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Research Article

The Hybrid and Interaction Solutions with Resonance Y-Type Solitons for the (2 + 1)-Dimensional KdV Equation

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Abstract: The hybrid solutions and interaction solutions with resonance Y-type solitons are mainly investigated for the (2 + 1)-dimensional Korteweg-de Vries (KdV) equation, concerning lumps, resonance Y-type solitons and interaction solutions. Firstly, by the Hirota bilinear form, we deduce the periodic solitary wave solutions through modified three-wave method. Consequently, the lumps are gained in virtue of the parameter limit approach. Secondly, when applying some constraint conditions to the N-solitons, the resonance Y-type solitons are obtained. They are new types of soliton solutions for this system. Thirdly, adding other constraint conditions to the resonance Y-type solitons, the novel interaction phenomena of resonance Y-type solitons with breathers, and resonance Y-type solitons with the lumps arise. Finally, two types of hybrid solutions are obtained for this system, including the solitons, breathers and lumps; resonance Y-type solitons, breathers and lumps, respectively.

Keywords: the degeneration solutions, resonance Y-type soliton solutions, lump solutions, interaction solutions, (2 + 1)-dimensional KdV equation

MSC: 35Q35, 35Q51

1. Introduction

Nonlinear sciences are very important for the development of researches about the nature phenomenon from linear to nonlinear. Recently, nonlinear problems have been paid more and more attention by many scholars. Then nonlinear partial differential equations (NLPDEs) had been studied for explaining nonlinear phenomena in nature. Researches on the exact solutions of NLPDEs are beneficial for people to grasp the pattern of phenomena over time, to understand the nature of these nonlinear phenomena and to solve practical problems better. As is well known, numerous methodologies have emerged for solving the exact solutions for NLPDEs, including the method of Hirota bilinear form [1, 2], homoclinic test technique [3, 4], three-wave method [5, 6], the parameter limit approach [7, 8], Painlevé expansion technique [9, 10], Lie group method [11, 12], etc. Various types of exact solutions have also been acquired, such as soliton solutions [9, 13, 14], breather solutions [15–17], lump solutions [18, 19], interaction solutions [7, 8, 20, 21], etc.

Recently, resonance Y-type solitons are extensively investigated by introducing appropriate constraints into N-solitons [22–24]. They are the novel soliton solutions. Moreover, the hybrid solutions generated by resonance Y-type

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solitons combining with breathers and lumps solutions for (2 + 1)-dimensional generalized Bogoyavlensky-Konopelchenko equation, generalized (2 + 1)-dimensional nonlinear wave equation, extended (3 + 1)-dimensional Hirota-Satsuma-Ito equation had been given in [22–24], respectively.

In 1895, the Korteweg-de Vries (KdV) equation had been firstly deduced by Korteweg and de Vries' in the processes of research about shallow water waves [25]. As an important model, the KdV equation has been extensively paid attention by lots of scholars in the world. And it can be extended to (2 + 1)-dimensional KdV equation [26]

$$\begin{cases} u_t - u_{xxx} + 3(uv)_x = 0, \\ u_x = v_y. \end{cases} \tag{1}$$

Notice that the above system (1) will reduce to classical KdV equation in the case $x = y$. Concerning the classical KdV equation, many exact solutions are given. For example, in [27], the soliton solutions had been obtained through the Bäcklund transform. The solitons and the periodic solutions were gotten in [28] by cosh ansatz, tanh-coth or Exp-function method. In [6], the multi-soliton solutions have been given by the three-wave method. In addition, in [15] according to parameter limit technique, some multi-breathing solutions, N-solitons, M-lump solutions had been yielded. As for the system (1) under consideration, some particular solutions and the Painlevé property are studied in [10]. Currently, the system (1) is still being studied to yield some new exact solutions. On the one hand, the degeneration solutions have few studied about the (2 + 1)-dimensional KdV equation, which is a novel idea to considering the nonlinear wave solutions. Moreover, the resonance Y-type solitons and interactions arising from them are new types of nonlinear wave solutions. They are active thesis of researches in recent years.

In this paper, the degeneration phenomena been firstly investigated. Then the interaction solutions involving resonance Y-type solitons are also majorly studied for the (2 + 1)-dimensional KdV equation. In section 2, lump solutions are given via degeneration of the periodic solitary wave solutions, which are derived from the modified three-wave approach. In section 3, resonance Y-type solitons are yielded through adding some constraints into the N-solitons. Moreover, the 2-resonance, 3-resonance, and two 2-resonance Y-type soliton solutions will be provided. In the next section 4, four different types of novel interaction solutions can be considered, which include breathers and the resonance Y-type solitons; lumps and the resonance Y-type solitons; the solitons, breathers and lumps; breathers, lumps and the resonance Y-type solitons. The process of their interaction is also presented in the figures. In section 5, it is devoted to conclusions.

2. The degenerated solutions of the periodic solitary wave

Through Painlevé analysis, the following transformation of the function [15]

$$\begin{cases} u(x,\ y,\ t) = u_0 - 2[\ln \mathscr{H}(x,\ y,\ t)]_{xy}, \\ v(x,\ y,\ t) = v_0 - 2[\ln \mathscr{H}(x,\ y,\ t)]_{xx}, \end{cases} \tag{2}$$

can be considered, where $\mathscr{H}(x,\ y,\ t)$ is a function required to be determined, u_0 and v_0 are initial values. Then the system (1) has the following Hirota bilinear form

$$(D_yD_t - D_x^3D_y + 3v_0D_xD_y + 3u_0D_x^2)\mathscr{H} \cdot \mathscr{H} = 0. \tag{3}$$

Via the idea relating three-wave approach in [5] one can assume that an undetermined function $\mathscr{H}(x,\ y,\ t)$ is

$$\mathscr{H} = \exp(-\phi_1) + \delta_2 \cos\phi_2 + \delta_3 \tanh\phi_3 + \delta_1 \exp(\phi_1), \tag{4}$$

where $\phi_i = k_i(a_i x + b_i y + c_i t + r_i)$, and $k_i,\ a_i,\ b_i,\ c_i,\ r_i,\ \delta_i,\ (i = 1,\ 2,\ 3)$ are parameters needed to be determined below. Adding (4) into (3), some polynomials of $\exp(-\phi_1)$, $\exp(\phi_1)$, $\cos\phi_2$, $\tanh\phi_3$ are deduced. Take coefficients to be 0. So we can obtain some equations about the parameters. Through solving equations, there are some different cases.

Case I

$$a_2 = 0,\ a_3 = 0,\ b_1 = 0,\ c_1 = k_1^2 a_1^3 - 3a_1 v_0,\ c_2 = 0,\ c_3 = 0,\ u_0 = 0. \tag{5}$$

Adding the relations of (5) into (4), one has

$$\mathscr{H} = \exp(-\phi_1) + \delta_2 \cos\phi_2 + \delta_3 \tanh\phi_3 + \delta_1 \exp(\phi_1), \tag{6}$$

where $\phi_1 = k_1(a_1 x + (a_1^3 k_1^2 - 3a_1 v_0)t + r_1),\ \phi_2 = k_2(b_2 y + r_2),\ \phi_3 = k_3(b_3 y + r_3)$ when $a_1 b_i k_j \delta_j \neq 0$ for $i = 2,\ 3$ and $j = 1,\ 2,\ 3$.

Adding (6) into (2), one has

$$u_{21} = 2a_1 k_1[-\exp(-\phi_1) + \delta_1 \exp(\phi_1)] \times \frac{[-\delta_2 k_2 b_2 \sin\phi_2 + \delta_3 k_3 b_3 (1 - \tanh^2\phi_3)]}{[\exp(-\phi_1) + \delta_2 \cos\phi_2 + \delta_3 \tanh\phi_3 + \delta_1 \exp(\phi_1)]^2},$$

$$v_{21} = v_0 - 2a_1^2 k_1^2 \times \frac{[(\exp(-\phi_1) + \delta_1 \exp(\phi_1))(\delta_2 \cos\phi_2 + \delta_3 \tanh\phi_3) + 4\delta_1]}{[\exp(-\phi_1) + \delta_2 \cos\phi_2 + \delta_3 \tanh\phi_3 + \delta_1 \exp(\phi_1)]^2}. \tag{7}$$

The solutions u_{21} and v_{21} show the periodic properties, which are periodic solitary wave solutions. As the properties of $u(x,\ y,\ t)$ are similar to $v(x,\ y,\ t)$, this section only discusses $u(x,\ y,\ t)$. By analyzing the evolutionary behaviors of u_{21} in Figure 1, the solution's position is moving along the positive direction in the x-axis with time t increasing.

