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Research Article

# $H$-Amalgamation Property in the Class of Torsion Free Abelian Groups

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**Abstract:** It is proved that the classes of torsion free abelian groups and divisible torsion free abelian groups have unique positive existentially closed model up to isomorphism. we give complete description of $h$-amalgamation bases in the class of non-trivial torsion free abelian groups.

***Keywords***: positive logic, $h$-inductive theory, existentially closed, $h$-amalgamation, torsion free abelian groups, semantics, mathematical model

**MSC:** 03C52, 08B25

## 1. Introduction

The notions of existentially closed and amalgamation bases in mathematics are closely tied to the development of model theory and algebra.

The study of existentially closed structures has led to the development of the concept of amalgamation bases, where the amalgamation property is regarded as a fundamental characteristic of existentially closed structures.

In group theory, the concept of existentially closed groups has been a subject of significant interest and has a rich history of evolving relationships with other mathematical disciplines. Initially, it was a part of the theory of infinite groups, as explored by Neumann and Scott [1, 2]. Later, Macintyre [3] and others incorporated it into model theory, while Ziegler [4] transformed it into recursion theory through a fundamental work. However, recent research has revisited the spirit of the early years of the subject, focusing on exploring the behaviors and characteristics of special categories of groups [5, 6]. While logic techniques are employed as needed, the primary goal remains understanding the nature of infinite groups.

In this paper we will study the notions of existentially closed and h-amalgamation bases in the class of free torsion groups in the framework of positive logic. The particularity of positive model theory lies essentially in the study of $h$-inductive theories, and it prohibits the use of the negation operator in the construction of formulas. It focuses on positive formulas rather than general formulas, and on homomorphisms rather than embeddings. This approach creates new scenarios in the study of existentially closed structures and the amalgamation property, which go beyond the traditional framework of first-order logic. For instance, in the framework of positive logic, the class of existentially closed groups is reduced to the trivial group and each group has the property of amalgamation. However, in the framework of the first order logic, the class of existentially closed groups is not axiomatizable. So, to prevent such undesirable implications, we


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need make some modifications to the language of the theory in order to eliminate the trivial structures from the class of models of the theory.

In this paper, we investigate the classes of positive existentially closed and $h$-amalgamation bases of the theory of torsion-free groups. We accomplish our approach by adding a symbol of constant to the language of the theory.

The article is structured as follows. The Section 2 provides an overview of the subject, exploring the fundamental tools positive logic without extensive elaboration. Section 3 contains a description of the class of positive existentially closed torsion free groups. Section 4 is devoted to the examination of the $h$-amalgamation property. Specifically, it establishes a characterization of $h$-amalgamation bases of the theory of torsion-free groups.

## 2. Preliminaries

In this section, we summarize the essential concepts of positive mathematical logic in the framework of the theory of abelian groups that we will need in this paper.

Let $L$ be a first order language (ie. a set of symbols of functions, of relations and constants). An $L$-structure is defined as a set A called the domain of the structure that satisfies the following conditions:

- Every symbol of $n$-ary function $f$ of $L$ is interpreted by a function $f_A$ defined from $A^n$ to $A$.
- For every symbol of $m$-ary symbol of relation $R$ of $L$ there is an interpretation of $R_A$ in $A$ given by a subset of $A^m$.
- Every symbol of constant $c$ of $L$ is interpreted by an element $c_A$ of $A$.

Let $L$ be a language, and let $A$ and $B$ two $L$-structures. A function $f$ from $A$ to $B$ is said to be a $L$-homomorphism if it fulfils the following properties:

- For every symbol of $n$-ary function $h$ of $L$, and for every $\overline{a} = (a_1, \cdots, a_n) \in A^n$; $f(h_A(\overline{a})) = h_B(\overline{f_A(a_i)})$.
- For every symbol of $n$-ary relation $R$ of $L$, and for every $\overline{a} \in A^n$; if $A$ satisfies $R_A(\overline{a})$ then $B$ satisfies $R_B(\overline{f(a_i)})$.
- For every symbols of constant $c$ in $L$, we have $f(c_A) = c_B$.

