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Research Article

A New Higher-Order Scheme for Multiple Roots with Unknown Multiplicity

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Abstract: In this work, we propose a new efficient iterative method to find multiple roots of nonlinear equations with unknown multiplicity n. The new scheme is free from the second derivative and consists of three steps derived from Enhanced Halley's method introduced by the contributors and a Newton step. To increase efficiency, the first derivatives were approximated using forward difference, central difference, and Hermite interpolation techniques. It is demonstrated that the implemented method achieves sixth order of convergence. As an application, we apply the new method to chemical engineering problem (volume from van der Waals), biomedical engineering (blood rheology model) and ten academic problems. Comparisons and examples clarify that the new method outperforms existing methods with the same order of convergence.

Keywords: multiple roots, nonlinear equations, unknown multiplicity, high-order convergence, iterative method

MSC: 65B99, 65H04

1. Introduction

Finding approximate solutions is the only way to resolve most of real-world applications, such as chemical engineering problems [1–3], fluid mechanics [4, 5] and mathematical modeling [6, 7]. Consequently, finding a solution to the non-linear equation $f(x) = 0$, is a fundamental inquiry. One of the basic and important methods for solving such equations is Newton method [8]

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}, \tag{1}$$

which converges quadratically to the simple root. A lot of researchers endeavor to modify Newton's method to achieve greater precision and a higher degree of convergence, for example, [9–12]. Very recently, Solaiman and Hashim [13]

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presented an optimal eighth-order iterative scheme of three steps using modified Halley's method and Newton's method. All of these modifications of different methods and techniques are only applicable for simple roots, not multiple roots.

The conventional Newton's method exhibits linear convergence when locating multiple roots, prompting Schroder's proposal of the modified Newton method [14]

$$x_{i+1} = x_i - n\frac{f(x_i)}{f'(x_i)}, \tag{2}$$

which achieves second-order convergence but necessitates prior knowledge of the root multiplicity n. In case of known multiplicity n, numerous methods with optimal higher-order have been introduced in the scholarly literature, for instance, see [15–18] and non-optimal see [19–22]. Conversely, in cases where the multiplicity n is not known explicitly, Traub [8] proposed a straightforward transformation:

$$F(x) = \begin{cases} \dfrac{f(x)}{f'(x)}, & \text{if } f(x) \neq 0, \\ \\ 0, & \text{if } f(x) = 0. \end{cases} \tag{3}$$

Applying this transformation to Newton's method in equation (1), we obtain:

$$x_{i+1} = x_i - \frac{f(x_i)\, f'(x_i)}{f'(x_i)^2 - f(x_i)\, f''(x_i)}. \tag{4}$$

This method transformed the nonlinear equation with multiple roots into a simple root problem. Since this transformation (3) preserves the original order of convergence, so scheme (4) has a second order of convergence as well. Despite the independent knowledge of the multiplicity, scheme (4) has limited efficiency, as it requires the evaluation of three nonlinear functions per iteration $f(x)$, $f'(x)$ and $f''(x)$.

To overcome this problem, some scholars proposed derivative-free transformations with unknown multiplicity to develop more efficient multiple root finders. King [23] introduced secant method of order 1.618 for finding multiple roots of nonlinear equations, by using another free-derivative transformation. This method requires only two functional evaluations and has higher efficiency than the method in equation (4). Parida and Gupta [24] proposed an alternative transformation which simplifies the task of solving multiple zeros to that of solving a simple zero. They obtained a quadratically convergent derivative-free Newton-like iterative method. Yun [25] has implemented a new Steffensen-type iterative method based on another transformation, which does not require any derivatives of $f(x)$. Yun showed that the proposed scheme is second-order convergence. By using transformation (3), Qudsi et al. [26] established sixth order iterative method, based on three-step Newton-like method. They approximated the derivatives by using central difference and divided difference. Sharma and Bahl [27] developed a transformation method of sixth order. More recently, Sharma and Bahl [28] have improved the efficiency of the method in [27] using divided differences of order one and two. Although the method in [27] requires fewer function evaluations (a total of 4) compared to the method in [28] (which requires a total of 5), the method in [28] still achieved greater efficiency.

Construction of iterative schemes that do not essentially require the value of the second-order derivative in case multiplicity n is not known explicitly is one of the difficult issues in the region of numerical analysis. Some high-order variants and modifications of Newton's method that can solve multiple roots are proposed and studied in the literature. Out of them, only three iterative methods by [26, 27, 29] have a high order of convergence that satisfies the above requirements.

Therefore, we have limited methods that are capable of solving multiple roots with unknown multiplicity and are free of second derivatives.

Inspired by the ongoing efforts in this direction, we introduce a new transformation method of order six for solving multiple roots without prior knowledge of multiplicity and does not depend on second derivative evaluation. The paper is organized as follows. In Section 2, the new algorithm is derived and explained. In Section 3, the order of convergence is analyzed. In Section 4, a variety of numerical tests are conducted to compare the performance of the new algorithm with the same order transformation methods. Finally, conclusion of the research work is presented in Section 5.

2. The new method

We consider the following iteration scheme proposed by [30]:

$$\begin{cases} y_i = x_i - \dfrac{f(x_i)}{f'(x_i)}, \\ \\ x_{i+1} = y_i - \dfrac{f(y_i)}{f'(y_i)} - \dfrac{2(f(y_i))^2 f'(y_i) Q(x_i, y_i)}{4(f'(y_i))^4 - 4f(y_i)(f'(y_i))^2 Q(x_i, y_i) + (f(y_i))^2 (Q(x_i, y_i))^2}, \end{cases} \tag{5}$$

where $Q(x_i, y_i)$ is Hermite's approximation of the second derivative $f''(y_i)$

$$f''(y_i) \approx \frac{2}{x_i - y_i}\left[3\frac{f(x_i) - f(y_i)}{x_i - y_i} - 2f'(y_i) - f'(x_i)\right] = Q(x_i, y_i). \tag{6}$$

Said Solaiman et al. [13] named Algorithm (5) as enhanced Hally's method (MH2). They derived MH2 by using Taylor's expansion of $f(x)$, incorporating Newton's and Halley's methods. Method of [30] has sixth order of convergence. Furthermore, it is suitable exclusively for solving non-linear equations with simple roots.

