Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

Research Article

Advancements in Integral Inequalities Through Hattaf Fractional Operators

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Received: 12 November 2024; **Revised:** 2 January 2025; **Accepted:** 3 January 2025

Abstract: In studies of inequality theory, integral identities are developed to support numerous inequalities. Various fractional integral and derivative operators have been employed recently to accomplish these identities. In this article, we first establish an integral identity by employing Hattaf fractional integral operators. Then, we use this identity to give some novel generalizations of integral inequalities for the convexity of $|\aleph|$ using the Jensen integral inequality, Young's inequality, power-mean inequality, and Hölder inequality. The main motivating goal of this study is to use Hattaf-fractional integral operators with strong kernel structure to derive new and general form of integral inequalities.

***Keywords*:** young inequality, convex function, power-mean inequality, fractional operators

MSC: 26A33, 26A51, 26D10

1. Introduction

In many scientific fields nowadays, the fractional calculus is frequently used due to its numerous applications in dynamical problems. In [1], the authors presented existence results for connected systems of hybrid boundary value problems with hybrid conditions. In [2], the results for the existence of super-linear fractional differential equations are presented. Baleanu et al. [3] gave the set of solutions for a class of fractional-based sequential differential equations. The generalization of fractional integral and derivative operators are recently presented by the researchers [4–6]. Samraiz et al. [7] presented certain fractional operators and their applications in mathematical physics. Samraiz et al. [8] gave Riemann-type weighted fractional operators and Cauchy problem solutions.

A novel formula for fractional derivatives and integrals has been presented by Dumitru and Arran [9], who have used the Mittag-Leffler kernel. In [10, 11], more theoretical concepts regarding fractional operators with Mittag-Leffler kernels (Atangana-Baleanu operators) and the higher-order case were discussed. The generalization of the Mittag-Leffler kernels to provide a semigroup property was examined in [12, 13].

On the other hand, noninteger-order calculus, sometimes referred to as fractional calculus, is used to generalize integrals and derivatives, particularly integrals containing inequalities. Agarwal et al. [14] used generalized k-fractional

DOI: https://doi.org/10.37256/cm.6120256060

integrals to demonstrate Hermite-Hadamard-type inequalities. Shuang and Qi [15] investigated particular methods for a class of s-convex functions and proved a variety of Hermite-Hadamard-type inequalities. In [16], the authors presented different notions of convexity. By using the traditional Hermite-Hadamard inequalities, Mehrez and Agarwal [17] developed new integral inequalities and examined particular scenarios of their findings with application to special means. New generalized inequalities were studied and then used in stability analysis by Park et al. [18]. Applying the local fractional technique, Sarikaya et al. [19] produced fractional integral inequalities that went beyond what was discovered in the classical literature. A generalization of the integral inequalities has recently been analyzed using Mittag-Leffler functions. Fernandez and Mohammed [20] established Hermite-Hadamard by using Atangana-Baleanu fractional integral operators. Farid et al. [21] evaluated particular implementations of their findings and generalized several classical inequalities using the generalized Mittag-Leffler function. The researchers examined some prominent integral inequalities using several types of fractional operators. Readers are directed to the study by [22–26].

2. Preliminaries

In this section, we consider some known results.

Definition 2.1 A function $\hbar : [x_1, x_2] \subseteq \mathbb{R} \to \mathbb{R}$ is considered convex if

$$\hbar(\eta u + (1-\eta)v) \leq \eta \hbar(u) + (1-\eta)\hbar(v),$$

for all $u, v \in [x_1, x_2]$ and $\eta \in [0, 1]$.

One often used concept in inequality theory is that of convex functions. Using averages of the mean value of a convex function, the Hermite-Hadamard inequality provides upper and lower bounds.

Definition 2.2 Given a convex mapping $\hbar : I \subseteq \mathbb{R} \to \mathbb{R}$, let $x_1 < x_2$ be on the interval I of $\mathbb{R}$. If the following inequality holds, then the Hermite-Hadamard inequality is defined by

$$\hbar\left(\frac{x_1+x_2}{2}\right) \leq \frac{1}{x_2-x_1}\int_{x_1}^{x_2} \hbar(x)dx \leq \frac{\hbar(x_1)+\hbar(x_2)}{2}.$$

Definition 2.3 [27] The ABC-fractional derivative is defined by

$$^{ABC}\mathfrak{D}^{\zeta}_{x_1,\,\xi}\hbar(\xi) = \frac{M(\zeta)}{1-\zeta}\int_{x_1}^{\xi} \hbar'(\eta)\mathbb{E}_{\zeta}\left(\frac{-\zeta(\xi-\eta)^{\zeta}}{1-\zeta}\right)d\eta,$$

where $0 < \zeta < 1$, $\hbar' \in L^1(r, T]$ and $M(\zeta)$ is normalization function which satisfies the condition $M(0) = M(1) = 1$.

Definition 2.4 [28, 29] The AB-fractional operator for $\hbar \in L^1(r, T]$ and $0 < \zeta < 1$ is defined by

$$^{AB}\mathcal{I}^{\zeta}_{x_1,\,\xi}\hbar(\xi) = \frac{1-\zeta}{M(\zeta)}\hbar(\xi) + \frac{\zeta}{M(\zeta)\Gamma(\zeta)}\int_{x_1}^{\xi}(\xi-\eta)^{\zeta-1}\hbar(\eta)d\eta. \quad (1)$$

Definition 2.5 [30] The Hattaf fractional derivative $\hbar$ is defined by

$$\mathfrak{D}_{x_1,\xi}^{\zeta,\varrho,\delta}\hbar(\xi)=\frac{M(\zeta)}{1-\zeta}\frac{1}{\omega(\xi)}\int_{x_1}^{\xi}\mathbb{E}_{\varrho}\left(\frac{-\zeta(\xi-\eta)^{\delta}}{1-\zeta}\right)\frac{d}{dt}(\omega\hbar)(\eta)d\eta,$$

where $0<\zeta<1$, $\hbar'\in L^1(r,T]$ $\omega\in C^1(a,b)$, ω, $\omega'>0$ on $[a,b]$.