We say that $B$ is a continuation of $A$ if there exists a $L$-homomorphism from $A$ to $B$.

The $L$-sentences are constructed by combining a finite set of expressions of the form below using the conjunction operator $\wedge$.

$$\forall \overline{y};\ \exists \overline{x} \varphi(\overline{x},\ \overline{y}) \rightarrow \exists \overline{z} \psi(\overline{z},\ \overline{y})$$

where $\varphi(\overline{x},\ \overline{y})$ and $\psi(\overline{z},\ \overline{y})$ are positive existential $L$-formulas (see Example 1).

A $L$-theory $T$ consists of a set of $L$-sentences that can be satisfied by an $L$-structure. Every $L$-structure that satisfies all sentences of $T$ is called a model of $T$.

For further details in positive logic and models theory, [7–9] are sufficiently complete references.

**Example 1**

**Groups:** The language of groups is defined by the set $L_g = \{e,\ \cdot,\ ^{-1}\}$, where $e$ is a symbol of constant, $\cdot$, a symbol of function of arity 2, and, $^{-1}$, a symbol of function of arity 1.

A positive $L_g$-formula $\varphi(\overline{x},\ \overline{y})$ is a finite combination of conjunctions $\vee$ and disjunctions $\wedge$ of formulas in the following form:

$$\exists \overline{y};\ x_{\sigma(1)}^{n_1} \cdot y_{\delta(1)}^{m_1} \cdot x_{\sigma(2)}^{n_2} \cdot y_{\delta(2)}^{m_2} \cdots x_{\sigma(p)}^{n_p} \cdot y_{\delta(q)}^{m_q} = e$$

where $\overline{x} = (x_1,\ x_2,\ \cdots,\ x_n)$, $\overline{y} = (y_1,\ y_2,\ \cdots,\ y_m)$ are variables, $\sigma$ (resp. $\delta$) runs overs the set of mapping defined from a finite interval of $\mathbb{N}$ into the set $\{1,\ \cdots,\ n\}$ (resp, $\{1,\ \cdots,\ m\}$), and $\{n_1,\ \cdots,\ n_p,\ m_1 \cdots,\ m_q\} \subset \mathbb{Z}$. The variables $(x_1,\ x_2,\ \cdots,\ x_n)$ are in the expression of the formula above are said to be free.

A group $G$ is a $L$-structure that satisfies the theory $T_g$, where $T_g$ is the set of the following $L$-sentences:

- $\forall x,\ y,\ z;\ x\cdot(y\cdot z)=(x\cdot y)\cdot z.$
- $\forall x;\ \ e\cdot x=x\cdot e=x.$
- $\forall x;\ \ x\cdot x^{-1}=x^{-1}\cdot x=e.$

where $x,\ y,\ z$ are symbols of variables.

Note that, every variable in the expression of any $L$-sentence it follows the universal quantifier $\forall$ or the existential quantifier $\exists$. In other therms, there are no free variables in the expression of $L$-sentences.

The $L_g$-homomorphisms are the homomorphisms of groups in the usual algebraic sense.

**Abelian groups:** Let $L_{ab}=\{0,\ +,\ -\}$. The abelian groups are the models of $T_{ab}$ the set of the following $L_{ab}$ sentences:

- $\forall x,\ y,\ z;\ x+(y+z)=(x+y)+z.$
- $\forall x,\ y;\ x+y=y+x.$
- $\forall x;\ x+0=x.$
- $\forall x;\ x+(-x)=0.$

The $L_{ab}$-homomorphisms are the usual homomorphisms of groups.

**Non trivial abelian groups:** Let $L^{*}=L_{ab}\cup\{g\}$ be the language of non trivial abelian group, where $g$ is a symbol of constant and $L_{ab}$ the language of abelian groups. Let

$$T_{ab}^{*}=T_{ab}\cup\{g\neq 0\}$$

For every abelien group $G$ and for every $g\in G$, the pair $(G,\ g)$ is a model of $T_{ab}^{*}$ if and only if $g\neq 0$.

