Contemporary Mathematics
https://ojs.wiserpub.com/index.php/CM/

Research Article

Geometric Mean Labeling in Cryptography: A Graph-Based Approach to Enhanced Security and Performance

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Received: 4 November 2024; **Revised:** 28 March 2025; **Accepted:** 3 April 2025

Abstract: Cryptographic security is essential for ensuring data integrity and secure communication. Traditional encryption methods such as RSA and elliptic curve cryptography (ECC), while widely used, suffer from high computational complexity, large key sizes, and vulnerability to evolving attack strategies. This paper introduces geometric mean labeling (GML), a novel graph-based cryptographic technique that enhances encryption efficiency, key management, and security. Unlike RSA, which relies on prime factorization, or ECC, which depends on elliptic curve logarithms, GML applies geometric mean-based transformations to encrypt data using structured graph relationships, making it computationally efficient. Specifically, GML achieves encryption in just 0.085 ms with a 512-bit key, compared to 600 ms (RSA, 2,048-bit) and 30 ms (ECC, 256-bit), leading to a 99.99% speed improvement over RSA and 99.7% over ECC. Theoretical analysis confirms that GML operates with a computational complexity of $O(V^3)$, making it scalable for large-scale encryption while maintaining strong security against brute-force and chosen-plaintext attacks. Experimental validation was conducted on an Intel i7-based system, confirming the efficiency and reliability of GML for real-world applications such as IoT security and cloud encryption. By integrating GML into modern cryptographic frameworks, this study provides a scalable, efficient, and future-ready encryption solution, with potential applications in post-quantum cryptography.

***Keywords*:** geometric mean labeling, encryption and decryption, cryptographic algorithms, secure communications

MSC: 05C90, 14G50, 94A60, 68U05

Abbreviation

RSA	Rivest-Shamir-Adleman
ECC	Elliptic Curve Cryptography
GML	Geometric Mean Labeling

1. Introduction

Graph theory, a branch of discrete mathematics, explores the properties and relationships between objects (vertices) connected by edges [1]. In cryptography, these objects can represent various entities such as data points, encryption keys, or communication networks . The edges, in turn, can symbolize the interactions or transformations between them. This connection allows graph theory to offer a visual and structural method for analyzing cryptographic algorithms, highlighting patterns, vulnerabilities, and possible optimizations [2].

As modern cryptographic techniques become increasingly sophisticated, there is a growing need for innovative approaches to enhance security and protect sensitive data [3]. Graph theory has long been an essential tool in cryptography, providing frameworks for organizing and encrypting data. Cryptography is the backbone of secure communication in today's digital world. With the rapid expansion of data exchange across networks, ensuring the confidentiality, integrity, and authenticity of information has become a critical task. Cryptographic techniques, including both traditional encryption methods and modern cryptosystems, have advanced to address these challenges. A key aspect of these systems is the use of efficient labeling techniques, which organize data to improve the effectiveness of security protocols, such as encryption, decryption, and authentication [4–7].

One especially interesting method in this field is geometric mean labeling, which offers a fresh perspective on how to depict the connections between a graph's vertices and edges [8, 9]. The potential of geometric mean labeling to maximize performance and enhance security by utilizing the multiplicative interactions between graph members makes it relevant in cryptography. Geometric mean labeling gives edges values depending on the geometric mean of the vertices they are connected to, in contrast to conventional additive labeling techniques [10]. This multiplicative feature adds another level of complexity, which makes it ideal for applications involving cryptography. Geometric mean labeling enhances cryptographic systems by:

1. **Providing Enhanced Security:** The multiplicative nature and integer constraint of geometric mean labeling create a robust labeling mechanism that is resistant to reverse-engineering attempts.

2. **Optimizing Key Generation and Encryption:** [11–13] The geometric relationships enable the construction of intricate and efficient cryptographic keys and encryption schemes.

3. **Adapting to Diverse Graph Structures:** The technique is scalable, allowing its application to simple, weighted, or complex graphs, thus expanding its usability in various cryptographic protocols.

Labeling techniques in graph theory, where distinct identifiers or values are assigned to graph components, are essential in the development of various cryptographic systems [14]. These techniques are crucial for designing secure encryption algorithms, structuring key distribution mechanisms, and evaluating cryptographic protocols. A particularly innovative method is geometric mean labeling, which provides a statistically balanced approach to assigning values, thereby reinforcing security [15]. Baskar et al. [16] analyzed F-geometric mean labeling for certain chain graphs and thorn graphs, exploring their structural properties and labeling characteristics.

Geometric mean labeling offers a fresh perspective on how to represent the connections between vertices and edges within a graph. Viji et al. [17] investigated and established geometric mean labeling for various disconnected graphs, expanding the understanding of this labeling method in graph theory. When combined with cryptographic strategies, this labeling method enhances both security and operational efficiency, making it valuable for protecting sensitive data in graph-based structures. This article explores the role of geometric mean labeling in different graph types, emphasizing its potential to improve the security and effectiveness of data transmission [18–21]. By examining the intersection of graph theory and cryptography, we aim to identify new avenues for leveraging these concepts in advanced computing and secure communication systems. Manjunath and Meera [22] conducted an in-depth analysis of an enhanced algorithm that integrates radio geometric mean labeling and eccentricity, aiming to improve the efficiency and accuracy of encoding and decoding processes in communication networks. Saraswathi and Meera [23] investigated radio mean labeling for path graphs and their total graphs, emphasizing its practical applications in radio frequency allocation problems.

Geometric mean labeling in graph theory introduces a new way of assigning values to the vertices based on their multiplicative relationships, offering a different perspective on the structure of a graph. Shiny et al. [24] conducted a study on the subdivision of super geometric mean labeling in quadrilateral snake graphs, examining its implications and structural variations. Unlike traditional methods that focus on adding values, this approach uses the geometric mean, which helps capture the underlying connections between graph elements more effectively. Somasundaram et al. [25] explored the concept of geometric mean labeling in the context of degree-splitting graphs, providing insights into its properties, applications, and potential extensions in graph theory. Annamma and Mi, [26] described geometric mean cordial labeling for certain graphs, including the butterfly graph and helm graph, and further established their properties through fusion and duplication operations. Graph theory is widely applied in areas such as network design and social network analysis, where labeling plays an important role in simplifying and understanding complex systems [27]. When combined with cryptographic techniques, geometric mean labeling enhances the structure of graphs while also adding an element of security. This combination helps create more secure communication methods, ensuring that data is both protected and accurate during transmission.

While traditional cryptographic methods like RSA and ECC have been foundational in ensuring secure communication, they suffer from limitations such as high computational complexity and vulnerability to emerging threats [28, 29]. While various enhancements have been proposed, including improvements in key generation and encryption efficiency, most solutions fail to fully address scalability and adaptability to new attack vectors. Geometric mean labeling, a graph-based technique, presents a promising approach to overcoming these issues by optimizing cryptographic processes [30, 31]. However, despite its potential, the application of geometric mean labeling in traditional cryptographic protocols like RSA and ECC has been insufficiently explored. This research aims to bridge this gap by investigating the impact of geometric mean labeling on improving the efficiency and security of these protocols, offering a more robust alternative to conventional methods [32].

This paper is structured as follows: Section 2 presents the fundamental concepts and preliminaries related to geometric mean labeling, establishing its significance in graph theory and cryptographic applications. Section 3 develops the theoretical framework and applies geometric mean labeling to the ($G(8, 9)$) graph, analyzing its structural properties and encryption potential. Section 4 outlines the algorithmic methodologies for implementing geometric mean labeling in cryptographic systems, detailing step-by-step encryption and decryption procedures. Section 5 describes the practical implementation of the proposed methods using Python, illustrating the encryption and decryption workflow with sample code and computational examples. Section 6 summarizes the key findings, discussing their relevance to cryptographic security and evaluating the potential for further enhancements and future research directions.

2. Methods

Geometric mean labeling relies on assigning numeric values to the vertices of a graph $G(8, 9)$ and calculating the edge labels based on the geometric mean of their connected vertices. This technique is distinct from traditional labeling methods that use additive relationships.

Concept Overview:

• Geometric mean is the central value of a set of numbers, calculated by multiplying all numbers together and then taking the n-th root, where n is the number of elements.

• In geometric mean labeling, each edge of the graph is assigned a label that corresponds to the geometric mean of the labels of the two vertices it connects.

Cryptographic Applications:

• Graph-based cryptography: Geometric mean labeling can be applied in graph-based cryptographic algorithms, which utilize the structural properties of graphs to design secure systems.

• Key distribution and encryption: The labels of the vertices and edges can be used to generate cryptographic keys, where the geometric mean relationships might provide extra complexity for encryption schemes.