Definition 2.6 [30] The left sided Hattaf fractional operator for $\hbar\in L^1(r,T]$ and $0<\zeta<1$ is defined by

$$\mathcal{I}_{x_1,\xi}^{\zeta,\varrho}\hbar(\xi)=\frac{1-\zeta}{M(\zeta)}\hbar(\xi)+\frac{\zeta}{M(\zeta)\Gamma(\varrho)\omega(\xi)}\int_{x_1}^{\xi}(\xi-\eta)^{\varrho-1}\omega(\eta)\hbar(\eta)d\eta. \tag{2}$$

Definition 2.7 [30] The right sided Hattaf fractional operator for $\hbar\in L^1(x_1,x_2)$ and $0<\zeta<1$ is defined by

$$\mathcal{I}_{x_2,\xi}^{\zeta,\varrho}\hbar(\xi)=\frac{1-\zeta}{M(\zeta)}\hbar(\xi)+\frac{\zeta}{\omega(\xi)M(\zeta)\Gamma(\varrho)}\int_{\xi}^{x_2}(\eta-\xi)^{\varrho-1}\omega(\eta)\hbar(\eta)d\eta. \tag{3}$$

Remark 2.8 *i*. If we consider $\varrho=\zeta$ in (2) and (3), then we get AB-operator defined in (1).
ii. If we consider $\varrho=\zeta$ and $\omega=1$ in (2) and (3), then we get AB-operator defined in (1).

Definition 2.9 The left sided Hattaf fractional operator for $\omega=1$, $\hbar\in L^1(r,T]$ and $0<\zeta<1$ is defined by

$$\mathcal{I}_{x_1,\xi}^{\zeta,\varrho}\hbar(\xi)=\frac{1-\zeta}{M(\zeta)}\hbar(\xi)+\frac{\zeta}{M(\zeta)\Gamma(\varrho)}\int_{x_1}^{\xi}(\xi-\eta)^{\varrho-1}\hbar(\eta)d\eta. \tag{4}$$

Definition 2.10 The right sided Hattaf fractional operator for $\omega=1$, $\hbar\in L^1(x_1,x_2)$ and $0<\zeta<1$ is defined by

$$\mathcal{I}_{x_2,\xi}^{\zeta,\varrho}\hbar(\xi)=\frac{1-\zeta}{M(\zeta)}\hbar(\xi)+\frac{\zeta}{M(\zeta)\Gamma(\varrho)}\int_{\xi}^{x_2}(\eta-\xi)^{\varrho-1}\hbar(\eta)d\eta. \tag{5}$$

The aim of this paper is to prove an integral identity by utilizing Hattaf fractional integral operators. Then we use this identity to give some novel generalizations of Hermite-Hadamard type integral inequalities for differentiable convex functions.

The structure of the paper follows: In section 1, a brief discussion is given. Section 2 is devoted to some auxiliary results. In section 3, we prove an integral identity and some novel generalizations of integral inequalities via Hattaf-fractional integral operators. The concluding remarks are given in section 4.

3. Main result

Here, first we will prove the following fractional integral identity which will be used in our main findings.

Lemma 3.1 Assume that $\hbar:[x_1,x_2]\to\mathbb{R}$ represents a differentiable function on (x_1,x_2), where $x_1<x_2$. Next, for Hattaf-fractional operators, we have the identity given below:

$$\Im_{x_1,t}^{\zeta,\varrho}\hbar(t)+\Im_{x_2,t}^{\zeta,\varrho}\hbar(t)-\zeta\frac{(t-x_1)^{\varrho}\hbar(x_1)+(x_2-t)^{\varrho}\hbar(x_2)}{H(\zeta)\Gamma(\varrho+1)}-\frac{2(1-\zeta)\hbar(t)}{H(\zeta)}$$

$$=\frac{\zeta(t-x_1)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)}\int_0^1(1-\eta)^{\varrho}\hbar'(\eta t+(1-\eta)x_1)d\eta$$

$$-\frac{\zeta(x_2-t)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)}\int_0^1\eta^{\varrho}\hbar'(\eta x_2+(1-\eta)t)d\eta,$$

where $\zeta\in(0,\,1]$, $t\in[x_1,\,x_2]$ and $\varrho>0$.