Let $(G,\ g)$ and $(K,\ k)$ be two models of $T_{ab}^{*}$ and $f$ a homomorphism of groups defined from $G$ to $K$. $f$ is a $L^{*}$-homomorphism if and only if $f(g)=k$.

**Non trivial torsion free abelian groups:** Let $L^{*}$ be the language of non trivial abelian group. We denote by $T_{tf}$ and $T_{tfd}$ the $L^{*}$-theories of free torsion abelian groups and torsion free divisible abelian groups respectively, where

$$T_{tf}=T_{ab}^{*}\cup\{\forall x,\ \ nx=0\rightarrow x=0|\ n\in\mathbb{N}^{*}\}$$

and

$$T_{tfd}=T_{tf}\cup\{\forall y,\ \exists x;\ nx=y|\ n\in\mathbb{N}^{*}\}.$$

We denote by $(G,\ g)$ the model of $T_{tf}$ (resp. $T_{tfd}$) where $G$ is a non trivial torsion free abelian (resp. divisible) group and $g$ in the interpretation of the constant $c$ of $L^{*}$.

Note that, in the theories $T_{ab}^{*}$, $T_{tf}$ and $T_{tfd}$, the positive $L^{*}$-formulas are of the form:

$$\exists\bar{y}\bigwedge_{i=1}^{p}\bigvee_{j=1}^{q}\phi_{i,\ j}(\bar{x},\ \bar{y}),$$

where $\phi_{i,\ j}(x_1,\ \cdots,\ x_n,\ y_1,\ \cdots,\ y_m)$ are linear equations of the form:

$$\sum_{a=1}^{n}n_{a,\ i}x_a+\sum_{b=1}^{m}n_{b,\ j}y_b=t_{i,\ j}g$$

where $n_{a,\ i}$, $n_{b,\ j}$ and $t_{i,\ j}$ are in $\mathbb{Z}$, and $g$ is the interpretation of the constant $c$ of $L^*$.

**Definition 1** Let $A$ and $B$ be two models of a $L$-theory. A $L$-homomorphism $f$ from $A$ to $B$ is said to be an immersion if and only if; for every $\bar{a}$ of $A$ and every positive formula $\varphi$, if $B$ satisfies $\varphi(\bar{a},\ \bar{b})$ for some $\bar{b} \in B$, then there is $\bar{c} \in A$ such that $A$ satisfies $\varphi(\bar{a},\ \bar{c})$.

**Remark 1** An $L^*$-homomorphism of abelian groups from $(G,\ g)$ to $(K,\ k)$ is an immersion if and only if every finite linear system of equations of the form

$$\sum_{i=1}^{n} n_{i,\ p} a_{i,\ p} + \sum_{j=1}^{m} n_{j,\ p} y_{j,\ p} = t_p k$$

where $a_{i,\ p} \in A$ and $n_{i,\ p}$, $n_{j,\ p}$, $t_p \in \mathbb{Z}$, has a solution in $K$ implies that the system

$$\sum_{i=1}^{n} n_{i,\ p} a_{i,\ p} + \sum_{j=1}^{m} n_{j,\ p} y_{j,\ p} = t_p g$$

has a solution in $G$.

## 3. Existentially closed torsion free abelian groups

Positive existentially closed models of a theory $T$ forms a special non empty class of models of $T$ that satisfy certain properties related to existential formulas. In particular, every positive existential formula that has a solution in any continuation of the model also has a solution within the model itself.

Although not axiomatizable in general, positive existentially closed models are considered to be the most representative models of a theory. This is because each model in this class represents a specific subset of the space of types (see [7, 8]) of the theory.

**Definition 2** A model $A$ of a $L$-theory $T$ is said to be positively existentially closed (pc in short) if and only if every $L$-homomorphism from $A$ to $B$ is an immersion.

Two $L$-theories are said to be companion if every model of one of them can be continued in a model of the other, or equivalently, if they have the same pc models.

