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Research Article

Some Properties of Analytic Functions Associated with Erdély-Kober Integral Operator

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Abstract: The target of this paper is to discuss a new subclass $\tilde{\mathscr{T}}\mathscr{S}_{\vartheta}^{p,\,q}\,(\hbar,\ \sigma,\ \varsigma)$ of schlicht mappings with negative coefficients correlated to Erdély-Kober Integral Operator, in the unit disk $U = \{w \in \mathbb{C} : |w| < 1\}$. For mappings in our class, we learn fundamental properties like the Hadamard product, the coefficient inequality, the distortion and covering theorem, the radii of starlikeness, the convexity and close-to-convexity, the extreme points, and the closure theorems.

Keywords: starlike, convex, integral operator, coefficient estimates, convolution

MSC: 30C45, 30C50

1. Introduction

The theory of regular mapping undermines a field that is still actively reviewed today despite being an old subject. Using the classes of analytical mappings, numerous studies on the privileged topic of inequalities in complex analysis have been carried out. The interaction of geometry and analysis in complex mapping theory is its most attractive characteristic. These connections between geometric behaviour and analytical structure have been the key area of attention for rapid development. The current work, which developed a new subclass of regular mappings related to the Erdély-Kober Integral Operator, was inspired by this tactic.Many researchers have looked into the characteristics of regular mapping subclasses and shown how their research has numerous uses in signal theory, engineering and hydrodynamics. The extremal difficulties are one of the main issues with geometric mapping theory. Geometric mapping theory, the discovery of coefficient bounds, sharp estimates, and an extremal mapping all depend heavily on extremal problems. Understanding the theory of analytical schlicht mappings is crucial to comprehending the time development of the free boundary of a viscous fluid for planar flows in Hele-Shaw cells under injection. The findings we came to in this study could potentially be applicable in other pure and applied disciplines of mathematics.

The objectives of the Erdély-Kober integral operator in the context of analytic functions are wide-ranging and include generalization of classical transforms, advancing the theory of fractional calculus, and providing a robust framework for the study of special functions. It allows mathematicians to explore deeper connections in analysis and to solve complex problems in areas such as mathematical physics, engineering, and applied mathematics.

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TheErdély-Kober integral operator is a powerful and versatile tool that has applications across a broad spectrum of fields, including special functions, mathematical physics, control theory, signal processing, finance, and engineering. Its ability to generalize classical operators and provide fractional-order solutions makes it indispensable for modeling complex phenomena that traditional integer-order models cannot adequately describe. The operator is fundamental in advancing research in these areas and in developing more accurate and effective models for real-world systems.

Let A indicate the class of all mappings $\eta(w)$ of the type

$$\eta(w) = w + \sum_{v\geq 2} a_v w^v, \ (a_v \in \mathbb{C}) \tag{1}$$

in the open unit disc $U = \{w \in \mathbb{C} : |w| < 1\}$. Assume S is the subclass of A that only contains schlicht mapping and fulfils the normal normalization condition $\eta(0) = \eta'(0) - 1 = 0$. By S, we designate the subclass of A made up of mapping $\eta(w)$ that are all schlicht in U. A mapping $\eta \in A$ is a starlike mapping of the order $\wp$, $0 \leq \wp < 1$, if it fulfils

$$\Re\left\{\frac{w\eta'(w)}{\eta(w)}\right\} > \wp, \ w \in U. \tag{2}$$

We denote this class with $S^*(\wp)$. A mapping $\eta \in A$ is a convex mapping of the order $\wp$, $0 \leq \wp < 1$, if it fulfils

$$\Re\left\{1 + \frac{w\eta''(w)}{\eta'(w)}\right\} > \wp, \ w \in U. \tag{3}$$

We use $K(\wp)$ to represent this class. Keep in mind that the typical classes of starlike and convex mapping in U are $S^*(0) = S^*$ and $K(0) = K$, accordingly. For $\eta \in A$ provided by (1) and $g(w)$ provided by

$$g(w) = w + \sum_{v\geq 2} b_v w^v, \tag{4}$$

their convolution, specified by $(\eta * g)$, is described as

$$(\eta * g)(w) = w + \sum_{v\geq 2} a_v b_v w^v = (g * \eta)(w), \ (w \in U). \tag{5}$$

Note that $\eta * g \in A$.

Denote by $\tilde{\mathscr{T}}$ the subclass of A consisting of mappings of the form

$$\eta(w) = w - \sum_{v\geq 2} a_v w^v, \ \ a_v \geq 0 \ (w \in U), \tag{6}$$

and let $\tilde{\mathscr{T}} \cap S^*(\wp) = \tilde{\mathscr{T}}^*(\wp)$, $\tilde{\mathscr{T}} \cap K(\wp) = C(\wp)$. The class $\tilde{\mathscr{T}}^*(\wp)$ and related classes have been significantly deliberated for their intriguing properties reviewed by Silverman [1].

The Erdély-Kober type ([2] Ch 5) integral operator definition should be recalled, and it will be used throughout the paper as indicated below.