Proof. By using integration by parts for the right hand side, we have

$$\frac{\zeta(t-x_1)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)}\int_0^1(1-\eta)^{\varrho}\hbar'(\eta t+(1-\eta)x_1)d\eta+\frac{1-\zeta}{H(\zeta)}\hbar(t)+\frac{\zeta(t-x_1)^{\zeta}\hbar(x_1)}{H(\zeta)\Gamma(\varrho+1)}$$

$$=\frac{\zeta(t-x_1)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)}\left[\frac{(1-\eta)^{\varrho}\hbar(\eta t+(1-\eta)r)}{t-x_1}\Big|_0^1+\varrho\int_0^1\frac{(1-\eta)^{\varrho-1}\hbar(\eta t+(1-\eta)x_1)}{t-x_1}d\eta\right]$$

$$+\frac{1-\zeta}{H(\zeta)}\hbar(t)+\frac{\zeta(t-x_1)^{\varrho}\hbar(x_1)}{H(\zeta)\Gamma(\varrho+1)}.$$

By substituting $\eta=\frac{u-x_1}{t-x_1}$, we have

$$\frac{\zeta(t-x_1)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)}\int_0^1(1-\eta)^{\varrho}\hbar'(\eta t+(1-\eta)x_1)d\eta+\frac{1-\zeta}{H(\zeta)}\hbar(t)+\frac{\zeta(t-x_1)^{\varrho}\hbar(x_1)}{H(\zeta)\Gamma(\varrho+1)}$$

$$=-\frac{\zeta(t-x_1)^{\varrho}}{H(\zeta)\Gamma(\varrho+1)}\hbar(x_1)+\frac{\zeta}{H(\zeta)\Gamma(\varrho)}\int_{x_1}^t(t-u)^{\varrho-1}\hbar(u)\,du$$

$$+\frac{1-\zeta}{H(\zeta)}\hbar(t)+\frac{\zeta(t-x_1)^{\varrho}\hbar(x_1)}{H(\zeta)\Gamma(\varrho+1)}$$

$$=\frac{\zeta}{H(\zeta)\Gamma(\varrho)}\int_{x_1}^t(t-u)^{\varrho-1}\hbar(u)\,du+\frac{1-\zeta}{H(\zeta)}\hbar(t)$$

$$=\Im_{x_1,t}^{\zeta,\varrho}\hbar(t).\tag{6}$$

Similarly, by integrating the following integral, we obtain

$$\frac{\zeta(x_2-t)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)}\int_0^1 \eta^{\varrho}\hbar'(\eta x_2+(1-\eta)t)d\eta$$

$$-\frac{1-\zeta}{H(\zeta)}\hbar(t)-\frac{\zeta(x_2-t)^{\varrho}\hbar(x_2)}{H(\zeta)\Gamma(\varrho+1)}=-\Im^{\zeta,\varrho}_{x_2,t}\hbar(t). \tag{7}$$

Adding (6) and (7), we get the desired Lemma 3.1. □

Remark 3.2 If we choose $\zeta=\varrho$ in Lemma 3.1, we get the identity derived by [31].

Theorem 3.3 Assume that a differentiable function $\hbar$: $[x_1, x_2]\to\mathbb{R}$ on (x_1, x_2), $\hbar'\in L_1[x_1, x_2]$ and $x_1<x_2$. The following inequality holds for Hattaf-fractional operators if $|\hbar'|$ is a convex function.

$$\left|\Im^{\zeta,\varrho}_{x_1,t}\hbar(t)+\Im^{\zeta,\varrho}_{x_2,t}\hbar(t)-\zeta\frac{(t-x_1)^{\varrho}\hbar(x_1)+(x_2-t)^{\varrho}\hbar(x_2)}{H(\zeta)\Gamma(\varrho+1)}-\frac{2(1-\zeta)\hbar(t)}{H(\zeta)}\right|$$

$$\leq\frac{\zeta(t-x_1)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)}\left[\frac{|\hbar'(t)|}{(\varrho+1)(\varrho+2)}+\frac{|\hbar'(x_1)|}{\varrho+2}\right]$$

$$+\frac{\zeta(x_2-t)^{\lambda+1}}{H(\zeta)\Gamma(\varrho+1)}\left[\frac{|\hbar'(x_2)|}{(\varrho+2)}+\frac{|\hbar'(t)|}{(\varrho+1)(\varrho+2)}\right],$$

where $t\in[x_1, x_2]$, $\zeta\in(0, 1]$ and $H(\zeta)>0$.

Proof. By using Lemma 3.1, we have

$$\left|\Im^{\zeta,\varrho}_{x_1,t}\hbar(t)+\Im^{\zeta,\varrho}_{x_2,t}\hbar(t)-\zeta\frac{(t-x_1)^{\varrho}\hbar(x_1)+(x_2-t)^{\varrho}\hbar(x_2)}{H(\zeta)\Gamma(\varrho+1)}-\frac{2(1-\zeta)\hbar(t)}{H(\zeta)}\right|$$

$$=\left|\frac{\zeta(t-x_1)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)}\int_0^1(1-\eta)^{\varrho}\hbar'(\eta t+(1-\eta)x_1)d\eta-\frac{\zeta(x_2-t)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)}\int_0^1\eta^{\varrho}\hbar'(\eta x_2+(1-\eta)t)d\eta\right|$$

$$\leq\frac{\zeta(t-x_1)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)}\int_0^1(1-\eta)^{\varrho}\left|\hbar'(\eta t+(1-\eta)x_1)\right|d\eta+\frac{\zeta(x_2-t)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)}\int_0^1\eta^{\varrho}\left|\hbar'(\eta x_2+(1-\eta)t)\right|d\eta.$$

By applying the convexity of $|\hbar'|$, we get

$$\left|\Im_{x_1,t}^{\zeta,\varrho}\hbar(t)+\Im_{x_2,t}^{\zeta,\varrho}\hbar(t)-\zeta\frac{(t-x_1)^{\varrho}\hbar(x_1)+(x_2-t)^{\varrho}\hbar(x_2)}{H(\zeta)\Gamma(\varrho+1)}-\frac{2(1-\zeta)\hbar(t)}{H(\zeta)}\right|$$

$$\leq\frac{\zeta(t-x_1)^{\lambda+1}}{H(\zeta)\Gamma(\varrho+1)}\int_0^1(1-\eta)^{\varrho}\left[\eta\left|\hbar'(t)\right|+(1-\eta)\left|\hbar'(x_1)\right|\right]d\eta$$

$$+\frac{\zeta(x_2-t)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)}\int_0^1\eta^{\varrho}\left[\eta\left|\hbar'(x_2)\right|+(1-\eta)\left|\hbar'(t)\right|\right]d\eta.$$