Yaacov and Poizat [8] proves the existence of pc models for every $h$-inductive theories (ie, the theories according to the definition adopted in this Section 1). The subsequent Lemma proves to be beneficial in the remainder of the article.

**Lemma 1** (Theorem 1, [8]) Every model of an $h$-inductive theory $T$ is continued in a pc model of $T$.

**Example 2**

1. The trivial group $\{e\}$ is the unique pc model of $T_{ab}$ in the language $L_{ab}$.

2. The pc models of $T^*_{ab}$ are the groups $(\mathbb{Z}(p^\infty),\ z_p)$, where $p$ is a prime number and $z_p$ is a $p$-th root of unity (Proposition 2.13. [10]), where $\mathbb{Z}(p^\infty)$ is the group of all complex $p^n$-th roots of unity.

The following definition and remark introduce some notations and preliminaries which will be useful throughout this paper.

**Definition 3**

• Let $G$ be an abelian group and $H$ a subgroup of $G$. We denote by $\overline{H}$ the local divisible closure of $H$ in $G$, defined by

$$\overline{H} = \{g |\ \exists n \in \mathbb{Z}^*,\ ng \in H\}.$$

• For every torsion free abelian group $G$, we denote by $[G]$ the divisible hull of $G$ (Lemma 3.1.8, [11]), which is the divisible torsion free abelian group defined by

$$[G] = \left\{ \frac{a}{n} \mid n \in \mathbb{N}^*,\ a \in G \right\}.$$

**Remark 2**

1. Let $H$ be a subgroup of a torsion free abelian group. The local divisible closure $\overline{H}$ is a subgroup of $G$, and the quotient group $G/H$ is torsion free if and only if $H = \overline{H}$.

2. For every subgroup $H$ of $G$ we have $\overline{\overline{H}} = \overline{H}$.

3. If $G$ is torsion free divisible group and $H \leq G$, then $\overline{H}$ is divisible. Indeed, let $a \in \overline{H}$ and $m \in \mathbb{Z}$ such that $ma \in H$. For every $n \in \mathbb{N}^*$, we have, $mn\dfrac{a}{n} = ma \in H$. Then $\dfrac{a}{n} \in \overline{H}$.

4. For every torsion free abelian groups $G$ and $K$, and $f$ a homomorphism from $G$ to $K$. There is a unique homomorphism $\bar{f}$ from $[G]$ to $[K]$ such that

$$\bar{f}_{|G} = f.$$

5. Given that every torion free abelian group is embedded in its divisible hull group, and the divisible hull is torsion free abelian group, it follows that $T^*_{tf}$ and $T^*_{tfd}$ are companion theories. In other words, $T^*_{tf}$ and $T^*_{tfd}$ has the same pc models.

**Theorem 1** Every pc model of $T^*_{tf}$ is isomorphic to $\mathbb{Q}$.

**Proof.** Let $(G,\ g)$ be a pc model of $T^*_{tf}$. Recall that a homomorphism $f$ from $G$ into a group $K$ is a $L^*_g$-homomorphism if and only if $f(g) \neq 0$ where 0 is the unit element of $K$.

Let $a \in G$ and let the subgroup

$$[a] = \left\{ \frac{n \cdot a}{m} \mid n \in \mathbb{Z},\ m \in \mathbb{N}^\star \right\}.$$

The quotient group $G/[a]$ is torsion free group (bullet 1, Remark 2). Consider the natural homomorphism of groups $\pi:\ G \to G/[a]$. Since $\pi$ is not an immersion because is not injective, then $\pi(g) = \overline{0}$. Which implies that $g = \dfrac{na}{m}$, so $a = \dfrac{mg}{n}$, then $a \in [g]$. Thereby $G = [g] \approx \mathbb{Q}$. □

**Corollary 1** The class of pc models of theory $T^*_{tf}$ is not $L^*$-axiomatizable.

## 4. *H*-amalgamation property

**Definition 4** A model $A$ of a $L$-theory $T$ is said to be an $h$-amalgamation basis of $T$ if for every models $B$ and $C$ of $T$, $f$ a homomorphism from $A$ to $B$ and $g$ a homomorphism from $A$ to $C$, there are a model $D$ of $T$ and homomorphisms $f'$ and $g'$, such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "g'"] \\ B \arrow[r, "f'"'] & D \end{tikzcd}$$

**Lemma 2** Let $(G,\ g)$ be an $h$-amalgamation basis of $T^*_{tf}$. Then for every subgroups $H_1$ and $H_2$ of $G,$ if $g \in \overline{H_1} + \overline{H_2}$ then $g \in \overline{H}_1$ or $g \in \overline{H}_2$.