Definition 1 Let $\mathscr{I}_{\vartheta}^{p,\, q}: A \to A$, an Erdély- Kober type integral operator be such that for $\vartheta > 0,\ p,\ q \in \mathbb{C}, \Re(q-p) \geq 0$, and $\Re(p) > -\vartheta$ be specified by

$$\mathscr{I}_{\vartheta}^{p,\, q}\eta(w) = \frac{\Gamma(q+\vartheta)}{\Gamma(p+\vartheta)} \frac{1}{\Gamma(q-p)} \int_0^1 (1-t)^{q-p-1}\eta(wt^{\vartheta})\mathrm{d}t,\ \vartheta > 0. \tag{7}$$

For $\vartheta > 0,\ \Re(q-p) \geq 0,\ \Re(p) > -\vartheta$ and $\eta \in A$ of the type (1) we have

$$\mathscr{I}_{\vartheta}^{p,\, q}\eta(w) = w + \sum_{\nu \geq 2} \mathscr{B}_{\vartheta}^{p,\, q}(\nu) a_{\nu} w^{\nu}. \tag{8}$$

where

$$\mathscr{B}_{\vartheta}^{p,\, q}(\nu) = \frac{\Gamma(q+\vartheta)\Gamma(p+\nu\vartheta)}{\Gamma(p+\vartheta)\Gamma(q+\nu\vartheta)} \text{ and } \mathscr{B}_{\vartheta}^{p,\, q}(2) = \frac{\Gamma(q+\vartheta)\Gamma(p+2\vartheta)}{\Gamma(p+\vartheta)\Gamma(q+2\vartheta)} \tag{9}$$

Note that $\mathscr{I}_{\vartheta}^{p,\, p}\eta(w) = \eta(w)$. The operator $\mathscr{I}_{\vartheta}^{p,\, q}\eta(w)$ summarises to a number of well-known operators that have already been discussed, and some of the most intriguing particular cases are presented below:

(i). We acquire the operator $Q_{\kappa}^{\varsigma}\eta(w)(\varsigma \geq 0;\ \kappa > 1)$ reviewed by Jung et al. [3], for $p = \kappa;\ q = \varsigma + \kappa$ and $\vartheta = 1$.

(ii). We acquire the operator $\mathscr{L}_{\varsigma,\, \kappa}\eta(w)$ $(\varsigma;\ \kappa \in \mathbb{C}\backslash\mathbb{Z}_0;\ \mathbb{Z}_0 = \{0;\ -1;\ -2;\ \cdots\})$ reviewed by Carlson and Shafer [4] for $p = \varsigma - 1;\ q = \kappa - 1$ and $\vartheta = 1$.

(iii). We acquire the operator $\mathscr{I}_{\varsigma,\, \ell}$ $(\varsigma > 0;\ \ell > 0)$ reviewed by Choi et al [5] for $p = \varsigma - 1;\ q = \ell$ and $\vartheta = 1$.

(iv). we acquire the operator $\mathscr{D}^{\varsigma}$ $(\varsigma > -1)$ reviewed by Ruschweyh [6] for $p = \varsigma;\ q = 0$ and $\vartheta = 1$.

(v). we acquire the operator $\mathscr{I}_n$ $(n > \mathbb{N}_0$ reviewed by Noor [7], Noor and Noor [8] for $p = 1;\ q = n$ and $\vartheta = 1$.

(vi). We acquire the integral operator $\mathscr{I}_{\kappa,\, 1}$ which reviewed by Bernardi [9] for $p = \kappa;\ q = \kappa + 1$ and $\vartheta = 1$.

(vii). We acquire the integral operator $\mathscr{I}_{1,\, 1} = I$ which reviewed by Libera [10] and Livingston [11] for $p = 1;\ q = 2$ and $\vartheta = 1$.

Inspried by the work of several researchers [12–19], we describe a new subclass of mappings belonging to the class A.

Definition 2 For $0 \leq \hbar < 1,\ 0 \leq \sigma < 1$ and $0 < \varsigma < 1$, we let $\tilde{\mathscr{T}}\mathscr{S}_{\vartheta}^{p,\, q}$ $(\hbar,\ \sigma,\ \varsigma)$ be the subclass of η comprising of mappings of the type (6) and its geometrical condition satisfy

$$\left| \frac{\hbar\left((\mathscr{I}_{\vartheta}^{p,\, q}\eta(w))' - \dfrac{\mathscr{I}_{\vartheta}^{p,\, q}\eta(w)}{w} \right)}{\sigma(\mathscr{I}_{\vartheta}^{p,\, q}\eta(w))' + (1-\hbar)\dfrac{\mathscr{I}_{\vartheta}^{p,\, q}\eta(w)}{w}} \right| < \varsigma,\ w \in U,$$

where $\tilde{\mathscr{T}}\mathscr{S}_{\vartheta}^{p,\, q}$ is provided by (8).

2. Coefficient inequality

We acquire the necessary and adequate requirements for act being assigned to the class $\tilde{\mathscr{T}}\mathscr{S}_{\vartheta}^{p,\, q}$ $(\hbar,\ \sigma,\ \varsigma)$ in the subsequent theorem.

Theorem 1 Let $\eta \in \tilde{\mathscr{T}}\mathscr{S}_{\vartheta}^{p,\, q}$ $(\hbar,\ \sigma,\ \varsigma) \Leftrightarrow$

$$\sum_{\nu\geq 2}[\hbar(\nu-1)+\varsigma(\nu\sigma+1-\hbar)]\mathscr{B}_{\vartheta}^{p,\,q}(\nu)a_{\nu}\leq\varsigma(\sigma+(1-\hbar)), \tag{10}$$

where $0<\varsigma<1,\ 0\leq\hbar<1$ and $0\leq\sigma<1$. The result (10) is sharp for the mapping

$$\eta(w)=w-\frac{\varsigma(\sigma+(1-\hbar))}{[\hbar(\nu-1)+\varsigma(\nu\sigma+1-\hbar)]\mathscr{B}_{\vartheta}^{p,\,q}(\nu)}w^{\nu},\ \nu\geq 2.$$