After solving the above integrals, we get

$$\left|\Im_{x_1,t}^{\zeta,\varrho}\hbar(t)+\Im_{x_2,t}^{\zeta,\varrho}\hbar(t)-\zeta\frac{(t-x_1)^{\varrho}\hbar(x_1)+(x_2-t)^{\varrho}\hbar(x_2)}{H(\zeta)\Gamma(\varrho+1)}-\frac{2(1-\zeta)\hbar(t)}{H(\zeta)}\right|$$

$$\leq\frac{\zeta(t-x_1)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)}\left[\frac{1}{(\varrho+1)(\varrho+2)}|\hbar'(t)|+\frac{1}{(\varrho+2)}|\hbar'(x_1)|\right]$$

$$+\frac{\zeta(x_2-t)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)}\left[\frac{1}{(\varrho+2)}|\hbar'(x_2)|+\frac{1}{(\varrho+1)(\varrho+2)}|\hbar'(t)|\right],$$

which completes the desired proof. □

Corollary 3.4 Applying Theorem 3.3 for $t=\frac{x_1+x_2}{2}$, we get the following inequality

$$\left|\Im_{x_1,\frac{x_1+x_2}{2}}^{\zeta,\varrho}\hbar\left(\frac{x_1+x_2}{2}\right)+\Im_{x_2,\frac{x_1+x_2}{2}}^{\zeta,\varrho}\hbar\left(\frac{x_1+x_2}{2}\right)-\zeta\frac{(x_1+x_2)^{\varrho}}{2^{\varrho}H(\zeta)\Gamma(\varrho+1)}\left[\hbar(x_1)+\hbar(x_2)\right]-\frac{2(1-\zeta)\hbar\left(\frac{x_1+x_2}{2}\right)}{H(\zeta)}\right|$$

$$\leq\frac{\zeta(x_2-x_1)^{\varrho+1}}{2^{\varrho+1}H(\zeta)\Gamma(\varrho+1)}\left[2\frac{\left|\hbar'\left(\frac{x_1+x_2}{2}\right)\right|}{(\varrho+1)(\varrho+2)}+\frac{|\hbar'(x_1)|+|\hbar'(x_2)|}{\varrho+2}\right].$$

Remark 3.5 Applying Theorem 3.3 for $\zeta=\varrho$, we get Theorem 2.2 proved by Set et al. [31].

Remark 3.6 Applying Theorem 3.4 for $\zeta=\varrho$, we get Corollary 2.3 proved by Set et al. [31].

Remark 3.7 Applying Theorem 3.3 for $\zeta=\varrho=1$, we get the result given in [32].

Theorem 3.8 Let $\hbar:[x_1, x_2]\to\mathbb{R}$ be a differentiable function on (x_1, x_2) with $x_1<x_2$ and $\hbar'\in L_1[x_1, x_2]$. For Hattaf-fractional operators, we have the following inequality if $|\hbar'|^q$ is a convex function.

$$\left|\Im_{x_1,t}^{\zeta,\varrho}\hbar(t)+\Im_{x_2,t}^{\zeta,\varrho}\hbar(t)-\zeta\frac{(t-x_1)^{\varrho}\hbar(x_1)+(x_2-t)^{\varrho}\hbar(x_2)}{H(\zeta)\Gamma(\varrho+1)}-\frac{2(1-\zeta)\hbar(t)}{H(\zeta)}\right|$$

$$\leq\frac{\zeta(t-x_1)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)}\left(\frac{1}{\varrho p+1}\right)^{\frac{1}{p}}\left[\frac{|\hbar'(t)|^q+|\hbar'(x_1)|^q}{2}\right]^{\frac{1}{q}}$$

$$+\frac{\zeta(x_2-t)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)}\left(\frac{1}{\varrho p+1}\right)^{\frac{1}{p}}\left[\frac{|\hbar'(x_2)|^q+|\hbar'(t)|^q}{2}\right]^{\frac{1}{q}},$$

where $\frac{1}{p}+\frac{1}{q}=1$, $t\in[x_1,x_2]$, $\zeta\in(0,1]$ and $H(\zeta)>0$.

Proof. By utilizing Lemma 3.1, we have

$$\left|\Im_{x_1,t}^{\zeta,\varrho}\hbar(t)+\Im_{x_2,t}^{\zeta,\varrho}\hbar(t)-\zeta\frac{(t-x_1)^{\varrho}\hbar(x_1)+(x_2-t)^{\varrho}\hbar(x_2)}{H(\zeta)\Gamma(\varrho+1)}-\frac{2(1-\zeta)\hbar(t)}{H(\zeta)}\right|$$

$$\leq\frac{\zeta(t-x_1)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)}\int_0^1(1-\eta)^{\varrho}\,|\,\hbar'(\eta t+(1-\eta)x_1)\,|\,d\eta$$

$$+\frac{\zeta(x_2-t)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)}\int_0^1\eta^{\varrho}\,|\,\hbar'(\eta x_2+(1-\eta)t)\,|\,d\eta.$$