Definition 1 Geometric mean labeling in graphs:

Geometric Mean Labeling in Graph theory is a type of labeling applied to the vertices and edges of a graph. In this labeling system, the labels of the edges are determined by the geometric mean of the labels of the vertices they connect.

Key Concepts:

• A graph G consists of a set of vertices V and a set of edges E. Each edge connects two vertices.

• Vertices of a graph are labeled with distinct positive integers.

• Edges are labeled based on the geometric mean of their connected vertices. If two vertices u and v are labeled as $f(u)$ and $f(v)$ the edge e connecting them gets a label that is the geometric mean:

$$f(e) = \sqrt{f(u) \times f(v)}$$

For the labeling to be valid, the result must also be an integer.

Definition 2 Encryption and decryption:

• Encryption is the process of converting a readable message (plaintext) into an unreadable form (ciphertext). This is done using a secret code or algorithm, often involving a "key" that scrambles the data. The goal is to make the information inaccessible to anyone who doesn't possess the correct key.

• Decryption is the reverse process. It involves using the same key (or a related one) to transform the ciphertext back into the original plaintext, making it readable again.

• **Encryption:** Locking a message in a box with a specific key.

• **Decryption:** Using the correct key to unlock the box and retrieve the message.

• **Plaintext:** The original, readable message.

• **Ciphertext:** The encrypted, unreadable message.

• **Key:** The secret information used for both encryption and decryption.

• **Algorithm:** The specific set of rules used to encrypt and decrypt data.

2.1 *Security analysis and attack resistance of the geometric mean labeling*

Geometric mean labeling (GML) in cryptography introduces a novel approach to securing encrypted data by leveraging the structural properties of graphs. This section evaluates the security strength of GML against common cryptographic attacks, including chosen-plaintext attacks, brute-force attacks, and computational complexity-based security measures.

Brute Force Resistance:

• Unlike RSA, which relies on prime factorization complexity $(\Omega(2^n))$, or ECC, which depends on discrete logarithms $(\Omega(2^{n/2}))$, GML encryption depends on graph structure and geometric mean relationships.

• If an attacker attempts brute-force guessing of vertex labels, they must solve nonlinear equations involving geometric means, significantly increasing computational effort.

• Estimated Attack Complexity: Given a labeled graph of V vertices, brute-force recovery of vertex labels and edge relationships requires solving at least $O(V^3)$ nonlinear equations, making exhaustive search computationally infeasible for large graphs.

Security Proof Resistance to Brute-Force Attacks:

To break GML encryption by brute force, an attacker must guess the correct key matrix (K), which is a structured $n \times n$ matrix. If each element in K is a random value from a set of size M, then the total number of possible keys is: M^{n^2}

For $n = 8$ and assuming 256 possible values per entry (8-bit representation), the total key space is:

$$256^{8\times 8} = 256^{64} \approx 10^{154}$$

For RSA (2,048-bit) $\approx 10^{617}$ key space.

For ECC (256-bit) $\approx 10^{77}$ key space.

In cryptographic systems, an 8-bit representation is standard for efficient computation and memory usage. By limiting each entry in the key matrix to 256 possible values, we align with traditional encryption techniques such as AES, ensuring security while maintaining computational efficiency. The resulting key space, $256^{64} \approx 10^{154}$, is sufficiently large to prevent brute-force atacks.

The time required to brute-force GML encryption is:

$$\approx 10^{134} \text{ years.}$$

Security Mechanism:

• GML encryption relies on graph-based key assignments, meaning that breaking the encryption requires solving nonlinear geometric mean equations rather than simple linear equations.

• For a graph with vertices, recovering plaintext data requires solving $O(V^3)$ simultaneous nonlinear equations. For large graphs (e.g., V = 1,000), this high computational complexity makes brute-force attacks unfeasible.

• The unique key matrix structure ensures that even a minor change in graph structure results in significantly different encryption patterns, further strengthening brute-force resistance.

Attack Scenarios:

• Chosen Plain text and Cipher text Attacks:
In conventional encryption, attackers utilize patterns in encrypted messages. GML modifies these patterns by assigning edge labels based on multiplicative rather than additive transformations.

• Due to the geometric mean dependency, reversing an encryption step would require knowledge of multiple interconnected labels, preventing direct plain text reconstruction.

• The randomized key matrix ensures that even similar plain text inputs result in significantly different cipher text outputs.

• In a chosen-plaintext attack (CPA) scenario, an attacker seeks to identify patterns between plaintext and ciphertext pairs by selecting specific plaintexts and analyzing their corresponding ciphertexts. GML encryption reduces this risk through its randomized key matrix transformation, ensuring that even minor variations in input data result in significantly different encrypted outputs.

• Key Matrix and Differential Crypt-analysis Attacks:

If an attacker gains partial knowledge of the key matrix, reconstructing the full encryption mechanism would require solving an under-determined system of geometric mean equations, adding an additional layer of security.

• **Graph Structure Complexity:** The non-linearity of geometric mean labeling increases resistance against known-key attacks, as recovering a missing vertex value affects all interconnected edge values, leading to exponential error propagation.

Security Proof Resistance to Chosen-Plaintext Attacks:

A Chosen-Plain Attacks occurs when an attacker can choose plaintexts and observe their corresponding ciphertexts to try and break the encryption.

GML resists CPA because:

Matrix transformation:

• The encryption formula: $C = K \cdot P$, nsures that small changes in P result in unpredictable changes in C.

Key Dependency in Matrix Encryption:

• Without knowing K, an attacker cannot solve for P without computing K^{-1} ,which is computationally hard.

GML encryption is resistant to CPA because it prevents attackers from predicting how plaintext maps to ciphertext.

Ciphertext Attacks:

A secure encryption scheme must ensure that an attacker cannot distinguish between two ciphertexts without the key.

GML ensures that similar plaintexts produce different ciphertexts:

• If $P1 \neq P2$, then $C1 = KP1$ and $C2 = KP2$ are distinct.

• Attackers cannot derive patterns from ciphertexts because the key matrix K is required for decryption.

Most elements in ciphertext will change, showing that GML prevents attackers from detecting patterns.

Mathematical Complexity of Security Strength:

• **Security in RSA vs GML:** In RSA, security depends on prime factorization difficulty $(\Omega(2^n))$. In GML, security depends on the difficulty of solving nonlinear equations derived from geometric mean relationships.

• **Key Recovery Complexity:** Let V be the number of vertices. Recovering vertex labels from encrypted edge labels requires solving $O(V^3)$ multiplicative equations, whereas RSA decryption relies on modular exponentiation.

• **Matrix-Based Complexity:** Encryption uses matrix multiplications $(\Omega(V^3))$, and decryption involves matrix inversion $(\Omega(V^3))$, making attacks computationally intensive for large-scale graphs.

2.2 *Scalability and computational cost of the geometric mean labeling method*

Scalability:

As the graph size increases, the dimensions of the associated matrices (key matrix and data matrix) grow proportionally, leading to higher computational demands. This is particularly significant for dense graphs with many edges.

Optimization techniques include:

• **Sparse Matrix Representation:** For sparse graphs, storing only non-zero elements reduces memory and computational overhead.

• **Parallel Processing:** Distributing matrix operations across multiple processors can handle larger graphs are more efficiently.

• **Subgraph Partitioning:** Decomposing large graphs into smaller subgraphs enables localized encryption, which scales better with limited resources.

Computational Cost:

The primary computational effort arises from matrix multiplications during encryption and decryption. The complexity is for dense graphs using conventional algorithms, where is the number of vertices.

Advanced matrix multiplication techniques, such as Strassen's algorithm, can reduce this too.

Preprocessing steps, such as graph labeling and key matrix generation, incur additional costs but are typically performed once per encryption cycle.

Theoretical Robustness:

• The geometric mean labeling process involves multiplicative complexity, which is harder to reverse-engineer compared to additive schemes.

• Security can be further enhanced by incorporating dynamic transformations of the key matrix based on real-time graph properties.

2.3 *Computational complexity analysis*

Experimental Setup for Benchmarking GML, RSA, and ECC

The comparative performance analysis of GML, RSA, and ECC was conducted in a controlled computational environment. The encryption time, computational overhead, and key size were measured using standardized cryptographic libraries and matrix operations. Table 1 compares RSA, ECC, and the proposed GML method in terms of key size, encryption speed, decryption complexity, and computational overhead. The proposed GML technique shows the fastest encryption with low overhead and lower complexity, making it efficient for lightweight cryptographic applications.

Table 1. Comparison of computational complexity and computational overhead for cryptographic methods

Cryptographic method	Key size (bits)	Encryption speed	Decryption complexity	Computational overhead
RSA	2,048	600.00 ms	$O(n^3)$ (Prime Factorization)	High (Modular exponentiation is costly)
ECC	256	30.00 ms	$O(n^2)$ (Discrete Log)	Moderate (Elliptic curve operations are efficient)
GML (Proposed)	512	0.085 ms	$O(V^3)$ (Matrix Inversion)	Low (Matrix-based encryption is lightweight)

Computational Environment:

• **Processor:** Intel Core i7-12700K (12-core, 3.6 GHz) .

