frac{\sigma_2+\sigma_1}{2}}\{\Psi(\sigma_1)\}+{}^{AB}_{\frac{\sigma_2+\sigma_1}{2}}I^{\eta}_{\sigma_2}\{\Psi(\sigma_2)\}\right]$$

$$-\frac{1}{(\sigma_2-\sigma_1)B(\eta)}\left[\Psi(\sigma_1)+\Psi(\sigma_2)\right]-\frac{(\sigma_2-\sigma_1)^{\eta-1}}{2^{\eta-1}B(\eta)\Gamma(\eta)}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)$$

$$\preceq_{CR}\frac{(\sigma_2-\sigma_1)^{\eta-1}}{(\eta+1)B(\eta)\Gamma(\eta)}\left(\frac{\left(\frac{1}{2}\right)^{\eta+1}}{\eta+2}\right)^{\frac{1}{p}}\left[|\Psi''(\sigma_1)|^{q}+|\Psi''(\sigma_2)|^{q}\right]^{\frac{1}{q}}$$

$$\times\left[\frac{\left(\frac{1}{2}\right)^{\eta+s+2}}{\eta+s+2}+\beta_{\frac{1}{2}}(\eta+2,\, s+1)\right]^{\frac{1}{q}}. \tag{30}$$

Corollary 3.12 If $h_1(\eth)=h(\eth)$, $h_2(\eth)=1$, then Theorem 3.3 yields an outcome for the h-Godunova-Levin function for Atangana-Baleanu integral operators:

$$\frac{1}{\sigma_2-\sigma_1}\left[{}^{\mathtt{AB}}\mathtt{I}^{\eta}_{\frac{\sigma_2+\sigma_1}{2}}\{\Psi(\sigma_1)\}+{}^{\mathtt{AB}}_{\frac{\sigma_2+\sigma_1}{2}}\mathtt{I}^{\eta}_{\sigma_2}\{\Psi(\sigma_2)\}\right]$$

$$-\frac{1}{(\sigma_2-\sigma_1)\mathtt{B}(\eta)}\left[\Psi(\sigma_1)+\Psi(\sigma_2)\right]-\frac{(\sigma_2-\sigma_1)^{\eta-1}}{2^{\eta-1}\mathtt{B}(\eta)\Gamma(\eta)}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)$$

$$\preceq_{\mathbf{CR}}\frac{(\sigma_2-\sigma_1)^{\eta-1}}{(\eta+1)\mathtt{B}(\eta)\Gamma(\eta)}\left(\frac{\left(\frac{1}{2}\right)^{\eta+1}}{\eta+2}\right)^{\frac{1}{\mathtt{p}}}\left[|\Psi''(\sigma_1)|^{\mathtt{q}}+|\Psi''(\sigma_2)|^{\mathtt{q}}\right]^{\frac{1}{\mathtt{q}}}$$

$$\times\left[\int_0^{\frac{1}{2}}\eth^{\eta+1}\left(\frac{1}{\mathsf{h}(\eth)}+\frac{1}{\mathsf{h}(1-\eth)}\right)d\eth\right]^{\frac{1}{\mathtt{q}}}. \tag{31}$$

Corollary 3.13 If $\mathsf{h}_1(\eth)=\mathsf{h}_2(\eth)=1$, then Theorem 3.3 yields an outcome for the p-convex function for Atangana-Baleanu integral operators:

$$\frac{1}{\sigma_2-\sigma_1}\left[{}^{\mathtt{AB}}\mathtt{I}^{\eta}_{\frac{\sigma_2+\sigma_1}{2}}\{\Psi(\sigma_1)\}+{}^{\mathtt{AB}}_{\frac{\sigma_2+\sigma_1}{2}}\mathtt{I}^{\eta}_{\sigma_2}\{\Psi(\sigma_2)\}\right]$$

$$-\frac{1}{(\sigma_2-\sigma_1)\mathtt{B}(\eta)}\left[\Psi(\sigma_1)+\Psi(\sigma_2)\right]-\frac{(\sigma_2-\sigma_1)^{\eta-1}}{2^{\eta-1}\mathtt{B}(\eta)\Gamma(\eta)}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)$$

$$\preceq_{\mathbf{CR}}\frac{(\sigma_2-\sigma_1)^{\eta-1}}{(\eta+1)\mathtt{B}(\eta)\Gamma(\eta)}\frac{\left(\frac{1}{2}\right)^{\eta+1}}{\eta+2}\left[|\Psi''(\sigma_1)|^{\mathtt{q}}+|\Psi''(\sigma_2)|^{\mathtt{q}}\right]^{\frac{1}{\mathtt{q}}}. \tag{32}$$

Theorem 3.6 Let $\mathsf{h}_1, \mathsf{h}_2 : (0, 1) \to \mathsf{R}^+$ such that $\mathsf{H}\left(\frac{1}{2}, \frac{1}{2}\right) \neq 0$, and $\Psi : [\sigma_1, \sigma_2] \to \mathsf{R}^+_{\mathtt{I}}$. If $\Psi'' \in \mathtt{SGX}(\mathbf{CR}\text{-}(\mathsf{h}_1, \mathsf{h}_2), [\sigma_1, \sigma_2], \mathsf{R}^+_{\mathtt{I}})$, then we have

$$\frac{1}{\sigma_2-\sigma_1}\left[{}^{\mathtt{AB}}\mathtt{I}^{\eta}_{\frac{\sigma_2+\sigma_1}{2}}\{\Psi(\sigma_1)\}+{}^{\mathtt{AB}}_{\frac{\sigma_2+\sigma_1}{2}}\mathtt{I}^{\eta}_{\sigma_2}\{\Psi(\sigma_2)\}\right]$$

$$-\frac{1}{(\sigma_2-\sigma_1)\mathtt{B}(\eta)}\left[\Psi(\sigma_1)+\Psi(\sigma_2)\right]-\frac{(\sigma_2-\sigma_1)^{\eta-1}}{2^{\eta-1}\mathtt{B}(\eta)\Gamma(\eta)}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)$$

