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Research Article

Moore-Penrose Pseudo-Inverse Applied to the Decoding of 5G and Beyond (B5G) Communication Systems

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Abstract: This work uses the Moore-Penrose pseudo-inverse (PINV) properties to improve the decoding of signals in the uplink channel of a 5th-Generation mobile communication technology (5G) and Beyond (B5G) communication system. The results of computational simulations show an improvement in the bit error rate (BER) and in the Error Vector Magnitude (EVM) under adverse conditions of the wireless communication channel, which is of great importance to meet the high transmission rate and high reliability requirements of 5G and B5G systems.

***Keywords*:** 5G, pseudo-inverse, Singular Value Decomposition (SVD), machine learning

MSC: 15A09, 15A10, 15A18, 68P30, 94A14

1. Introduction

Sparse code multiple access (SCMA) is one of the non-orthogonal multiple access (NOMA) techniques [1–4], used in 5G and B5G Communications systems [5]; initially, SCMA encodes two user bits [6, 7] into multidimensional codewords using more than one radio subcarrier. In [8, 9]. Singular Value Decomposition (SVD) is used to optimize the distance between points of the IQ constellation. Later, in [10] the authors use 8-APSK to increase the spectral efficiency, transmitting 3 user bits simultaneously. When there are 6 users, 4 radio carriers and SVD-SCMA to encode two user bits, a 64-point IQ constellation is obtained; and by encoding 3 user bits, an IQ constellation with up to 512 possible points to represent the data is obtained. On the receiver side, signals are passed directly to the decoder, to decode SCMA signals, Message Passing Algorithm (MPA) was initially used [11], and later Supervised Machine Learning (ML) [12].

2. Contribution

This work proposes a method to preprocess the received SCMA signals using the PINV; this matrix transformation is applied before the signal enters the decoding module. The results obtained through computational simulations show a

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3 dB improvement in signal decoding of the proposed communications system (BER). PINV is a matrix transformation used in different fields of engineering. In this work, we extend its use in signal decoding in 5G and B5G communications systems.

The rest of the paper is organized as follows: first, the test scenario used is described, followed by a summary of how user data is encoded using SVD-SCMA, and then the radio resource allocation is defined as well as the wireless channel considerations used in the computational simulation. In the receiver side, it is defined how PINV is used to pre-process the received data, which is then decoded by the neural network module, in this section the parameters used for machine learning are listed. Then, the computational simulation results are shown as well as a discussion of them, in the final section the conclusions of this work are given.

3. Testing scenario

The uplink is the connection that goes from the user terminal to the base station, Figure 1 shows the test scenario used in this work, from left to right it can be seen that there are six users ($J = 6$), the user data is encoded using three bits using eight code words ($M = 8$) (000, 001, ..., 110, 111), to encode the data SVD-SCMA is used using four radio subcarriers ($SC = k = 4$), to later be transmitted by a wireless channel affected by attenuation, interference and noise. At the base station, the data are pre-processed by PINV and are sent to the neural network module for decoding, as performance indicators the bit error rate (BER) and the error vector magnitude (EVM) were used. For the computational simulation, Python was used as the programming language, and Tensor Flow for machine learning.