Proof. Suppose that the inequality (10) holds true and $|w|=1$. Then

$$\left|\hbar\left((\mathscr{I}_{\vartheta}^{p,\,q}\eta(w))'-\frac{\mathscr{I}_{\vartheta}^{p,\,q}\eta(w)}{w}\right)\right|-\varsigma\left|\sigma\left(\mathscr{I}_{\vartheta}^{p,\,q}\eta(w))'+(1-\hbar)\frac{\mathscr{I}_{\vartheta}^{p,\,q}\eta(w)}{w}\right)\right|$$

$$=\left|-\hbar\sum_{\nu\geq 2}(\nu-1)\mathscr{B}_{\vartheta}^{p,\,q}(\nu)a_{\nu}w^{\nu-1}\right|$$

$$-\varsigma\left|\sigma+(1-\hbar)-\sum_{\nu\geq 2}(\nu\sigma+1-\hbar)\mathscr{B}_{\vartheta}^{p,\,q}(\nu)a_{\nu}w^{\nu-1}\right|$$

$$\leq\sum_{\nu\geq 2}[\hbar(\nu-1)+\varsigma(\nu\sigma+1-\hbar)]\mathscr{B}_{\vartheta}^{p,\,q}(\nu)a_{\nu}-\varsigma(\sigma+(1-\hbar))$$

$$\leq 0.$$

Hence, by maximum modulus principle, $\eta\in\tilde{\mathscr{T}\mathscr{S}}_{\vartheta}^{p,\,q}\ (\hbar,\ \sigma,\ \varsigma)$. Now adopt that $\nu\in\tilde{\mathscr{T}\mathscr{S}}_{\vartheta}^{p,\,q}\ (\hbar,\ \sigma,\ \varsigma)$ so that

$$\left|\frac{\hbar\left((\mathscr{I}_{\vartheta}^{p,\,q}\eta(w))'-\dfrac{\mathscr{I}_{\vartheta}^{p,\,q}\eta(w)}{w}\right)}{\sigma(\mathscr{I}_{\vartheta}^{p,\,q}\eta(w))'+(1-\hbar)\dfrac{\mathscr{I}_{\vartheta}^{p,\,q}\eta(w)}{w}}\right|<\varsigma,\ w\in U.$$

Hence,

$$\left|\hbar\left((\mathscr{I}_{\vartheta}^{p,\,q}\eta(w))'-\frac{\mathscr{I}_{\vartheta}^{p,\,q}\eta(w)}{w}\right)\right|<\varsigma\left|\sigma\left(\mathscr{I}_{\vartheta}^{p,\,q}\eta(w))'+(1-\hbar)\frac{\mathscr{I}_{\vartheta}^{p,\,q}\eta(w)}{w}\right)\right|.$$

Therefore, we get

$$\left| -\sum_{\nu\geq 2} \hbar(\nu-1)\mathscr{B}_{\vartheta}^{p,\, q}(\nu)a_\nu w^{\nu-1} \right|$$

$$< \varsigma \left| \sigma + (1-\hbar) - \sum_{\nu\geq 2} \nu\sigma + 1 - \hbar)\mathscr{B}_{\vartheta}^{p,\, q}(\nu)a_\nu w^{\nu-1} \right|.$$

Thus,

$$\sum_{\nu\geq 2} [\hbar(\nu-1) + \varsigma(\nu\sigma + 1 - \hbar)]\mathscr{B}_{\vartheta}^{p,\, q}(\nu)a_\nu \leq \varsigma(\sigma + (1-\hbar)).$$

Therefore, we get the required inequality (2.1) of Theorem 1. □

Corollary 1 Let $\eta \in \tilde{\mathscr{T}}\mathscr{S}_{\vartheta}^{p,\, q}(\hbar,\, \sigma,\, \varsigma)$. Then

$$a_\nu \leq \frac{\varsigma(\sigma + (1-\hbar))}{[\hbar(\nu-1) + \varsigma(\nu\sigma + 1 - \hbar)]\mathscr{B}_{\vartheta}^{p,\, q}(\nu)}.$$

3. Distortion properties

In this section, we present the growth and distortion theorems for mapping η belonging to class $\tilde{\mathscr{T}}\mathscr{S}_{\vartheta}^{p,\, q}(\hbar,\, \sigma,\, \varsigma)$.

Theorem 2 Let $0 < \varsigma < 1$, $0 \leq \hbar < 1$ and $0 \leq \sigma < 1$. If the the mapping η given by (6) is in the class $\tilde{\mathscr{T}}\mathscr{S}_{\vartheta}^{p,\, q}(\hbar,\, \sigma,\, \varsigma)$. Then

$$|w| - \frac{\varsigma(\sigma + (1-\hbar))}{\mathscr{B}_{\vartheta}^{p,\, q}(2)[\hbar + \varsigma(2\sigma + 1 - \hbar)]}|w|^2 \leq |\eta(w)| \leq |w| + \frac{\varsigma(\sigma + (1-\hbar))}{\mathscr{B}_{\vartheta}^{p,\, q}(2)[\hbar + \varsigma(2\sigma + 1 - \hbar)]}|w|^2.$$

The result is sharp and attained

$$\eta(w) = w - \frac{\varsigma(\sigma + (1-\hbar))}{\mathscr{B}_{\vartheta}^{p,\, q}(2)[\hbar + \varsigma(2\sigma + 1 - \hbar)]}w^2.$$