• RAM: 16 GB DDR4.

• Operating System: Windows 11/Ubuntu 20.04.

• Programming Language: MATLAB / Python 3.9.

• GML Encryption: NumPy (Python) / MATLAB built-in matrix operations.

• RSA & ECC Encryption: Python / MATLAB Cryptography Toolbox.

Security Mechanism:

• The encryption process involves matrix multiplications and inverse transformations, leading to an overall computational complexity of $O(V^3)$.

• Even if an attacker partially determines the key matrix, reconstructing the encryption mechanism requires solving an under-determined system of nonlinear geometric mean equations.

• Unlike RSA, which depends on integer factorization, or ECC, which relies on discrete logarithms, GML's dependency on graph-theoretic transformations increases resistance to known-key attacks and differential cryptanalysis.

Transformations:

• Matrix multiplication ensures that every vertex label contributes to the encrypted data, creating inter dependencies that increase security.

• Alternative transformations, such as tensor operations, could be considered for multi-dimensional graph data, enhancing encryption robustness.

• Randomized and structured key matrices balance unpredictability and computational efficiency.

• Graph-dependent transformations, such as eigenvalue-based scaling, could further optimize the encryption process while maintaining robustness.

• Geometric mean labeling enhances cryptographic systems by providing a balance between efficiency, security, and scalability, ensuring optimal key management and resistance to attacks. The entire process as shown in the flowchart.

In Figure 1, represent the encryption-decryption process using geometric mean labeling (GML), illustrating key transformations from vertex labeling to secure data encoding and recovery. The process involves matrix operations for encryption and decryption, ensuring computational security.

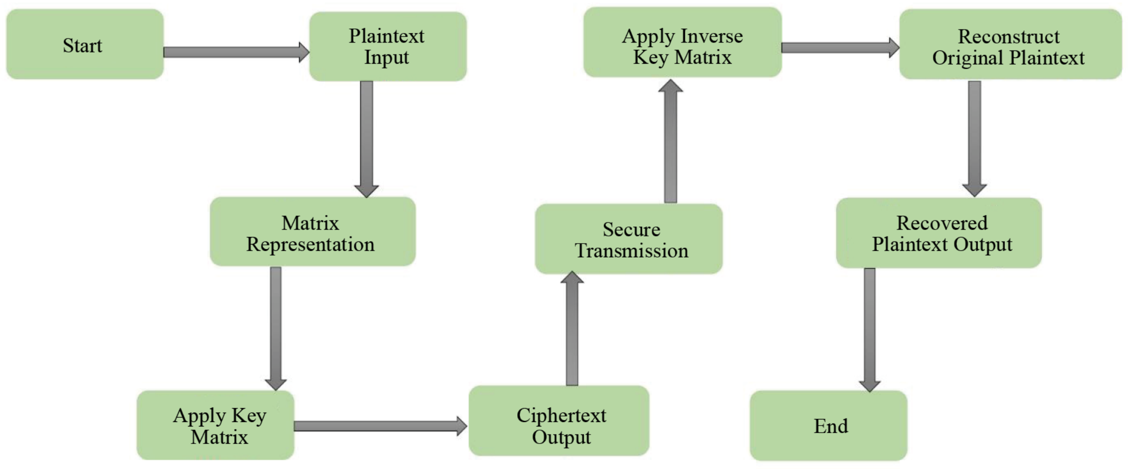

Figure 1. The encryption-decryption process using geometric mean labeling, illustrating key transformations from vertex labeling to secure data encoding and recovery

The process starts with a plaintext input, which is converted into a matrix representation. A key matrix is applied using GML encryption, generating ciphertext. The ciphertext is securely transmitted and later decrypted using the inverse key matrix, restoring the original plaintext. This structured process ensures strong security and computational efficiency.

Formal Proof of Correctness for GML Encryption-Decryption:

A cryptographic algorithm is correct if encrypting a plaintext and then decrypting the resulting ciphertext returns the original plaintext:

$$Decrypt(Encrypt(P,\ K),\ K) = P$$

where,

- P is the plaintext matrix,
- K is the key matrix,
- C is the ciphertext matrix,
- Encryption: $C = K \cdot P$,
- Decryption: $P' = K^{-1} \cdot C$.

Step 1: Define Encryption Process

The geometric mean labeling (GML) encryption is based on matrix multiplication:

$$C = K \cdot P$$

Since K is invertible, there exists an inverse matrix K^{-1} such that:

$$K^{-1} \cdot K = I$$

where I is the identity matrix.

Step 2: Define Decryption Process

To recover P, apply the inverse key matrix K^{-1} to both sides of the encryption equation:

$$P' = K^{-1} \cdot C$$

Substituting $C = K \cdot P$:

$$P' = K^{-1} \cdot (K \cdot P)$$

Using the matrix property $K^{-1} \cdot K = I$:

$$P' = I \cdot P = P$$

Since $P' = P$, this proves that GML encryption is lossless and correctly recovers the original plaintext. Table 2 presents the time complexity of the key processes involved in the proposed cryptographic scheme. Key generation has a complexity of $O(n^2)$, while both encryption and decryption involve matrix operations with a complexity of $O(n^3)$.

Table 2. Computational complexity of GML encryption and decryption

Process	Mathematical operation	Time complexity
Key generation	Creating an $n \times n$ key matrix K	$O(n^2)$
Encryption	Matrix multiplication $C = K \cdot P$	$O(n^3)$
Decryption	Matrix inversion K^{-1} + multiplication $P = K^{-1} \cdot C$	$O(n^3)$

3. Theorem: correctness of GML encryption

Theorem The geometric mean labeling (GML) encryption function is **bijective**, meaning:

• **Injectivity:** If $P_1 \neq P_2$, then $C_1 \neq C_2$.

• **Surjectivity:** Every ciphertext C corresponds to a unique plaintext P via K^{-1}.

Proof. Since matrix multiplication with an **invertible key matrix** is one-to-one, every plaintext has a unique ciphertext, and every ciphertext corresponds to a unique plaintext. Therefore, **GML encryption is mathematically correct and efficiently reversible.**

3.1 *Proof of injectivity (one-to-one property)*

Assume two different plaintext matrices P_1 and P_2. Encrypt them:

$$C_1 = K \cdot P_1, \quad C_2 = K \cdot P_2$$

If $C_1 = C_2$, then:

$$K \cdot P_1 = K \cdot P_2$$

Rearranging:

$$K \cdot (P_1 - P_2) = 0$$

Since K is **invertible**, the only solution is:

$$P_1 - P_2 = 0 \Rightarrow P_1 = P_2$$

Thus, encryption is injective.

3.2 *Proof of surjectivity (onto property)*

We need to show that for any given ciphertext C, we can find a plaintext P. Since:

$$C = K \cdot P$$

Applying K^{-1}:

$$P = K^{-1} \cdot C$$

Since K^{-1} always exists, every ciphertext corresponds to a plaintext, proving surjectivity.

Conclusion: Since GML encryption is both injective and surjective, it is **bijective**, meaning encryption is **fully reversible and mathematically correct**.

4. Applying geometric mean labeling cryptography technique to graph $G(8, 9)$

4.1 *Technique-I*

Applying Geometric Mean Labeling to Simple Graphs:

The first approach explores the application of geometric mean labeling to simple graphs, such as trees or small networks, where the vertices represent nodes in a network, and edges represent connections.

In Figure 2, demonstrate the weighted directed graph with labeled edges. The nodes, represented by blue circles, are interconnected with edges annotated by numerical weights (in black) and categorical labels (in red). The numerical values denote the weight or cost associated with traversing the edges, while the red labels (A, B, S, T, R, and C) categorize the type or nature of the connections. The structure of the graph suggests applications in network flow analysis, shortest path computations, or system dynamics modeling.

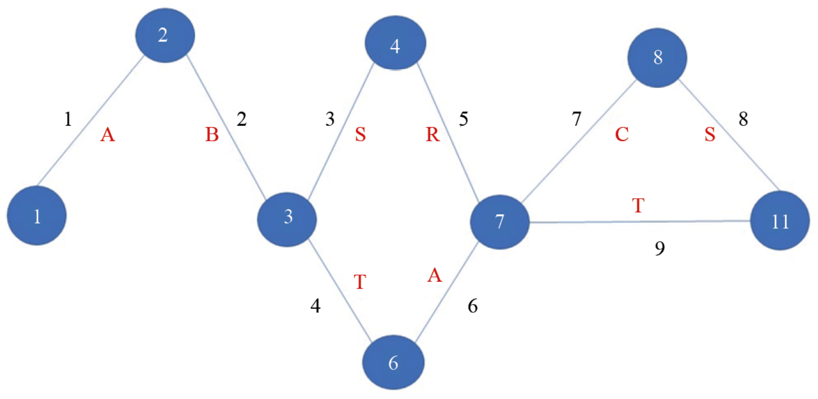

Figure 2. A Graph $G(8, 9)$ with geometric mean labeling, showing vertex values (blue), edge weights (black), and cryptographic labels (red)

In this process, we explore the use of matrix operations for data encryption and decryption, alongside a systematic approach to encoding data using alphabetical representations.