By utilizing Hölder inequality, we obtain

$$\left|\Im_{x_1,t}^{\zeta,\varrho}\hbar(t)+\Im_{x_2,t}^{\zeta,\varrho}\hbar(t)-\zeta\frac{(t-x_1)^{\varrho}\hbar(x_1)+(x_2-t)^{\varrho}\hbar(x_2)}{H(\zeta)\Gamma(\varrho+1)}-\frac{2(1-\zeta)\hbar(t)}{H(\zeta)}\right|$$

$$\leq\frac{\zeta(t-x_1)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)}\left[\left(\int_0^1(1-\eta)^{\varrho p}d\eta\right)^{\frac{1}{p}}\left(\int_0^1|\,\hbar'(\eta t+(1-\eta)x_1)\,|^q\,d\eta\right)^{\frac{1}{q}}\right]$$

$$+\frac{\zeta(x_2-t)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)}\left[\left(\int_0^1\eta^{\varrho p}d\eta\right)^{\frac{1}{p}}\left(\int_0^1|\,\hbar'(\eta x_2+(1-\eta)t)\,|^q\,d\eta\right)^{\frac{1}{q}}\right]. \tag{8}$$

By utilizing the convexity of $|\,\hbar'\,|^q$, we get

$$\int_0^1|\,\hbar'(\eta t+(1-\eta)x_1)\,|^q\,d\eta\leq\int_0^1[\eta\,|\,\hbar'(t)\,|^q+(1-\eta)\,|\,\hbar'(x_1)\,|^q]d\eta \tag{9}$$

and

$$\int_0^1 | \hbar'(\eta x_2+(1-\eta)t) |^q \, d\eta \leq \int_0^1 [\eta \mid \hbar'(x_2) |^q +(1-\eta) \mid \hbar'(t) |^q] d\eta. \tag{10}$$

Substituting (9) and (10) in (8) and then solving the integrals, we get the required inequality. □

Corollary 3.9 Applying Theorem 3.8 for $t=\frac{x_1+x_2}{2}$, we get the following inequality

$$\left| \Im^{\zeta,\varrho}_{x_1,\frac{x_1+x_2}{2}} \hbar\left(\frac{x_1+x_2}{2}\right) + \Im^{\zeta,\varrho}_{x_2,\frac{x_1+x_2}{2}} \hbar(t) - \zeta \frac{(x_2-x_1)^\varrho}{H(\zeta)\Gamma(\varrho+1)} [\hbar(x_1)+\hbar(x_2)] - \frac{2(1-\zeta)\hbar\left(\frac{x_1+x_2}{2}\right)}{H(\zeta)} \right|$$

$$\leq \frac{\zeta(x_2-x_1)^{\varrho+1}}{2^{\varrho+1}H(\zeta)\Gamma(\varrho+1)} \left(\frac{1}{\varrho p+1}\right)^{\frac{1}{p}} \left[\left(\frac{\left|\hbar'\left(\frac{x_1+x_2}{2}\right)\right|^q + | \hbar'(x_1) |^q}{2} \right)^{\frac{1}{q}} \right.$$

$$\left. + \left(\frac{| \hbar'(x_2) |^q + \left|\hbar'\left(\frac{x_1+x_2}{2}\right)\right|^q}{2} \right)^{\frac{1}{q}} \right].$$

Remark 3.10 Applying Theorem 3.8 for $\zeta=\varrho$, we get Theorem 2.5 proved by Set et al. [31].

Remark 3.11 Applying Theorem 3.9 for $\zeta=\varrho$, we get Corollary 2.6 proved by Set et al. [31].

Remark 3.12 Applying Theorem 3.8 for $\zeta=\varrho=1$, we get Theorem 5 given in [32].

Theorem 3.13 Let $\hbar : [x_1, x_2] \to \mathbb{R}$ is a differentiable function on (x_1, x_2), where $\hbar' \in L_1[x_1, x_2]$ and $x_1 < x_2$. The following inequality holds for Hattaf-fractional operators if $|\hbar'|^q$ is a convex function.

$$\left| \Im^{\zeta,\varrho}_{x_1,t} \hbar(t) + \Im^{\zeta,\varrho}_{x_2,t} \hbar(t) - \zeta \frac{(t-x_1)^\varrho \hbar(x_1)+(x_2-t)^\varrho \hbar(x_2)}{H(\zeta)\Gamma(\varrho+1)} - \frac{2(1-\zeta)\hbar(t)}{H(\zeta)} \right|$$

$$\leq \frac{\zeta(t-x_1)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)} \left(\frac{1}{p(\varrho p+1)} + \frac{| \hbar'(t) |^q + | \hbar'(x_1) |^q}{2q} \right)$$

$$+ \frac{\zeta(x_2-t)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)} \left(\frac{1}{p(\varrho p+1)} + \frac{| \hbar'(x_2) |^q + | \hbar'(t) |^q}{2q} \right),$$

where $\frac{1}{p}+\frac{1}{q}=1$, $t \in [x_1, x_2]$, $\zeta \in (0, 1]$ and $H(\zeta) > 0$.