Proof. By Theorem 1, we have

$$\sum_{\nu\geq 2} [\hbar(\nu-1) + \varsigma(\nu\sigma + 1 - \hbar)]\mathscr{B}_{\vartheta}^{p,\, q}(\nu)a_\nu \leq \varsigma(\sigma + (1-\hbar)),$$

then, we have

$$[\hbar + \varsigma(2\sigma + 1 - \hbar)]\mathscr{B}_{\vartheta}^{p,\, q}(2) \leq \sum_{\nu\geq 2} [\hbar(\nu-1) + \varsigma(\nu\sigma + 1 - \hbar)]\mathscr{B}_{\vartheta}^{p,\, q}(\nu)a_\nu \leq \varsigma(\sigma + (1-\hbar)),$$

then,

$$\sum_{\nu \geq 2} a_\nu \leq \frac{\varsigma(\sigma+(1-\hbar))}{[\hbar+\varsigma(2\sigma+1-\hbar)]\mathscr{B}_{\vartheta}^{p,\,q}(2)}.$$

Hence,

$$\begin{aligned} |\eta(w)| &\leq |w| + \sum_{\nu \geq 2} a_\nu w^2 \\ &\leq |w| + |w|^2 \sum_{\nu \geq 2} a_\nu \\ &\leq |w| + \frac{\varsigma(\sigma+(1-\hbar))}{\mathscr{B}_{\vartheta}^{p,\,q}(2)[\hbar+\varsigma(2\sigma+1-\hbar)]}|w|^2. \end{aligned}$$

Also,

$$\begin{aligned} |\eta(w)| &\geq |w| - \sum_{\nu \geq 2} a_\nu w^2 \\ &\geq |w| + |w|^2 \sum_{\nu \geq 2} a_\nu \\ &\geq |w| + \frac{\varsigma(\sigma+(1-\hbar))}{\mathscr{B}_{\vartheta}^{p,\,q}(2)[\hbar+\varsigma(2\sigma+1-\hbar)]}|w|^2. \end{aligned}$$

□

Theorem 3 Let $0 < \varsigma < 1$, $0 \leq \hbar < 1$ and $0 \leq \sigma < 1$. If the the mapping η given by (6) is in the class $\mathscr{T}\mathscr{S}_{\vartheta}^{p,\,q}\,(\hbar,\ \sigma,\ \varsigma)$. Then

$$1 - \frac{2\varsigma(\sigma+(1-\hbar))}{\mathscr{B}_{\vartheta}^{p,\,q}(2)[\hbar+\varsigma(2\sigma+1-\hbar)]}|w| \leq |\eta'(w)| \leq 1 + \frac{2\varsigma(\sigma+(1-\hbar))}{\mathscr{B}_{\vartheta}^{p,\,q}(2)[\hbar+\varsigma(2\sigma+1-\hbar)]}|w|$$

with equality for

$$\eta(w) = w - \frac{2\varsigma(\sigma+(1-\hbar))}{\mathscr{B}_{\vartheta}^{p,\,q}(2)[\hbar+\varsigma(2\sigma+1-\hbar)]}w^2.$$

Proof. Notice that

$$\mathscr{B}_{\vartheta}^{p,\,q}(2)[\hbar+\varsigma(2\sigma+1-\hbar)]\sum_{\nu\geq 2}\nu a_{\nu}$$

$$\leq\sum_{\nu\geq 2}\nu[\hbar(\nu-1)+\varsigma(\nu\sigma+1-\hbar)]\mathscr{B}_{\vartheta}^{p,\,q}(\nu)a_{\nu}$$

$$\leq\varsigma(\sigma+(1-\hbar)), \tag{11}$$

from Theorem 1. Thus,

$$|\eta'(w)|=\left|1-\sum_{\nu\geq 2}\nu a_{\nu}z^{\nu-1}\right|$$

$$\leq 1+\sum_{\nu\geq 2}\nu a_{\nu}|w|^{\nu-1}$$

$$\leq 1+|w|\sum_{\nu\geq 2}\nu a_{\nu}$$

$$\leq 1+|w|\frac{2\varsigma(\sigma+(1-\hbar))}{\mathscr{B}_{\vartheta}^{p,\,q}(2)[\hbar+\varsigma(2\sigma+1-\hbar)]}. \tag{12}$$

On the other hand,

$$|\eta'(w)|=\left|1-\sum_{\nu\geq 2}\nu a_{\nu}z^{\nu-1}\right|$$

$$\geq 1-\sum_{\nu\geq 2}\nu a_{\nu}|w|^{\nu-1}$$

$$\geq 1-|w|\sum_{\nu\geq 2}\nu a_{\nu}$$

$$\geq 1-|w|\frac{2\varsigma(\sigma+(1-\hbar))}{\mathscr{B}_{\vartheta}^{p,\,q}(2)[\hbar+\varsigma(2\sigma+1-\hbar)]}. \tag{13}$$

Combining (12) and (13), we get the result. □

4. Radii properties

The radius of starlikeness, convexity and close-to-convexity for the class $\mathscr{T}\mathscr{S}_{\vartheta}^{p,\, q}\,(\hbar,\, \sigma,\, \varsigma)$ is given by the following theorems.