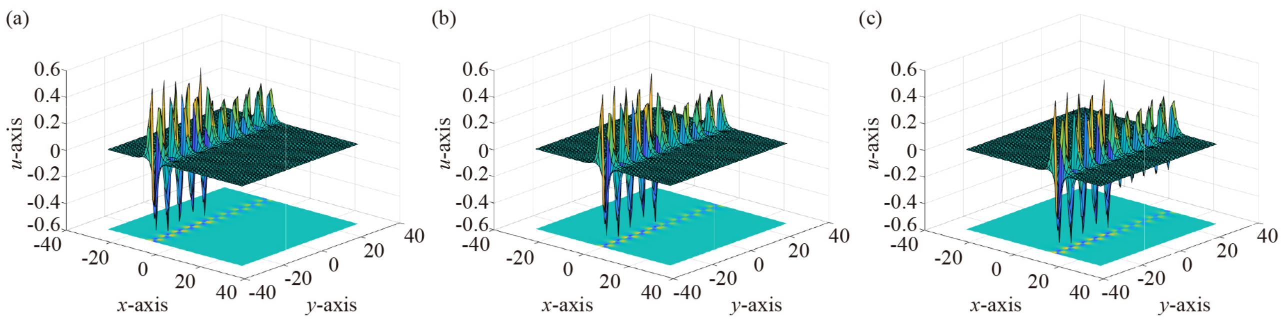

Figure 1. u_{21} evolutionary behaviors when $a_1 = b_1 = b_2 = b_3 = k_1 = k_2 = \delta_2 = 1,\ k_3 = \delta_1 = 2,\ \delta_3 = v_0 = 0.5,\ r_1 = r_2 = r_3 = 0$. (a) $t = -20$, (b) $t = 0$, (c) $t = 20$

Assume $k_1 = lk_3$, $k_2 = nk_3$, $\delta_1 = 1$, $\delta_2 = -2\cos(sk_3)$, $\delta_3 = \tan(mk_3)$, where $\delta_i (i = 1,\ 2,\ 3)$ are parameters such that $\lim_{k_i \to 0}(1 + \delta_1 + \delta_2 + \delta_3) = 0$. When $k_3 \to 0$, the lump solutions are obtained, which are

$$u_{22}=\frac{4a_1l^2\theta_1(2n^2b_2\theta_2+mb_3)}{(l^2\theta_1^2+s^2+n^2\theta_2^2+m\theta_3)^2},$$

$$v_{22}=v_0-\frac{4a_1^2l^2(s^2-l^2\theta_1^2+n^2\theta_2^2+m\theta_3)}{(l^2\theta_1^2+s^2+n^2\theta_2^2+m\theta_3)^2}, \tag{8}$$

where $\lim_{k_i\to 0}\frac{\phi_i}{k_i}=\theta_i$, and $\theta_1=a_1x-3a_1v_0t+r_1,\ \theta_2=b_2y+r_2,\ \theta_3=b_3y+r_3$. The evolutionary behaviors of the u_{22} is demonstrated in Figure 2, where the position of the solutions is moving along with the positive direction in the x-axis with time t increasing.

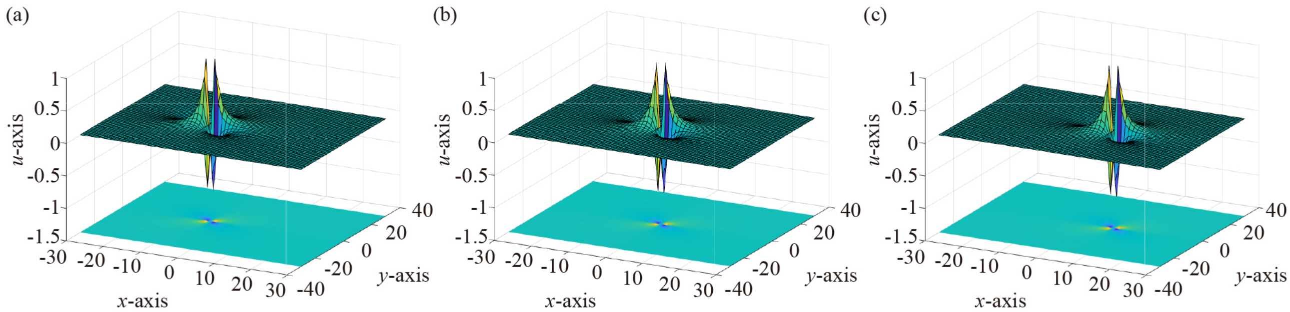

Figure 2. The lump solutions u_{22} when $a_1=b_2=b_3=m=n=l=s=1$, $v_0=0.5$, $r_1=r_2=r_3=0$. (a) $t=-4$, (b) $t=0$, (c) $t=4$

Moreover, when t goes to infinity, the solutions u_{21} and u_{22} tend to 0, v_{21} and v_{22} tend to v_0. The degenerated solutions have the same trends as the lump solutions at infinity, because the lump solutions are directly constructed by rational functions through the polynomial test functions. A stable point is given when time goes infinity. Therefore, when considering the property of solutions with t goes to infinity, it suffices to discuss the lump solutions.

Case II

$$a_1=0,\ a_3=0,\ b_2=0,\ c_2=-a_2^3k_2^2-3a_2v_0,\ c_3=0,\ k_1=0,\ u_0=0. \tag{9}$$

Adding the relations of (9) into (4), one has

$$\mathscr{H}=1+\delta_1+\delta_2\cos\phi_2+\delta_3\tanh\phi_3, \tag{10}$$

where $a_2b_3k_i\delta_j\neq 0$, $\phi_2=k_2(a_2x+(-a_2^3k_2^2-3a_2v_0)t+r_2)$ and $\phi_3=k_3(b_3y+r_3)$, $(i=2,\ 3,\ j=1,\ 2,\ 3)$.

Adding (10) into (2), one has

$$u_{23} = -\frac{2k_2k_3a_2b_3\delta_2\delta_3 \sin\phi_2(1-\tanh^2\phi_3)}{(1+\delta_1+\delta_2\cos\phi_2+\delta_3\tanh\phi_3)^2},$$

$$v_{23} = v_0 + \frac{2k_2^2a_2^2\delta_2[\cos\phi_2(1+\delta_1+\delta_3\tanh\phi_3)+\delta_2]}{(1+\delta_1+\delta_2\cos\phi_2+\delta_3\tanh\phi_3)^2}. \tag{11}$$

Evidently, the solutions u_{23} and v_{23} are also the periodic solitary wave solutions.

Similarly, we take $k_2 = nk_3$, $\delta_1 = 1$, $\delta_2 = -2\cos(sk_3)$, $\delta_3 = \tan(mk_3)$. When $k_3 \to 0$, one has

$$u_{24} = \frac{4mn^2a_1b_3\theta_2}{(s^2+n^2\theta_2^2+m\theta_3)^2},$$

$$v_{24} = v_0 - \frac{4a_2^2n^2(s^2-n^2\theta_2^2+m\theta_3)}{(s^2+n^2\theta_2^2+m\theta_3)^2}, \tag{12}$$

where $\theta_2 = a_2x - 3a_2v_0t + r_2,\ \theta_3 = b_3y + r_3$.

Case III

$$a_1 = 0,\ a_3 = 0,\ b_2 = 0,\ c_1 = 0,\ c_2 = -a_2^3k_2^2 - 3a_2v_0,\ c_3 = 0,\ u_0 = 0. \tag{13}$$

Adding the relations of (13) into (4), one has

$$\mathscr{H} = \exp(-\phi_1) + \delta_2\cos\phi_2 + \delta_3\tanh\phi_3 + \delta_1\exp(\phi_1), \tag{14}$$

where $\phi_1 = k_1(b_1y + r_1),\ \phi_2 = k_2(a_2x + (-a_2^3 - 3a_2v_0)t + r_2),\ \phi_3 = k_3(b_3y + r_3)$ with the constraints $a_2b_ik_j\delta_j \neq 0$ for $i = 1,\ 3$ and $j = 1,\ 2,\ 3$.

Adding (14) into (2), one has

$$u_{25} = -2a_2k_2\delta_2\sin\phi_2 \times \frac{[k_1b_1(-\exp(-\phi_1)+\delta_1\exp(\phi_1)) + \delta_3k_3b_3(1-\tanh^2\phi_3)]}{[\exp(-\phi_1)+\delta_2\cos\phi_2+\delta_3\tanh\phi_3+\delta_1\exp(\phi_1)]^2},$$

$$v_{25} = v_0 - 2a_2^2k_2^2\delta_2^2\cos\phi_2 \times \frac{[\exp(-\phi_1)+\delta_3\tanh\phi_3+\delta_1\exp(\phi_1)]}{[\exp(-\phi_1)+\delta_2\cos\phi_2+\delta_3\tanh\phi_3+\delta_1\exp(\phi_1)]^2}. \tag{15}$$

Similarly, the solutions u_{25} and v_{25} are also the periodic solitary wave solutions. It differs from (7) in that ϕ_2 contains the two independent variables about $x,\ t$ and the ϕ_1 contains the independent variable y.