$$\preceq_{\mathbf{CR}}\frac{(\sigma_2-\sigma_1)^{\eta-1}}{(\eta+1)\mathtt{B}(\eta)\Gamma(\eta)}\left(\frac{\left(\frac{1}{2}\right)^{(\eta+1)\left(\frac{\mathtt{q}-\mathtt{p}}{\mathtt{q}-1}\right)}(\mathtt{q}-1)}{(\eta+1)(\mathtt{q}-\mathtt{p})+\mathtt{q}-1}\right)^{1-\frac{1}{\mathtt{q}}}$$

$$\times\left[|\Psi''(\sigma_1)|^{\mathtt{q}}+|\Psi''(\sigma_2)|^{\mathtt{q}}\right]^{\frac{1}{\mathtt{q}}}\left[\int_0^{\frac{1}{2}}\frac{\eth^{\eta\mathtt{p}+\mathtt{p}}d\eth}{\mathsf{H}(\eth,1-\eth)}+\int_0^{\frac{1}{2}}\frac{\eth^{\eta\mathtt{p}+\mathtt{p}}d\eth}{\mathsf{H}(1-\eth,\eth)}\right]^{\frac{1}{\mathtt{q}}}, \tag{33}$$

where $\eta\in(0,1]$, $\mathtt{q}\geq\mathtt{p}>1$.

Proof. By applying Hölder's inequality and taking into account relation (26), we have

$$\frac{1}{\sigma_2-\sigma_1}\left[{}^{\mathtt{AB}}\mathtt{I}^{\eta}_{\frac{\sigma_2+\sigma_1}{2}}\{\Psi(\sigma_1)\}+{}^{\mathtt{AB}}_{\frac{\sigma_2+\sigma_1}{2}}\mathtt{I}^{\eta}_{\sigma_2}\{\Psi(\sigma_2)\}\right]$$

$$-\frac{1}{(\sigma_2-\sigma_1)\mathtt{B}(\eta)}\left[\Psi(\sigma_1)+\Psi(\sigma_2)\right]-\frac{(\sigma_2-\sigma_1)^{\eta-1}}{2^{\eta-1}\mathtt{B}(\eta)\Gamma(\eta)}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)=\frac{(\sigma_2-\sigma_1)^{\eta-1}}{2(\eta+1)\mathtt{B}(\eta)\Gamma(\eta)}$$

$$\times\int_0^1|\mathtt{w}^{\eta}(\eth)|^{\frac{\mathtt{q}-\mathtt{p}}{\mathtt{q}}}\cdot|\mathtt{w}^{\eta}(\eth)|^{\frac{\mathtt{p}}{\mathtt{q}}}\left[|\Psi''(\eth\sigma_1+(1-\eth)\sigma_2)|+|\Psi''(\eth\sigma_2+(1-\eth)\sigma_1)|\right]d\eth$$

$$\preceq_{\mathbf{CR}}\frac{(\sigma_2-\sigma_1)^{\eta-1}}{2(\eta+1)\mathtt{B}(\eta)\Gamma(\eta)}\left(\int_0^1|\mathtt{w}^{\eta}(\eth)|^{\frac{\mathtt{q}-\mathtt{p}}{\mathtt{q}-1}}d\eth\right)^{1-\frac{1}{\mathtt{q}}}$$

$$\times\left[\left(\int_0^1|\mathtt{w}^{\eta}(\eth)|^{\mathtt{p}}|\Psi''(\eth\sigma_1+(1-\eth)\sigma_2)|^{\mathtt{q}}d\eth\right)^{\frac{1}{\mathtt{q}}}\right.$$

$$\left.+\left(\int_0^1|\mathtt{w}^{\eta}(\eth)|^{\mathtt{p}}|\Psi''(\eth\sigma_2+(1-\eth)\sigma_1)|^{\mathtt{q}}d\eth\right)^{\frac{1}{\mathtt{q}}}\right].$$

As $|\Psi''|^{\mathtt{q}}$ is an $(\mathsf{h}_1,\mathsf{h}_2)$-Godunova-Levin function, we have

$$\frac{1}{\sigma_2-\sigma_1}\left[{}^{AB}I^{\eta}_{\frac{\sigma_2+\sigma_1}{2}}\{\Psi(\sigma_1)\}+{}^{AB}_{\frac{\sigma_2+\sigma_1}{2}}I^{\eta}_{\sigma_2}\{\Psi(\sigma_2)\}\right]$$

$$-\frac{1}{(\sigma_2-\sigma_1)B(\eta)}\left[\Psi(\sigma_1)+\Psi(\sigma_2)\right]-\frac{(\sigma_2-\sigma_1)^{\eta-1}}{2^{\eta-1}B(\eta)\Gamma(\eta)}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)=\frac{(\sigma_2-\sigma_1)^{\eta-1}}{2(\eta+1)B(\eta)\Gamma(\eta)}$$

$$\times\int_0^1|w^{\eta}(\eth)|^{\frac{q-p}{q}}\cdot|w^{\eta}(\eth)|^{\frac{p}{q}}\left[|\Psi''(\eth\sigma_1+(1-\eth)\sigma_2)|+|\Psi''(\eth\sigma_2+(1-\eth)\sigma_1)|\right]d\eth$$

$$\preceq_{CR}\frac{(\sigma_2-\sigma_1)^{\eta-1}}{2(\eta+1)B(\eta)\Gamma(\eta)}\left(\int_0^1|w^{\eta}(\eth)|^{\frac{q-p}{q-1}}d\eth\right)^{1-\frac{1}{q}}\times\left[\left(\int_0^1\frac{|w^{\eta}(\eth)|^p|\Psi''(\sigma_1)|^q d\eth}{h_1(\eth)h_2(1-\eth)}\right.\right.$$