Now to develop an efficient method for solving non-linear equations with multiple roots without the knowledge of its multiplicity n. We consider the transformation (3) and the composition of algorithm (5) with Newton's method:

$$\begin{cases} y_i = x_i - \dfrac{F(x_i)}{F'(x_i)}, \\ \\ w_i = y_i - \dfrac{F(y_i)}{F'(y_i)} - \dfrac{2(F(y_i))^2 F'(y_i) Q(x_i, y_i)}{4(F'(y_i))^4 - 4F(y_i)(F'(y_i))^2 Q(x_i, y_i) + (F(y_i))^2 (Q(x_i, y_i))^2}, \\ \\ x_{i+1} = w_i - \dfrac{F(w_i)}{F'(w_i)}. \end{cases} \tag{7}$$

To avoid computing the first derivatives of the function $F'(x_i)$, $F'(y_i)$ and $F'(w_i)$ we approximate them by using forward difference, central difference, and Hermite's interpolating respectively as follows:

$$F'(x_i) \approx \frac{F(x_i + F(x_i)) - F(x_i)}{F(x_i)} = S(x_i), \tag{8}$$

$$F'(y_i) \approx \frac{F(y_i + F(y_i)) - F(y_i - F(y_i))}{2F(y_i)} = A(y_i), \tag{9}$$

$$F'(w_i) \approx F[w_i, x_i]\left(2 + \frac{x_i - w_i}{y_i - w_i}\right) - \frac{(x_i - w_i)^2}{(x_i - y_i)(y_i - w_i)} F[x_i, y_i] + S(x_i)\frac{y_i - w_i}{x_i - y_i} = R(w_i) \tag{10}$$

(see [13] for the detailed discussions of (10)).

Replacing the approximation from (8), (9) and (10) in Algorithm (7) and (6) respectively, we propose the following new iterative method:

$$\begin{cases} y_i = x_i - \dfrac{F(x_i)}{S(x_i)}, \\ w_i = y_i - \dfrac{F(y_i)}{A(y_i)} - \dfrac{2(F(y_i))^2 A(y_i) Q(x_i, y_i)}{4(A(y_i))^4 - 4F(y_i)(A(y_i))^2 Q(x_i, y_i) + (F(y_i))^2 (Q(x_i, y_i))^2}, \\ x_{i+1} = w_i - \dfrac{F(w_i)}{R(w_i)}. \end{cases} \tag{11}$$

We refer to scheme (11) the new Multiple roots method, denoted MR, which demonstrates sixth-order convergence, as detailed in the next section.

3. Convergence analysis

In this section, we will demonstrate the convergence properties by utilizing Mathematica code and proving the following convergence theorem of the developed Algorithm 11.

Theorem 1 Assuming $F \in C^2(L)$ (where $L \subseteq \mathbb{R} \to \mathbb{R}$) has a simple zero $\alpha \in L$, with L being an open interval. If the initial guess x_0 is close enough to α, then the proposed iterative scheme given by (11) is of sixth-order convergence.

Proof. We suppose that $f(x)$ can be written as

$$f(x) = (x - \alpha)^n h(x), \tag{12}$$

where α is a multiple root of equation (12) with multiplicity n, $h(x)$ is a continuous function with $h(\alpha) \neq 0$. Differentiating f with respect to x, we have

$$f'(x) = n(x - \alpha)^{n-1} h(x) + (x - \alpha)^n h'(x). \tag{13}$$

Dividing (12) by (13), we obtain

$$F(x)=\frac{f(x)}{f'(x)}=\frac{(x-\alpha)h(x)}{nh(x)+(x-\alpha)h'(x)}. \tag{14}$$

Therefore, the problem of computing multiple zero of $f(x)=0$ has been transformed into an equivalent problem of computing a simple zero α of $F(x)=0$.

Let $e_i=x_i-\alpha$. Using Taylor's expansion about $x_i=\alpha$, we have

$$h(x_i)=h(\alpha)\left[1+A_1e_i+A_2e_i^2+A_3e_i^3+A_4e_i^4+A_5e_i^5+A_6e_i^6+A_7e_i^7+O(e_i^8)\right], \tag{15}$$

$$h'(x_i)=h(\alpha)\left[A_1+2A_2e_i+3A_3e_i^2+4A_4e_i^3+5A_5e_i^4+6A_6e_i^5+7A_7e_i^6+O(e_i^7)\right], \tag{16}$$

where $A_k=\dfrac{h^{(k)}(\alpha)}{k!h(\alpha)}$, $k=1, 2, \ldots$. Using (14), (15) and (16) and simplifying, we get

$$F(x_i)=C_1e_i+C_2e_i^2+C_3e_i^3+C_4e_i^4+C_5e_i^5+C_6e_i^6+O(e_i^7), \tag{17}$$

where

$$C_1=\frac{1}{n},$$

$$C_2=\frac{-A_1}{n^2},$$

$$C_3=\frac{1}{n^3}\left[A_1^2(1+n)-2A_2n\right],$$

$$C_4=\frac{1}{n^4}\left[-A_1^3(1+2n+n^2)+A_1A_2(4n+3n^2)-3A_3n^2\right],$$

$$C_5=\frac{1}{n^5}\left[A_1^4(1+3n+3n^2+n^3)\right.$$

$$\left.-A_1^2A_2(6n+10n^2+4n^3)+A_2^2(4n^2+2n^3)+A_1A_3(6n^2+4n^3)-4A_4n^3\right],$$

$$C_6=\frac{1}{n^6}[-A_1^5(1+4n+6n^2+4n^3+n^4)+A_1^3A_2(8n+21n^2+18n^3+5n^4)$$

$$-A_1A_2^2(12n^2+16n^3+5n^4)-A_1^2A_3(9n^2+14n^3+5n^4)+A_2A_3(12n^3+5n^4)$$

$$+A_1A_4(8n^3+5n^4)-5A_5n^4].$$

Substituting (17) in (8) and using geometric series of $F(x_i)$ and $F(x_i+F(x_i))$, we get

$$S(x_i)=\frac{1}{n}+B_1e_i+B_2e_i^2+B_3e_i^3+B_4e_i^4+B_5e_i^5+B_6e_i^6+O(e_i^7), \tag{18}$$

where

$$B_1=\frac{1}{n^3}[-(2n+1)A_1],$$

$$B_2=\frac{1}{n^5}[(3n^3+6n^2+5n+1)A_1^2-2n(3n^2+3n+1)A_2],$$

$$B_3=\frac{1}{n^7}[(2n+1)(2n^4+6n^3+9n^2+6n+1)A_1^3-n(6n^3+14n^2+15n+4)A_1A_2$$
$$+3n^2(2n^2+2n+1)A_3],$$