We take $k_1 = lk_3$, $k_2 = nk_3$, $\delta_1 = 1$, $\delta_2 = -2\cos(sk_3)$, $\delta_3 = \tan(mk_3)$, with real numbers l, n, s, m. Taking $k_3 \to 0$, the degenerated solutions are obtained

$$u_{26}=\frac{4a_2n^2\theta_2(2l^2b_1\theta_1+mb_3)}{(l^2\theta_1^2+s^2+n^2\theta_2^2+m\theta_3)^2},$$

$$v_{26}=v_0-\frac{4a_2^2n^2(s^2-n^2\theta_2^2+l^2\theta_1^2+m\theta_3)}{(l^2\theta_1^2+s^2+n^2\theta_2^2+m\theta_3)^2}, \tag{16}$$

where $\theta_1=k_1(b_1y+r_1),\ \theta_2=a_2x+(-a_2^3-3a_2v_0)t+r_2,\ \theta_3=b_3y+r_3$.

3. Resonance Y-type soliton solutions

Let $\mathscr{H}_N$ [15] be

$$\mathscr{H}_N=\sum_{\mathscr{A}_i=0,1}\exp\Big(\sum_{i=1}^{N}\mathscr{A}_i\Psi_i+\sum_{i<j}^{N}\mathscr{K}_{ij}\mathscr{A}_i\mathscr{A}_j\Big). \tag{17}$$

Here $\Psi_i=\alpha_i\left(x+\rho_iy+\left(\alpha_i^2-3v_0-\dfrac{3u_0}{\rho_i}\right)t\right)+\Psi_i^{(0)},\ \alpha_i\neq 0,\ \rho_i\neq 0$ and $\Psi_i^{(0)}$ are arbitrary constants, and

$$\exp(\mathscr{K}_{ij})=\frac{\rho_i\rho_j(\alpha_i-\alpha_j)(\alpha_i\rho_i-\alpha_j\rho_j)+u_0(\rho_i-\rho_j)^2}{\rho_i\rho_j(\alpha_i+\alpha_j)(\alpha_i\rho_i+\alpha_j\rho_j)+u_0(\rho_i-\rho_j)^2}. \tag{18}$$

Substituting (17) into (2), N-solitons are yielded. Due to the limitation of space variables, only $u(x, y, t)$ are concerned in the following paper.

Based on the same idea in studying the resonance Y-type solitons [23], similar constraints as in (18)

$$N=P+Q,\ \ \exp(\mathscr{K}_{ij})=0,\ \ 1\leq i<j\leq P,\ \ P<i<j\leq Q, \tag{19}$$

$$\rho_j=\frac{\rho_i\left[\alpha_i\rho_i(\alpha_i-\alpha_j)-2u_0\pm\sqrt{\rho_i(\alpha_i^2\rho_i-4u_0)(\alpha_i-\alpha_j)^2}\right]}{2\alpha_j\rho_i(\alpha_i-\alpha_j)-2u_0}, \tag{20}$$

and

$$\rho_i(\alpha_i^2\rho_i-4u_0)>0 \tag{21}$$

are employed. Then, one can deduce the interaction phenomena of P-resonance and Q-resonance Y-type solitons.

When $N=2$, we acquire

$$
\begin{aligned}
u_{31} &= u_0 - 2[\ln \mathscr{H}_2]_{xy} = u_0 - 2[\ln(1+\exp(\Psi_1)+\exp(\Psi_2))]_{xy} \\
&= u_0 - 2\frac{(\alpha_1+\alpha_2)(\alpha_1\rho_1+\alpha_2\rho_2)\exp(\Psi_1+\Psi_2)}{[1+\exp(\Psi_1)+\exp(\Psi_2)]^2} \\
&\quad - 2\frac{\alpha_1^2\rho_1\exp(\Psi_1)(1+2\exp(\Psi_1))+\alpha_2^2\rho_2\exp(\Psi_2)(1+2\exp(\Psi_2))}{[1+\exp(\Psi_1)+\exp(\Psi_2)]^2},
\end{aligned} \tag{22}
$$

where ρ_2 satisfies (20), Ψ_1 and Ψ_2 satisfy (17). It is worth noting that the two cases "+" and "−" are included in (20). So, we speculate that the different values of ρ_j can lead to different shapes for Y-type soliton solutions when other parameters are the same. Figure 1 exhibits two Y-shaped solitons with openings in opposite directions. This speculation is confirmed by Figure 3, which shows the two cases of (22) in Figure 3a and b, respectively.

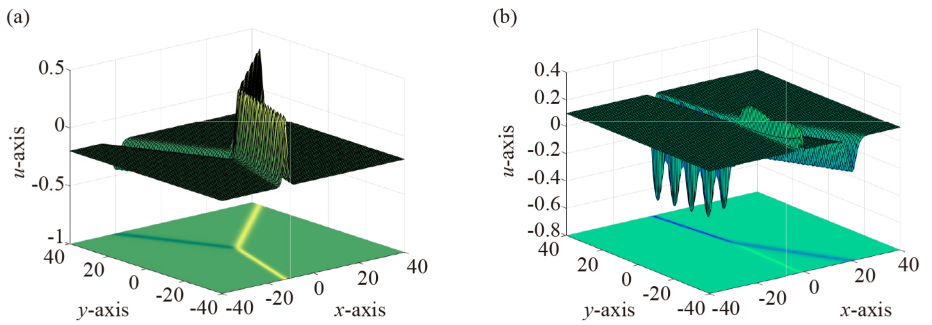

Figure 3. Two different 2-resonance Y-type solitons when $t=0$, $u_0=0.1$, $v_0=-0.1$, $\alpha_1=1$, $\alpha_2=-3$, $\rho_1=0.5$, $\Psi_1^{(0)}=\Psi_2^{(0)}=0$. (a) $\rho_2=-0.1104$, (b) $\rho_2=-0.0371$

When $N=3$, we obtain [24]

$$
\mathscr{H}_3 = 1+\exp(\Psi_1)+\exp(\Psi_2)+\exp(\Psi_3)+\exp(\Psi_2+\Psi_3+\mathscr{K}_{23}), \tag{23}
$$

where ρ_2 and ρ_3 satisfy (20), $\exp(\mathscr{K}_{23})$ satisfies (18). The 3-resonance Y-type solitons are depicted in Figure 4. Due to the similarity of its shape to the letter "X", it is also referred to as an X-type soliton.

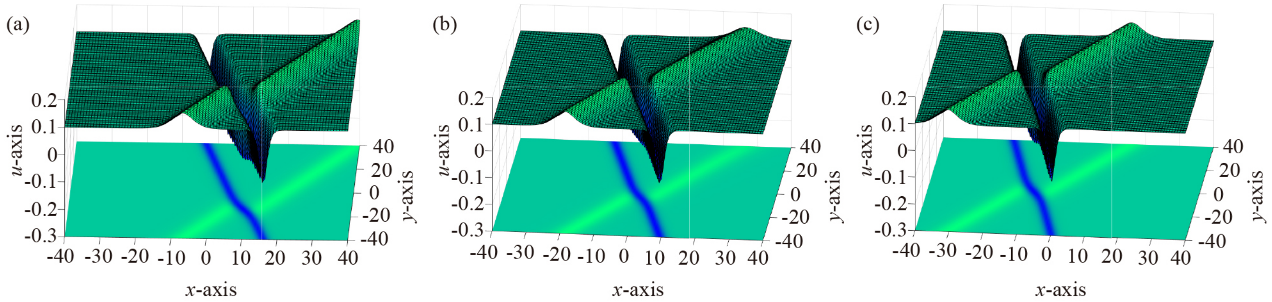

Figure 4. The 3-resonance Y-type solitons when $u_0 = 0.1,\ v_0 = -0.1,\ \alpha_1 = 1,\ \alpha_2 = -0.5,\ \alpha_3 = 1.4,\ \rho_1 = 0.5,\ \rho_2 = -0.4660,\ \rho_3 = 0.2043,\ \Psi_1^{(0)} = \Psi_2^{(0)} = \Psi_3^{(0)} = 0$. (a) $t = -10$, (b) $t = 0$, (c) $t = 10$

When $N = 4$, we gain

$$\mathscr{H}_4 = 1 + \exp(\Psi_1) + \exp(\Psi_2) + \exp(\Psi_3) + \exp(\Psi_4) + \exp(\Psi_1 + \Psi_3 + \mathscr{K}_{13})$$

$$+ \exp(\Psi_1 + \Psi_4 + \mathscr{K}_{14}) + \exp(\Psi_2 + \Psi_3 + \mathscr{K}_{23}) + \exp(\Psi_2 + \Psi_4 + \mathscr{K}_{24}), \tag{24}$$

where ρ_2 and ρ_4 satisfy (20), $\exp(\mathscr{K}_{ij})$ satisfy (18) with $i = 1,\ 2,\ j = 3,\ 4$, and $i < j$. In Figure 5, it indicates the separation to fusion for re-separation of two 2-resonance Y-type solitons along opposite directions with time. In addition, two Y-type solitons represent opposite behaviors. Thus, they are elastic collisions.