Proof. By utilizing Lemma 3.1, we have

$$\left|\Im_{x_1,t}^{\zeta,\varrho}\hbar(t)+\Im_{x_2,t}^{\zeta,\varrho}\hbar(t)-\zeta\frac{(t-x_1)^{\varrho}\hbar(x_1)+(x_2-t)^{\varrho}\hbar(x_2)}{H(\zeta)\Gamma(\varrho+1)}-\frac{2(1-\zeta)\hbar(t)}{H(\zeta)}\right|$$

$$\leq\frac{\zeta(t-x_1)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)}\int_0^1(1-\eta)^{\varrho}\mid\hbar'(\eta t+(1-\eta)x_1)\mid d\eta$$

$$+\frac{\zeta(x_2-t)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)}\int_0^1\eta^{\varrho}\mid\hbar'(\eta x_2+(1-\eta)t)\mid d\eta.$$

By employing Young inequality as $uv\leq\frac{u^p}{p}+\frac{v^q}{q}$ in above, we have

$$\left|\Im_{x_1,t}^{\zeta,\varrho}\hbar(t)+\Im_{x_2,t}^{\zeta,\varrho}\hbar(t)-\zeta\frac{(t-x_1)^{\varrho}\hbar(x_1)+(x_2-t)^{\varrho}\hbar(x_2)}{H(\zeta)\Gamma(\varrho+1)}-\frac{2(1-\zeta)\hbar(t)}{H(\zeta)}\right|$$

$$\leq\frac{\zeta(t-x_1)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)}\left[\frac{1}{p}\int_0^1(1-\eta)^{\varrho p}d\eta+\frac{1}{q}\int_0^1\mid\hbar'(\eta t+(1-\eta)x_1)\mid^q d\eta\right]$$

$$+\frac{\zeta(x_2-t)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)}\left[\frac{1}{p}\int_0^1\eta^{\varrho p}d\eta+\frac{1}{q}\int_0^1\mid\hbar'(\eta x_2+(1-\eta)t)\mid^q d\eta\right].$$

After solving the integrals, we get the desired inequality. □

Corollary 3.14 Applying Theorem 3.13 for $t=\frac{x_1+x_2}{2}$, we get the following inequality

$$\left|\Im_{x_1,\frac{x_1+x_2}{2}}^{\zeta,\varrho}\hbar\left(\frac{x_1+x_2}{2}\right)+\Im_{x_2,\frac{x_1+x_2}{2}}^{\zeta,\varrho}\hbar\left(\frac{x_1+x_2}{2}\right)-\zeta\frac{(x_2-x_1)^{\varrho}}{2^{\varrho}H(\zeta)\Gamma(\varrho+1)}[\hbar(x_1)+\hbar(x_2)]-\frac{2(1-\zeta)\hbar\left(\frac{x_1+x_2}{2}\right)}{H(\zeta)}\right|$$

$$\leq\frac{(x_2-x_1)^{\varrho+1}}{2^{\varrho+1}H(\zeta)\Gamma(\varrho+1)}\left(\frac{2}{p(\varrho p+1)}+\frac{2\left|\hbar'\left(\frac{x_1+x_2}{2}\right)\right|^q+\mid\hbar'(x_1)\mid^q+\mid\hbar'(x_2)\mid^q}{2q}\right).$$

Remark 3.15 Applying Theorem 3.13 for $\zeta=\varrho$, we get Theorem 2.8 proved by Set et al. [31].
Remark 3.16 Applying Theorem 3.14 for $\zeta=\varrho$, we get Corollary 2.9 proved by Set et al. [31].
Remark 3.17 Applying Theorem 3.13 for $\zeta=\varrho=1$, we get the result proved by [32].

Theorem 3.18 Let $\hbar:[x_1,x_2]\to\mathbb{R}$ is a differentiable function on (x_1,x_2), where $\hbar'\in L_1[x_1,x_2]$ and $x_1<x_2$. The following inequality holds for Hattaf-fractional operators if $|\hbar'|^q$ is a convex function.

$$\left|\Im^{\zeta,\varrho}_{x_1,t}\hbar(t)+\Im^{\zeta,\varrho}_{x_2,t}\hbar(t)-\zeta\frac{(t-x_1)^{\varrho}\hbar(x_1)+(x_2-t)^{\varrho}\hbar(x_2)}{H(\zeta)\Gamma(\varrho+1)}-\frac{2(1-\zeta)\hbar(t)}{H(\zeta)}\right|$$

$$\leq \frac{\zeta(t-x_1)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)}\left(\frac{1}{\varrho+1}\right)^{1-\frac{1}{q}}\left[\frac{|\hbar'(t)|^q}{(\varrho+1)(\varrho+2)}+\frac{|\hbar'(x_1)|^q}{(\varrho+2)}\right]^{\frac{1}{q}}$$

$$+\frac{(x_2-t)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)}\left(\frac{1}{\varrho+1}\right)^{1-\frac{1}{q}}\left[\frac{|\hbar'(x_2)|^q}{(\varrho+2)}+\frac{|\hbar'(t)|^q}{(\varrho+1)(\varrho+2)}\right]^{\frac{1}{q}},$$

where $q \geq 1$, $t \in [x_1, x_2]$, $\zeta \in (0, 1]$ and $H(\zeta) > 0$.

Proof. By utilizing Lemma 3.1, we have

$$\left|\Im^{\zeta,\varrho}_{x_1,t}\hbar(t)+\Im^{\zeta,\varrho}_{x_2,t}\hbar(t)-\zeta\frac{(t-x_1)^{\varrho}\hbar(x_1)+(x_2-t)^{\varrho}\hbar(x_2)}{H(\zeta)\Gamma(\varrho+1)}-\frac{2(1-\zeta)\hbar(t)}{H(\zeta)}\right|$$

$$\leq \frac{(t-x_1)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)}\int_0^1(1-\eta)^{\varrho}\,|\hbar'(\eta t+(1-\eta)x_1)|\,d\eta$$

$$+\frac{(x_2-t)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)}\int_0^1\eta^{\varrho}\,|\hbar'(\eta x_2+(1-\eta)t)|\,d\eta.$$