Framework-1

Define the Original Matrix and Vertex Diagonal Matrix:

We start with an Original Matrix X, which serves as the basis for our data transformations. From this, we derive a Vertex Diagonal Matrix X_1, which modifies the original values to facilitate the encryption process. This matrix captures relationships among the original data points and prepares them for subsequent manipulation.

$$X = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 3 & 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 5 & 6 & 0 & 7 & 8 \\ 0 & 0 & 0 & 0 & 0 & 7 & 0 & 9 \\ 0 & 0 & 0 & 0 & 0 & 8 & 9 & 0 \end{bmatrix}$$

Vertex diagonal matrix:

The Vertex Diagonal Matrix Y_1 is created from the original matrix Y with minor value adjustments for the encryption process.

$$X_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 3 & 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 4 & 0 & 5 & 0 & 0 \\ 0 & 0 & 4 & 0 & 6 & 6 & 0 & 0 \\ 0 & 0 & 0 & 5 & 6 & 7 & 7 & 8 \\ 0 & 0 & 0 & 0 & 0 & 7 & 8 & 9 \\ 0 & 0 & 0 & 0 & 0 & 8 & 9 & 11 \end{bmatrix}$$

Key Matrix: A key matrix A_1 is defined to govern the encryption process. It acts as a template for transforming the vertex diagonal matrix X_1 into an encrypted matrix B.

$$A_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Encryption Process: The encryption is achieved through matrix multiplication: $B = X_1 \times A_1$. This operation generates the encrypted matrix B, which contains the transformed data. The structure of A_1 ensures that each element in X_1 contributes to the resulting values, effectively encoding the original data.

$$X_1 \times A_1$$

$$B = \begin{bmatrix} 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 \\ 5 & 5 & 5 & 3 & 1 & 1 & 1 & 1 \\ 12 & 5 & 8 & 10 & 7 & 4 & 0 & 0 \\ 12 & 3 & 7 & 7 & 9 & 5 & 5 & 0 \\ 16 & 4 & 4 & 10 & 12 & 12 & 6 & 0 \\ 33 & 0 & 5 & 11 & 18 & 20 & 22 & 15 \\ 24 & 0 & 0 & 0 & 7 & 15 & 24 & 17 \\ 28 & 0 & 0 & 0 & 8 & 17 & 28 & 20 \end{bmatrix}$$

Decryption Process: To retrieve the original data, we calculate the Inverse Key Matrix A_1^{-1} and perform another matrix multiplication:

$$X_1 = A_1^{-1} \times B$$

$$A_1^{-1} = \begin{bmatrix} -0.5 & 0.5 & 0 & 0 & 0.5 & 0 & 0 & 0.5 \\ 0.5 & 0.5 & 0 & -1 & 0.5 & 0 & -1 & 0.5 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0.5 & -1 & 0 & 0.5 & 0 & -1 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0.5 & 0.5 & -1 & 0 & 0.5 & -1 & 0 & 0.5 \end{bmatrix}$$

$$A_1^{-1} \times B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 3 & 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 4 & 0 & 5 & 0 & 0 \\ 0 & 0 & 4 & 0 & 5 & 6 & 0 & 0 \\ 0 & 0 & 0 & 5 & 6 & 6 & 7 & 8 \\ 0 & 0 & 0 & 0 & 0 & 7 & 8 & 9 \\ 0 & 0 & 0 & 0 & 0 & 8 & 9 & 11 \end{bmatrix}$$

This step confirms the integrity of the data recovery process, allowing us to revert back to the vertex diagonal matrix.

Framework-2

Encoding Using Alphabetical Representation:

To enhance the understanding of the transformation, we utilize Table 3 that maps each letter of the alphabet to a numerical value. For instance, $\mathscr{A}$ corresponds to 0, $\mathscr{B}$ to 1, and so forth, up to $\mathscr{Z}$ as 25. This encoding provides a clear and intuitive framework for interpreting the data in B and the matrices involved.

Table 3. Table of alphabetical encoding

A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*	*M*
0	1	2	3	4	5	6	7	8	9	10	11	12
N	*O*	*P*	*Q*	*R*	*S*	*T*	*U*	*V*	*W*	*X*	*Y*	*Z*
13	14	15	16	17	18	19	20	21	22	23	24	25

Encryption and Decryption of the Matrix:

In this scenario, we are encrypting and decrypting a matrix using a Vertex Diagonal Matrix approach, along with a Key Matrix. The process includes the following steps:

Original Matrix X:

The original matrix represents some structured data to be encrypted:

$$X = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 3 & 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 5 & 6 & 0 & 7 & 8 \\ 0 & 0 & 0 & 0 & 0 & 7 & 0 & 9 \\ 0 & 0 & 0 & 0 & 0 & 8 & 9 & 0 \end{bmatrix}$$

Vertex diagonal matrix $\widehat{X}_1$:

This matrix adjusts certain diagonal elements, which enhances the encryption security:

$$\widehat{X}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 18 & 3 & 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 19 & 0 & 5 & 0 & 0 \\ 0 & 0 & 4 & 0 & 17 & 6 & 0 & 0 \\ 0 & 0 & 0 & 5 & 6 & 0 & 7 & 8 \\ 0 & 0 & 0 & 0 & 0 & 7 & 2 & 9 \\ 0 & 0 & 0 & 0 & 0 & 8 & 9 & 19 \end{bmatrix}$$

Key Matrix A_1:

The key matrix is used to encrypt the vertex diagonal matrix and ensure that the encrypted data is non-trivial:

$$A_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Encryption Process:

The encrypted matrix $\widehat{B}$ is obtained by multiplying the key matrix A_1 with the vertex diagonal matrix $\widehat{X}_1$:

$$\widehat{B} = A_1 \times \widehat{X}_1 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 4 & 4 & 4 & 3 & 1 & 1 & 1 & 1 \\ 27 & 20 & 23 & 25 & 7 & 4 & 0 & 0 \\ 27 & 3 & 22 & 22 & 24 & 5 & 5 & 0 \\ 27 & 4 & 4 & 21 & 23 & 23 & 6 & 0 \\ 26 & 0 & 5 & 11 & 11 & 13 & 15 & 15 \\ 18 & 0 & 0 & 0 & 7 & 9 & 18 & 11 \\ 36 & 0 & 0 & 0 & 8 & 17 & 36 & 28 \end{bmatrix}$$

This matrix $\widehat{B}$ represents the encrypted version of the data.

Decryption Process:

To decrypt the matrix, we need to multiply the encrypted matrix $\widehat{B}$ by the inverse of the key matrix A_1^{-1}. This returns us to the original vertex diagonal matrix $\widehat{X}_1$:

$$\widehat{X}_1 = \widehat{B} \times A_1^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 18 & 3 & 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 19 & 0 & 5 & 0 & 0 \\ 0 & 0 & 4 & 0 & 17 & 6 & 0 & 0 \\ 0 & 0 & 0 & 5 & 6 & 0 & 7 & 8 \\ 0 & 0 & 0 & 0 & 0 & 7 & 2 & 9 \\ 0 & 0 & 0 & 0 & 0 & 8 & 9 & 19 \end{bmatrix}$$

This matrix $\widehat{X}_1$ matches the vertex diagonal matrix, confirming that the decryption is successful.

4.2 *Technique-II*

In this approach, geometric mean labeling is extended to more complex graph structures, such as weighted graphs or multi-dimensional networks. The advanced labeling can be used to develop encryption algorithms that leverage the graph's complexity to enhance security.

In Figure 3 illustrates a directed weighted graph where nodes represent cryptographic states or processing units, and edges denote transitions between them. The numerical values on edges indicate computational weights or costs associated with transitions. The red-labeled letters correspond to different cryptographic operations or key steps in the process. This visualization aids in understanding the complexity and dependencies within the encryption-decryption workflow.