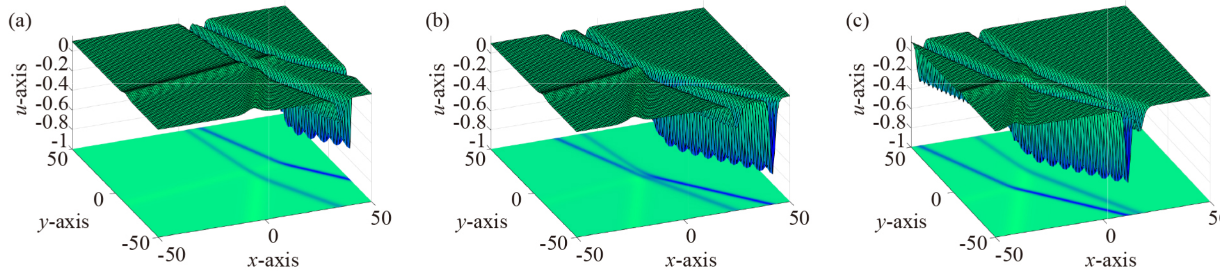

Figure 5. Two 2-resonance Y-type solitons when $u_0 = 0.1,\ v_0 = -0.1,\ \alpha_1 = 1,\ \alpha_2 = -0.5,\ \alpha_3 = 1.4,\ \alpha_4 = -0.1,\ \rho_4 = -5.7232,\ \rho_2 = -0.4660,\ \rho_1 = 0.5,\ \rho_3 = 0.8,\ \Psi_1^{(0)} = \Psi_2^{(0)} = \Psi_3^{(0)} = \Psi_4^{(0)} = 0$. (a) $t = -15$, (b) $t = 0$, (c) $t = 15$

4. New interaction phenomena

4.1 *Interactions of breathers and resonance Y-type solitons*

Aiming at the interactions phenomena of R-breathers and P-resonance Y-type solitons of (1), some new constraint conditions in (19) are introduced as following

$$N = P + 2R,\ \ \exp(\mathscr{K}_{ij}) = 0,\ \ \Psi_{P+2r-1} = \Psi^*_{P+2r},\ \ 1 \le i < j \le P,\ \ 1 \le r \le R, \tag{25}$$

where $*$ denotes conjugate. In the range $1 \le r \le R$, the parameters need to satisfy

$$\begin{cases} \alpha_{P+2r-1} = \alpha^*_{P+2r} = \alpha_{P+2r-11} + \mathrm{i}\alpha_{P+2r-12}, \\ \rho_{P+2r-1} = \rho^*_{P+2r} = \rho_{P+2r-11} + \mathrm{i}\rho_{P+2r-12}, \\ \Psi^{(0)}_{P+2r-1} = \Psi^{(0)}_{P+2r} = 0, \ \ r = 1,\ 2,\ ...,\ R, \end{cases} \tag{26}$$

where i indicates imaginary number, $\alpha_{P+2r-11},\ \alpha_{P+2r-12},\ \rho_{P+2r-11}$ and $\rho_{P+2r-12}$ are real numbers. Substituting (26) into Eq.(17), we get

$$\begin{cases} \Psi_{P+2r-1} = \Psi^*_{P+2r} = \Psi_{P+2r-11} + \mathrm{i}\Psi_{P+2r-12}, \\ \Psi_{P+2r-11} = \alpha_{P+2r-11}x + (\alpha_{P+2r-11}\rho_{P+2r-11} - \alpha_{P+2r-12}\rho_{P+2r-12})y \\ \qquad - \left[\alpha_{P+2r-11}(2\alpha^2_{P+2r-11} + 3v_0) + \dfrac{3u_0(\alpha_{P+2r-11}\rho_{P+2r-11} + \alpha_{P+2r-12}\rho_{P+2r-12})}{\rho^2_{P+2r-11} + \rho^2_{P+2r-12}}\right] t, \\ \Psi_{P+2r-12} = \alpha_{P+2r-12}x + (\alpha_{P+2r-11}\rho_{P+2r-12} + \alpha_{P+2r-12}\rho_{P+2r-11})y \\ \qquad + \left[\alpha_{P+2r-12}(2\alpha^2_{P+2r-12} - 3v_0) + \dfrac{3u_0(\alpha_{P+2r-11}\rho_{P+2r-12} - \alpha_{P+2r-12}\rho_{P+2r-11})}{\rho^2_{P+2r-11} + \rho^2_{P+2r-12}}\right] t, \\ r = 1,\ 2,\ ...,\ R \end{cases} \tag{27}$$

Case I Take $N = 4,\ P = 2,\ R = 1.$

The interactions of 1-breather and 2-resonance Y-type solitons are yielded by

$$\begin{aligned} \mathscr{H}_4 = {} & 1 + \exp(\Psi_1) + \exp(\Psi_2) + \exp(\Psi_3) + \exp(\Psi_4) + \exp(\Psi_1 + \Psi_3 + \mathscr{K}_{13}) \\ & + \exp(\Psi_1 + \Psi_4 + \mathscr{K}_{14}) + \exp(\Psi_2 + \Psi_3 + \mathscr{K}_{23}) + \exp(\Psi_2 + \Psi_4 + \mathscr{K}_{24}) \\ & + \exp(\Psi_3 + \Psi_4 + \mathscr{K}_{34}) + \exp(\Psi_1 + \Psi_3 + \Psi_4 + \mathscr{K}_{13} + \mathscr{K}_{14} + \mathscr{K}_{34}) \\ & + \exp(\Psi_2 + \Psi_3 + \Psi_4 + \mathscr{K}_{23} + \mathscr{K}_{24} + \mathscr{K}_{34}). \end{aligned} \tag{28}$$

Substituting (27) into (18) and (28), we obtain

$$
\begin{aligned}
\hat{\mathscr{H}}_4 = {} & 1+\exp(\Psi_1)+\exp(\Psi_2)+2\cos\Psi_{32}\exp(\Psi_{31})+\cos\Psi_{32}\exp(\Psi_1+\Psi_{31}) \\
& \times[\exp(\mathscr{K}_{13})+\exp(\mathscr{K}_{14})]+\cos\Psi_{32}\exp(\Psi_2+\Psi_{31})[\exp(\mathscr{K}_{23})+(\mathscr{K}_{24})] \\
& +\mathrm{i}\sin\Psi_{32}\exp(\Psi_1+\Psi_{31})[\exp(\mathscr{K}_{13})-\exp(\mathscr{K}_{14})]+\mathrm{i}\sin\Psi_{32}\exp(\Psi_2 \\
& +\Psi_{31})[\exp(\mathscr{K}_{23})-\exp(\mathscr{K}_{24})]+\exp(2\Psi_{31}+\mathscr{K}_{34})+\exp(\Psi_1+2\Psi_{31} \\
& +\mathscr{K}_{13}+\mathscr{K}_{14}+\mathscr{K}_{34})+\exp(\Psi_2+2\Psi_{31}+\mathscr{K}_{23}+\mathscr{K}_{24}+\mathscr{K}_{34}), \qquad (29)
\end{aligned}
$$

where

$$
\begin{aligned}
& \Psi_i=\alpha_i(x+\rho_i y+(\alpha_i^2-3v_0-\frac{3u_0}{\rho_i})t)+\Psi_i^{(0)}, \\
& \rho_2=\frac{\rho_1\left[\alpha_1\rho_1(\alpha_1-\alpha_2)-2u_0+\sqrt{\rho_1(\alpha_1^2\rho_1-4u_0)(\alpha_1-\alpha_2)^2}\right]}{2\alpha_2\rho_1(\alpha_1-\alpha_2)-2u_0}, \\
& \Psi_{31}=\alpha_{31}x+(\alpha_{31}\rho_{31}-\alpha_{32}\rho_{32})y-[2\alpha_{31}(\alpha_{31}^3+3v_0)+\frac{3u_0(\alpha_{31}\rho_{31}+\alpha_{32}\rho_{32})]}{\rho_{31}^2+\rho_{32}^2}t, \\
& \Psi_{32}=\alpha_{32}x+(\alpha_{31}\rho_{32}+\alpha_{32}\rho_{31})y+\left[\alpha_{32}(2\alpha_{32}^2-3v_0)+\frac{3u_0(\alpha_{31}\rho_{32}-\alpha_{32}\rho_{31})}{\rho_{31}^2+\rho_{32}^2}\right]t, \\
& \exp(\mathscr{K}_{ij})=\frac{\rho_i\rho_j(\alpha_i-\alpha_j)(\alpha_i\rho_i-\alpha_j\rho_j)+u_0(\rho_i-\rho_j)^2}{\rho_i\rho_j(\alpha_i+\alpha_j)(\alpha_i\rho_i+\alpha_j\rho_j)+u_0(\rho_i-\rho_j)^2}, \\
& i=1,\ 2;\ \ j=3,\ 4. \qquad (30)
\end{aligned}
$$

Depending on (29), obviously, the period of the solution is controlled with Ψ_{32} and the trajectory of the solution is determined by $\Psi_1,\ \Psi_2,\ \Psi_{31}$. The evolution trend of the 1-breather and the interactions of 2-resonance Y-type soliton is described by Figure 6, wtih abc being 3d plots and def being the corresponding projections, respectively. The breather and Y-type solitons have evolved in opposite directions with time t increasing.