$$B_4=\frac{1}{n^9}[(5n^7+25n^6+65n^5+100n^4+90n^3+45n^2+11n+1)A_1^4$$
$$-n(20n^6+90n^5+203n^4+252n^3+165n^2+52n+6)A_1^2A_2$$
$$+n^2(20n^5+70n^4+121n^3+104n^2+43n+6)A_1A_3$$
$$+2n^2(5n^5+20n^4+36n^3+29n^2+11n+2)A_2^2-4n^3(5n^4+10n^3+10n^2+5n+1)A_4],$$

$$B_5=\frac{1}{n^{11}}[-(n+1)^2(6n^7+27n^6+76n^5+120n^4+110n^3+53n^2+12n+1)A_1^5$$
$$+n(30n^8+183n^7+585n^6+1{,}137n^5+1{,}365n^4+1{,}000n^3+424n^2+93n+8)A_1^3A_2$$
$$-n^2(30n^7+159n^6+435n^5+693n^4+651n^3+348n^2+92n+9)A_1^2A_3$$
$$-n^2(30n^7+171n^6+488n^5+791n^4+726n^3+373n^2+104n+12)A_1A_2^2$$
$$+n^3(30n^6+147n^5+334n^4+375n^3+228n^2+77n+12)A_2A_3$$
$$+n^3(30n^6+123n^5+264n^4+315n^3+210n^2+69n+8)A_1A_4$$
$$-5n^4(6n^5+15n^4+20n^3+15n^2+6n+1)A_5],$$

$$B_6 = \frac{1}{n^{13}}[(n+1)^2(7n^9 + 42n^8 + 154n^7 + 350n^6 + 511n^5 + 476n^4 + 272n^3 + 89n^2 + 15n + 1)A_1^6$$

$$-n(42n^{10} + 322n^9 + 1{,}328n^8 + 3{,}498n^7 + 6{,}115n^6 + 7{,}186n^5 + 5{,}627n^4 + 2{,}846n^3 + 875n^2$$

$$+146n + 10)A_1^4A_2 + n^2(42n^9 + 294n^8 + 1{,}092n^7 + 2{,}526n^6 + 3{,}775n^5 + 3{,}676n^4 + 2{,}273n^3$$

$$+832n^2 + 159n + 12)A_1^3A_3 + n^2(63n^9 + 462n^8 + 1{,}776n^7 + 4{,}212n^6 + 6{,}342n^5$$

$$+6{,}112n^4 + 3{,}717n^3 + 1{,}377n^2 + 282n + 24)A_1^2A_2^2 - n^3(84n^8 + 560n^7 + 1{,}888n^6$$

$$+3{,}728n^5 + 4{,}447n^4 + 3{,}248n^3 + 1{,}437n^2 + 356n + 36)A_1A_2A_3 - n^3(42n^8 + 252n^7$$

$$+804n^6 + 1{,}560n^5 + 1{,}909n^4 + 1{,}456n^3 + 641n^2 + 142n + 12)A_1^2A_4$$

$$-2n^3(7n^8 + 49n^7 + 173n^6 + 355n^5 + 423n^4 + 299n^3 + 129n^2 + 32n + 4)A_2^3$$

$$+n^4(42n^7 + 196n^6 + 500n^5 + 760n^4 + 701n^3 + 372n^2 + 101n + 10)A_1A_5$$

$$+2n^4(21n^7 + 119n^6 + 325n^5 + 473n^4 + 403n^3 + 205n^2 + 59n + 8)A_2A_4$$

$$+3n^4(7n^7 + 42n^6 + 116n^5 + 164n^4 + 135n^3 + 70n^2 + 22n + 3)A_2^3$$

$$-6n^5(7n^6 + 21n^5 + 35n^4 + 35n^3 + 21n^2 + 7n + 1)A_6].$$

Using (17) and (18) in first substep of (11), we obtain

$$\tilde{e}_i = y_i - \alpha = M_2e_i^2 + M_3e_i^3 + M_4e_i^4 + M_5e_i^5 + M_6e_i^6 + \mathcal{O}(e_i^7), \tag{19}$$

where

$$M_2 = \frac{1}{n^2}[-(n+1)A_1],$$

$$M_3 = \frac{1}{n^3}[(2n^2 + 3n + 2)A_1^2 - (4n^2 + 6n + 2)A_2],$$

$$M_4 = \frac{1}{n^4}[-(3n^3+5n^2+6n+3)A_1^3+(9n^3+16n^2+16n+5)A_1A_2$$

$$-(9n^3+18n^2+12n+3)A_3],$$

$$M_5 = \frac{2}{n^5}[(2n^4+3n^3+5n^2+5n+2)A_1^4-(8n^4+13n^3+19n^2+15n+4)A_1^2A_2$$

$$+(4n^4+6n^3+6n^2+n-1)A_2^2+(8n^4+17n^3+23n^2+16n+5)A_1A_3$$

$$-(8n^4+20n^3+20n^2+10n+2)A_4],$$

$$M_6 = \frac{1}{n^6}[-(5n^5+5n^4+11n^3+18n^2+15n+5)A_1^5+(25n^5+28n^4+52n^3$$

$$+71n^2+46n+10)A_1^3A_2-(25n^5+27n^4+36n^3+29n^2-2n-9)A_1A_2^2$$

$$-(25n^5+42n^4+72n^3+85n^2+58n+20)A_1^2A_3$$

$$+(25n^5+63n^4+108n^3+115n^2+70n+17)A_1A_4$$

$$+(25n^5+33n^4+28n^3-3n^2-18n-7)A_2A_3$$

$$-(25n^5+75n^4+100n^3+75n^2+30n+5)A_5].$$

Moreover in (14), expanding $h(y_i)$ and $h'(y_i)$ about α and using (19), we get

$$F(y_i) = D_2e_i^2+D_3e_i^3+D_4e_i^4+D_5e_i^5+D_6e_i^6+O(e_i^7), \tag{20}$$

where

$$D_2 = \frac{1}{n^3}[-(n+1)A_1],$$

$$D_3 = \frac{1}{n^4}[(2n^2+3n+2)A_1^2-2(2n^2+3n+1)A_2],$$

$$D_4 = \frac{1}{n^6}[-(3n^4+5n^3+7n^2+5n+1)A_1^3 + n(9n^3+16n^2+16n+5)A_1A_2$$

$$-3n(3n^3+6n^2+4n+1)A_3],$$

$$D_5 = \frac{1}{n^7}[2(2n^5+3n^4+7n^3+10n^2+7n+2)A_1^4 - 2(8n^5+13n^4+23n^3+25n^2+12n+2)A_1^2A_2$$