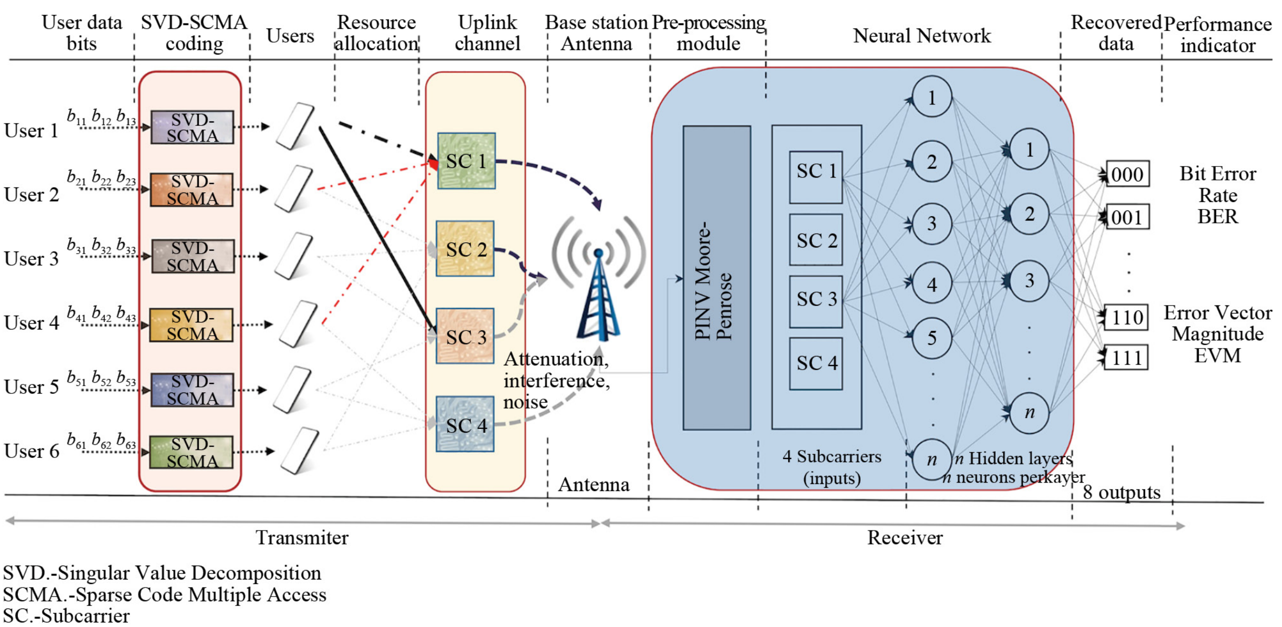

Figure 1. Testing scenario

4. Coding SCMA signals using singular value decomposition

As mentioned in the previous section, NOMA systems allow overloading, which means that there are more users than radio resources, in this work six users ($J = 4$) and four radio subcarriers ($SC = k = 4$) are considered, using 8-APSK [9, 10] it is possible to encode three user bits using eight code words, two code dimensions are used ($N = 2$), three users sharing the same subcarrier ($d_f = 3$), and each user uses two subcarriers to transmit its information ($d_v = 3$).

The mother constellation M_c, is an $N \times M$ matrix, where $N = 2$ and $M = 8$ (code dimensions and codewords respectively). Using 8-APSK [10], the first vector S_1 of the mother constellation M_c, is given by (1)

$$S_{11} = 1 + 1i,\ S_{12} = -1 + 1i,\ S_{13} = -1 - 1i,\ S_{14} = 1 - 1i$$
$$S_{15} = 3 + 3i,\ S_{16} = -3 + 3i,\ S_{17} = -3 - 3i,\ S_{18} = 3 - 3i \tag{1}$$

The second vector is obtained from (2):

$$S_2 = U_N S_1 \tag{2}$$

where: $U_N = diag\ \left(1e^{i\theta_{l-1}}\right) \in C^{N\ \times\ M}$ and $\theta_{l-1} = \dfrac{(l-1)\pi}{M\ N},\ l = 1,\ \ldots,\ N$ which results in (3)

$$M_C = (S_1,\ S_2)^T = \begin{bmatrix} s_{11}s_{12}\ldots s_{18} \\ s_{21}s_{22}\ldots s_{28} \end{bmatrix} \tag{3}$$

In [9], it is proposed to use SVD to encode user data. SVD [8, 13] is the factorization of a real or complex rectangular matrix of $n \times m$, where $n \neq m$, as shown in (4)

$$A_{n\times m} = U\Sigma V^* \tag{4}$$

where:

U is a $n \times p$ unitary matrix,

V^* is an $m \times p$ unitary matrix,

Σ is a $p \times p$ diagonal matrix with non-negative values sorted in descending order and

p is the rank of matrix A.

So, (4) can be rewritten as:

$$A_{n\times m} = U_{n\times p}\Sigma_{p\times p}V^*_{m\times p}. \tag{5}$$

The elements of the diagonal matrix Σ are known as the singular values σ_i and determine the rank of the matrix A. The columns U are known as left singular vectors and the columns V are the right singular vectors. When SVD is applied to SCMA codebooks, the resulting matrices are orthogonal with orthonormal vectors, which produces IQ constellations with an optimal distance between adjacent points [10]. The singular value decomposition (SVD) is applied to the matrix defined in (5), as shown in (6):

$$M_{SVD} = SVD\ [M_c] = U_{n\times p}\Sigma_{p\times p}V^*_{m\times p}. \tag{6}$$

V^* is a 2×8 matrix like M_c, except that V^* is a set of orthonormal eigenvectors.