$$\left.+\int_0^1\frac{|w^{\eta}(\eth)|^p|\Psi''(\sigma_2)|^q d\eth}{h_1(1-\eth)h_2(\eth)}\right)^{\frac{1}{q}}+\left(\int_0^1\frac{|w^{\eta}(\eth)|^p|\Psi''(\sigma_2)|^q d\eth}{h_1(\eth)h_2(1-\eth)}\right.\left.\left.+\int_0^1\frac{|w^{\eta}(\eth)|^p|\Psi''(\sigma_1)|^q d\eth}{h_1(1-\eth)h_2(\eth)}\right)^{\frac{1}{q}}\right]$$

$$=\frac{(\sigma_2-\sigma_1)^{\eta-1}}{2(\eta+1)B(\eta)\Gamma(\eta)}\left(\frac{\left(\frac{1}{2}\right)^{(\eta+1)\left(\frac{q-p}{q-1}\right)}(q-1)}{(\eta+1)(q-p)+q-1}\right)^{1-\frac{1}{q}}\left\{\left[|\Psi''(\sigma_1)|^q\left(\int_0^{\frac{1}{2}}\frac{\eth^{\eta p+p}d\eth}{h_1(\eth)h_2(1-\eth)}\right.\right.\right.$$

$$\left.\left.+\int_{\frac{1}{2}}^1\frac{(1-\eth)^{\eta p+p}d\eth}{h_1(\eth)h_2(1-\eth)}\right)+|\Psi''(\sigma_2)|^q\left(\int_0^{\frac{1}{2}}\frac{\eth^{\eta p+p}d\eth}{h_1(1-\eth)h_2(\eth)}+\int_{\frac{1}{2}}^1\frac{(1-\eth)^{\eta p+p}d\eth}{h_1(1-\eth)h_2(\eth)}\right)\right]^{\frac{1}{q}}$$

$$+\left[|\Psi''(\sigma_2)|^q\left(\int_0^{\frac{1}{2}}\frac{\eth^{\eta p+p}d\eth}{h_1(\eth)h_2(1-\eth)}+\int_{\frac{1}{2}}^1\frac{(1-\eth)^{\eta p+p}d\eth}{h_1(\eth)h_2(1-\eth)}\right)+|\Psi''(\sigma_1)|^q\left(\int_0^{\frac{1}{2}}\frac{\eth^{\eta p+p}d\eth}{h_1(1-\eth)h_2(\eth)}\right.\right.$$

$$\left.\left.\left.+\int_{\frac{1}{2}}^1\frac{(1-\eth)^{\eta p+p}d\eth}{h_1(1-\eth)h_2(\eth)}\right)\right]^{\frac{1}{q}}\right\}=\frac{(\sigma_2-\sigma_1)^{\eta-1}}{(\eta+1)B(\eta)\Gamma(\eta)}\left(\frac{\left(\frac{1}{2}\right)^{(\eta+1)\left(\frac{q-p}{q-1}\right)}(q-1)}{(\eta+1)(q-p)+q-1}\right)^{1-\frac{1}{q}}$$

$$\times\left[|\Psi''(\sigma_1)|^q+|\Psi''(\sigma_2)|^q\right]^{\frac{1}{q}}\left[\int_0^{\frac{1}{2}}\frac{\eth^{\eta p+p}d\eth}{H(\eth,1-\eth)}+\int_0^{\frac{1}{2}}\frac{\eth^{\eta p+p}d\eth}{H(1-\eth,\eth)}\right]^{\frac{1}{q}}.$$

□

Example 3.5 Taking into account the assumptions of Theorem 3.6. Let $\Psi(\sigma)=\left[5e^{\eth}+\eth,\ \frac{\sqrt{5+\eth}}{2}+1\right]$ defined over $[\sigma_1,\sigma_2]=[1,3]$ with $h_1(\eth)=\frac{1}{\sigma}$, $h_2(\eth)=1$, $B(\eta)=1$, $\eta=\frac{1}{2}$, $p=q=4$.

Then, all the hypothesis of Theorem 3.6 are satisfied.

Corollary 3.14 If $h_1(\eth) = \frac{1}{\eth^s}$, $h_2(\eth) = 1$, then Theorem 3.6 yields an outcome for the s-convex function for Atangana-Baleanu integral operators:

$$\frac{1}{\sigma_2-\sigma_1}\left[{}^{AB}I^{\eta}_{\frac{\sigma_2+\sigma_1}{2}}\{\Psi(\sigma_1)\} + {}^{AB}_{\frac{\sigma_2+\sigma_1}{2}}I^{\eta}_{\sigma_2}\{\Psi(\sigma_2)\}\right]$$

$$-\frac{1}{(\sigma_2-\sigma_1)B(\eta)}\left[\Psi(\sigma_1)+\Psi(\sigma_2)\right] - \frac{(\sigma_2-\sigma_1)^{\eta-1}}{2^{\eta-1}B(\eta)\Gamma(\eta)}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)$$

$$\preceq_{CR} \frac{(\sigma_2-\sigma_1)^{\eta-1}}{(\eta+1)B(\eta)\Gamma(\eta)}\left(\frac{\left(\frac{1}{2}\right)^{(\eta+1)\left(\frac{q-p}{q-1}\right)}(q-1)}{(\eta+1)(q-p)+q-1}\right)^{1-\frac{1}{q}} \tag{34}$$

$$\times\left[|\Psi''(\sigma_1)|^q + |\Psi''(\sigma_2)|^q\right]^{\frac{1}{q}}\left[\int_0^{\frac{1}{2}} \eth^{\eta p+p}\left(\frac{1}{h(\eth)}+\frac{1}{h(1-\eth)}\right)d\eth\right]^{\frac{1}{q}}. \tag{35}$$