$$+2n(8n^4+17n^3+23n^2+16n+5)A_1A_3 + 2n(4n^4+6n^3+6n^2+n-1)A_2^2$$

$$-4n(4n^4+10n^3+10n^2+5n+1)A_4],$$

$$D_6 = \frac{1}{n^9}[-(5n^7+5n^6+21n^5+47n^4+58n^3+41n^2+14n+1)A_1^5$$

$$-n(25n^6+27n^5+52n^4+77n^3+50n^2+15n+4)A_1A_2^2$$

$$+n(25n^6+28n^5+86n^4+171n^3+176n^2+94n+20)A_1^3A_2$$

$$-n(25n^6+42n^5+90n^4+139n^3+118n^2+50n+6)A_1^2A_3$$

$$+n^2(25n^5+33n^4+28n^3-3n^2-18n-7)A_2A_3$$

$$+n^2(25n^5+63n^4+108n^3+115n^2+70n+17)A_1A_4$$

$$-5n^2(5n^5+15n^4+20n^3+15n^2+6n+1)A_5].$$

Now, using (20) and (19) in (9) and simplifying, we get

$$A(y_i) = \frac{F(y_i+F(y_i)) - F(y_i-F(y_i))}{2F(y_i)} = \frac{1}{n} + K_2e_i^2 + K_3e_i^3 + K_4e_i^4 + K_5e_i^5 + K_6e_i^6 + O(e_i^7), \tag{21}$$

where

$$K_2 = \frac{1}{n^4}[2(n+1)A_1^2],$$

$$K_3 = \frac{1}{n^5}[A_1^3(-4-6n-4n^2) + A_1A_2(4+12n+8n^2)],$$

$$K_4 = \frac{1}{n^9}[A_1^2 A_2(-2n - 4n^2 - 18n^3 - 44n^4 - 38n^5 - 18n^6)$$

$$+ A_1^4(1 + 3n + 6n^2 + 16n^3 + 21n^4 + 13n^5 + 6n^6)$$

$$+ A_1 A_3(6n^3 + 24n^4 + 36n^5 + 18n^6)],$$

$$K_5 = \frac{1}{n^{10}} A_1^2 A_3(-20n^3 - 64n^4 - 92n^5 - 68n^6 - 32n^7)$$

$$+ A_1^5(-4 - 14n - 32n^2 - 64n^3 - 84n^4 - 62n^5 - 24n^6 - 8n^7)$$

$$+ A_1^3 A_2(4 + 28n + 68n^2 + 148n^3 + 244n^4 + 220n^5 + 100n^6 + 32n^7)$$

$$+ A_1 A_2^2(-8n - 32n^2 - 60n^3 - 116n^4 - 144n^5 - 72n^6 - 16n^7)$$

$$+ A_4(8n^3 + 40n^4 + 80n^5 + 80n^6 + 32n^7).$$

(refer to Appendix for full computation of K_6).

Substituting equations (17)-(21) in (6) we obtain

$$Q(x_i, y_i) = \frac{2}{n^3} A_1(n-1) + Q_1 e_i + Q_2 e_i^2 + Q_3 e_i^3 + Q_4 e_i^4 + Q_5 e_i^5 + Q_6 e_i^6 + O(e_i^7), \tag{22}$$

where

$$Q_1 = \frac{1}{n^5}[A_1^2(-4 - 12n - 6n^2) + A_2(4n + 12n^2)],$$

$$Q_2 = \frac{1}{n^7}[A_1 A_2(-16n - 74n^2 - 96n^3 - 32n^4 - 6n^5)$$

$$+ A_1^3(6 + 36n + 60n^2 + 40n^3 + 10n^4 + 2n^5)]$$

$$+ A_3(6n^2 + 24n^3 + 36n^4 + 6n^5),$$

$$Q_3 = \frac{1}{n^9}[A_1A_3(-24n^2 - 140n^3 - 304n^4 - 284n^5 - 92n^6 - 16n^7)$$

$$+A_2^2(-16n^2 - 92n^3 - 180n^4 - 120n^5 - 8n^6 - 8n^7)$$

$$+A_1^4(-12 - 84n - 246n^2 - 352n^3 - 266n^4 - 100n^5 - 16n^6 - 4n^7)$$

$$+A_1^2A_2(44n + 276n^2 + 676n^3 + 720n^4 + 348n^5 + 60n^6 + 16n^7)$$

$$+A_4(8n^3 + 40n^4 + 80n^5 + 80n^6 + 16n^7)],$$

$$Q_4 = \frac{1}{n^{11}}[A_1^3A_2(-88n - 778n^2 - 2{,}756n^3 - 5{,}126n^4 - 5{,}318n^5 - 3{,}138n^6 - 974n^7 - 120n^8 - 30n^9)$$

$$+A_2A_3(-48n^3 - 322n^4 - 876n^5 - 1{,}146n^6 - 632n^7 - 66n^8 - 30n^9)$$

$$+A_1^5(14 + 152n + 644n^2 + 1{,}460n^3 + 1{,}946n^4 + 1{,}580n^5 + 780n^6 + 212n^7 + 26n^8 + 6n^9)$$

$$+A_1^2A_3(54n^2 + 464n^3 + 1{,}486n^4 + 2{,}354n^5 + 1{,}926n^6 + 806n^7 + 148n^8 + 30n^9)$$

$$+A_5(10n^4 + 60n^5 + 150n^6 + 200n^7 + 150n^8 + 30n^9)$$

$$+A_1A_4(-32n^3 - 226n^4 - 620n^5 - 870n^6 - 632n^7 - 190n^8 - 30n^9)$$

$$+A_2^2(104n^2 + 704n^3 + 2{,}054n^4 + 3{,}116n^5 + 2{,}394n^6 + 872n^7 + 86n^8 + 30n^9)],$$

(refer to Appendix for full computation of Q_5 and Q_6).