As described in [6], the codebooks for the different six users ($J = 6$), are obtained from (7),

$$X_j = V_j \Delta_j V_{SVD}^*, \mid j = 1,\ 2,\ \ldots\ J. \tag{7}$$

According to [10] V_j is a scattering matrix given by (8)

$$
\begin{aligned}
&V_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} V_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} V_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \\
&V_4 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} V_5 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} V_6 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned} \tag{8}
$$

and Δ_j is the rotation operator

$$
\begin{aligned}
&\Delta_1 = \begin{bmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{bmatrix} \Delta_2 = \begin{bmatrix} \varphi_2 & 0 \\ 0 & \varphi_3 \end{bmatrix} \Delta_3 = \begin{bmatrix} \varphi_1 & 0 \\ 0 & \varphi_3 \end{bmatrix} \\
&\Delta_4 = \begin{bmatrix} \varphi_3 & 0 \\ 0 & \varphi_1 \end{bmatrix} \Delta_5 = \begin{bmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{bmatrix} \Delta_6 = \begin{bmatrix} \varphi_2 & 0 \\ 0 & \varphi_3 \end{bmatrix}
\end{aligned} \tag{9}
$$

and V^* is the result of applying SVD to the mother matrix M_c. X_j is a four-row by eight-column (4×8) matrix representing the 6 codebooks, where the rows define the subcarriers used $(SC = k = 4)$ and the columns the 8 possible words $(M = 8)$ $(000,\ 001,\ \ldots,\ 110,\ 111)$.

5. Radio resource allocation

In NOMA systems there are more users than radio subcarriers, the process of defining which carrier will use which radio resource is called resource allocation, there are dynamic algorithms that change the subcarriers to be used depending on some channel variable [14, 15]. In this work, in order to have the same test scenario, static allocation of radio resources is used; therefore, user data are sent over two subcarriers (SC) and each subcarrier carries data from three users. As is shown in Figure 2a SC1 carries the data of users 1, 3, and 5. If user 1 transmits codeword 000, this codeword could be combined with the 8 codewords of user 3 and the 8 codewords of user 5, producing an IQ constellation of 64 possible points like the one shown in Figure 2b. The full constellation of 512 points appears in Figure 2c.

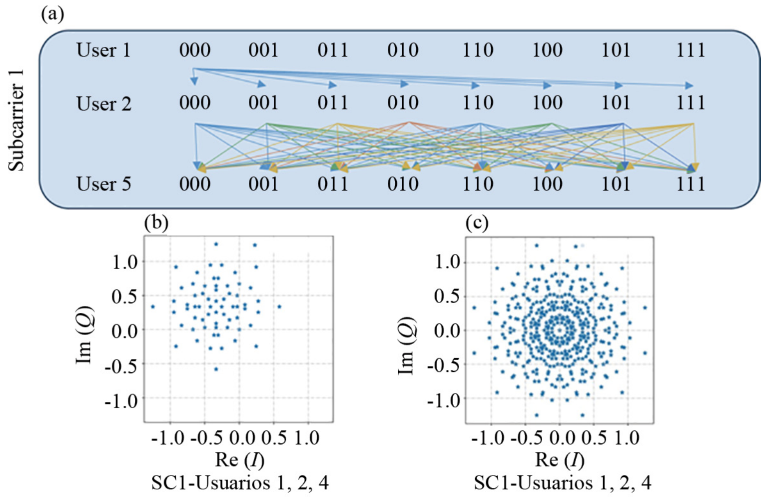

Figure 2. (a) SC1 carries data from users 1, 3, and 5; (b) *IQ* constellation for 64 points; (c) *IQ* constellation for 512 points

Once the user data is assigned to the radio subcarriers, it is transmitted over the air, so the signal received at the base station $y_k \in Y$ is described by (10):

$$y_k = \sum_{j \in J} x_{kj} + g_k \tag{10}$$

where: $x_{kj} \in X$ is the signal transmitted by the mobile device with k radio subcarriers, j users and $g_k \in G$ is the SNR channel matrix with normal distribution (from -20 dB up to +20 dB).