Theorem 4 Let $0 < \varsigma < 1$, $0 \leq \hbar < 1$ and $0 \leq \sigma < 1$. If the the mapping η given by (6) is in the class $\mathscr{T}\mathscr{S}_{\vartheta}^{p,\, q}\,(\hbar,\, \sigma,\, \varsigma)$. Then η is starlike in $|w| < R_1$ of order ϖ, $0 \leq \varpi < 1$,

$$R_1 = \inf_{\nu}\left\{\frac{(1-\varpi)(\hbar(\nu-1)+\varsigma(\nu\sigma+1-\hbar))\mathscr{B}_{\vartheta}^{p,\, q}(\nu)}{(\nu-\varpi)\varsigma(\sigma+(1-\hbar))}\right\}^{\frac{1}{\nu-1}},\ \nu \geq 2. \tag{14}$$

Proof. η is starlike of order ϖ, $0 \leq \varpi < 1$ if

$$\Re\left\{\frac{w\eta'(w)}{\eta(w)}\right\} > \varpi.$$

Therefore, demonstrating that

$$\left|\frac{w\eta'(w)}{\eta(w)}-1\right| = \left|\frac{-\sum\limits_{\nu\geq 2}(\nu-1)a_\nu w^{\nu-1}}{1-\sum\limits_{\nu\geq 2}a_\nu w^{\nu-1}}\right| \leq \frac{\sum\limits_{\nu\geq 2}(\nu-1)a_\nu|w|^{\nu-1}}{1-\sum\limits_{\nu\geq 2}a_\nu|w|^{\nu-1}}.$$

Thus,

$$\left|\frac{w\eta'(w)}{\eta(w)}-1\right| \leq 1-\varpi \ if \ \sum_{\nu\geq 2}\frac{(\nu-\varpi)}{(1-\varpi)}a_\nu|w|^{\nu-1} \leq 1. \tag{15}$$

Hence, by Theorem 1, (15) will be true if

$$\frac{\nu-\varpi}{1-\varpi}|w|^{\nu-1} \leq \frac{(\hbar(\nu-1)+\varsigma(\nu\sigma+1-\hbar))\mathscr{B}_{\vartheta}^{p,\, q}(\nu)}{\varsigma(\sigma+(1-\hbar)},$$

or if

$$|w| \leq \left[\frac{(1-\varpi)(\hbar(\nu-1)+\varsigma(\nu\sigma+1-\hbar))\mathscr{B}_{\vartheta}^{p,\, q}(\nu)}{(\nu-\varpi)\varsigma(\sigma+(1-\hbar))}\right]^{\frac{1}{\nu-1}},\ \nu \geq 2. \tag{16}$$

The theorem follows easily from (16). □

Theorem 5 Let $0 < \varsigma < 1$, $0 \leq \hbar < 1$ and $0 \leq \sigma < 1$. If the the mapping η given by (6) is in the class $\mathscr{T}\mathscr{S}_{\vartheta}^{p,\, q}\,(\hbar,\, \sigma,\, \varsigma)$. Then η is convex in $|w| < R_2$ of order ϖ, $0 \leq \varpi < 1$, where

$$R_2 = \inf_{\nu}\left\{\frac{(1-\varpi)(\hbar(\nu-1)+\varsigma(\nu\sigma+1-\hbar))\mathscr{B}_{\vartheta}^{p,\,q}(\nu)}{\nu(\nu-\varpi)\varsigma(\sigma+(1-\hbar))}\right\}^{\frac{1}{\nu-1}},\ \nu\geq 2. \tag{17}$$

Proof. η is convex of order ϖ, $0\leq\varpi<1$ if

$$\Re\left\{1+\frac{w\eta''(w)}{\eta'(w)}\right\}>\varpi.$$

Therefore, demonstrate that

$$\left|\frac{w\eta''(w)}{\eta'(w)}\right| = \left|\frac{-\sum\limits_{\nu\geq 2}\nu(\nu-1)a_{\nu}w^{\nu-1}}{1-\sum\limits_{\nu\geq 2}\nu a_{\nu}w^{\nu-1}}\right| \leq \frac{\sum\limits_{\nu\geq 2}\nu(\nu-1)a_{\nu}|w|^{\nu-1}}{1-\sum\limits_{\nu\geq 2}\nu a_{\nu}|w|^{\nu-1}}.$$

Thus,

$$\left|\frac{w\eta''(w)}{\eta'(w)}\right|\leq 1-\varpi \ if \ \sum_{\nu\geq 2}\frac{\nu(\nu-\varpi)}{(1-\varpi)}a_{\nu}|w|^{\nu-1}\leq 1. \tag{18}$$

Hence, by Theorem 1, (18) will be true if

$$\frac{\nu(\nu-\varpi)}{1-\varpi}|w|^{\nu-1}\leq\frac{(\hbar(\nu-1)+\varsigma(\nu\sigma+1-\hbar))\mathscr{B}_{\vartheta}^{p,\,q}(\nu)}{\varsigma(\sigma+(1-\hbar)},$$

or if

$$|w|\leq\left[\frac{(1-\varpi)(\hbar(\nu-1)+\varsigma(\nu\sigma+1-\hbar))\mathscr{B}_{\vartheta}^{p,\,q}(\nu)}{\nu(\nu-\varpi)\varsigma(\sigma+(1-\hbar))}\right]^{\frac{1}{\nu-1}},\ \nu\geq 2. \tag{19}$$

The theorem follows easily from (19). □

Theorem 6 Let $0<\varsigma<1$, $0\leq\hbar<1$ and $0\leq\sigma<1$. If the the mapping η given by (6) is in the class $\mathscr{T}\mathscr{S}_{\vartheta}^{p,\,q}(\hbar,\ \sigma,\ \varsigma)$. Then η is close-to-convex in $|w|<R_3$ of order ϖ, $0\leq\varpi<1$, where

$$R_3 = \inf_{\nu}\left\{\frac{(1-\varpi)(\hbar(\nu-1)+\varsigma(\nu\sigma+1-\hbar))\mathscr{B}_{\vartheta}^{p,\,q}(\nu)}{\nu\,\varsigma(\sigma+(1-\hbar))}\right\}^{\frac{1}{\nu-1}},\ \nu\geq 2. \tag{20}$$