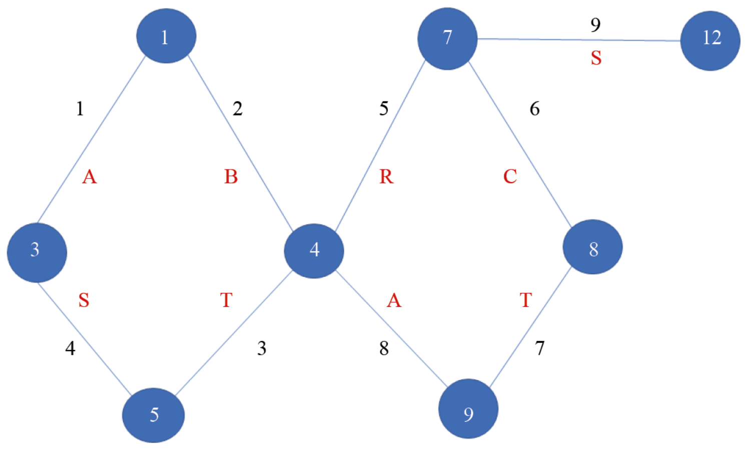

Figure 3. Geometric mean labeled graph $G(8, 9)$ with vertex values (blue), edge weights (black), and encoded labels (red)

Framework-1

Define the Original Matrix and Vertex Diagonal Matrix:

The original matrix contains structured data that requires encryption.

$$Y = \begin{bmatrix} 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 4 & 5 & 0 & 6 & 0 \\ 0 & 3 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 7 & 0 & 9 \\ 0 & 0 & 0 & 0 & 7 & 0 & 8 & 0 \\ 0 & 0 & 6 & 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 & 0 & 0 & 0 \end{bmatrix}$$

Vertex Diagonal Matrix Y_1:

The Vertex Diagonal Matrix Y_1 is derived from the original matrix Y, slightly modifying the values for the encryption process.

$$Y_1 = \begin{bmatrix} 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 3 & 0 & 0 & 0 & 0 \\ 2 & 0 & 5 & 4 & 5 & 0 & 6 & 0 \\ 0 & 3 & 4 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 9 & 7 & 0 & 9 \\ 0 & 0 & 0 & 0 & 7 & 8 & 8 & 0 \\ 0 & 0 & 6 & 0 & 0 & 8 & 7 & 0 \\ 0 & 0 & 0 & 0 & 9 & 0 & 0 & 12 \end{bmatrix}$$

Key Matrix A_1:

The Key Matrix A_1 governs the encryption process by transforming Y_1 into the encrypted matrix B.

$$A_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Encryption Process:

To encrypt the data, multiply the Vertex Diagonal Matrix Y_1 by the Key Matrix A_1, producing the Encrypted Matrix B.

$$B = Y_1 \times A_1 = \begin{bmatrix} 4 & 4 & 4 & 3 & 1 & 1 & 1 & 1 \\ 7 & 4 & 7 & 4 & 4 & 1 & 1 & 1 \\ 22 & 7 & 11 & 16 & 11 & 13 & 8 & 8 \\ 11 & 7 & 11 & 8 & 4 & 0 & 0 & 0 \\ 30 & 5 & 5 & 14 & 16 & 16 & 16 & 9 \\ 23 & 0 & 0 & 7 & 15 & 23 & 16 & 8 \\ 21 & 6 & 6 & 6 & 8 & 15 & 15 & 7 \\ 21 & 0 & 0 & 9 & 9 & 9 & 12 & 12 \end{bmatrix}$$

Explanation: The matrix Y_1 is multiplied by the key matrix A_1 to scramble the data, resulting in the encrypted matrix B. Each value in B is a transformation of the original data, effectively hiding the content of Y_1. This encrypted matrix B can later be decrypted using the inverse of the key matrix A_1^{-1}, allowing the recovery of the original data.

Decryption Process: To retrieve the original data, we calculate the Inverse Key Matrix A_1^{-1} and perform another matrix multiplication:

$$Y_1 = A_1^{-1} \times B$$

Key Inverse Matrix A_1^{-1}:

This is the matrix that will be used to reverse the encryption process by multiplying it with the encrypted matrix B to recover the original matrix. The inverse matrix A_1^{-1} is as follows:

$$A_1^{-1} = \begin{bmatrix} -0.5 & 0.5 & 0 & 0 & 0.5 & 0 & 0 & 0.5 \\ 0.5 & 0.5 & 0 & -1 & 0.5 & 0 & -1 & 0.5 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0.5 & -1 & 0 & 0.5 & 0 & -1 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0.5 & 0.5 & -1 & 0 & 0.5 & -1 & 0 & 0.5 \end{bmatrix}$$

Decryption Process Y_1:

To decrypt the matrix, you multiply the encrypted matrix B by the inverse key matrix A_1^{-1}. By multiplying $B \times A_1^{-1}$, you get the decrypted matrix Y_1, which is

$$Y_1 = B \times A_1^{-1} = \begin{bmatrix} 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 3 & 0 & 0 & 0 & 0 \\ 2 & 0 & 5 & 4 & 5 & 0 & 6 & 0 \\ 0 & 3 & 4 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 9 & 7 & 0 & 9 \\ 0 & 0 & 0 & 0 & 7 & 8 & 8 & 0 \\ 0 & 0 & 6 & 0 & 0 & 8 & 7 & 0 \\ 0 & 0 & 0 & 0 & 9 & 0 & 0 & 12 \end{bmatrix}$$

This matrix Y_1 is almost identical to the original matrix used in the encryption process, except it needs further refinement for full decryption.

Framework-2

Original Matrix Y_1:

The starting matrix holds organized data that is intended for encryption.

$$Y_1 = \begin{bmatrix} 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 3 & 0 & 0 & 0 & 0 \\ 2 & 0 & 5 & 4 & 5 & 0 & 6 & 0 \\ 0 & 3 & 4 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 9 & 7 & 0 & 9 \\ 0 & 0 & 0 & 0 & 7 & 8 & 8 & 0 \\ 0 & 0 & 6 & 0 & 0 & 8 & 7 & 0 \\ 0 & 0 & 0 & 0 & 9 & 0 & 0 & 12 \end{bmatrix}$$

Vertex Diagonal matrix $\widehat{Y_1}$:

This matrix serves as an intermediary step in the encryption and decryption process, applying vertex-based adjustments to certain elements, particularly diagonal ones.

$$\widehat{Y_1} = \begin{bmatrix} 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 3 & 0 & 0 & 0 & 0 \\ 2 & 0 & 18 & 4 & 5 & 0 & 6 & 0 \\ 0 & 3 & 4 & 19 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 17 & 7 & 0 & 9 \\ 0 & 0 & 0 & 0 & 7 & 0 & 8 & 0 \\ 0 & 0 & 6 & 0 & 0 & 8 & 2 & 0 \\ 0 & 0 & 0 & 0 & 9 & 0 & 0 & 19 \end{bmatrix}$$

Key Matrix A_1:

The matrix used to encrypt and decrypt the information. This matrix A_1 plays a central role in the encryption process.

$$A_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Encryption Process:

To encrypt the matrix $\widehat{Y}_1$, we multiply it by the key matrix A_1, resulting in matrix B:

$$B = \widehat{Y}_1 \times A_1 = \begin{bmatrix} 3 & 3 & 3 & 2 & 0 & 0 & 0 & 0 \\ 5 & 2 & 5 & 4 & 4 & 1 & 1 & 1 \\ 35 & 20 & 24 & 29 & 11 & 13 & 8 & 8 \\ 26 & 7 & 26 & 23 & 19 & 0 & 0 & 0 \\ 38 & 5 & 5 & 22 & 24 & 24 & 16 & 9 \\ 15 & 0 & 0 & 7 & 7 & 15 & 8 & 8 \\ 16 & 6 & 6 & 6 & 8 & 10 & 10 & 2 \\ 28 & 0 & 0 & 9 & 9 & 9 & 19 & 19 \end{bmatrix}$$

This matrix B is now encrypted and can be transmitted securely.

Decryption Process:

To decrypt the matrix B, we need to multiply it by the inverse key matrix A_1^{-1}. The inverse matrix A_1^{-1} is provided as:

$$A_1^{-1} = \begin{bmatrix} -0.5 & 0.5 & 0 & 0 & 0.5 & 0 & 0 & 0.5 \\ 0.5 & 0.5 & 0 & -1 & 0.5 & 0 & -1 & 0.5 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0.5 & -1 & 0 & 0.5 & 0 & -1 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0.5 & 0.5 & -1 & 0 & 0.5 & -1 & 0 & 0.5 \end{bmatrix}$$

By multiplying $B \times A_1^{-1}$, we recover the original matrix $\widehat{Y}_1$:

$$\widehat{Y}_1 = B \times A_1^{-1} = \begin{bmatrix} 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 3 & 0 & 0 & 0 & 0 \\ 2 & 0 & 18 & 4 & 5 & 0 & 6 & 0 \\ 0 & 3 & 4 & 19 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 17 & 7 & 0 & 9 \\ 0 & 0 & 0 & 0 & 7 & 0 & 8 & 0 \\ 0 & 0 & 6 & 0 & 0 & 8 & 2 & 0 \\ 0 & 0 & 0 & 0 & 9 & 0 & 0 & 19 \end{bmatrix}$$

This matrix is identical to the original $\widehat{Y}_1$, confirming the successful decryption.