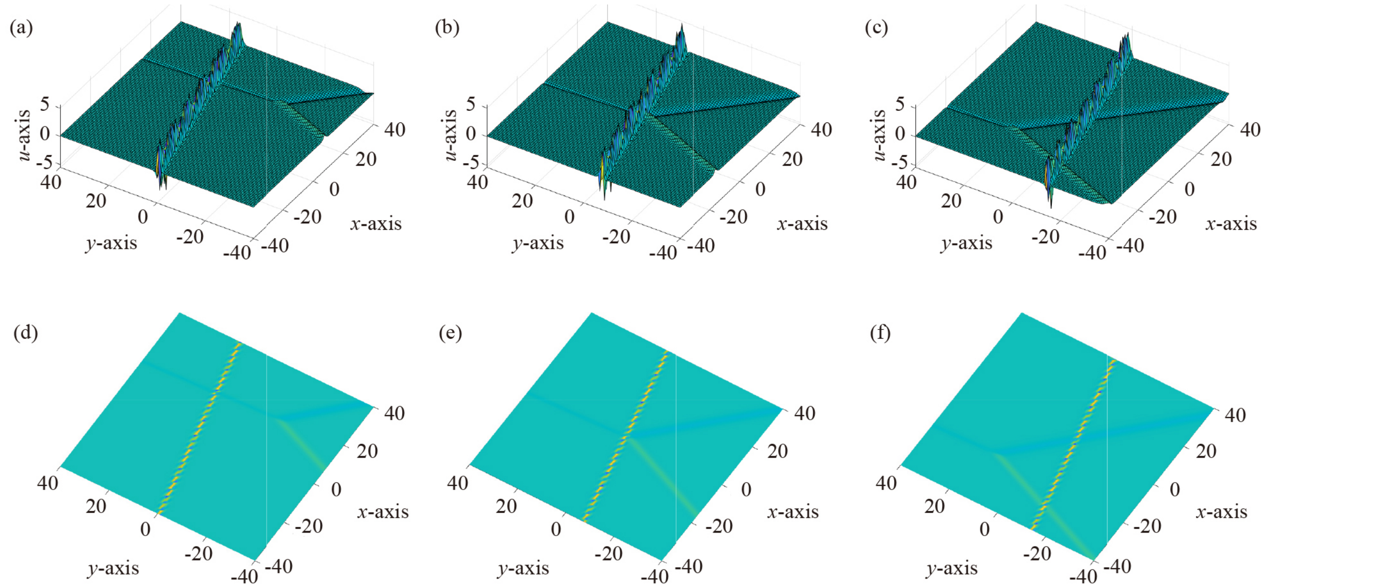

Figure 6. Interaction solution of 1-breather and 2-resonance Y-type solitons as $u_0 = 0.1,\ v_0 = 0,\ \alpha_1 = 1,\ \alpha_2 = -2,\ \alpha_{31} = 0.5,\ \alpha_{32} = 1.5,\ \alpha_1 = 1,\ \rho_{31} = 1,\ \rho_{32} = 2,\ \Psi_1^{(0)} = \Psi_2^{(0)} = \Psi_3^{(0)} = \Psi_4^{(0)} = 0$. (a) (d) $t = -5$, (b) (e) $t = 0$, (c) (f) $t = 5$

Case II Take $N = 6,\ P = 4,\ R = 1$.

Adding constraints (25) and (27) into (17) and (18), $\mathscr{H}_6$ is gained. With $u_0 = 0.1,\ v_0 = -0.1,\ \alpha_2 = -0.5,\ \alpha_1 = 1,\ \alpha_3 = 1.4,\ \alpha_4 = -0.1,\ \alpha_{51} = 0.6,\ \alpha_{52} = 0.7,\ \rho_2 = -0.4660,\ \rho_1 = 0.5,\ \rho_3 = 0.8,\ \rho_4 = -5.3272,\ \rho_{51} = 0.6,\ \rho_{52} = 0.7,\ \Psi_1^{(0)} = \Psi_2^{(0)} = \Psi_3^{(0)} = \Psi_4^{(0)} = \Psi_5^{(0)} = \Psi_6^{(0)} = 0$, the trend for the interaction of 1-breather and two 2-resonance Y-type solitons is shown by Figure 7.

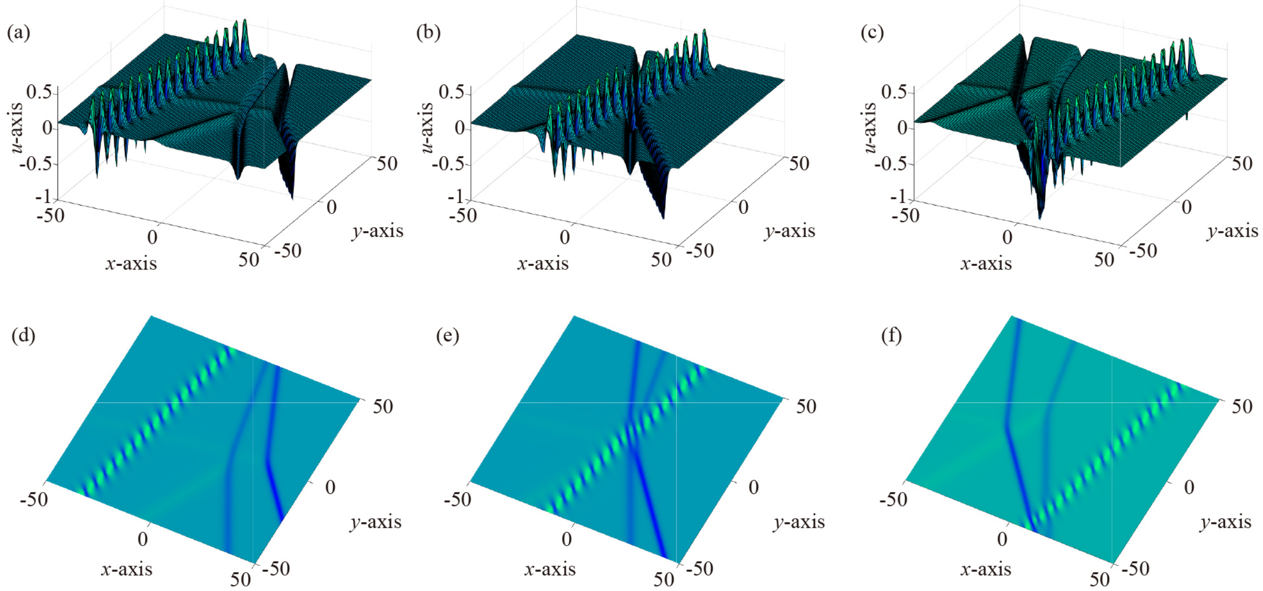

Figure 7. Interactions of 1-breather and two 2-resonance Y-type solitons. (a) (d) $t = -15$, (b) (e) $t = 0$, (c) (f) $t = 15$

Case III Take $N=6,\ P=2,\ R=2$.

Similarly, assuming $v_0=0,\ u_0=0.1,\ \alpha_1=1,\ \alpha_2=-2,\ \alpha_{31}=0.5,\ \alpha_{32}=1.5,\ \alpha_{51}=1,\ \alpha_{52}=1.8,\ \rho_1=0.6,\ \rho_2=-0.2140,\ \rho_{32}=1,\ \rho_{32}=2,\ \rho_{51}=1.4,\ \rho_{52}=2.2,\ \Psi_1^{(0)}=\Psi_2^{(0)}=\Psi_3^{(0)}=\Psi_4^{(0)}=\Psi_5^{(0)}=\Psi_6^{(0)}=0$, the interactions of 2-breathers and 2-resonance Y-type solitons are described by Figure 8.