Proof. η is close-to-convex of order ϖ, $0\leq\varpi<1$ if

$$\Re\left\{\eta'(w)\right\}>\varpi.$$

Consequently, proving that

$$|\eta'(w)-1| = \left| -\sum_{\nu\geq 2} \nu a_\nu w^{\nu-1} \right| \leq \sum_{\nu\geq 2} \nu a_\nu |w|^{\nu-1}.$$

Thus,

$$|\eta'(w)-1| \leq 1-\varpi \; if \; \sum_{\nu\geq 2} \frac{\nu}{(1-\varpi)} a_\nu |w|^{\nu-1} \leq 1. \tag{21}$$

Hence, by Theorem 2, (21) will be true if

$$\frac{\nu}{1-\varpi}|w|^{\nu-1} \leq \frac{(\hbar(\nu-1)+\varsigma(\nu\sigma+1-\hbar))\mathscr{B}_{\vartheta}^{p,\,q}(\nu)}{\varsigma(\sigma+(1-\hbar)},$$

or if

$$|w| \leq \left[\frac{(1-\varpi)(\hbar(\nu-1)+\varsigma(\nu\sigma+1-\hbar))\mathscr{B}_{\vartheta}^{p,\,q}(\nu)}{\nu\varsigma(\sigma+(1-\hbar))}\right]^{\frac{1}{\nu-1}}, \; \nu \geq 2. \tag{22}$$

Theorem easily implies from (22). □

5. Extreme points

In the following theorem, we acquire extreme points for the class $\tilde{\mathscr{T}\mathscr{S}}_{\vartheta}^{p,\,q}(\hbar,\,\sigma,\,\varsigma)$.
Theorem 7 Let $\eta_1(w) = w$ and

$$\eta_\nu(w) = w - \frac{\varsigma(\sigma+(1-\hbar))}{[\hbar(\nu-1)+\varsigma(\nu\sigma+1-\hbar)]\mathscr{B}_{\vartheta}^{p,\,q}(\nu)}w^\nu, \text{ for } \nu = 2,\,3,\,\cdots.$$

Then $\eta \in \tilde{\mathscr{T}\mathscr{S}}_{\vartheta}^{p,\,q}(\hbar,\,\sigma,\,\varsigma)$, $(0<\varsigma<1,\; 0\leq\hbar<1 \text{ and } 0\leq\sigma<1)$, $\Leftrightarrow$ it can be described by the type of

$$\eta(w) = \sum_{\nu=1}^{\infty}\theta_\nu\eta_\nu(w), \text{ where } \theta_\nu \geq 0 \text{ and } \sum_{\nu=1}^{\infty}\theta_\nu = 1.$$

Proof. Assume that $\eta(w) = \sum_{\nu=1}^{\infty}\theta_\nu\eta_\nu(w)$, hence we get

$$\eta(w) = w - \sum_{\nu\geq 2}\frac{\varsigma(\sigma+(1-\hbar))\theta_\nu}{[\hbar(\nu-1)+\varsigma(\nu\sigma+1-\hbar)]\mathscr{B}_{\vartheta}^{p,\,q}(\nu)}w^\nu.$$

Now, $\eta \in \tilde{\mathscr{T}\mathscr{S}}_{\vartheta}^{p,\, q}\,(\hbar,\, \sigma,\, \varsigma)$, since

$$\sum_{\nu\geq 2}\frac{[\hbar(\nu-1)+\varsigma(\nu\sigma+1-\hbar)]\mathscr{B}_{\vartheta}^{p,\, q}(\nu)}{\varsigma(\sigma+(1-\hbar))}$$

$$\times\frac{\varsigma(\sigma+(1-\hbar))\theta_{\nu}}{[\hbar(\nu-1)+\varsigma(\nu\sigma+1-\hbar)]\mathscr{B}_{\vartheta}^{p,\, q}(\nu)}$$

$$=\sum_{\nu\geq 2}\theta_{\nu}=1-\theta_{1}\leq 1.$$

Conversely, suppose $\eta \in \tilde{\mathscr{T}\mathscr{S}}_{\vartheta}^{p,\, q}\,(\hbar,\, \sigma,\, \varsigma)$. Then we show that η can be written in the type $\sum_{\nu=1}^{\infty}\theta_{\nu}\eta_{\nu}(w)$. Now $\eta \in \tilde{\mathscr{T}\mathscr{S}}_{\vartheta}^{p,\, q}\,(\hbar,\, \sigma,\, \varsigma)$ implies from Theorem 1

$$a_{\nu}\leq\frac{\varsigma(\sigma+(1-\hbar))}{[\hbar(\nu-1)+\varsigma(\nu\sigma+1-\hbar)]\mathscr{B}_{\vartheta}^{p,\, q}(\nu)}.$$

Setting $\theta_{\nu}=\frac{[\hbar(\nu-1)+\varsigma(\nu\sigma+1-\hbar)]\mathscr{B}_{\vartheta}^{p,\, q}(\nu)}{\varsigma(\sigma+(1-\hbar))}a_{\nu},\ \nu=2,\ 3,\ \cdots$ and $\theta_{1}=1-\sum_{\nu\geq 2}\theta_{\nu}$, we acquire $\eta(w)=\sum_{\nu=1}^{\infty}\theta_{\nu}\eta_{\nu}(w)$. □

6. Hadamard product

We receive the convolution result for mapping that belongs to the class $\tilde{\mathscr{T}\mathscr{S}}_{\vartheta}^{p,\, q}\,(\hbar,\, \sigma,\, \varsigma)$ in the subsequent theorem.