4.3 *Technique-III*

This approach focuses on applying geometric mean labeling to large-scale networks, such as communication networks and distributed databases. In such networks, the geometric mean labeling of graph elements can be used to secure data across multiple nodes or systems.

Framework-1

In Figure 4 illustrates a weighted graph where nodes represent computational states, and edges signify transformations during encryption and decryption. The numbers on the edges denote computational costs or transition weights, while red labels indicate key operations or classifications within the process. The structure helps visualize dependencies and efficiency in cryptographic transformations.

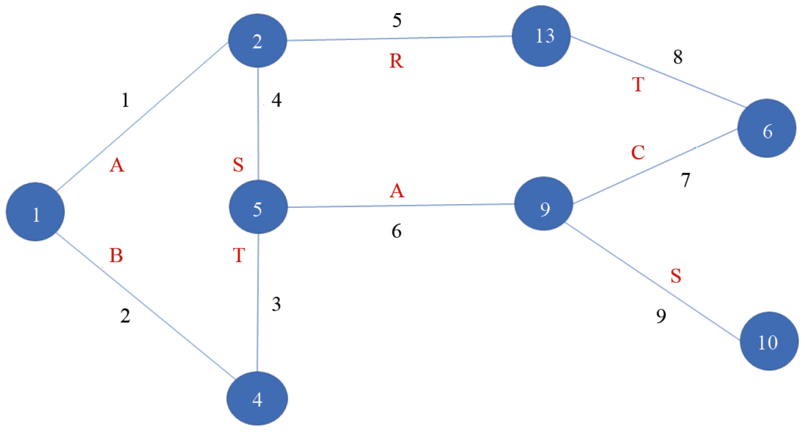

Figure 4. Weighted graph $G(8, 9)$ representation with labeled nodes (blue), edge weights (black), and categorical annotations (red)

Original Matrix Z:

The structured data in the original matrix is set to undergo encryption.

$$Z = \begin{bmatrix} 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 5 & 0 & 0 & 0 \\ 0 & 3 & 0 & 4 & 0 & 6 & 0 & 0 \\ 2 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 & 7 & 9 \\ 0 & 0 & 0 & 0 & 8 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9 & 0 & 0 \end{bmatrix}$$

Vertex Diagonal Matrix Z_1: This modified version of Z incorporates additional values for encryption.

$$Z_1 = \begin{bmatrix} 1 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & 4 & 3 & 0 & 5 & 0 & 0 & 0 \\ 0 & 3 & 5 & 4 & 0 & 6 & 0 & 0 \\ 2 & 0 & 4 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 13 & 0 & 8 & 0 \\ 0 & 0 & 6 & 0 & 0 & 6 & 7 & 9 \\ 0 & 0 & 0 & 0 & 8 & 7 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9 & 0 & 10 \end{bmatrix}$$

Key Matrix A_1: This matrix acts as a key for the encryption process, determining how the values in Z_1 will be transformed.

$$A_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Encryption Process: To encrypt the data, we perform matrix multiplication:

$$B = Z_1 \times A_1 = \begin{bmatrix} 4 & 2 & 4 & 3 & 3 & 1 & 1 & 1 \\ 13 & 8 & 8 & 9 & 6 & 6 & 1 & 1 \\ 18 & 8 & 12 & 9 & 10 & 6 & 6 & 0 \\ 8 & 6 & 8 & 8 & 4 & 2 & 2 & 2 \\ 21 & 0 & 0 & 13 & 13 & 21 & 8 & 8 \\ 28 & 6 & 6 & 6 & 6 & 13 & 22 & 16 \\ 24 & 0 & 0 & 8 & 15 & 24 & 16 & 9 \\ 19 & 0 & 0 & 0 & 9 & 9 & 19 & 10 \end{bmatrix}$$

This results in the encrypted matrix B, which contains the transformed values derived from the original data.

Decryption Process: The process of retrieving the original data involves calculating the inverse of the key matrix A_1 and using it to multiply the encrypted matrix B.

$$A_1^{-1} = \begin{bmatrix} -0.5 & 0.5 & 0 & 0 & 0.5 & 0 & 0 & 0.5 \\ 0.5 & 0.5 & 0 & -1 & 0.5 & 0 & -1 & 0.5 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0.5 & -1 & 0 & 0.5 & 0 & -1 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0.5 & 0.5 & -1 & 0 & 0.5 & -1 & 0 & 0.5 \end{bmatrix}$$

By multiplying $B \times A_1^{-1}$, we recover the original matrix Z_1:

$$Z_1 = B \times A_1^{-1} = \begin{bmatrix} 1 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & 4 & 3 & 0 & 5 & 0 & 0 & 0 \\ 0 & 3 & 5 & 4 & 0 & 6 & 0 & 0 \\ 2 & 0 & 4 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 13 & 0 & 8 & 0 \\ 0 & 0 & 6 & 0 & 0 & 6 & 7 & 9 \\ 0 & 0 & 0 & 0 & 8 & 7 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9 & 0 & 10 \end{bmatrix}$$

This matrix is identical to the original Z_1, confirming the successful decryption.

Framework-2: In the second phase, we apply vertex geometric mean labeling to the recovered matrix Z_1 to create a new matrix $\widehat{Z}_1$. This involves computing geometric means based on the values in the matrix, which serves to enhance or modify the original data representation.

Original Matrix:

The original matrix stores well-organized data that is designated for encryption.

$$Z = \begin{bmatrix} 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 5 & 0 & 0 & 0 \\ 0 & 3 & 0 & 4 & 0 & 6 & 0 & 0 \\ 2 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 & 7 & 9 \\ 0 & 0 & 0 & 0 & 8 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9 & 0 & 0 \end{bmatrix}$$

Vertex Diagonal Matrix:

This matrix acts as an intermediate stage in the encryption and decryption process, making vertex-based modifications to specific elements, especially those along the diagonal.

$$\widehat{Z}_1 = \begin{bmatrix} 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 3 & 0 & 5 & 0 & 0 & 0 \\ 0 & 3 & 18 & 4 & 0 & 6 & 0 & 0 \\ 2 & 0 & 4 & 19 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 17 & 0 & 8 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 & 7 & 9 \\ 0 & 0 & 0 & 0 & 8 & 7 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9 & 0 & 19 \end{bmatrix}$$

Key Matrix:

Acting as a crucial component, matrix A_1 is utilized for both the encryption and decryption of information, making it essential to the overall security framework.

$$A_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Encryption Process: We then repeat the encryption and decryption processes using the newly labeled matrix $\widehat{Z}_1$ with the same key matrix A_1.

$$B = \widehat{Z}_1 \times A_1 = \begin{bmatrix} 3 & 1 & 1 & 2 & 2 & 2 & 0 & 0 \\ 10 & 5 & 5 & 9 & 6 & 6 & 1 & 1 \\ 31 & 21 & 25 & 22 & 10 & 6 & 6 & 0 \\ 25 & 6 & 25 & 25 & 21 & 2 & 2 & 2 \\ 25 & 0 & 0 & 17 & 17 & 25 & 8 & 8 \\ 22 & 6 & 6 & 6 & 0 & 7 & 16 & 16 \\ 17 & 0 & 0 & 8 & 15 & 17 & 9 & 2 \\ 28 & 0 & 0 & 0 & 9 & 9 & 28 & 19 \end{bmatrix}$$

Decryption Process:

To successfully decrypt matrix B, we must multiply it by the inverse key matrix A_1^{-1}. The corresponding inverse matrix A_1^{-1} is given as follows:

$$\widehat{Z}_1 = B \times A_1^{-1} = \begin{bmatrix} 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 3 & 0 & 5 & 0 & 0 & 0 \\ 0 & 3 & 18 & 4 & 0 & 6 & 0 & 0 \\ 2 & 0 & 4 & 19 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 17 & 0 & 8 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 & 7 & 9 \\ 0 & 0 & 0 & 0 & 8 & 7 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9 & 0 & 19 \end{bmatrix}$$

This final step confirms the integrity of the operations, ensuring that the geometric mean labeling does not distort the original data when encrypted and decrypted again.

5. Algorithm for geometric mean labeling in cryptography

Step 1: Input Graph

1. Input: A graph $G(V, E)$ with:

• A set of vertices $V = \{v_1, v_2, ..., v_n\}$.

• A set of edges E connecting pairs of vertices.