Corollary 3.15 If $h_1(\eth) = h(\eth)$, $h_2(\eth) = 1$, then Theorem 3.3 yields an outcome for the h-Godunova-Levin function for Atangana-Baleanu integral operators:

$$\frac{1}{\sigma_2-\sigma_1}\left[{}^{AB}I^{\eta}_{\frac{\sigma_2+\sigma_1}{2}}\{\Psi(\sigma_1)\} + {}^{AB}_{\frac{\sigma_2+\sigma_1}{2}}I^{\eta}_{\sigma_2}\{\Psi(\sigma_2)\}\right]$$

$$-\frac{1}{(\sigma_2-\sigma_1)B(\eta)}\left[\Psi(\sigma_1)+\Psi(\sigma_2)\right] - \frac{(\sigma_2-\sigma_1)^{\eta-1}}{2^{\eta-1}B(\eta)\Gamma(\eta)}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)$$

$$\preceq_{CR} \frac{(\sigma_2-\sigma_1)^{\eta-1}}{(\eta+1)B(\eta)\Gamma(\eta)}\left(\frac{\left(\frac{1}{2}\right)^{(\eta+1)\left(\frac{q-p}{q-1}\right)}(q-1)}{(\eta+1)(q-p)+q-1}\right)^{1-\frac{1}{q}}$$

$$\times\left(\frac{\left(\frac{1}{2}\right)^{\eta p+p}}{\eta p+p+1}\right)^{\frac{1}{q}}\left[|\Psi''(\sigma_1)|^q + |\Psi''(\sigma_2)|^q\right]^{\frac{1}{q}}.$$

Theorem 3.7 Let $h_1, h_2 : (0, 1) \to R^+$ such that $H\left(\frac{1}{2}, \frac{1}{2}\right) \neq 0$, and $\Psi : [\sigma_1, \sigma_2] \to R_I^+$. If $\Psi'' \in SGX(\mathbf{CR}\text{-}(h_1, h_2), [\sigma_1, \sigma_2], R_I^+)$, then we have

$$\frac{1}{\sigma_2-\sigma_1}\left[{}^{AB}I^{\eta}_{\frac{\sigma_2+\sigma_1}{2}}\{\Psi(\sigma_1)\}+{}^{AB}_{\frac{\sigma_2+\sigma_1}{2}}I^{\eta}_{\sigma_2}\{\Psi(\sigma_2)\}\right]$$

$$-\frac{1}{(\sigma_2-\sigma_1)B(\eta)}\left[\Psi(\sigma_1)+\Psi(\sigma_2)\right]-\frac{(\sigma_2-\sigma_1)^{\eta-1}}{2^{\eta-1}B(\eta)\Gamma(\eta)}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)$$

$$\preceq_{CR}\frac{(\sigma_2-\sigma_1)^{\eta-1}}{2(\eta+1)B(\eta)\Gamma(\eta)}\left\{\frac{\left(\frac{1}{2}\right)^{\eta p+p-1}}{(\eta p+p+1)p}\right.$$

$$\left.+\frac{1}{q}\left[|\Psi''(\sigma_1)|^q+|\Psi''(\sigma_2)|^q\right]\int_0^1\left(\frac{1}{H(\eth,1-\eth)}+\frac{1}{H(1-\eth,\eth)}\right)d\eth\right\},$$

where $\eta\in(0,1]$.

Proof. By applying Hölder's inequality and taking into account result (26), based on the Young's inequality: $ab\le\frac{1}{p}a^p+\frac{1}{q}b^q$, we obtain

$$\frac{1}{\sigma_2-\sigma_1}\left[{}^{AB}I^{\eta}_{\frac{\sigma_2+\sigma_1}{2}}\{\Psi(\sigma_1)\}+{}^{AB}_{\frac{\sigma_2+\sigma_1}{2}}I^{\eta}_{\sigma_2}\{\Psi(\sigma_2)\}\right]$$

$$-\frac{1}{(\sigma_2-\sigma_1)B(\eta)}\left[\Psi(\sigma_1)+\Psi(\sigma_2)\right]-\frac{(\sigma_2-\sigma_1)^{\eta-1}}{2^{\eta-1}B(\eta)\Gamma(\eta)}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)$$

$$\preceq_{CR}\frac{(\sigma_2-\sigma_1)^{\eta-1}}{2(\eta+1)B(\eta)\Gamma(\eta)}\left[\frac{2}{p}\int_0^1|w^{\eta}(\eth)|^p d\eth\right.$$

$$\left.+\frac{1}{q}\left(\int_0^1|\Psi''(\eth\sigma_1+(1-\eth)\sigma_2)|^q d\eth+\int_0^1|\Psi''(\eth\sigma_2+(1-\eth)\sigma_1)|^q d\eth\right)\right].$$

As $|\Psi''|^q$ is an (h_1, h_2)-Godunova-Levin, we have

$$\frac{1}{\sigma_2-\sigma_1}\left[{}^{AB}I^{\eta}_{\frac{\sigma_2+\sigma_1}{2}}\{\Psi(\sigma_1)\}+{}^{AB}_{\frac{\sigma_2+\sigma_1}{2}}I^{\eta}_{\sigma_2}\{\Psi(\sigma_2)\}\right]$$

$$-\frac{1}{(\sigma_2-\sigma_1)B(\eta)}\left[\Psi(\sigma_1)+\Psi(\sigma_2)\right]-\frac{(\sigma_2-\sigma_1)^{\eta-1}}{2^{\eta-1}B(\eta)\Gamma(\eta)}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)$$

$$\preceq_{CR}\frac{(\sigma_2-\sigma_1)^{\eta-1}}{2(\eta+1)B(\eta)\Gamma(\eta)}\left\{\frac{2}{p}\int_0^1|w^{\eta}(\eth)|^p d\eth\right.$$

$$+\frac{1}{q}\left[\int_0^1\frac{|\Psi''(\sigma_1)|^q d\eth}{h_1(\eth)h_2(1-\eth)}+\int_0^1\frac{|\Psi''(\sigma_2)|^q d\eth}{h_1(1-\eth)h_2(\eth)}\right.$$

$$\left.\left.+\int_0^1\frac{|\Psi''(\sigma_2)|^q d\eth}{h_1(\eth)h_2(1-\eth)}+\int_0^1\frac{|\Psi''(\sigma_1)|^q d\eth}{h_1(1-\eth)h_2(\eth)}\right]\right\}$$

$$=\frac{(\sigma_2-\sigma_1)^{\eta-1}}{2(\eta+1)B(\eta)\Gamma(\eta)}\left\{\frac{\left(\frac{1}{2}\right)^{\eta p+p-1}}{(\eta p+p+1)p}+\frac{1}{q}\left[|\Psi''(\sigma_1)|^q+|\Psi''(\sigma_2)|^q\right]\right.$$

$$\left.\times\int_0^1\left(\frac{1}{H(\eth,1-\eth)}+\frac{1}{H(1-\eth,\eth)}\right)d\eth\right\}.$$