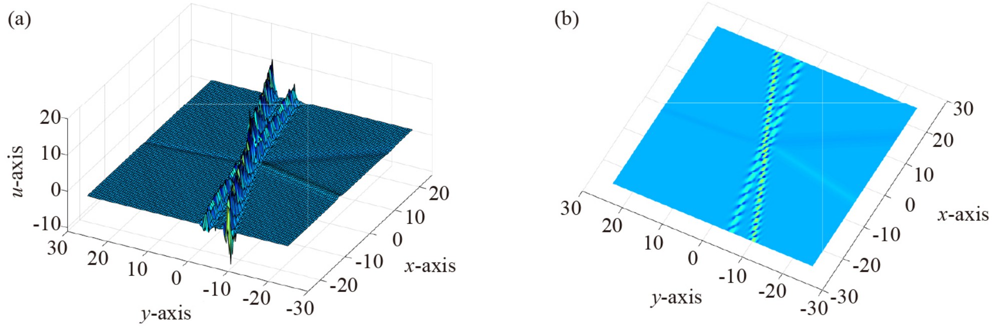

Figure 8. Interactions of 2-breather and 2-resonance Y-type solitons as $t=0$

4.2 *Interaction phenomena of lumps combined with resonance Y-type solitons*

Applying the following constraints (31) to (19), the interactions of the L-lumps and the P-resonance Y-type solitons arise.

$$N=2L+P,\ \ \alpha_{2l-1},\alpha_{2l}\to 0,\ \ \rho_l=\rho_{L+l}^*,\ \ \exp(\Psi_{2l-1}^{(0)})=\exp(\Psi_{2l}^{(0)})=-1,$$

$$\exp(\mathscr{K}_{ij})=0,\ 1\le l\le L,\ \ 2L<i<j\le N. \tag{31}$$

Case I Take $N=4,\ L=1,\ P=2$.

In such case, $\mathscr{H}_4$ is

$$\mathscr{H}_4=\Phi_1\Phi_2+\mathscr{M}_{12}+\exp(\Psi_3)[\mathscr{M}_{12}+(\mathscr{Q}_{13}+\Phi_1)(\mathscr{Q}_{23}+\Phi_2)]$$

$$+\exp(\Psi_4)[\mathscr{M}_{12}+(\mathscr{Q}_{14}+\Phi_1)(\mathscr{Q}_{24}+\Phi_2)], \tag{32}$$

where

$$\Phi_i = x + \rho_i y - \left(3v_0 + \frac{3u_0}{\rho_i}\right), \quad \mathscr{M}_{12} = -\frac{2\rho_1\rho_2(\rho_1+\rho_2)}{u_0(\rho_1-\rho_2)^2},$$

$$\Psi_j = \alpha_j\left(x + \rho_j y + \left(\alpha_j^2 - 3v_0 - \frac{3u_0}{\rho_j}\right)t\right) + \Psi_j^{(0)},$$

$$\mathscr{Q}_{ij} = -\frac{2\rho_i\rho_j\alpha_j(\rho_i+\rho_j)}{\rho_i\rho_j^2\alpha_j^2 + u_0(\rho_i-\rho_j)^2}, \quad \rho_1 = \rho_2^*,$$

$$\rho_4 = \frac{\rho_3\left[\alpha_3\rho_3(\alpha_3-\alpha_4) - 2u_0 + \sqrt{\rho_3(\alpha_3^2\rho_3 - 4u_0)(\alpha_3-\alpha_4)^2}\right]}{2\alpha_4\rho_3(\alpha_3-\alpha_4) - 2u_0}, \tag{33}$$

with $i = 1,\ 2$, and $j = 3,\ 4$. The interactions of 1-lump and 2-resonance Y-type soliton arise. The interaction phenomena are exhibited in Figure 9 when $v_0 = 0,\ u_0 = 0.1,\ \rho_1 = 1+2\mathrm{i},\ \rho_2 = 1-2\mathrm{i},\ \rho_3 = 0.5,\ \rho_4 = -0.2037,\ \alpha_3 = -1,\ \alpha_4 = 1.5,\ \Psi_3^{(0)} = \Psi_4^{(0)} = 0$.

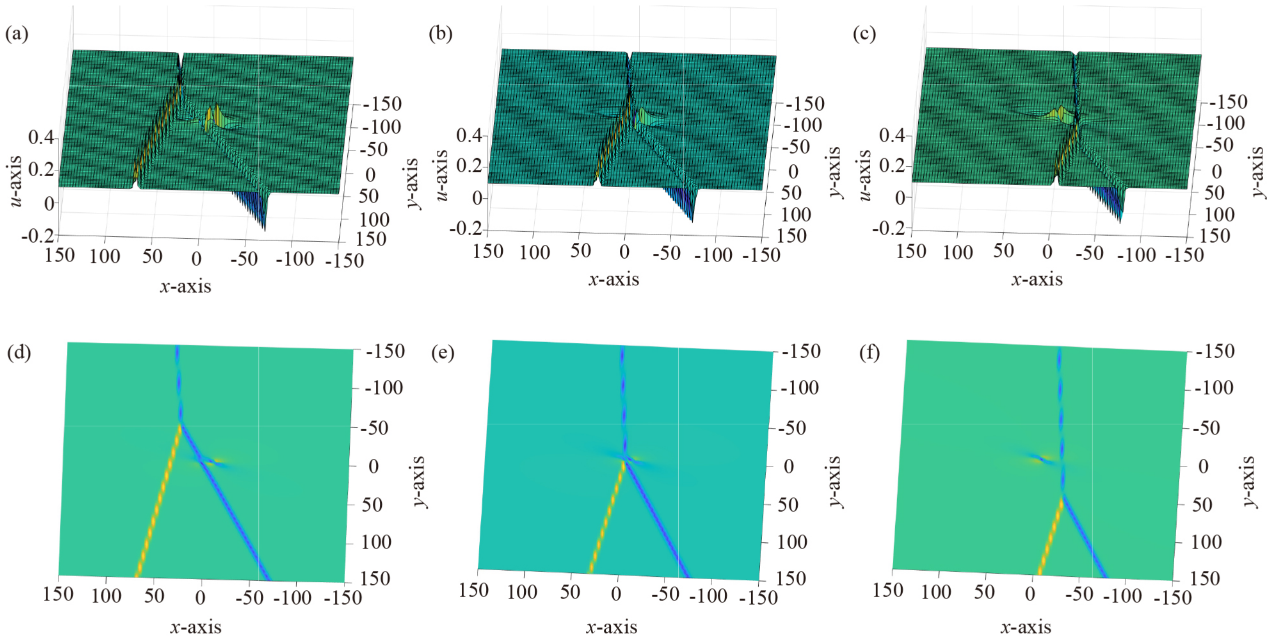

Figure 9. Interactions of 1-lump and 2-resonance Y-type solitons. (a) (d) $t = -10$, (b) (e) $t = 0$, (c) (f) $t = 10$

Case II Take $N = 6,\ L = 1,\ P = 4$.
With this case, $\mathscr{H}_6$ is

$$
\begin{aligned}
\mathscr{H}_6 = {} & \mathscr{H}_4 + \exp(\Psi_5)[\mathscr{M}_{12} + (\mathscr{Q}_{15} + \Phi_1)(\mathscr{Q}_{25} + \Phi_2)] + \exp(\Psi_6)[\mathscr{M}_{12} + (\mathscr{Q}_{16} + \Phi_1)(\mathscr{Q}_{26} + \Phi_2)] \\
& + \exp(\Psi_3 + \Psi_5 + \mathscr{K}_{35})[\mathscr{M}_{12} + (\mathscr{Q}_{13}\mathscr{Q}_{15} + \Phi_1)(\mathscr{Q}_{23}\mathscr{Q}_{25} + \Phi_2)] \\
& + \exp(\Psi_4 + \Psi_5 + \mathscr{K}_{45})[\mathscr{M}_{12} + (\mathscr{Q}_{14}\mathscr{Q}_{15} + \Phi_1)(\mathscr{Q}_{24}\mathscr{Q}_{25} + \Phi_2)] \\
& + \exp(\Psi_3 + \Psi_6 + \mathscr{K}_{36})[\mathscr{M}_{12} + (\mathscr{Q}_{13}\mathscr{Q}_{16} + \Phi_1)(\mathscr{Q}_{23}\mathscr{Q}_{26} + \Phi_2)] \\
& + \exp(\Psi_4 + \Psi_6 + \mathscr{K}_{46})[\mathscr{M}_{12} + (\mathscr{Q}_{14}\mathscr{Q}_{16} + \Phi_1)(\mathscr{Q}_{24}\mathscr{Q}_{26} + \Phi_2)], \qquad (34)
\end{aligned}
$$