Invocation of equations (19)-(22) in second substep of (11) leads to

$$\tilde{e}_i = w_i - \alpha = L_4e_i^4 + L_5e_i^5 + L_6e_i^6 + \mathcal{O}(e_i^7), \tag{23}$$

where

$$L_4 = \frac{1}{n^6}[A_1^3(1+n)^2],$$

$$L_5 = \frac{1}{n^8}[A_1^2A_2(-6n - 26n^2 - 34n^3 - 14n^4) + A_1^4(2 + 14n + 27n^2 + 22n^3 + 7n^4)],$$

and

$$L_6 = \frac{1}{n^{10}}[A_1A_2^2(-12n^2 - 80n^3 - 188n^4 - 184n^5 - 64n^6)$$

$$+ A_1^2A_3(-9n^2 - 48n^3 - 105n^4 - 105n^5 - 42n^6 - 3n^7)$$

$$+ A_1^5(-2 - 30n - 125n^2 - 230n^3 - 225n^4 - 121n^5 - 31n^6 - n^7)$$

$$+ A_1^3A_2(18n + 141n^2 + 410n^3 + 551n^4 + 369n^5 + 108n^6 + 3n^7)].$$

Again, expanding $h(w_i)$ and $h'(w_i)$ about α, and using in (14), we have

$$\tilde{e}_i = F(w_i) = N_4e_i^4 + N_5e_i^5 + N_6e_i^6 + \mathcal{O}(e_i^7), \tag{24}$$

where

$$N_4 = \frac{1}{n^7}[A_1^3(1+n)^2],$$

$$N_5 = \frac{1}{n^9}[A_1^2A_2(-6n - 26n^2 - 34n^3 - 14n^4) + A_1^4(2 + 14n + 27n^2 + 22n^3 + 7n^4)],$$

and

$$N_6 = \frac{1}{n^{11}}[A_1A_2^2(-12n^2 - 80n^3 - 188n^4 - 184n^5 - 64n^6)$$

$$+ A_1^2A_3(-9n^2 - 48n^3 - 105n^4 - 105n^5 - 42n^6 - 3n^7)$$

$$+ A_1^5(-2 - 30n - 125n^2 - 230n^3 - 225n^4 - 121n^5 - 31n^6 - n^7)$$

$$+ A_1^3A_2(18n + 141n^2 + 410n^3 + 551n^4 + 369n^5 + 108n^6 + 3n^7)].$$

Using equations (17)-(21), (23) and (24) we can write $R(w_i)$ in Algorithm (11) as

$$R(w_i) = N_4e_i^4 + N_5e_i^5 + N_6e_i^6 + \mathcal{O}(e_i^7), \tag{25}$$

Employing (23), (24) and (25) in third formula of (11), and implementing the computations by Mathematica [31] software, we have

$$e_{i+1} = -\frac{A_1^5(1+n)e^6}{n^{10}} + O[e]^7$$

The following code confirms the order of convergence using Mathematica

$$h[e\_] := dh[\alpha](1 + A_1 e + A_2 e^2 + A_3 e^3 + A_4 e^4 + A_5 e^5 + A_6 e^6 + A_7 e^7);$$

$$f[e\_] := \frac{e \times h[e]}{n \times h[e] + e \times h'[e]};$$

$$G[x\_, y\_] := \frac{(f[x] - f[y])}{(x - y)}; \text{ (*This is the finite difference*)}$$

$$S[e\_] := \frac{f[e + f[e]] - f[e]}{f[e]}; \text{ (*}F'(x)\text{ approximation*)}$$

$$A[y\_] := \frac{f[y + f[y]] - f[y - f[y]]}{2f[y]}; \text{ (*}F'(y)\text{ approximation*)}$$

$$R[x\_, y\_, w\_] := G[w, x]\left(2 + \frac{(w - x)}{(w - y)}\right) - \frac{(w - x)^2}{(w - y)(y - x)}G[x, y] + S[e]\frac{(w - y)}{(y - x)}; \text{ (*}F'(w)\text{ approximation*)}$$

$$Q[x\_, y\_] := (3G[y, x] - 2A[y] - S[e])\frac{2}{(x - y)}; \text{ (*}F''(y)\text{ approximation*)}$$

$$y = e - Series\left[\frac{f[e]}{S[e]}, e, 0, 6\right];$$

$$w = y - \frac{f[y]}{A[y]} - \frac{2f[y]^2 A[y] Q[e, y]}{4A[y]^4 - 4f[y]A[y]^2 Q[e, y] + f[y]^2 Q[e, y]^2};$$

$$e[n+1] = \text{Full Simplify}\left[w - \frac{f[w]}{R[e, y, w]}\right]$$

$$Out[9] = -\frac{A_1^5(1+n)e^6}{n^{10}} + O[e]^7$$

Hence, MR technique given in (11) is sixth-order of convergence. □

4. Applications and numerical results

In this section, we try to confirm the efficiency of the new sixth-order method (11). We present two types of examples when the nonlinear equation has a single multiplicity n of the roots. And when the multiplicity is the highest. We compare the efficiency of the new method with the transformed Newton method (NM) of the second order, sixth order (QM) method proposed by [26] and sixth order transformation methods for multiple roots (SH1) and (SH2) presented respectively by [27, 28].

We set $\varepsilon = 10^{-15}$ as the stopping criterion for the computer programs, where $|x_i - x_{i-1}| < \varepsilon$. The computations were conducted using Mathematica 14 with precision set to 10,000 significant digits. Tables 1-4 display the total number of iterations i required to satisfy the stopping criterion, x_i is the approximate root, the absolute difference between consecutive approximations $|x_i - x_{i-1}| < 10^{-15}$, $f(x_i)$ is the value of approximate zero, and (COC) indicates the computational order of convergence, which was suggested by [32] in the following approximated formula

$$\text{COC} \approx \frac{\ln\left|\dfrac{x_{i+1}-x_i}{x_i-x_{i-1}}\right|}{\ln\left|\dfrac{x_i-x_{i-1}}{x_{i-1}-x_{i-2}}\right|},$$

To do so, we consider the following test examples:

Example 1 (Volume from Van Der Waal's equation) Van der Waal's ideal gas equation is given by

$$\left(P + \frac{an^2}{V^2}\right)(V - nb) = nRT. \tag{26}$$

This equation shows the behavior of a real gas with two van Der Waals parameters a and b, which vary depending on the type of gas. Using these parameters a and b, we can calculate the values of P (pressure), T (absolute temperature) and n (number of moles). Therefore, equation (26) can be written as a nonlinear function to find the volume of gas (V) in terms of x as

$$f_1(x) = x^3 - \frac{261}{50}x^2 + \frac{3{,}633}{400}x - \frac{2{,}107}{400}.$$

The above equation has three roots; our desired root is a multiple root $x = 1.75$ of multiplicity $n = 2$. We considered the initial approximation $x_0 = 2$. Table 1 shows the results and comparisons obtained.