**Proof.** Assume that $g \notin \overline{H}_1 \cup \overline{H}_2$. Then $(G/\overline{H}_1,\ \bar{g})$ and $(G/\overline{H}_2,\ \bar{g})$ are models of $T^*_{tf}$, and the natural homomorphisms $\pi_i$ defined from $G$ into $G/\overline{H}_1$ where $1 = 1,\ 2$ are $L^*$ homomorphisms. Given that $(G,\ g)$ is an $h$-amalgamation basis of $T^*_{tf}$, let $(D,\ d)$ be a model of $T^*_{tf}$ such that the following diagram commutes:

$$\begin{tikzcd} (G,\ g) \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & (G/\overline{H}_1,\ \pi_1(g)) \arrow[d, "f"] \\ (G/\overline{H}_2,\ \pi_2(g)) \arrow[r, "h"'] & (D,\ d) \end{tikzcd}$$

where $f$ and $g$ are $L^*$-homomorphisms. □

We claim that $\pi_1(g) \notin \pi_1(\overline{H_2})$. If not, there is $a \in \overline{H_2}$ such that $\pi_1(g) = \pi_1(a)$. Then,

$$\begin{aligned} d =& f \circ \pi_1(g) \\ =& f \circ \pi_1(a) \\ =& h \circ \pi_2(a) \\ =& 0. \end{aligned}$$

A contradiction. So for every $a \in \overline{H_2}$, we have $g - a \notin \overline{H_1}$. So $g \notin \overline{H_1} + \overline{H_2}$.

**Lemma 3** Let $G$ be a torsion free divisible group. Let $g \in G - \{0\}$ such that; for every subgroups $H_1$ and $H_2$ of $G,$ if $g \in \overline{H_1} + \overline{H_2}$ then $g \in \overline{H}_1$ or $g \in \overline{H}_2$. Then $G$ is an $h$-amalgamation basis of $T^*_{tf}$.

**Proof.** Let $(K,\ k)$ and $(L,\ l)$ be two models of $T^*_{tf}$. Let $f$ a $L^*$-homomorphism from $(G,\ g)$ to $(K,\ k)$, and $h$ a $L^*$-homomorphism from $(G,\ g)$ to $(L,\ l)$. Let $H$ be the subgroup of $K \times L$ defined by;

$$H = \{(f(x),\ h(-x)) |\ x \in G\}.$$

We claim that $K \times L/H$ is a torsion free group. Indeed, let $(x,\ y) \in K \times L$ and $n \in \mathbb{N}^*$ such that $(nx,\ ny) \in H$. So there is $a \in G$ such that $nx = f(a)$ and $ny = h(-a)$. Given that $G$ is divisible, let $b \in G$ such that $a = nb$. Thus, we have $(x,\ y) = (f(b),\ h(-b)) \in H$. □

Now, let the diagram:

$$\begin{tikzcd} (G, g) \arrow[r, "f"] \arrow[d, "h"'] & (K, k) \arrow[d, "f'"] \\ (L, l) \arrow[r, "h'"'] & (K \times L/H, \overline{(k, l)}) \end{tikzcd}$$

where $f'$ and $g'$ are the homomorphisms of groups defined as follows:

$$\begin{cases} f'(x) = \overline{(x,\ 0)} & x \in K \\ g'(y) = \overline{(0,\ y)} & y \in L \end{cases}$$

It is clear that the diagram commutes. To complete the proof, it is sufficient to show that $f'$ and $g'$ are $L^*$-homomorphisms.