By employing power mean inequality, we obtain

$$\left|\Im^{\zeta,\varrho}_{x_1,t}\hbar(t)+\Im^{\zeta,\varrho}_{x_2,t}\hbar(t)-\zeta\frac{(t-x_1)^{\varrho}\hbar(x_1)+(x_2-t)^{\varrho}\hbar(x_2)}{H(\zeta)\Gamma(\varrho+1)}-\frac{2(1-\zeta)\hbar(t)}{H(\zeta)}\right|$$

$$\leq \frac{\zeta(t-x_1)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)}\left(\int_0^1(1-\eta)^{\varrho}d\eta\right)^{1-\frac{1}{q}}\left(\int_0^1(1-\eta)^{\varrho}\,|\hbar'(\eta t+(1-\eta)x_1)|^q\,d\eta\right)^{\frac{1}{q}}$$

$$+\frac{(x_2-t)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)}\left(\int_0^1\eta^{\varrho}d\eta\right)^{1-\frac{1}{q}}\left(\int_0^1\eta^{\varrho}\,|\hbar'(\eta x_2+(1-\eta)t)|^q\,d\eta\right)^{\frac{1}{q}}.$$

Now, by utilizing the convexity of $|\hbar'|^q$, we get

$$\left|\Im_{x_1,t}^{\zeta,\varrho}\hbar(t)+\Im_{x_2,t}^{\zeta,\varrho}\hbar(t)-\zeta\frac{(t-x_1)^{\varrho}\hbar(x_1)+(x_2-t)^{\varrho}\hbar(x_2)}{H(\zeta)\Gamma(\varrho+1)}-\frac{2(1-\zeta)\hbar(t)}{H(\zeta)}\right|$$

$$\leq\frac{\zeta(t-x_1)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)}\left(\int_0^1(1-\eta)^{\varrho}d\eta\right)^{1-\frac{1}{q}}\left(\int_0^1(1-\eta)^{\varrho}[\eta\mid\hbar'(t)\mid^q+(1-\eta)\mid\hbar'(x_1)\mid^q]d\eta\right)^{\frac{1}{q}}$$

$$+\frac{\zeta(x_2-t)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)}\left(\int_0^1\eta^{\varrho}d\eta\right)^{1-\frac{1}{q}}\left(\int_0^1\eta^{\varrho}[\eta\mid\hbar'(x_2))\mid^q+(1-\eta)\mid\hbar'(t)\mid^q]d\eta\right)^{\frac{1}{q}}$$

$$\leq\frac{\zeta(t-x_1)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)}\left(\frac{1}{\varrho+1}\right)^{1-\frac{1}{q}}\left[\frac{\mid\hbar'(t)\mid^q}{(\varrho+1)(\varrho+2)}+\frac{\mid\hbar'(x_1)\mid^q}{(\varrho+2)}\right]^{\frac{1}{q}}$$

$$+\frac{\zeta(x_2-t)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)}\left(\frac{1}{\varrho+1}\right)^{1-\frac{1}{q}}\left[\frac{\mid\hbar'(x_2)\mid^q}{(\varrho+2)}+\frac{\mid\hbar'(t)\mid^q}{(\varrho+1)(\varrho+2)}\right]^{\frac{1}{q}},$$

which completes the proof. □

Corollary 3.19 Applying Theorem 3.18 for $t=\dfrac{x_1+x_2}{2}$, we get the following inequality

$$\left|\Im_{x_1,\frac{x_1+x_2}{2}}^{\zeta,\varrho}\hbar\left(\frac{x_1+x_2}{2}\right)+\Im_{x_2,\frac{x_1+x_2}{2}}^{\zeta,\varrho}\hbar\left(\frac{x_1+x_2}{2}\right)-\zeta\frac{(x_2-x_1)^{\varrho}}{H(\zeta)\Gamma(\varrho+1)}[\hbar(x_1)+\hbar(x_2)]-\frac{2(1-\zeta)\hbar\left(\frac{x_1+x_2}{2}\right)}{H(\zeta)}\right|$$

$$\leq\frac{(x_2-x_1)^{\varrho+1}}{2^{\varrho+1}H(\zeta)\Gamma(\varrho+1)}\left(\frac{1}{\varrho+1}\right)^{1-\frac{1}{q}}\left[\frac{\left|\hbar'\left(\frac{x_1+x_2}{2}\right)\right|^q}{(\varrho+1)(\varrho+2)}+\frac{\mid\hbar'(x_1)\mid^q}{(\varrho+2)}\right]^{\frac{1}{q}}$$

$$+\frac{\zeta(x_2-x_1)^{\varrho+1}}{2^{\varrho+1}H(\zeta)\Gamma(\varrho+1)}\left(\frac{1}{\varrho+1}\right)^{1-\frac{1}{q}}\left[\frac{\mid\hbar'(x_2)\mid^q}{(\varrho+2)}+\frac{\left|\hbar'\left(\frac{x_1+x_2}{2}\right)\right|^q}{(\varrho+1)(\varrho+2)}\right]^{\frac{1}{q}}.$$

Remark 3.20 Applying Theorem 3.18 for $\zeta=\varrho$, we get Theorem 2.10 proved by Set et al. [31].

Remark 3.21 Applying Theorem 3.19 for $\zeta=\varrho$, we get Corollary 2.11 proved by Set et al. [31].

Remark 3.22 Applying Theorem 3.18 for $\zeta=\varrho=1$, we get Theorem 7 given in [32].