6. Pre-processing the data received with the PINV

Considering a matrix A of dimensions $n \times m$, and $n \neq m$ the SVD [16–20] of matrix A is defined by (11):

$$A_{n\times m} = U\Sigma V^* \tag{11}$$

So, the PINV of a matrix $A^\dagger$ [16–18] turns out to be:

$$A^\dagger_{m\times n} = V\Sigma^\dagger U^* \tag{12}$$

where: $\Sigma^\dagger = diag\left(\sigma_1^{-1},\ \ldots,\ \sigma_r^{-1},\ 0,\ \ldots,\ 0\right) = \begin{pmatrix} \Sigma_r^{-1} & 0 \\ 0 & 0 \end{pmatrix}$.

As can be seen, SVD is closely related to PINV [19], since it is possible to calculate PINV from singular values. Once the signal Y was received at the base station, the PINV is applied as shown below.

$$F \ = \ XY^\dagger \tag{13}$$

where:

X is the transmitted signal with k radio subcarriers and j users.

$Y^{\dagger}$ is the PINV of the received signal Y.

$Y = X + G$ is the received signal (original signal plus noise), where G is the noise matrix.

Therefore, the signal that is delivered to the machine learning module is given by:

$$FY = XY^{\dagger}Y. \tag{14}$$

The setup used in the machine learning module is described in the next section.

7. Signal decoding

As previously mentioned, SVD-SCMA codebooks are 4×8 matrices, the rows represent the subcarriers used, each codebook occupies only 2 subcarriers to transmit its data and two subcarriers are not used by that user, the above gives the codebooks the sparsity characteristic, which is exploited by MPA-based detectors [21, 22], which is an iterative decoding algorithm, which factors the global function of many variables into the product of simpler local functions, whose arguments are the subset of variables. The computational complexity of MPA can be high, especially when the system allows overload. Machine Learning (ML) has been used both in the decoding of optical communications signals [23, 24], as well as in SCMA signals from B5G communications [10, 12], this method can reduce the computational complexity with respect to MPA although it requires a training stage for the ML module to make predictions on new data. Both methods (MPA and ML) perform their tasks by taking the received data directly without pre-processing the signals. In this work, the properties of PINV are applied to the received signals to reduce the effects produced by the wireless channel.

The decoding module uses neural networks with supervised machine learning. The ML module has 4 inputs (1 for each subcarrier) and 8 outputs (1 for each three-bit codeword). The configuration parameters of the neural network used are shown in Table 1.

Table 1. Neural network configuration parameters used for the decoding module

Inputs	4 (Subcarriers)
Outputs	8 (3 bits)
Hidden dense layers	6 (1,024, 512, 512, 256, 64 and 32 neurons)
Optimization algorithm	Adam
Batch size	50
Validation interval	0.08
β_1	0.9
β_2	0.999
learning rate of $\mathbf{l_r}$	0.002
Activation function	Relu (except for the last layer where SOFTMAX was used)
Learning precision during the training stage	95%

The training data set was 100,000, of which 80% was used to train the neural network and 20% for validation. The neural network was trained with data that is affected by Rayleigh fading with an Eb/No between -20 dB and +20 dB and with 100 iterations to complete the training stage.

8. Simulation results and discussion

To verify the performance of the receiver at the base station, one million new data were generated randomly, which were pre-processed by the PINV to later feed the neural network for data decoding. Figure 3 shows the reference constellation and the received data (Considering the first 64 points of the IQ constellation and $Eb/No = 9$).

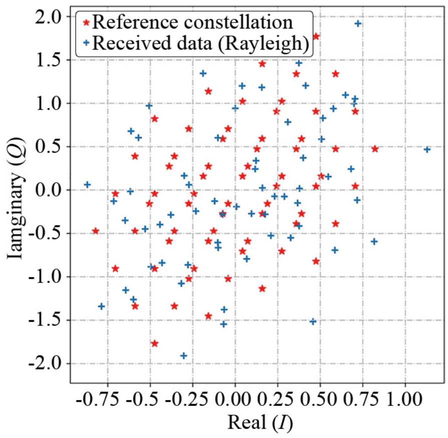

Figure 3. Reference IQ constellation and received data (Eb/No = 9)

For easy data visualization, Figure 4 shows the first 16 data considering the reference constellation, the received data, and the data pre-processed with the Moore-Penrose pseudo-inverse.