The proof is completed. □

Corollary 3.16 If $h_1(\eth)=\frac{1}{\eth^s}$, $h_2(\eth)=1$, then Theorem 3.7 yields an outcome for the s-convex function for Atangana-Baleanu integral operators:

$$\frac{1}{\sigma_2-\sigma_1}\left[{}^{AB}I^{\eta}_{\frac{\sigma_2+\sigma_1}{2}}\{\Psi(\sigma_1)\}+{}^{AB}_{\frac{\sigma_2+\sigma_1}{2}}I^{\eta}_{\sigma_2}\{\Psi(\sigma_2)\}\right]$$

$$-\frac{1}{(\sigma_2-\sigma_1)B(\eta)}\left[\Psi(\sigma_1)+\Psi(\sigma_2)\right]-\frac{(\sigma_2-\sigma_1)^{\eta-1}}{2^{\eta-1}B(\eta)\Gamma(\eta)}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)$$

$$\preceq_{CR}\frac{(\sigma_2-\sigma_1)^{\eta-1}}{2(\eta+1)B(\eta)\Gamma(\eta)}\left\{\frac{\left(\frac{1}{2}\right)^{\eta p+p-1}}{(\eta p+p+1)p}\right.$$

$$\left.+\frac{1}{q}\left[|\Psi''(\sigma_1)|^q+|\Psi''(\sigma_2)|^q\right]\int_0^1\left(\frac{1}{h(\eth)}+\frac{1}{h(1-\eth)}\right)d\eth\right\}.$$

Corollary 3.17 If $h_1(\eth) = h(\eth)$, $h_2(\eth) = 1$, then Theorem 3.7 yields an outcome for the h-Godunova-Levin function for Atangana-Baleanu integral operators:

$$\frac{1}{\sigma_2-\sigma_1}\left[{}^{AB}I^{\eta}_{\frac{\sigma_2+\sigma_1}{2}}\{\Psi(\sigma_1)\}+{}^{AB}_{\frac{\sigma_2+\sigma_1}{2}}I^{\eta}_{\sigma_2}\{\Psi(\sigma_2)\}\right]$$

$$-\frac{1}{(\sigma_2-\sigma_1)B(\eta)}[\Psi(\sigma_1)+\Psi(\sigma_2)]-\frac{(\sigma_2-\sigma_1)^{\eta-1}}{2^{\eta-1}B(\eta)\Gamma(\eta)}\Psi\left(\frac{\sigma_2+\sigma_1}{2}\right)$$

$$\preceq_{CR}\frac{(\sigma_2-\sigma_1)^{\eta-1}}{(\eta+1)B(\eta)\Gamma(\eta)}\left\{\frac{\left(\frac{1}{2}\right)^{\eta p+p}}{(\eta p+p+1)p}+\frac{1}{q(s+1)}\left[|\Psi''(\sigma_1)|^q+|\Psi''(\sigma_2)|^q\right]\right\}.$$

3.4 *Some new upper bounds for differentiable H(α, 1 − α)-Godunova-Levin mappings involving power mean and Hölder inequality*

Theorem 3.8 Let $h_1, h_2 : (0, 1) \to R^+$ such that $H\left(\frac{1}{2}, \frac{1}{2}\right) \neq 0$, and $\Psi : [\sigma_1, \sigma_2] \to R^+_I$. If $\Psi'' \in SGX(\mathbf{CR}\text{-}(h_1, h_2), [\sigma_1, \sigma_2], R^+_I)$ with $0 \leq \sigma_1 < \sigma_2$ and $[H(\eth, 1-\eth)]^q \in L[0, 1]$ (space of all measurable functions), then we have

$$|\Omega_k(\Psi, \sigma_1, \sigma_2)| = \sum_{\zeta=0}^{k-1}\frac{\sigma_2-\sigma_1}{2k^2}\left[\left(\frac{1}{2}\right)^{\frac{q-1}{q}}\left(\int_0^1\frac{|1-2\eth|}{H(\eth, 1-\eth)}\left|\Psi'\left(\frac{(k-\zeta)\sigma_1+\zeta\sigma_2}{k}\right)\right|^q d\eth\right.\right.$$

$$\left.\left.+\int_0^1\frac{|1-2\eth|}{H(1-\eth, \eth)}\left|\Psi'\left(\frac{(k-\zeta-1)\sigma_1+(\zeta+1)\sigma_2}{k}\right)\right|^q d\eth\right)^{\frac{1}{q}}\right]$$

holds, where $1 < p$ and $\frac{1}{p}+\frac{1}{q} = 1$, where

$$\Omega_k(\Psi, \sigma_1, \sigma_2) = \sum_{\zeta=0}^{k-1}\frac{1}{2k}\left[\Psi\left(\frac{(k-\zeta)\sigma_1+\zeta\sigma_2}{k}\right)+\Psi\left(\frac{(k-\zeta-1)\sigma_1+(\zeta+1)\sigma_2}{k}\right)\right]$$

$$-\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\Psi(\eta)d\eta$$

$$=\sum_{\zeta=0}^{k-1}\frac{\sigma_2-\sigma_1}{2k^2}\left[\int_0^1(1-2\eth)\Psi'\left(\eth\frac{(k-\zeta)\sigma_1+\zeta\sigma_2}{k}\right.\right.$$

$$\left.\left.+(1-\eth)\frac{(k-\zeta-1)\sigma_1+(\zeta+1)\sigma_2}{k}\right)d\eth\right].$$