Table 1. Comparisons between different methods on test function $f_1(x)$

Method	i	x_i	$\lvert x_i - x_{i-1}\rvert$	$f(x_i)$	COC
		$f_1(x)$, $x_0 = 2$			
NM	7	1.7500000000000000000000	2.30E-24	2.34E-94	2
QM	4	1.7500000000000000000000	4.18E-57	1.28E-664	6
SH1	4	1.7500000000000000000000	3.38E-63	8.58E-739	6
SH2	4	1.7500000000000000000000	3.26E-40	1.16E-461	6
MR	4	1.7500000000000000000000	1.17E-71	2.28E-841	6

Example 2 (Blood rheology model [5]) The study of blood rheology frequently models blood as a "Casson Fluid", a non-Newtonian fluid, to accurately represent its behavior. According to this model, fluids such as blood and water flow through tubes with a velocity gradient near the walls and a central core that moves as a plug with minimal deformation. To analyze this, a nonlinear fractional equation is used to measure the flow rate drop, represented as

$$B = -\frac{x^4}{21} + \frac{4x}{3} - \frac{16\sqrt{x}}{7} + 1, \tag{27}$$

with $B = 0.40$, is used to compute the flow rate reduction. Using B value, we formulate equation (27) as a nonlinear function with multiple roots given by

$$f_2(x) = \left(\frac{x^8}{441} - \frac{8x^5}{63} - \frac{2{,}857{,}144{,}357x^4}{50{,}000{,}000{,}000} + \frac{16x^2}{9} - \frac{906{,}122{,}449x}{250{,}000{,}000} + \frac{3}{10}\right)^4.$$

One of the solutions of this equation is $x = 0.08643355805196205950555 7$. As a starting point, we selected $x_0 = 0.22$. The numerical comparisons are provided in Table 2.

Table 2. Comparisons between different methods on test function $f_2(x)$

Method	i	x_i	$\lvert x_i - x_{i-1}\rvert$	$f(x_i)$	COC
		$f_2(x)$, $x_0 = 0.22$			
NM	5	0.0864335580519620595505557	2.38E-18	1.02E-140	2
QM	3	0.0864335580519620595505557	7.58E-37	1.48E-868	6
SH1	3	0.0864335580519620595505557	9.63E-36	8.20E-842	6
SH2	3	0.0864335580519620595505557	2.74E-35	1.34E-830	6
MR	3	0.0864335580519620595505557	1.91E-43	5.44E-1,033	6

Example 3 To examine the established method on some nonlinear functions, consider the following ten test functions:

$$f_3(x) = (\sin^2(x) - 2x + 1)^5,$$

$$f_4(x) = \frac{\left(2x\cos(x) + x^2 - 3\right)^{10}}{x^2 + 1},$$

$$f_5(x) = (x - 2.5)^{\frac{15}{4}} e^x,$$

$$f_6(x) = ((x-3)e^x)^5,$$

$$f_7(x) = \left(\sqrt{x} - \frac{1}{x} - 1\right)^7,$$

$$f_8(x) = (e^{-x} + 2\sin(x))^4 (x-2)^3,$$

$$f_9(x) = \frac{(2x\cos(x) + x^2 - 3)^{10}(x-3)^8}{x^2+1},$$

$$f_{10}(x) = \left(\sqrt{x^2+2x+5} - 2\sin(x) - x^2 + 3\right)^{20} \left(x - \sqrt{7}\right)^{11},$$

$$f_{11}(x) = (\ln(x) + \sqrt{5} - 5)^{15}(x-8)^{10},$$

$$f_{12}(x) = \left(\ln(x) + \sqrt{x^4+1} - 2\right)^{10} \left(x - \sqrt{\frac{3}{2}}\right)^{10} (x-1).$$

Tables 1-4 show the numerical results using two real-life applications and test functions with a single multiplicity $f_3(x) - f_7(x)$ and nonlinear functions with a higher multiplicity $f_8(x) - f_{12}(x)$. From the iteration number i in the second column, it is clear that our proposed MR method requires fewer iterations to reach the convergence criteria. Although in some cases MR was equivalent to other schemes compared with sixth-order in terms of iteration number, it outperforms them and shows higher efficiency, as $|x_i - x_{i-1}|$ and $f(x_i)$. Thus, considering either the number of iterations or the accuracy of the approximate root, the new method was found to be superior to the well-known existing methods.

Table 5 shows that with Setting the same convergence criterion for all methods $|x_i - x_{i+1}| < 10^{-200}$, the new method in five examples out of 10 requires the same or fewer iterations to reach the stopping criterion compared to other methods of the same order.

Table 3. Comparison of the efficiency among different iterative methods with single multiplicity

Method	i	x_i	$\lvert x_i - x_{i-1}\rvert$	$f(x_i)$	COC
		$f_3(x)$, $x_0 = 1.5$			
NM	7	0.714835825441389239763 04	6.46E-21	6.92E-207	2
QM	4	0.714835825441389239763 04	4.13E-39	1.85E-1,161	6
SH1	4	0.714835825441389239763 04	1.72E-25	2.25E-753	6
SH2	4	0.714835825441389239763 04	1.42E-45	2.92E-1,364	6
MR	3	0.714835825441389239763 04	9.26E-19	6.50E-569	6
		$f_4(x)$, $x_0 = 3.2$			
NM	6	2.980645279438536834 5949	5.44E-22	1.22E-421	2
QM	3	2.980645279438536834 5949	1.79E-21	8.60E-1,236	6
SH1	3	2.980645279438536834 5949	3.56E-20	4.96E -1,155	6
SH2	3	2.980645279438536834 5949	1.59E-21	2.96E-1,237	6
MR	3	2.980645279438536834 5949	3.64E-39	2.98E-2,319	6
		$f_5(x)$, $x_0 = 2.8$			
NM	5	2.5000 + 0.0E-23i	0.0E-21i	2.0E-129i	2
QM	3	2.5000 + 0.0E-23i	0.0E-49i	2.0E-1,034i	6
SH1	3	2.5000 + 0.0E-23i	0.0E-36i	2.0E-730i	6
SH2	3	2.5000 + 0.0E-23i	0.0E-37i	9.0E-766i	6
MR	3	2.5000 + 0.0E-23i	0.0E-51i	4.0E-1,072i	6
		$f_6(x)$, $x_0 = 3.4$			
NM	7	3	3.40E-26	6.80E-249	2
QM	3	3	1.13E-21	4.09E-628	6
SH1	4	3	1.19E-35	1.09E-1,037	6
SH2	4	3	1.34E-62	4.07E-1,847	6
MR	3	3	4.27E-26	4.19E-761	6
		$f_7(x)$, $x_0 = 4.5$			
NM	6	2.147899035704787354 0262	2.97E-19	4.45E-266	2
QM	3	2.147899035704787354 0262	4.24E-17	1.59E-704	6
SH1	3	2.147899035704787354 0262	9.81E-17	3.23E-688	6
SH2	3	2.147899035704787354 0262	1.28E-18	4.84E-771	6
MR	3	2.147899035704787354 0262	1.38E-29	3.21E-1,246	6