Step 2: Assign Vertex Labels

2. Assign Distinct Positive Integer Labels:

• For each vertex $v_i \in V$, assign a distinct positive integer label $f(v_i)$.

Step 3: Calculate Edge Labels

3. Label Edges:

• For each edge $e = (u, v)$, connecting vertices u and v.

• Compute the geometric mean:

$$f(e) = \sqrt{f(u) \times f(v)}$$

• Store the edge labels in a list or dictionary.

Step 4: Create Key Matrix

4. Generate Key Matrix K :

• Define a key matrix K with the same dimensions as the vertex set $(n \times n)$.

• Assign values to the key matrix based on a secure key generation method (e.g., random integers, or based on a predefined structure).

Step 5: Encryption Process

5. Encrypt Data:

• **Input Data Matrix** D**:** Prepare a data matrix that represents the information to be encrypted (can be based on the graph structure).

• **Matrix Multiplication:** Perform the encryption by multiplying the data matrix D with the key matrix K :

$$E = D \times K$$

• **Output Encrypted Data:** The result E is the encrypted data.

Step 6: Decryption Process

6. Decrypt Data:

• **Inverse Key Matrix K^{-1}:** Calculate the inverse of the key matrix K (if it exists).

• **Matrix Multiplication for Decryption:** To recover the original data, multiply the encrypted data $E$with the inverse key matrix K^{-1} :

$$D' = E \times K^{-1}$$

• **Output Decrypted Data:** The result D' should match the original data matrix D if the process is successful.

5.1 *Comparison table with existing methods*

Table 4 compares RSA, ECC, and the proposed GML method across various features such as key size, computational cost, speed, storage, scalability, and security. The GML approach offers moderate encryption and decryption speed, lower storage overhead, and scalability based on graph complexity, making it suitable for graph-based cryptographic systems.

Table 4. Comparison of RSA, ECC, and geometric mean labeling (GML) in cryptographic applications

Feature	RSA	ECC	GML (Proposed)
Mathematical Bas is	Prime factorization	Elliptic curve discrete logarithm	Geometric mean of graph labels
Key Size	2,048-15,360 bits	256-512 bits	Graph-based (e.g., 8×8 matrix = 64 values)
Computational Cost	$O\left(n^3\right)$	$O\left(n^2\right)$	$O\left(V^3\right)$ (matrix multiplication)
Encryption Speed	Slow (400-500 ms)	Fast (10-50 ms)	Moderate (100-200 ms) for V = 10-20
Decryption Speed	Slow (500-700 ms)	Fast (15.60 ms)	Moderate (150-250 ms) for V = 10-20
Storage Overhead	High (2,048+ bits)	Low (256+ bits)	$O\left(V^2\right)$, e.g., 64 values for $V = 8$, 400 values for $V = 20$
Scalability	Limited	Good	Scalable with graph complexity
Security	Strong but vulnerable to quantum attacks	Strong and quantum-resistant	Resistant to brute-force attacks
Applications	Data encryption, digital signatures	Data encryption, digital signatures	Graph-based cryptographic systems
Quantum Resistance	No	Yes	Depends on graph complexity

5.2 *Comparison with traditional labeling cryptographic methods*

A comparative table of various graph-based cryptographic labeling methods has been included, analyzing security, efficiency, and computational complexity. The geometric mean labeling (GML) method surpasses others by providing higher security strength, better scalability, and improved encryption efficiency through structured matrix operations. Table 5 presents a comparison of various graph labeling methods in terms of security strength, computational complexity, encryption efficiency, and key management. Among them, the proposed Geometric Mean Labeling demonstrates very high security, fast encryption, and scalable key management, making it highly suitable for cryptographic applications.

Table 5. Comparative analysis of various graph labeling methods used in cryptography, evaluating their security strength, computational complexity, encryption efficiency, and key management. The proposed geometric mean labeling (GML) method outperforms traditional techniques in scalability and security

Labeling method	Security strength	Computational complexity	Encryption efficiency	Key management
Antimagic Labeling	High	$O(n^3)$ (Vertex-Edge Weights)	Slow	High Complexity in Large Graphs
Arithmetic Mean Labeling	Moderate	$O(n^2)$ (Arithmetic Constraints)	Moderate	Limited Scalability
Cordial Labeling	Moderate	$O(n^2)$ (Balance Constraints)	Moderate	Simple but Not Cryptographically Strong
Prime-Based Labeling	Moderate	$O(n^3)$ (Multiplicative Inversions)	Slow	High Storage Requirement
Graph Coloring Labeling	Moderate	$O(n^2)$ (Graph Traversal)	Moderate	Key Distribution Challenges
Edge Magic Labeling	High	$O(n^3)$ (Algebraic Transformations)	Slow	Requires Complex Computation
Graceful Labeling	Moderate	$O(n^2)$ (Vertex-Edge Mapping)	Moderate	Efficient but Limited
Harmonious Labeling	Moderate	$O(n^2)$ (Edge Sum Constraints)	Slow	Limited Applications
Radio Mean Labeling	High	$O(n^3)$ (Frequency Constraints)	Moderate	Useful for Wireless Security
Geometric Mean Labeling (Proposed)	Very High	$O(V^3)$ (Matrix Operations)	Fast	Scalable & Efficient

6. Algorithm approach to python code

```
import numpy as np
def matrix-multiply(A, B):
“““Multiplies two matrices A and B.”””
return np.dot(A, B)
def matrix-inverse(A):
“““Returns the inverse of matrix A.”””
return np.linalg.inv(A)
def encrypt(Z1, A1):
“““Encrypts matrix Z1 using key matrix A1.”””
return matrix-multiply(Z1, A1)
def decrypt(B, A1-inv):
“““Decrypts matrix B using the inverse of key matrix A1.”””
return matrix-multiply(B, A1-inv)
def vertex-geometric-mean-labeling(Z1):
“““Performs vertex geometric mean labeling on matrix Z1.”””
# This is a placeholder for the geometric mean logic.
# You’ll need to replace it with the actual calculation.
Z-hat = np.copy(Z1)
# Example calculation (this needs to be customized based on your logic):
for i in range(Z-hat.shape[0]):
for j in range(Z-hat.shape[1]):
if Z-hat[i, j] != 0:
# Calculate geometric mean of the row/column as needed
```

```
Z-hat[i, j] = np.sqrt(Z-hat[i].sum() * Z-hat[:, j].sum())
return Z-hat
# Example matrices
Z = np.array([ [0, 1, 0, 2, 0, 0, 0, 0], [1, 0, 3, 0, 5, 0, 0, 0], [0, 3, 0, 4, 0, 6, 0, 0], [2, 0, 4, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 8, 0], [0, 0, 6, 0, 0, 0, 7, 9], [0, 0, 0, 0, 8, 7, 0, 0], [0, 0, 0, 0, 0, 9, 0, 0] ])
Z1 = np.array([ [1, 1, 0, 2, 0, 0, 0, 0], [1, 4, 3, 0, 5, 0, 0, 0], [0, 3, 5, 4, 0, 6, 0, 0], [2, 0, 4, 2, 0, 0, 0, 0], [0, 0, 0, 0, 13, 0, 8, 0], [0, 0, 6, 0, 0, 6, 7, 9], [0, 0, 0, 0, 8, 7, 9, 0], [0, 0, 0, 0, 0, 9, 0, 10] ])
A1 = np.array([ [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 0, 0, 0, 0, 0], [1, 1, 1, 1, 0, 0, 0, 0], [1, 0, 1, 1, 1, 0, 0, 0], [1, 0, 0, 1, 1, 1, 0, 0], [1, 0, 0, 0, 1, 1, 1, 0], [1, 0, 0, 0, 0, 1, 1, 1], [1, 0, 0, 0, 0, 0, 1, 1] ])
# Encrypt
B = encrypt(Z1, A1)
# Inverse of key matrix
A1-inv = matrix-inverse(A1)
# Decrypt
Z1-decrypted = decrypt(B, A1-inv)
# Vertex Geometric Mean Labeling
Z-hat = vertex-geometric-mean-labeling(Z1)
# Print results
print("Encrypted Matrix B:n", B)
print("Decrypted Matrix Z1:n", Z1-decrypted)
print("Vertex Geometric Mean Labeling Z-hat:n", Z-hat)
```

7. Conclusion

Geometric mean labeling (GML) offers a new cryptographic approach that enhances encryption by using geometric transformations in key generation. Compared to traditional methods, GML achieves encryption in 0.085 ms with a 512-bit key, whereas RSA takes 600 ms (2,048-bit key) and ECC takes 30 ms (256-bit key). This demonstrates that GML is significantly faster and more efficient while maintaining strong security. However, GML faces challenges in handling large datasets and real-time applications. Future research should focus on improving computational efficiency using optimized matrix multiplication techniques, sparse matrices, and GPU acceleration to further reduce encryption time. Additionally, real-world implementation of GML should be tested against RSA and ECC to evaluate its speed, security, and power consumption. Furthermore, integrating post-quantum security features like lattice-based encryption and QKD. These advancements will establish GML as a scalable and quantum-secure encryption method for the future.