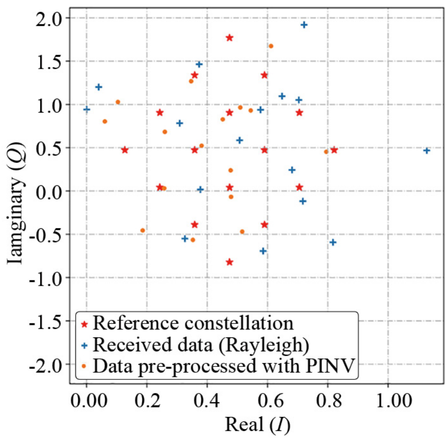

Figure 4. Reference constellation, received data and preprocessed data (Eb/No = 9)

From Figure 4, it can be observed that the data that were pre-processed with PINV appear closer to the reference constellation than those data that were not processed with PINV. For performance evaluation of a communications system, the bit error rate (BER) is frequently used, in this work, the error vector magnitude (EVM) is also used. Figure 5 shows the results of the BER analysis.

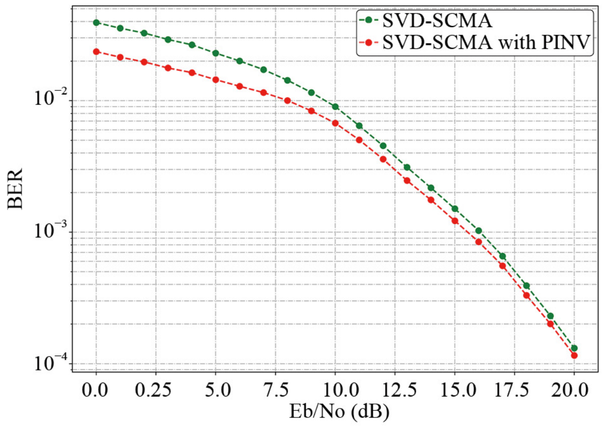

Figure 5. BER analysis when PINV is used to preprocess data and when PINV is not used

From the Figure 5, it is observed that preprocessing the received data with the PINV improves the BER between 2 dB and 3 dB, compared to not using the PINV. Figure 6 shows the BER analysis for the SVD-SCMA and phase rotation coding methods when PINV is not used at reception and when it is used.

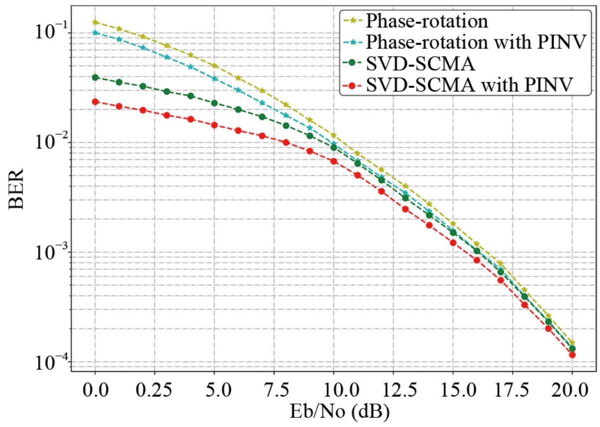

Figure 6. BER analysis when SVD-SCMA and Phase Rotation methods are used to encode the signals and when PINV is used to preprocess data and when PINV is not used

Related to the phase rotation method, from Figure 6 it is observed that using PINV improves the BER by approximately 1 dB when having $Eb/No = 0$ dB. When $Eb/No = +10$ dB, the phase rotation performance resembles that of SCMA-SVD.

If the received data is far from the IQ constellation reference point, that symbol could be decoded incorrectly. The greater the distance to the reference point, the greater the probability of bit errors. The most common way to quantify the distance between the received point and the reference point is the error vector magnitude (EVM). The higher the EVM value, the greater the distance between the received data and the reference.

The EVM magnitude is reported in terms of percentage or decibels (dB), comparing the error vector magnitude against the maximum power of the constellation, or concerning the root mean square (RMS) power. Considering the same set of data used for the BER analysis, Figure 7 shows the maximum, minimum, and average percentage values in relation to the Eb/No. Lower percentage values indicate greater demodulation accuracy.