Theorem 8 Let $\eta,\ g \in \tilde{\mathscr{T}\mathscr{S}}_{\vartheta}^{p,\, q}\,(\hbar,\, \sigma,\, \varsigma)$. Then $\eta * g \in \tilde{\mathscr{T}\mathscr{S}}_{\vartheta}^{p,\, q}\,(\hbar,\, \sigma,\, \varsigma)$ for

$$\eta(w)=w-\sum_{\nu\geq 2}a_{\nu}w^{\nu},\ g(w)=w-\sum_{\nu\geq 2}b_{\nu}w^{\nu}\ \text{and}\ (\eta * g)(w)=w-\sum_{\nu\geq 2}a_{\nu}b_{\nu}w^{\nu},$$

where

$$\zeta\geq\frac{\varsigma^{2}(\sigma+(1-\hbar))\hbar(\nu-1)}{[\hbar(\nu-1)+\varsigma(\nu\sigma+1-\hbar)]^{2}\mathscr{B}_{\vartheta}^{p,\, q}(\nu)-\varsigma^{2}(\sigma+(1-\hbar))(n\sigma+1-\hbar)}.$$

Proof. $\eta \in \tilde{\mathscr{T}\mathscr{S}}_{\vartheta}^{p,\, q}\,(\hbar,\, \sigma,\, \varsigma)$ and so

$$\sum_{\nu\geq 2}\frac{[\hbar(\nu-1)+\varsigma(\nu\sigma+1-\hbar)]\mathscr{B}_{\vartheta}^{p,\, q}(\nu)}{\varsigma(\sigma+(1-\hbar))}a_{\nu}\leq 1, \tag{23}$$

and

$$\sum_{\nu\geq 2}\frac{[\hbar(\nu-1)+\varsigma(\nu\sigma+1-\hbar)]\mathscr{B}_{\vartheta}^{p,\,q}(\nu)}{\varsigma(\sigma+(1-\hbar))}b_{\nu}\leq 1. \tag{24}$$

We must determine the smallest number ζ possible so that

$$\sum_{\nu\geq 2}\frac{[\hbar(\nu-1)+\zeta(\nu\sigma+1-\hbar)]\mathscr{B}_{\vartheta}^{p,\,q}(\nu)}{\zeta(\sigma+(1-\hbar))}a_{\nu}b_{\nu}\leq 1. \tag{25}$$

By Cauchy-Schwarz inequality

$$\sum_{\nu\geq 2}\frac{[\hbar(\nu-1)+\varsigma(\nu\sigma+1-\hbar)]\mathscr{B}_{\vartheta}^{p,\,q}(\nu)}{\varsigma(\sigma+(1-\hbar))}\sqrt{a_{\nu}b_{\nu}}\leq 1. \tag{26}$$

Consequently, proving that

$$\frac{[\hbar(\nu-1)+\zeta(\nu\sigma+1-\hbar)]\mathscr{B}_{\vartheta}^{p,\,q}(\nu)}{\zeta(\sigma+(1-\hbar))}a_{\nu}b_{\nu}$$

$$\leq\frac{[\hbar(\nu-1)+\varsigma(\nu\sigma+1-\hbar)]\mathscr{B}_{\vartheta}^{p,\,q}(\nu)}{\varsigma(\sigma+(1-\hbar))}\sqrt{a_{\nu}b_{\nu}}.$$

That is

$$\sqrt{a_{\nu}b_{\nu}}\leq\frac{[\hbar(\nu-1)+\varsigma(\nu\sigma+1-\hbar)]\zeta}{[\hbar(\nu-1)+\zeta(\nu\sigma+1-\hbar)]\varsigma}. \tag{27}$$

From (26)

$$\sqrt{a_{\nu}b_{\nu}}\leq\frac{\varsigma(\sigma+(1-\hbar))}{[\hbar(\nu-1)+\varsigma(\nu\sigma+1-\hbar)]\mathscr{B}_{\vartheta}^{p,\,q}(\nu)}.$$

Consequently, proving that

$$\frac{\varsigma(\sigma+(1-\hbar))}{[\hbar(\nu-1)+\varsigma(\nu\sigma+1-\hbar)]\mathscr{B}_{\vartheta}^{p,\,q}(\nu)}\leq\frac{[\hbar(\nu-1)+\varsigma(\nu\sigma+1-\hbar)]\zeta}{[\hbar(\nu-1)+\zeta(\nu\sigma+1-\hbar)]\varsigma},$$

which simplifies to

$$\zeta \geq \frac{\varsigma^2(\sigma+(1-\hbar))\hbar(\nu-1)}{[\hbar(\nu-1)+\varsigma(\nu\sigma+1-\hbar)]^2\mathscr{B}_{\vartheta}^{p,\, q}(\nu)-\varsigma^2(\sigma+(1-\hbar))(\nu\sigma+1-\hbar)}.$$

□

7. Closure theorems

The subsequent closure theorems will be demonstrated for the class $\tilde{\mathscr{T}\mathscr{S}}_{\vartheta}^{p,\, q}\,(\hbar,\ \sigma,\ \varsigma)$.

Theorem 9 Let the mapping η_j be in the class $\tilde{\mathscr{T}\mathscr{S}}_{\vartheta}^{p,\, q}\,(\hbar,\ \sigma,\ \varsigma)$ for every $j = 1,\ 2,\ \cdots,\ s$. Then the mapping g defined by

$$g(w) = \sum_{j=1}^{s} c_j \eta_j(w)$$

is also in the class $\tilde{\mathscr{T}\mathscr{S}}_{\vartheta}^{p,\, q}\,(\hbar,\ \sigma,\ \varsigma)$ where $\sum_{j=1}^{s} c_j = 1$, $(c_j \geq 0)$.