where

$$
\begin{aligned}
& \Phi_i = x + \rho_i y - \left(3v_0 + \frac{3u_0}{\rho_i}\right), \ \ \mathscr{M}_{12} = -\frac{2\rho_1\rho_2(\rho_1 + \rho_2)}{u_0(\rho_1 - \rho_2)^2}, \\
& \Psi_j = \alpha_j\left(x + \rho_j y + \left(\alpha_j^2 - 3v_0 - \frac{3u_0}{\rho_j}\right)t\right) + \Psi_j^{(0)}, \\
& \mathscr{Q}_{ij} = -\frac{2\rho_i\rho_j\alpha_j(\rho_i + \rho_j)}{\rho_i\rho_j^2\alpha_j^2 + u_0(\rho_i - \rho_j)^2}, \\
& \rho_n = \frac{\rho_{n-1}\left[\alpha_{n-1}\rho_{n-1}(\alpha_{n-1} - \alpha_n) - 2u_0\right]}{2\alpha_n\rho_{n-1}(\alpha_{n-1} - \alpha_n) - 2u_0} + \frac{\rho_{n-1}\left[\sqrt{\rho_{n-1}(\alpha_{n-1}^2\rho_{n-1} - 4u_0)(\alpha_{n-1} - \alpha_n)^2}\right]}{2\alpha_n\rho_{n-1}(\alpha_{n-1} - \alpha_n) - 2u_0}, \\
& \rho_1 = \rho_2^*, \ \ i = 1,\ 2; \ \ j = 3,\ 4,\ 5,\ 6; \ \ n = 4,\ 6. \qquad (35)
\end{aligned}
$$

Taking $v_0 = -0.1,\ u_0 = 0.1,\ \rho_2 = 1 - 2\mathrm{i},\ \rho_1 = 1 + 2\mathrm{i},\ \rho_3 = 0.5,\ \rho_4 = -0.4660,\ \rho_5 = 0.8,\ \rho_6 = -5.3272,\ \alpha_3 = 1,\ \alpha_4 = -0.5,\ \alpha_5 = 1.4,\ \alpha_6 = -0.1,\ \Psi_3^{(0)} = \Psi_4^{(0)} = \Psi_5^{(0)} = \Psi_6^{(0)} = 0$, the interactions of 1-lump and two 2-resonance Y-type solitons are described by Figure 10.

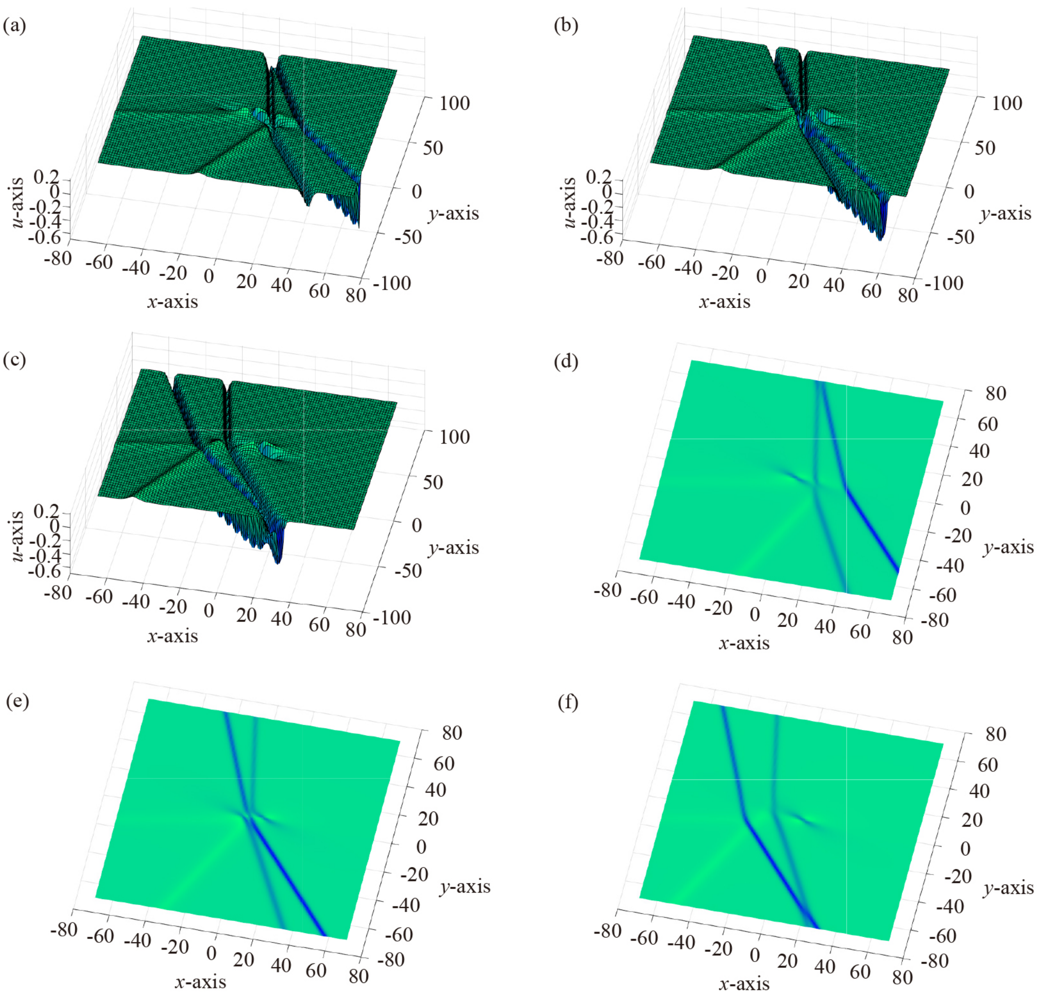

Figure 10. Interactions of 1-lump combined with two 2-resonance Y-type solitons. (a) (d) $t=-15$, (b) (e) $t=0$, (c) (f) $t=15$

Case III Take $N=6$, $L=2$, $P=2$.

The evolutionary behaviors of the interaction solution of 2-resonance Y-type soliton and 2-lumps are visualized in Figure 11 with $v_0=-0.1$, $u_0=0.1$, $\rho_1=-1+0.8\mathrm{i}$, $\rho_2=-0.3+0.6\mathrm{i}$, $\rho_3=-1-0.8\mathrm{i}$, $\rho_4=-0.3-0.6\mathrm{i}$, $\rho_5=-1.8$, $\rho_6=0.0198$, $\alpha_5=-1.5$, $\alpha_6=2$, $\Psi_5^{(0)}=\Psi_6^{(0)}=0$.

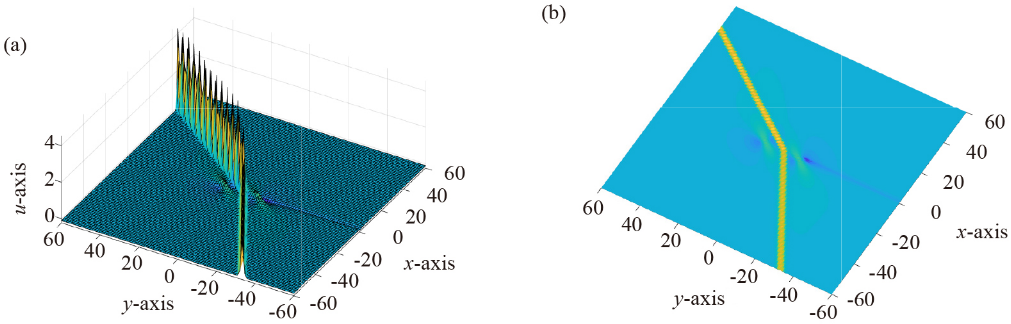

Figure 11. Interactions of 2-resonance Y-type solitons and 2-lumps as $t = 0$

4.3 *Hybrid solutions for solitons, lumps and breathers*

The hybrid solutions for R-breathers, L-lumps and P-solitons are acquired when the following constraints are satisfied by (17) and (18)

$$N = 2L+2R+P,\ \ \alpha_{2l-1},\ \alpha_{2l} \to 0, \quad \rho_l = \rho^*_{L+l},\ \exp(\Psi^{(0)}_{2l-1}) = \exp(\Psi^{(0)}_{2l}) = -1,$$

$$\Psi_{2L+2r-1} = \Psi^*_{2L+2r},\ \ 1 \le l \le L,\ \ 1 \le r \le R,\ \ 2L+2R < i < j \le N. \tag{36}$$

When $N = 6,\ L = 1,\ R = 1,\ P = 2$, letting

$$v_0 = 0,\ \ u_0 = 0.1,\ \ \rho_1 = \rho^*_2 = 0.5+\mathrm{i},\ \ \rho_3 = \rho^*_4 = -0.8+1.1\mathrm{i},$$

$$\rho_5 = 0.3,\ \ \rho_6 = 1.2,\ \ \alpha_3 = \alpha^*_4 = -1.5+2\mathrm{i},\ \ \alpha_5 = -2,\ \ \alpha_6 = -0.5,\ \ \Psi^{(0)}_5 = \Psi^{(0)}_6 = 0, \tag{37}$$

the interaction process of the hybrid solution for 1-breather, 1-lump combined with 2-solitons is presented by Figure 12.