Table 4. Comparison of the efficiency among different iterative methods with higher multiplicity

Method	i	x_i	$\lvert x_i - x_{i-1}\rvert$	$f(x_i)$	COC
		$f_8(x)$, $x_0 = 4$			
NM	6	3.1627488709263653591869	3.06E-17	3.16E-132	2
QM	4	3.1627488709263653591869	4.17E-69	1.32E-1,640	6
SH1	4	3.1627488709263653591869	9.08E-78	3.79E-1,859	6
SH2	3	3.1627488709263653591869	2.56E-16	1.24E-374	6
MR	3	3.1627488709263653591869	2.41E-18	7.16E-430	6
		$f_9(x)$, $x_0 = 3.1$			
NM	7	2.9806452794385368345949	1.28E-25	1.96E-49	2
QM	4	2.9806452794385368345949	7.11E-51	5.25E-2,932	6
SH1	4	2.9806452794385368345949	2.17E-52	3.02E-3,027	6
SH2	4	2.9806452794385368345949	6.96E-71	5.98E-4,135	6
MR	3	2.9806452794385368345949	1.44E-17	1.96E-595	6
		$f_{10}(x)$, $x_0 = 2$			
NM	6	2.3319676558839640103080	1.04E-18	1.77E-713	2
QM	3	2.3319676558839640103080	4.82E-16	8.41E-1,797	6
SH1	3	2.3319676558839640103080	1.39E-20	3.27E-2,375	6
SH2	3	2.3319676558839640103080	2.38E-18	1.55E-2,083	6
MR	3	2.3319676558839640103080	8.05E-28	8.63E-3,279	6
		$f_{11}(x)$, $x_0 = 7$			
NM	6	8	1.21E-18	3.65E-367	2
QM	3	8	3.83E-21	6.40E-1,257	6
SH1	4	8	4.37E-88	4.44E-5,266	6
SH2	3	8	7.78E-18	5.70E-1,053	6
MR	3	8	1.62E-35	1.40E-2,137	6
		$f_{12}(x)$, $x_0 = 1.3$			
NM	7	1.2228139636289731043280	7.53E-22	2.65E-419	2
QM	4	1.2228139636289731043280	6.29E-38	2.61E-2,114	6
SH1	4	1.2228139636289731043280	2.88E-36	2.89E-2,016	6
SH2	4	1.2228139636289731043280	1.54E-46	3.49E-2,632	6
MR	4	1.2228139636289731043280	3.67E-75	3.81E-4,373	6

Table 5. Comparisons of iterations number i needed for different algorithms such that $|x_i - x_{i+1}| < 10^{-200}$

	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$	$f_5(x)$	$f_6(x)$	$f_7(x)$	$f_8(x)$	$f_9(x)$	$f_{10}(x)$
x_0	1.5	3.2	2.8	3.4	4.5	4	3.1	2	7	1.3
NM	11	10	9	10	10	10	11	10	10	11
QM	5	5	4	5	5	5	5	5	5	5
SH1	6	5	5	5	5	5	5	5	5	6
SH2	5	5	4	5	5	5	5	5	5	5
MR	5	4	4	5	5	5	5	5	4	5

5. Conclusion

A new efficient transformation method for finding multiple roots has been established in this study. The principal advantage of the algorithm implemented is that neither the second-order derivative nor the knowledge of the root multiplicity is needed. The convergence of the new algorithms has been proven using Mathematica software, and it has been established to be in the sixth order. The numerical experiments illustrated the preferable performance of the algorithm developed in comparison to the existing sixth-order transformation methods. A qualitative analysis of the fractals of the basins of attraction [33–35] will be the subject of future work.

Conflict of interest

The authors declare no competing financial interest.

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Appendix

$$K_6 = \frac{1}{n^{12}}[A_2^3(-8n^2 - 48n^3 - 128n^4 - 240n^5 - 344n^6 - 288n^7 - 96n^8)$$

$$+A_1^4A_2(-38n - 188n^2 - 428n^3 - 784n^4 - 1{,}162n^5 - 1{,}148n^6 - 680n^7 - 218n^8 - 50n^9)$$

$$+A_1^6(6 + 38n + 96n^2 + 167n^3 + 259n^4 + 319n^5 + 267n^6 + 140n^7 + 40n^8 + 10n^9)$$

$$+A_1^3A_3(6n + 48n^2 + 144n^3 + 310n^4 + 506n^5 + 566n^6 + 372n^7 + 138n^8 + 50n^9)$$

$$+A_1A_2A_3(-12n^2 - 60n^3 - 142n^4 - 252n^5 - 390n^6 - 380n^7 - 174n^8 - 50n^9)$$

$$+A_5(10n^4 + 60n^5 + 150n^6 + 200n^7 + 150n^8 + 50n^9)$$

$$+A_1^2A_4(-34n^4 - 140n^5 - 230n^6 - 216n^7 - 126n^8 - 50n^9)$$

$$+A_2^2(4n + 64n^2 + 244n^3 + 522n^4 + 952n^5 + 1{,}186n^6 + 852n^7 + 306n^8 + 50n^9)]$$