Proof.

$$g(w) = \sum_{j=1}^{s} c_j \eta_j(w)$$

$$= w - \sum_{\nu \geq 2} \sum_{j=1}^{s} c_j a_{\nu,\, j} w^{\nu}$$

$$= w - \sum_{\nu \geq 2} e_{\nu} w^{\nu},$$

where $e_{\nu} = \sum_{j=1}^{s} c_j a_{\nu,\, j}$. Thus $g(z) \in \tilde{\mathscr{T}\mathscr{S}}_{\vartheta}^{p,\, q}\,(\hbar,\ \sigma,\ \varsigma)$ if

$$\sum_{\nu \geq 2} \frac{[\hbar(\nu-1)+\varsigma(\nu\sigma+1-\hbar)]\mathscr{B}_{\vartheta}^{p,\, q}(\nu)}{\varsigma(\sigma+(1-\hbar))} e_{\nu} \leq 1,$$

that is, if

$$\sum_{\nu\geq 2}\sum_{j=1}^{s}\frac{[\hbar(\nu-1)+\varsigma(\nu\sigma+1-\hbar)]\mathscr{B}_{\vartheta}^{p,\,q}(\nu)}{\varsigma(\sigma+(1-\hbar))}c_j a_{\nu,\,j}$$

$$=\sum_{j=1}^{s}c_j\sum_{\nu\geq 2}\frac{[\hbar(\nu-1)+\varsigma(\nu\sigma+1-\hbar)]\mathscr{B}_{\vartheta}^{p,\,q}(\nu)}{\varsigma(\sigma+(1-\hbar))}a_{\nu,\,j}$$

$$\leq\sum_{j=1}^{s}c_j=1.$$

□

Theorem 10 Let $\eta,\ g\in\tilde{\mathscr{T}\mathscr{S}}_{\vartheta}^{p,\,q}\ (\hbar,\ \sigma,\ \varsigma)$. Then

$$h(w)=w-\sum_{\nu\geq 2}(a_\nu^2+b_\nu^2)w^\nu\in\tilde{\mathscr{T}\mathscr{S}}_{\vartheta}^{p,\,q}\ (\hbar,\ \sigma,\ \varsigma),$$

where

$$\zeta\geq\frac{2\hbar(\nu-1)\varsigma^2(\sigma+(1-\hbar))}{[\hbar(\nu-1)+\varsigma(\nu\sigma+1-\hbar)]^2\mathscr{B}_{\vartheta}^{p,\,q}(\nu)-2\varsigma^2(\sigma+(1-\hbar))(\nu\sigma+1-\hbar)}.$$

Proof. Since $\eta,\ g\in\tilde{\mathscr{T}\mathscr{S}}_{\vartheta}^{p,\,q}\ (\hbar,\ \sigma,\ \varsigma)$, so Theorem 1 yields

$$\sum_{\nu\geq 2}\left[\frac{(\hbar(\nu-1)+\varsigma(\nu\sigma+1-\hbar))\mathscr{B}_{\vartheta}^{p,\,q}(\nu)}{\varsigma(\sigma+(1-\hbar))}a_\nu\right]^2\leq 1,$$

and

$$\sum_{\nu\geq 2}\left[\frac{(\hbar(\nu-1)+\varsigma(\nu\sigma+1-\hbar))\mathscr{B}_{\vartheta}^{p,\,q}(\nu)}{\varsigma(\sigma+(1-\hbar))}b_\nu\right]^2\leq 1.$$

We acquire from the last two inequalities

$$\sum_{\nu\geq 2}\frac{1}{2}\left[\frac{(\hbar(\nu-1)+\varsigma(\nu\sigma+1-\hbar))\mathscr{B}_{\vartheta}^{p,\,q}(\nu)}{\varsigma(\sigma+(1-\hbar))}\right]^2(a_\nu^2+b_\nu^2)\leq 1. \qquad (28)$$

But $h(w)\in\tilde{\mathscr{T}\mathscr{S}}_{\vartheta}^{p,\,q}\ (\hbar,\ \sigma,\ \varsigma),\Leftrightarrow$

$$\sum_{\nu \geq 2} \frac{[\hbar(\nu-1)+\zeta(\nu\sigma+1-\hbar)]\mathscr{B}_{\vartheta}^{p,\, q}(\nu)}{\zeta(\sigma+(1-\hbar))}(a_{\nu}^{2}+b_{\nu}^{2}) \leq 1, \tag{29}$$

where $0 < \zeta < 1$, however (28) implies (29) if

$$\frac{[\hbar(\nu-1)+\zeta(\nu\sigma+1-\hbar)]\mathscr{B}_{\vartheta}^{p,\, q}(\nu)}{\zeta(\sigma+(1-\hbar))}$$

$$\leq \frac{1}{2}\left[\frac{(\hbar(\nu-1)+\varsigma(\nu\sigma+1-\hbar))\mathscr{B}_{\vartheta}^{p,\, q}(\nu)}{\varsigma(\sigma+(1-\hbar))}\right]^{2}.$$

Simplifying, we get

$$\zeta \geq \frac{2\hbar(\nu-1)\varsigma^{2}(\sigma+(1-\hbar))}{[\hbar(\nu-1)+\varsigma(\nu\sigma+1-\hbar)]^{2}\mathscr{B}_{\vartheta}^{p,\, q}(\nu)-2\varsigma^{2}(\sigma+(1-\hbar))(\nu\sigma+1-\hbar)}.$$

□

8. Conclusions

This research has introduced a new subclass of analytic functions involving Erdely-Kober integral operator and studied some basic properties of geometric function theory. Accordingly, some results related to obtained coefficient inequalities, distortion theorem, extreme points, Hadamard product and closure theorems for this class have also been considered, inviting future research for this field of study.

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Confilict of Interest

The authors declare no competing financial interest.

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