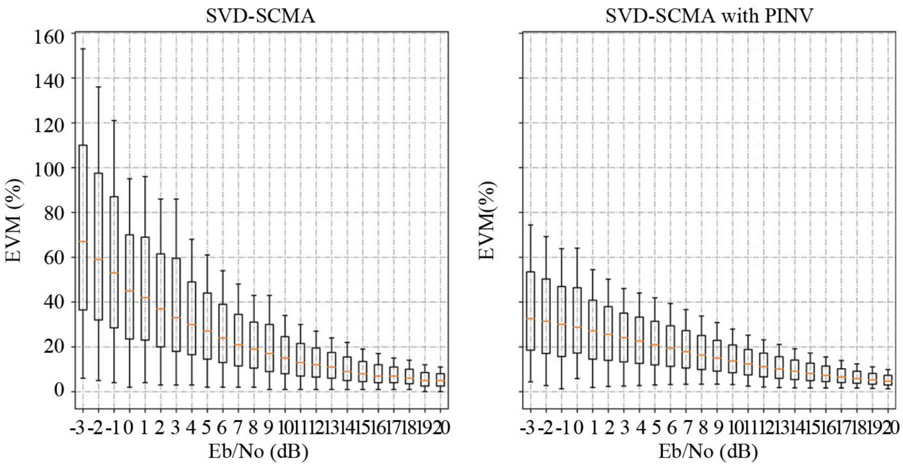

Figure 7. EVM analysis (in percentage) when PINV is used to preprocess data and when PINV is not used. (Maximum, minimum and average EVM values are shown)

Considering $\frac{Eb}{No} = 0$, when PINV is used the average EVM is equal to 28.2%, and 45.3% when not used. Figure 8 shows the maximum, minimum, and average values in terms of dB. Lower (more negative) dB values indicate better EVM.

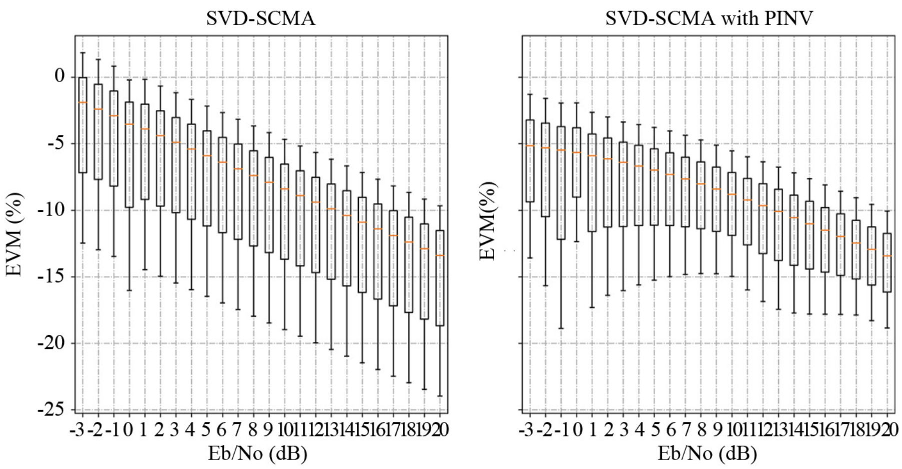

Figure 8. EVM analysis (in dB) when PINV is used to preprocess data and when PINV is not used. (Maximum, minimum and average EVM values are shown)

Considering the above Figure, there is a 2 dB difference in the average EVM when PINV is used, compared to not using it $\left(considering\ \frac{Eb}{No} = 0\right)$, while the average EVM is -5.66 dB when PINV is used, when not using it is -3.53 dB. Finally, Figure 9 shows the comparison of average EVM both in percentage and dB.

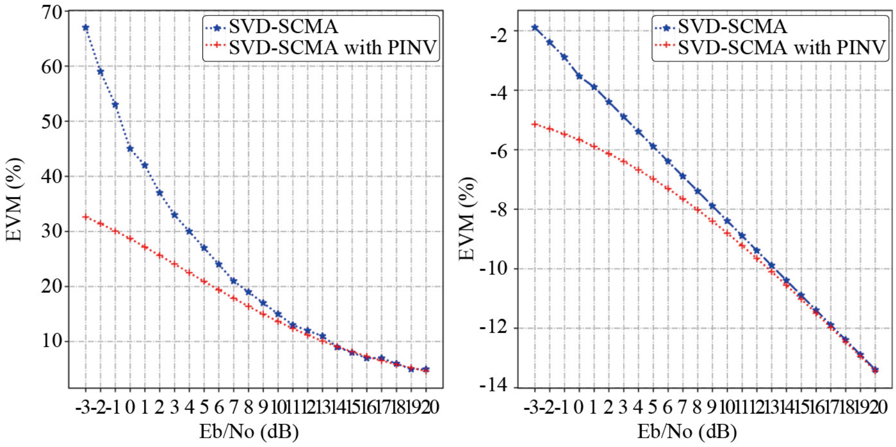

Figure 9. EVM analysis (in percentage and dB) when PINV is used to preprocess data and when PINV is not used (average EVM values are shown)

It can be observed in the image above that when channel conditions are adverse, the average EVM has a significant improvement when PINV is used to pre-process the received data.