$$Q_5 = \frac{1}{n^{13}}[-A_1^3A_3(-120n^2 - 1{,}296n^3 - 5{,}532n^4 - 12{,}624n^5 - 17{,}136n^6 - 14{,}228n^7 - 7{,}116n^8 - 1{,}972n^9$$

$$-260n^{10} - 48n^{11})$$

$$+A_2A_4(-64n^4 - 492n^5 - 1{,}604n^6 - 2{,}812n^7 - 2{,}740n^8 - 1{,}284n^9 - 156n^{10} - 48n^{11})$$

$$+A_3^2(-36n^4 - 276n^5 - 924n^6 - 1{,}692n^7 - 1{,}704n^8 - 804n^9 - 90n^{10} - 24n^{11})$$

$$+A_1^6(-16 - 248n - 1{,}410n^2 - 4{,}340n^3 - 8{,}306n^4 - 10{,}466n^5 - 8{,}848n^6 - 4{,}998n^7 - 1{,}838n^8$$

$$-400n^9 - 42n^{10} - 8n^{11})$$

$$+A_2^3(80n^3 + 592n^4 + 1{,}940n^5 + 3{,}652n^6 + 4{,}020n^7 + 2{,}404n^8 + 700n^9 + 36n^{10} + 16n^{11})$$

$$+A_1^4A_2(132n + 1{,}668n^2 + 8{,}120n^3 + 21{,}456n^4 + 34{,}592n^5 + 35{,}452n^6 + 23{,}244n^7 + 9{,}572n^8$$

$$+2{,}268n^9 + 236n^{10} + 48n^{11})$$

$$+A_6(12n^5+84n^6+252n^7+420n^8+420n^9+252n^{10}+48n^{11})$$

$$+A_1^2A_2^2(-256n^2-2{,}548n^3-10{,}504n^4-23{,}756n^5-31{,}880n^6-25{,}620n^7-12{,}244n^8-3{,}212n^9$$

$$-302n^{10}-72n^{11})$$

$$+A_4(72n^3+720n^4+2{,}772n^5+5{,}544n^6+6{,}416n^7+4{,}308n^8+1{,}600n^9+284n^{10}+48n^{11})$$

$$+A_1A_5(-40n^4-332n^5-1{,}104n^6-1{,}952n^7-2{,}020n^8-1{,}200n^9-332n^{10}-48n^{11})$$

$$+A_2A_3(264n^3+2{,}216n^4+8{,}060n^5+15{,}944n^6+18{,}048n^7+11{,}276n^8+3{,}624n^9+416n^{10}+96n^{11})]$$

$$Q_6=\frac{1}{n^{15}}[A_1^5A_2(-184n-2{,}990n^2-19{,}432n^3-68{,}458n^4-150{,}602n^5-221{,}342n^6-223{,}900n^7$$

$$-156{,}870n^8-75{,}180n^9-23{,}866n^{10}-4{,}648n^{11}-440n^{12}-70n^{13})$$

$$+A_3A_4(-96n^5-830n^6-3{,}184n^7-7{,}040n^8-9{,}568n^9-7{,}748n^{10}-3{,}184n^{11}-368n^{12}-70n^{13})$$

$$+A_2A_5(-80n^5-694n^6-2{,}632n^7-5{,}592n^8-7{,}184n^9-5{,}560n^{10}-2{,}264n^{11}-292n^{12}-70n^{13})$$

$$+A_1^7(18+332n+2{,}526n^2+10{,}350n^3+26{,}488n^4+45{,}764n^5+55{,}510n^6+48{,}048n^7+29{,}666n^8$$

$$+12{,}850n^9+3{,}772n^{10}+692n^{11}+66n^{12}+10n^{13})$$

$$+A_2^2A_3(312n^4+2{,}672n^5+10{,}462n^6+24{,}064n^7+34{,}524n^8+30{,}248n^9+15{,}040n^{10}+3{,}872n^{11}$$

$$+280n^{12}+70n^{13})$$

$$+A_7(14n^6+112n^7+392n^8+784n^9+980n^{10}+784n^{11}+392n^{12}+70n^{13})$$

$$+A_1^4A_3(186n^2+2{,}596n^3+14{,}874n^4+46{,}394n^5+89{,}342n^6+112{,}692n^7+95{,}142n^8+53{,}540n^9$$

$$+19{,}482n^{10}+4{,}176n^{11}+460n^{12}+70n^{13})$$

$$+A_1^2A_2A_3(-696n^3-8{,}254n^4-39{,}680n^5-106{,}034n^6-174{,}888n^7-184{,}194n^8-123{,}194n^9$$

$$-50{,}802n^{10}-11{,}898n^{11}-1{,}242n^{12}-210n^{13})$$

$$+A_5(90n^4+1{,}028n^5+4{,}638n^6+11{,}152n^7+16{,}086n^8+14{,}654n^9+8{,}338n^{10}+2{,}786n^{11}$$

$$+474n^{12}+70n^{13})$$

$$+A_1A_6(-48n^5-458n^6-1{,}792n^7-3{,}836n^8-4{,}984n^9-4{,}088n^{10}-2{,}056n^{11}-524n^{12}-70n^{13})$$

$$+A_2^3(-352n^3-3{,}608n^4-16{,}560n^5-43{,}554n^6-71{,}584n^7-74{,}552n^8-48{,}320n^9-19{,}180n^{10}$$

$$-4{,}376n^{11}-324n^{12}-70n^{13})$$

$$+A_3^2(162n^4+1{,}668n^5+7{,}244n^6+17{,}650n^7+26{,}114n^8+23{,}530n^9+12{,}326n^{10}+3{,}532n^{11}$$

$$+434n^{12}+70n^{13})$$

$$+A_2A_4(352n^4+3{,}360n^5+14{,}096n^6+33{,}416n^7+48{,}492n^8+43{,}800n^9+23{,}552n^{10}+6{,}880n^{11}$$

$$+792n^{12}+140n^{13})$$

$$+A_1^3A_4(-160n^3-1{,}890n^4-9{,}428n^5-25{,}718n^6-42{,}816n^7-45{,}942n^8-32{,}126n^9-14{,}250n^{10}$$

$$-3{,}666n^{11}-466n^{12}-70n^{13})$$

$$+A_2^2(512n^2+7{,}032n^3+38{,}536n^4+116{,}484n^5+219{,}930n^6+272{,}686n^7+225{,}056n^8+122{,}464n^9$$

$$+42{,}768n^{10}+8{,}942n^{11}+804n^{12}+140n^{13})]$$