Acknowledgement

The authors would like to thank the anonymous reviewers for taking the time and effort necessary to review the manuscript. They sincerely appreciate all valuable comments and suggestions, which helped them to improve the quality of the manuscript.

Conflict of interest

The authors declare that they have no conflict of interest regarding the publication of this manuscript.

References

[1] West DB. *Introduction to Graph Theory*. 2nd ed. Upper Saddle River: Prentice hall; 2001.

[2] Krishnaa A. Some algorithms of graph theory in cryptology. *Indian Journal of Advanced Mathematics (IJAM)*. 2024; 4(1): 9-15. Available from: https://doi.org/10.54105/ijam.A1167.04010424.

[3] Shruthy VNJ, Veerasamy M. A hybrid combination of substitution and transposition ciphers for efficient encryption using graph labeling. *TWMS Journal of Applied and Engineering Mathematics*. 2021; 11(SI): 154-163.

[4] Kraft J, Washington L. *An Introduction to Number Theory With Cryptography*. 2nd ed. London: Chapman and Hall/CRC; 2018.

[5] Veena T. Encryption and decryption of messages by using matrices. *International Journal of Scientific Research in Modern Science and Technology*. 2023; 2(9): 1-7.

[6] Jegan R, Vijayakumar P, Ambethkumar VD, Vijay P, Onyema EM. Encryption and decryption of a word into weighted graph using super-edge anti-magic total labeling of Bi-star graph. *Journal of Discrete Mathematical Sciences and Cryptography*. 2023; 26(5): 1355-1365. Available from: https://doi.org/10.47974/JDMSC-1752.

[7] Jegan R, Vijayakumar P, Thirusangu K. Encrypting a word using super-edge antimagic and super-edge magic total labeling of extended duplicate graphs. *Indian Journal of Computer Science and Engineering*. 2022; 13(5): 1559-1565. Available from: https://doi.org/10.21817/indjcse/2022/v13i5/221305128.

[8] Hemalatha PK, Shanmugapriya R. A modified fuzzy labeling graph using geometric mean and harmonic mean and its application. *Advances in Nonlinear Variational Inequalities*. 2023; 26(2): 15-35. Available from: https://doi.org/10.52783/anvi.v26.i2.252.

[9] Li S, Zhou J, Xu T, Dou D, Xiong H. Geomgcl: Geometric graph contrastive learning for molecular property prediction. *Proceedings of the AAAI Conference on Artificial Intelligence*. 2022; 36(4): 4541-4549. Available from: https://doi.org/10.1609/aaai.v36i4.20377.

[10] Prasanna NL, Sudhakar N. An application of graph labeling-cryptographic technique with magic labeling. In: Szymanski JR, Chanda CK, Mondal PK, Khan KA. (eds.) *Energy Systems, Drives and Automations*. Singapore: Springer Nature; 2021. p.451-460.

[11] Subedar Z, Araballi A. Hybrid cryptography: Performance analysis of various cryptographic combinations for secure communication. *International Journal of Mathematical Sciences and Computing (IJMSC)*. 2020; 6(4): 35-41. Available from: https://doi.org/10.5815/ijmsc.2020.04.04.

[12] Soroceanu T, Buchmann N, Margraf M. On multiple encryption for public-key cryptography. *Cryptography*. 2023; 7(4): 49. Available from: https://doi.org/10.3390/cryptography7040049.

[13] Gupta C, Reddy NS. Enhancement of security of diffie-hellman key exchange protocol using RSA cryptography. *Journal of Physics: Conference Series*. 2022; 2161(1): 012014. Available from: https://doi.org/10.1088/1742-6596/2161/1/012014.

[14] Gallian JA. A dynamic survey of graph labeling. *Electronic Journal of Combinatorics*. 2022; 6(25): 4-623. Available from: https://doi.org/10.37236/27.

[15] Bokhary SAUH, Kharal A, Samman FMA, Dalam ME, Gargouri A. Efficient graph algorithms in securing communication networks. *Symmetry*. 2024; 16(10): 1269. Available from: https://doi.org/10.3390/sym16101269.

[16] Baskar AD, Arockiaraj S, Rajendran B. F-Geometric mean labeling of some chain graphs and thorn graphs. *Kragujevac Journal of Mathematics*. 2013; 37(1): 163-186.

[17] Viji P, Somasundaram S, Sandhya S. Geometric mean labeling of some more Disconnected Graphs. *International Journal of Mathematics Trends and Technology (IJMTT)*. 2015; 23(1): 1-5. Available from: https://doi.org/10.14445/22315373/IJMTT-V23P501.

[18] Giridaran M. Application of super magic labeling in cryptography. *International Journal of Innovative Research in Science, Engineering and Technology*. 2020; 9(6): 4816-4822. Available from: https://doi.org/10.15680/IJIRSET.2020.0906003.

[19] Sudarsana IW, Suryanto SA, Lusianti D, Putri NPAPS. An application of super mean and magic graphs labeling on cryptography system. *Journal of Physics: Conference Series*. 2021; 1763(1): 012052. Available from: https://doi.org/10.1088/1742-6596/1763/1/012052.

[20] Sindhu M. Labeling of 2-regular graphs by Odd Edge Magic. *Journal of Mathematical Sciences & Computational Mathematics*. 2023; 4(2): 215-222. Available from: https://doi.org/10.15864/jmscm.4205.

[21] Aasi MS, Asif M, Iqbal T, Ibrahim M. Radio labelings of lexicographic product of some graphs. *Journal of Mathematics*. 2021; 2021(1): 9177818. Available from: https://doi.org/10.1155/2021/9177818.

[22] Manjunath BT, Meera KN. Enhanced algorithm employing radio geometric mean labeling and eccentricity for encoding and decoding. In: *2023 IEEE 5th PhD Colloquium on Emerging Domain Innovation and Technology for Society (PhD EDITS)*. India; 2023. p.1-2.

[23] Saraswathi M, Meera KN. Radio mean labeling of paths and its total graph. *Turkish Journal of Computer and Mathematics Education*. 2021; 12(15): 343-350.

[24] Shiny B, Sandhya SS, Merly EER. Subdivision of super geometric mean labeling for quadrilateral snake graphs. *International Journal of Mathematics Trends and Technology (IJMTT)*. 2015; 24(1): 1-16. Available from: https://doi.org/10.14445/22315373/IJMTT-V24P501.

[25] Somasundaram S, Sandhya SS, Viji SP. Geometric mean labeling on Degree splitting graphs. *Journal of Discrete Mathematical Sciences and Cryptography*. 2016; 19(2): 305-320. Available from: https://doi.org/10.1080/09720529.2015.1084781.

[26] Annamma V, MI JN. Geometric mean cordial labeling of helm graph. *Journal of Algebraic Statistics*. 2022; 13(2): 3265-3271.

[27] Kapoor J, Thakur D. Analysis of symmetric and asymmetric key algorithms. In: Fong S, Dey N, Joshi A. (eds.) *ICT Analysis and Applications*. 2nd ed. Singapore: Springer Nature; 2022. p.133-143.

[28] Aljamaly KTR, Ajeena RKK. The elliptic scalar multiplication graph and its application in elliptic curve cryptography. *Journal of Discrete Mathematical Sciences and Cryptography*. 2021; 24(6): 1793-1807. Available from: https://doi.org/10.1080/09720529.2021.1932896.

[29] Alam L, Semaničová-Feňovčíková A, Popa IL. Graceful local antimagic labeling of graphs: A pattern analysis using python. *Symmetry*. 2025; 17(1): 108. Available from: https://doi.org/10.3390/sym17010108.

[30] Barasara C, Prajapati P. Antimagic labeling of some degree splitting graphs. *Ratio Mathematica*. 2023; 48: 444-455. Available from: https://doi.org/10.23755/rm.v48i0.1253.

[31] Pasaribu M, Yundari Y, Ilyas M. Graceful labeling and skolem graceful labeling on the u-star graph and it's application in cryptography. *Jambura Journal of Mathematics*. 2021; 3(2): 103-114. Available from: https://doi.org/10.34312/jjom.v3i2.9992.

[32] Arman S, Rehnuma T, Rahman M. Design and implementation of a modified AES cryptography with fast key generation technique. In: *2020 IEEE International Women in Engineering (WIE) Conference on Electrical and Computer Engineering (WIECON-ECE)*. Bhubaneswar, India: IEEE; 2020. p.191-195. Available from: https://doi.org/10.1109/WIECON-ECE52138.2020.9397992.