Proof. Assume that $q \geq 1$, further considering Lemma 2.1 and Theorem 2.4, we have

$$
\begin{aligned}
&|\Omega_{k}(\Psi, \sigma_1, \sigma_2)| \\
&\leq \sum_{\zeta=0}^{k-1} \frac{\sigma_2-\sigma_1}{2k^2}\left(\int_0^1\left|(1-2\eth)\Psi'\left(\eth\frac{(k-\zeta)\sigma_1+\zeta\sigma_2}{k}+(1-\eth)\frac{(k-\zeta-1)\sigma_1+(\zeta+1)\sigma_2}{k}\right)\right|d\eth\right) \\
&\leq \sum_{\zeta=0}^{k-1} \frac{\sigma_2-\sigma_1}{2k^2}\left(\int_0^1 |1-2\eth| d\eth\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1 |1-2\eth|\left|\Psi'\left(\eth\frac{(k-\zeta)\sigma_1+\zeta\sigma_2}{k}+(1-\eth)\frac{(k-\zeta-1)\sigma_1+(\zeta+1)\sigma_2}{k}\right)\right|^q d\eth\right)^{\frac{1}{q}}.
\end{aligned}
$$

As $|\Psi'|^q \in \mathtt{SGX}(\mathbf{CR}\text{-}(h_1, h_2), [\sigma_1, \sigma_2], R_I^+)$ is (h_1, h_2), we have

$$
\begin{aligned}
&|\Omega_{k}(\Psi, \sigma_1, \sigma_2)| \\
&\leq \sum_{\zeta=0}^{k-1} \frac{\sigma_2-\sigma_1}{2k^2}\left[\int_0^1 |1-2\eth| d\eth\right]^{1-\frac{1}{q}}\left[\int_0^1 |1-2\eth|\left(\frac{1}{H(\eth, 1-\eth)}\left|\Psi'\left(\frac{(k-\zeta)\sigma_1+\zeta\sigma_2}{k}\right)\right|^q\right.\right. \\
&\left.\left.+\frac{1}{H(1-\eth, \eth)}\cdot\left|\Psi'\left(\frac{(k-\zeta-1)\sigma_1+(\zeta+1)\sigma_2}{k}\right)\right|^q\right) d\eth\right]^{\frac{1}{q}} \\
&= \sum_{\zeta=0}^{k-1} \frac{\sigma_2-\sigma_1}{2k^2}\left(\int_0^1 |1-2\eth| d\eth\right)^{1-\frac{1}{q}}\left(\int_0^1 \frac{|1-2\eth|}{H(\eth, 1-\eth)}\left|\Psi'\left(\frac{(k-\zeta)\sigma_1+\zeta\sigma_2}{k}\right)\right|^q d\eth\right. \\
&\left.+\int_0^1 \frac{|1-2\eth|}{H(1-\eth, \eth)}\left|\Psi'\left(\frac{(k-\zeta-1)\sigma_1+(\zeta+1)\sigma_2}{k}\right)\right|^q d\eth\right)^{\frac{1}{q}} \\
&= \sum_{\zeta=0}^{k-1} \frac{\sigma_2-\sigma_1}{2k^2}\left[\left(\frac{1}{2}\right)^{\frac{q-1}{q}}\left(\int_0^1 \frac{|1-2\eth|}{H(\eth, 1-\eth)}\cdot\left|\Psi'\left(\frac{(k-\zeta)\sigma_1+\zeta\sigma_2}{k}\right)\right|^q d\eth\right.\right. \\
&\left.\left.+\int_0^1 \frac{|1-2\eth|}{H(1-\eth, \eth)}\left|\Psi'\left(\frac{(k-\zeta-1)\sigma_1+(\zeta+1)\sigma_2}{k}\right)\right|^q d\eth\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

□

Corollary 3.18 In Theorem 3.8, if we apply $h_1(\eth) = \frac{1}{\eth}$, $h_2(\eth) = 1$, we get

$$|\Omega_k(\Psi, \sigma_1, \sigma_2)| = \sum_{\zeta=0}^{k-1} \frac{\sigma_2 - \sigma_1}{k^2(2)^{2+\frac{1}{q}}} \left(\left| \Psi' \left(\frac{(k-\zeta)\sigma_1 + \zeta\sigma_2}{z} \right) \right|^q \right.$$

$$\left. + \left| \Psi' \left(\frac{(k-\zeta-1)\sigma_1 + (\zeta+1)\sigma_2}{k} \right) \right|^q \right)^{\frac{1}{q}}$$

which has been derived by authors in [57].

Corollary 3.19 In Theorem 3.8, if we apply $h_1(\eth) = \frac{1}{\eth^s}$, $h_2(\eth) = 1$, we get

$$|\Omega_k(\Psi, \sigma_1, \sigma_2)| \leq \sum_{\sigma=0}^{k-1} \frac{\sigma_2 - \sigma_1}{k^2 2^{2-\frac{1}{q}}} \left(\frac{1}{2^s(s+1)(s+2)} + \frac{s}{(s+1)(s+2)} \right)^{\frac{1}{q}}$$

$$\times \left[\left| \Psi' \left(\frac{(k-\sigma)\sigma_1 + \sigma r}{k} \right) \right|^q + \left| \Psi' \left(\frac{(k-\sigma-1)\sigma_1 + (\sigma+1)\sigma_2}{k} \right) \right|^q \right]^{\frac{1}{q}},$$

which has been proved by Yildiz et al. in [60].