One of the most well-known applications of the Moore-Penrose pseudo-inverse (PINV) is related to the least squares method, however, the Moore-Penrose PINV is also used in other areas of scientific research such as robotics, inverse problems, digital image restoration, fuzzy optical imaging, and neural networks [16–19, 25]. This work hypothesizes that pre-processing the received data with PINV can improve data decoding in 5G and B5G systems. Figure 10 shows the distance between the first four points of the reference IQ constellation and the data with and without PINV pre-processing.

Figure 10 shows that the pre-processed data with the PINV are closer to the reference constellation than the data that were not pre-processed, which results, as shown in Figures 5-9, in improved BER and EVM compared to decoding methods such as MPA [21, 22] or ML [10, 12], which do not perform pre-processing of the information before passing it to the decoding stage.

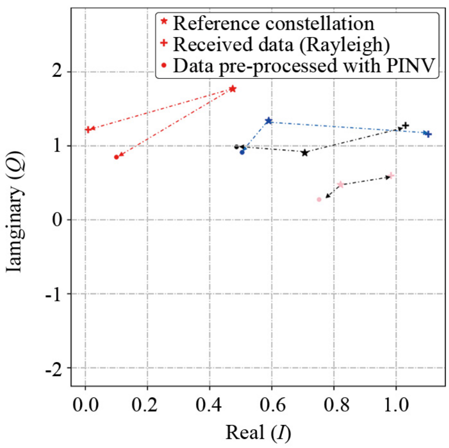

Figure 10. Distance between the reference IQ constellation points and the received data points with and without pre-processing

As shown in [10] when SVD is used to encode user data, the Euclidean distance between the IQ constellation points is improved which allows the decoding of signals to be more efficient even under adverse channel conditions. Using SVD makes it possible to increase the spectral efficiency from two to three bits per symbol using only one antenna. In [26, 27] the spectral efficiency is increased by using antenna arrays using SCMA or Phase Rotation. When channel conditions are adverse, SVD-SCMA [10] is more efficient, in terms of BER analysis, than other NOMA techniques such as Phase Rotation, SM-SCMA and RGSM-SCMA. As far as the receiver is concerned, applying PINV to pre-process the information before passing it to the decoding module is not used by other NOMA techniques (Phase Rotation, SM-SCMA or RGSM-SCMA), and these techniques use MPA as decoder and not ML. Therefore, the efficiency of using PINV, in terms of BER analysis and EVM analysis, is better than its predecessors.

9. Conclusions

The SVD and the PINV have been applied in several areas of engineering, in this work we propose to use the PINV to pre-process the data received from a B5G communications signal in the uplink. The results obtained in this work show that applying the PINV in the reception of signals that have been previously encoded by SVD improves the uplink BER between 2 dB and 3 dB. PINV improves signal decoding even in systems that do not use SVD as an encoding method, however, its best performance is obtained by using SVD in signal encoding at the transmitter and using PINV in signal decoding at the receiver at the base station.

When PINV is used under adverse channel conditions (-3 dB $< Eb/No < +10$ dB), the EVM decreases noticeably compared to when PINV is not used. Considering that $\frac{Eb}{No} = -3$ dB, when using PINV an EVM of 32% is obtained, while when not using it an EVM = 66% is obtained, in this case the magnitude of the error vector is reduced by half. Finally, it is concluded that the results obtained indicate that using PINV to pre-process the received data produces a decrease in both the BER and the EVM mainly when the channel conditions are adverse.

The next step of this research is to encode more than three bits using SVD-SCMA at the transmitter using spatial modulation (SM) or MIMO technologies, keeping the PINV pre-processing at the receiver and using GPU to accelerate the decoding of the received signals.

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Data availability

Data is available on request. The data presented in this study are available on request from the corresponding author. The data are not publicly available because this work is part of an ongoing funded project.

Conflicts of interest

The authors declare no conflict of interest.

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