Suppose that $f'(k) = \overline{(k,\ 0)} = \overline{(0,\ 0)}$, so there is $a \in G$ such that $(k,\ 0) = (f(a),\ h(-a))$. Then

$$\begin{cases} f(a) = k = f(g) \\ a \in ker(h) \end{cases}$$

which implies that $g - a \in ker(f)$ and $g \in ker(f) + ker(g)$. Given that $\overline{ker(f)} = ker(f)$ (ie, $\overline{ker(f)}$ the local divisible closure) and $\overline{ker(h)} = ker(h)$, it follows from the hypothesis of the lemma that $g \in ker(f)$ or $g \in ker(h)$, a contradiction. So $f'$ is a $L^*$-homomorphism. □

**Theorem 2** Let $G$ be a torsion free group and $g \in G - \{0\}$. The pair $(G,\ g)$ is an $h$-amalgamation basis of $T^*_{tf}$ if and only if for every subgroups $H_1$ and $H_2$ of $G$, if $g \in \overline{H_1} + \overline{H_2}$ then $g \in \overline{H}_1$ or $g \in \overline{H}_2$.

**Proof.** Let $G$ be a non trivial torsion free group. Let $[G]$ be the torsion free divisible hull of $H$. Let $g \in G - \{0\}$ such that $(G,\ g)$ satisfies the hypothesis of the Theorem. We claim that $([G],\ g)$ satisfies the hypothesis of the Theorem. Indeed, let $H_1$ and $H_2$ be two subgroups of $[G]$ such that $g \in \overline{H_1} + \overline{H_2}$. Assume that $g = \dfrac{a_1}{n_1} + \dfrac{a_2}{n_2}$, where $\dfrac{a_i}{n_i} \in H_i$ and $a_i \in G$. Then, $n_1 n_2 g \in \overline{< a_1 >} + \overline{< a_2 >}$ in $G$, where $< a_i >$ is the subgroup of $G$ generated by $a_i$. By the hypothesis of the Theorem, suppose that $n_1 n_2 g \in \overline{< a_1 >}$, then $n_1 g \in \overline{< a_1 >}$. Which implies that $g \in \overline{< \dfrac{a_1}{n_1} >} \le \overline{H_1}$. Thus, by Lemma 3, $([G],\ g)$ is an $h$-amalgamation basis of $T^*_{tf}$. □

Now, consider $(K,\ k)$ and $(L,\ l)$ two models of $T^*_{tf}$. Let $f$ and $g$ be two $L^*$-homomorphisms from $(G,\ g)$ to $(K,\ k)$ and $(L,\ l)$ respectively. Given that $([G],\ g)$ is an $h$-amalgamation basis, we get the following commutative diagram:

$$\begin{tikzcd}
 & (K,k) \arrow[r, "i_2"] & ([K],k) \arrow[dr, "g_1"] & \\
(G,g) \arrow[ur, "f"] \arrow[r, "i_1"] \arrow[dr, "h"'] & ([G],g) \arrow[ur, "\overline{f}"] \arrow[dr, "\overline{h}"'] & & (D,d) \\
 & (L,l) \arrow[r, "i_3"'] & ([L],l) \arrow[ur, "g_2"'] &
\end{tikzcd}$$

Wherez, $[K]$ and $[L]$ are the divisible closures of $K$ and $L$ respectively. $i_1$, $i_2$ and $i_3$ are the natural embeddings from the groups into their torsion free divisible hull. $\bar{f}$ and $\bar{h}$ are the $L^*$-homomorphisms induced by the $L^*$-homomorphisms $f$ and $g$ respectively (bullet 4, Remark 2). $(D,\ d)$, $g_1$ and $g_2$ are respectively the model of $T^*_{tf}$ and the $L^*$-homomorphisms given by the amalgamation property of $([G],\ g)$. Thereby, $(G,\ g)$ is an $h$-amalgamation basis of $T^*_{tf}$.

The proof on the other direction of the Theorem follows directly from Lemma 2.

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## Conflict of interest

The author declares no competing interests.

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