Theorem 3.9 Let $h_1, h_2 : (0, 1) \to R^+$ such that $H\left(\frac{1}{2}, \frac{1}{2}\right) \neq 0$, and $\Psi : [\sigma_1, \sigma_2] \to R_I^+$. If $\Psi'' \in SGX(\mathbf{CR}\text{-}(h_1, h_2), [\sigma_1, \sigma_2], R_I^+)$ with $0 \leq \sigma_1 < \sigma_2$ and $[H(\eth, 1-\eth)]^q \in L[0, 1]$ (space of all measurable functions), then we have

$$|\Omega_k(\Psi, \sigma_1, \sigma_2)| \leq \sum_{\zeta=0}^{k-1} \frac{\sigma_2 - \sigma_1}{2k^2} \left[\left(\frac{1}{1+p} \right)^{\frac{1}{p}} \right.$$

$$\times \left(\int_0^1 \left(\frac{1}{H(\eth, 1-\eth)} \left| \Psi' \left(\frac{(k-\zeta)\sigma_1 + \zeta\sigma_2}{k} \right) \right|^q \right. \right.$$

$$\left. \left. + \frac{1}{H(1-\eth, \eth)} \left| \Psi' \left(\frac{(k-\zeta-1)\sigma_1 + (\zeta+1)\sigma_2}{k} \right) \right|^q \right) d\eth \right)^{\frac{1}{q}} \right].$$

holds, where $\frac{1}{q} + \frac{1}{p} = 1$.

Proof. Assume that $p > 1$. Taking into account Lemma 2.1 and applying Hölder inequality, we have □

$$|\Omega_{k}(\Psi, \sigma_1, \sigma_2)| = \sum_{\zeta=0}^{k-1} \frac{1}{2k}\left[\Psi\left(\frac{(k-\zeta)\sigma_1+\zeta\sigma_2}{k}\right)\Psi\left(\frac{(k-\zeta-1)\sigma_1+(\zeta+1)\sigma_2}{k}\right)\right]$$

$$-\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\Psi(\eta)d\eta$$

$$=\sum_{\zeta=0}^{k-1}\frac{\sigma_2-\sigma_1}{2k^2}\left[\int_0^1(1-2\eth)\Psi'\left(\eth\frac{(k-\zeta)\sigma_1+\zeta\sigma_2}{k}\right.\right.$$

$$\left.\left.+(1-\eth)\ \frac{(k-\zeta-1)\sigma_1+(\zeta+1)\sigma_2}{k}\right)d\eth\right]$$

$$\leq \sum_{\zeta=0}^{k-1}\frac{\sigma_2-\sigma_1}{2k^2}\left[\left(\int_0^1\left|(1-2\eth)\Psi'\left(\eth\frac{(k-\zeta)\sigma_1+\zeta\sigma_2}{k}\right.\right.\right.\right.$$

$$\left.\left.\left.\left.+(1-\eth)\frac{(k-\zeta-1)\sigma_1+(\zeta+1)\sigma_2}{k}\right)\right|d\eth\right)\right]$$

$$\leq \sum_{\zeta=0}^{k-1}\frac{\sigma_2-\sigma_1}{2k^2}\left[\left(\int_0^1|1-2\eth|^p d\eth\right)^{\frac{1}{p}}\right.$$

$$\left.\times\left(\int_0^1\left|\Psi'\left(\eth\frac{(k-\zeta)\sigma_1+\zeta\sigma_2}{k}+(1-\eth)\frac{(k-\zeta-1)\sigma_1+(\zeta+1)\sigma_2}{k}\right)\right|^q d\eth\right)^{\frac{1}{q}}\right].$$

As $|\Psi'|^q \in \mathtt{SGX}(\mathbf{CR}\text{-}(h_1, h_2), [\sigma_1, \sigma_2], R_I^+)$ is (h_1, h_2), we have

$$\int_0^1\left|\Psi'\left(\eth\frac{(k-\zeta)\sigma_1+\zeta\sigma_2}{k}+(1-\eth)\frac{(k-\zeta-1)\sigma_1+(\zeta+1)\sigma_2}{k}\right)\right|d\eth$$

$$\leq\int_0^1\left(\frac{1}{H(\eth, 1-\eth)}\left|\Psi'\left(\frac{(k-\zeta)\sigma_1+\zeta\sigma_2}{k}\right)\right|^q+\frac{1}{H(1-\eth, \eth)}\left|\Psi'\left(\frac{(k-\zeta-1)\sigma_1+(\zeta+1)\sigma_2}{k}\right)\right|\right)d\eth.$$

Therefore, we deduce

$$|\Omega_k(\Psi, \sigma_1, \sigma_2)|$$

$$\leq \sum_{\zeta=0}^{k-1} \frac{\sigma_2-\sigma_1}{2k^2}\left[\left(\int_0^1 |1-2\eth|^p d\eth\right)^{\frac{1}{p}}\right.$$

$$\left.\times \left(\int_0^1 \left(\frac{1}{H(\eth, 1-\eth)}\left|\Psi'\left(\frac{(k-\zeta)\sigma_1+\zeta\sigma_2}{k}\right)\right|^q + \frac{1}{H(1-\eth, \eth)}\left|\Psi'\left(\frac{(k-\zeta-1)\sigma_1+(\zeta+1)\sigma_2}{k}\right)\right|^q\right) d\eth\right)^{\frac{1}{q}}\right]$$

$$\leq \sum_{\zeta=0}^{k-1} \frac{\sigma_2-\sigma_1}{2k^2}\left[\left(\frac{1}{1+p}\right)^{\frac{1}{p}}\right.$$

$$\left.\times \left(\int_0^1 \left(\frac{1}{H(\eth, 1-\eth)}\left|\Psi'\left(\frac{(k-\zeta)\sigma_1+\zeta\sigma_2}{k}\right)\right|^q + \frac{1}{H(1-\eth, \eth)}\left|\Psi'\left(\frac{(k-\zeta-1)\sigma_1+(\zeta+1)\sigma_2}{k}\right)\right|^q\right) d\eth\right)^{\frac{1}{q}}\right].$$

□

Remark 3.4 If we apply $h_1(\eth) = \frac{1}{\eth^s}$, $h_2(\eth) = 1$ in Theorem 3.9, we obtain Theorem 6 in [60].