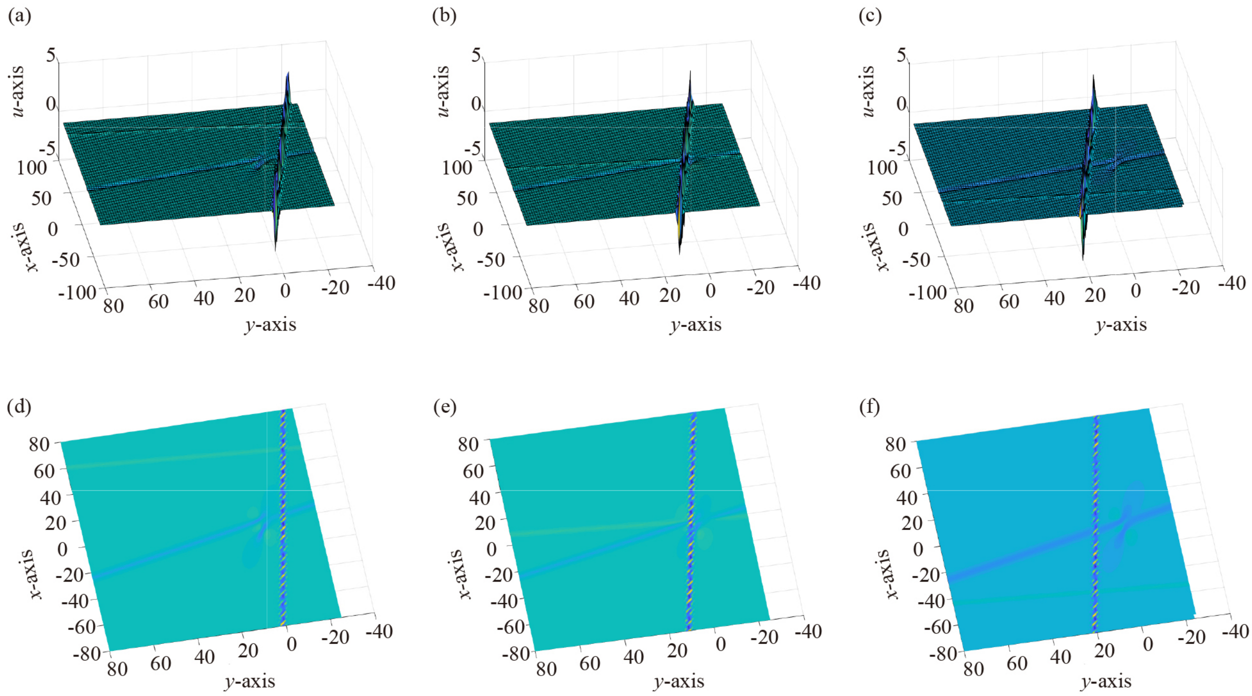

Figure 12. The hybrid solution for 1-breather, 1-lump and 2-solitons as (a) (d) $t=-10$, (b) (e) $t=0$, (c) (f) $t=10$

Based on above hybrid solutions, if we let

$$\exp(\mathscr{K}_{56})=0 \Rightarrow \rho_6=\frac{\rho_5\left[\alpha_5\rho_5(\alpha_5-\alpha_6)-2u_0+\sqrt{\rho_5(\alpha_5^2\rho_5-4u_0)(\alpha_5-\alpha_6)^2}\right]}{2\alpha_6\rho_5(\alpha_5-\alpha_6)-2u_0}. \tag{38}$$

The hybrid solutions for 1-breather, 1-lump combined with 2-Y-type-soliton arise. This interaction process is shown in Figure 13 with

$$v_0=0,\ \ u_0=0.1,\ \ \rho_1=\rho_2^*=1+2\mathrm{i},\ \ \rho_3=\rho_4^*=-0.9-2\mathrm{i},$$

$$\rho_5=-0.3,\ \ \rho_6=-0.3237,\ \ \alpha_3=\alpha_4^*=-0.4-1.6\mathrm{i},\ \ \alpha_5=-2,\ \ \alpha_6=1.5,\ \ \Psi_5^{(0)}=\Psi_6^{(0)}=0. \tag{39}$$

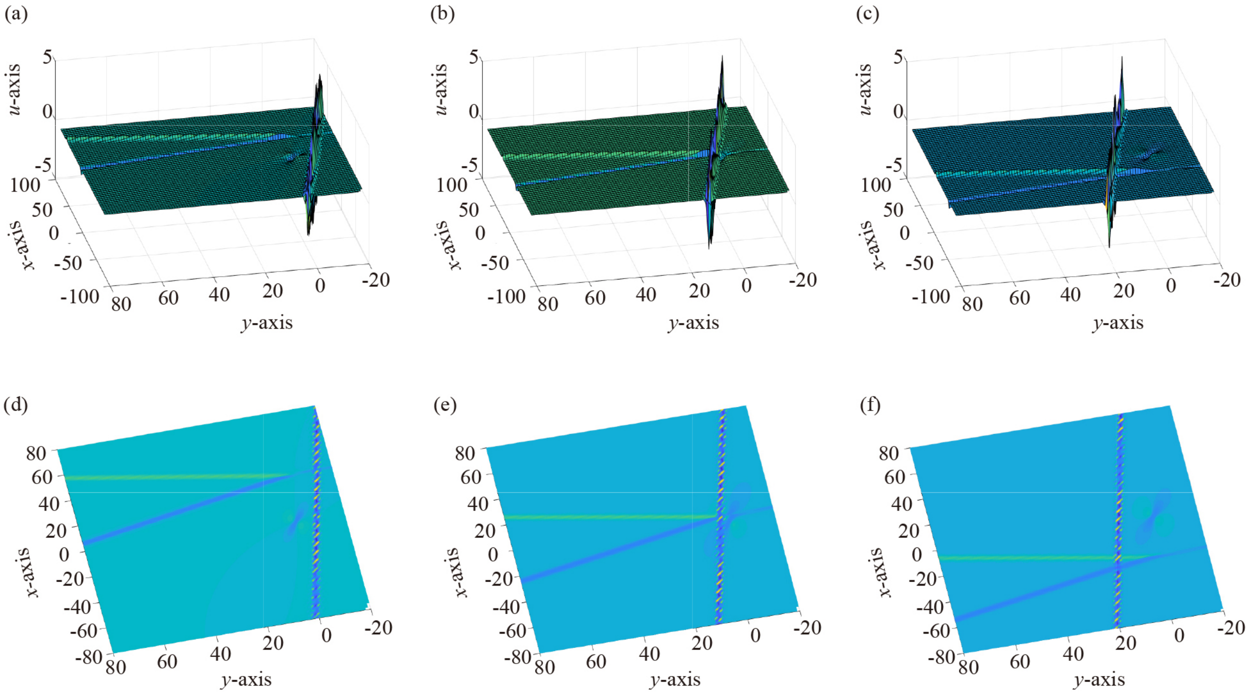

Figure 13. The hybrid solution for 1-breather, 1-lump combined with resonance Y-type soliton. (a) (d) $t=-10$, (b) (e) $t=0$, (c) (f) $t=10$

5. Conclusion

In the present paper, various kinds of exact solutions are investigated through different methods, which include the periodic solitary wave, the lumps, the singular periodic solitary wave, the resonance Y-type solitons, the interaction solutions. Especially, to our knowledge, novel exact solutions of (1) as studied here have been rarely explored yet, which includes the resonance Y-type solitons and interactions consisting of breathers, lumps and the resonance Y-type solitons. In section 2, we firstly yield three kinds of periodic solitary waves via the modified three-wave technique. Then by virtue of the parameter limit method, the periodic solitary wave solutions degenerate into the lump solutions when condition $\lim_{k_i\to 0}(1+\delta_1+\delta_2+\delta_3)=0$ holds. In section 3, the resonance Y-type soliton solutions are gained through applying certain different constraints to N-soliton solutions. Also, the 2-resonance, the 3-resonance, and the two 2-resonance Y-type solitons arise. In section 4, various of interaction solutions arise from imposing new constraint conditions on resonance Y-type solitons. Taking advantage of the complex conjugate technique to the resonance Y-type solitons partially, the interactions of the breathers combined with the resonance Y-type solitons are established. Employing the parametric limit approach, the interactions of the lumps combined with the resonance Y-type solitons arise from partial degeneration of the resonance Y-type solitons. Furthermore, combining the complex conjugate technique with the parametric limit approach, two kinds of hybrid solutions are yielded, which are the solitons, breathers and lumps; the breathers, lumps and resonance Y-type solitons.

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Conflict of interest

The authors declare no competing financial interest.

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