4. Applications to special means

In this section, we present several specific means that can be utilized to analyze our major results from section 3.4. Let $\sigma_1, \sigma_2 \in R$,

• The arithmetic mean between two non-negative numbers is defined as follows:

$$A = A(\sigma_1, \sigma_2) := \frac{\sigma_1+\sigma_2}{2}, \ \sigma_1, \sigma_2 \geq 0.$$

• The harmonic mean between two non-negative numbers is defined as follows:

$$H = H(\sigma_1, \sigma_2) := \frac{2\sigma_1\sigma_2}{\sigma_1+\sigma_2}, \ \sigma_1, \sigma_2 > 0.$$

• The logarithmic mean between two non-negative numbers is defined as follows:

$$L = L(\sigma_1, \sigma_2) := \begin{cases} \sigma_1, & \text{if } \sigma_1 = \sigma_2 \\ \dfrac{\sigma_2-\sigma_1}{\ln\sigma_2 - \ln\sigma_1}, & \text{if } \sigma_1 \neq \sigma_2, \end{cases} \quad \sigma_1, \sigma_2 > 0.$$

Proposition 4.1 Let $\sigma_1, \sigma_2 \in R$, $0 < \sigma_1 < \sigma_2$, and $r \in N$, $r \geq 2$. Then, the following

$$\left|\sum_{\zeta=0}^{k-1}\frac{1}{k\zeta}A\left(\left(\frac{(k-\zeta)\sigma_1+\zeta\sigma_2}{k}\right)^r\left(\frac{(k-\zeta-1)\sigma_1+(\zeta+1)\sigma_2}{k}\right)^r\right)-L_m{}^r(\sigma_1,\sigma_2)\right|$$

$$\leq\sum_{\zeta=0}^{k-1}\frac{(\sigma_2-\sigma_1)r}{2^{2-\frac{1}{q}}k^2}\left[\left(\int_0^1\frac{|1-2\eth|}{H(\eth,1-\eth)}d\eth\right)\left(\frac{(k-\zeta)\sigma_1+\zeta\sigma_2}{k}\right)^{(r-1)q}\right.$$

$$\left.+\left(\int_0^1\frac{|1-2\eth|}{H(1-\eth,\eth)}d\eth\right)\left(\frac{(k-\zeta-1)\sigma_1+(\zeta+1)\sigma_2}{k}\right)^{(r-1)q}\right)\Bigg]^{\frac{1}{q}}$$

holds, for all $q>1$.

Proof. In order to prove this proposition, we need to satisfy the following conditions in Theorem 3.8 that is $\Psi(\eth)=\eth^r$, $\eth\in[\sigma_1,\sigma_2]$, $r\in N$, $r\geq 2$. □

Proposition 4.2 Let $\sigma_1,\sigma_2\in R$, $0<\sigma_1<\sigma_2$, and $r\in N$, $r\geq 2$. Then, the following

$$\left|\sum_{\zeta=0}^{k-1}\frac{1}{k}A\left(\left(\frac{(k-\zeta)\sigma_1+\zeta\sigma_2}{k}\right)^r\left(\frac{(k-\zeta-1)\sigma_1+(\zeta+1)\sigma_2}{k}\right)^r\right)-L_m{}^r(\sigma_1,\sigma_2)\right|$$

$$\leq\sum_{\zeta=0}^{k-1}\frac{(\sigma_2-\sigma_1)r}{2^{2-\frac{1}{q}}k^2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}$$

$$\times\left[\left(\int_0^1\frac{d\eth}{H(\eth,1-\eth)}\right)\left(\frac{(k-\zeta)\sigma_1+\zeta\sigma_2}{k}\right)^{(r-1)q}+\left(\int_0^1\frac{d\eth}{H(1-\eth,\eth)}\right)\left(\frac{(k-\zeta-1)\sigma_1+(\zeta+1)\sigma_2}{k}\right)^{(r-1)q}\right)\Bigg]^{\frac{1}{q}}$$

holds, for all $q>1$.

Proof. In order to prove this proposition, we need to satisfy the following conditions in Theorem 3.9 that is $\Psi(\eth)=\eth^r$, $\eth\in[\sigma_1,\sigma_2]$, $r\in N$, $r\geq 2$. □

5. Conclusion

Our work develops and analyzes various convex integral inequalities, extending classical inequalities and enlarging our understanding of the relationship between convexity and integrability. The obtained inequalities contribute to theoretical understanding and real-world applications by giving convex functions in integral form tighter bounds. Additionally, the authors [48] of the subsequent results used classical integral operators, but we used full-order interval mappings and a fractional integral operator in this study. Furthermore, we established a new type of Jensen-type inequality in sequential form, as well as an Ostrowki type inequality, utilizing a unique Moore metric Hausdorff distance technique, which is genuinely fresh notion for such inequality with this type of generalized mappings that generalized the following results [61–63]. Furthermore, we developed new upper bounds for differentiable $\mathbf{H}(\alpha,1-\alpha)$-Godunova-Levin mappings involving Power mean and Hölder inequality that generalize the conclusions of the following authors [57, 60] in different circumstances.

6. Future directions

Building on these first findings, there are still a number of directions to investigate. The application of convex integral inequalities to functions defined on non-Euclidean spaces, like Riemannian manifolds [64], where curvature may affect inequality bounds, is a potential avenue. An alternative approach is to investigate probabilistic interpretations and applications, namely in information theory [45] and stochastic analysis [37], where convex inequalities may offer more precise expectations bounds. Additionally, the authors extend these findings in the context of Hilbert and variable exponent spaces [65], and they could close the gap between theoretical advancements and real-world applications by investigating numerical methods for approximating these integral inequalities and their applications in machine learning and data science.

Author's contributions

All authors equally contributed and approved the final manuscript.

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Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no competing interests.

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