Theorem 3.23 Let $\hbar:[x_1,x_2]\to\mathbb{R}$ be a differentiable function on (x_1,x_2), where $\hbar'\in L_1[x_1,x_2]$ and $x_1<x_2$. For Hattaf-fractional operators, we have the following inequality if $|\hbar'|$ is a concave function.

$$\left|\Im_{x_1,t}^{\zeta,\varrho}\hbar(t)+\Im_{x_2,t}^{\zeta,\varrho}\hbar(t)-\zeta\frac{(t-x_1)^{\varrho}\hbar(x_1)+(x_2-t)^{\varrho}\hbar(x_2)}{H(\zeta)\Gamma(\varrho+1)}-\frac{2(1-\zeta)\hbar(t)}{H(\zeta)}\right|$$

$$\leq\frac{\zeta(t-x_1)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)}\left(\frac{1}{\varrho+1}\right)\left|\hbar'\left(\frac{1}{\varrho+2}t+\frac{\varrho+1}{\varrho+2}r\right)\right|$$

$$+\frac{\zeta(x_2-t)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)}\left(\frac{1}{\varrho+1}\right)\left|\hbar'\left(\frac{\varrho+1}{\varrho+2}x_2+\frac{1}{\varrho+2}t\right)\right|,$$

where $t\in[x_1, x_2]$, $\zeta\in(0, 1]$ and $H(\zeta)>0$.

Proof. By utilizing Lemma 3.1, we have

$$\left|\Im_{x_1,t}^{\zeta,\varrho}\hbar(t)+\Im_{x_2,t}^{\zeta,\varrho}\hbar(t)-\zeta\frac{(t-x_1)^{\varrho}\hbar(x_1)+(x_2-t)^{\varrho}\hbar(x_2)}{H(\zeta)\Gamma(\varrho+1)}-\frac{2(1-\zeta)\hbar(t)}{H(\zeta)}\right|$$

$$\leq\frac{\zeta(t-x_1)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)}\int_0^1(1-\eta)^{\varrho}\mid\hbar'(\eta t+(1-\eta)x_1)\mid d\eta$$

$$+\frac{\zeta(x_2-t)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)}\int_0^1\eta^{\zeta}\mid\hbar'(\eta x_2+(1-\eta)t)\mid d\eta.$$

By employing the Jensen integral inequality, we get

$$\left|\Im_{x_1,t}^{\zeta,\varrho}\hbar(t)+\Im_{x_2,t}^{\zeta,\varrho}\hbar(t)-\zeta\frac{(t-x_1)^{\varrho}\hbar(x_1)+(x_2-t)^{\varrho}\hbar(x_2)}{H(\zeta)\Gamma(\varrho+1)}-\frac{2(1-\zeta)\hbar(t)}{H(\zeta)}\right|$$

$$\leq\frac{\zeta(t-x_1)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)}\left(\int_0^1(1-\eta)^{\varrho}d\eta\right)\left|\hbar'\left(\frac{\int_0^1(1-\eta)^{\varrho}(\eta t+(1-\eta)x_1)d\eta}{\int_0^1(1-\eta)^{\varrho}d\eta}\right)\right|$$

$$+\frac{\zeta(x_2-t)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)}\left(\int_0^1\eta^{\varrho}d\eta\right)\left|\hbar'\left(\frac{\int_0^1\eta^{\varrho}(\eta x_2+(1-\eta)t)d\eta}{\int_0^1\eta^{\varrho}d\eta}\right)\right|.$$

After simple calculation of above integrals, we get the required inequality. □

Corollary 3.24 Applying Theorem 3.23 for $t=\dfrac{x_1+x_2}{2}$, we get the following inequality

$$\left| \Im^{\zeta,\varrho}_{x_1,\frac{x_1+x_2}{2}} \hbar\left(\frac{x_1+x_2}{2}\right) + \Im^{\zeta,\varrho}_{x_2,\frac{x_1+x_2}{2}} \hbar\left(\frac{x_1+x_2}{2}\right) - \zeta \frac{(t-x_1)^{\varrho}}{2^{\varrho} H(\zeta)\Gamma(\varrho+1)} [\hbar(x_1)+\hbar(x_2)] - \frac{2(1-\zeta)\hbar\left(\frac{x_1+x_2}{2}\right)}{H(\zeta)} \right|$$

$$\leq \frac{\zeta(x_2-x_1)^{\varrho+1}}{2^{\varrho+1}H(\zeta)\Gamma(\varrho+1)} \left(\frac{1}{\varrho+1}\right) \left[\left| \hbar'\left(\frac{x_1+x_2}{2(\varrho+2)} + \frac{\varrho+1}{\varrho+2}a\right) \right| + \left| \hbar'\left(\frac{\varrho+1}{\varrho+2}b + \frac{x_1+x_2}{2(\varrho+2)}\right) \right| \right].$$

Remark 3.25 Applying Theorem 3.23 for $\zeta = \varrho$, we get Theorem 2.13 proved by Set et al. [31].

Remark 3.26 Applying Theorem 3.24 for $\zeta = \varrho$, we get Corollary 2.14 proved by Set et al. [31].

Remark 3.27 Applying Theorem 3.23 for $\varrho = \zeta = 1$, we get Theorem 8 given in [32].

Theorem 3.28 Let $\hbar : [x_1, x_2] \to \mathbb{R}$ be a differentiable function on (x_1, x_2), where $\hbar' \in L_1[x_1, x_2]$ and $x_1 < x_2$. For Hattaf-fractional operators, we have the following inequality if $|\hbar'|^q$ is a concave function.

$$\left| \Im^{\zeta,\varrho}_{x_1,t}\hbar(t) + \Im^{\zeta,\varrho}_{x_2,t}\hbar(t) - \zeta \frac{(t-x_1)^{\varrho}\hbar(x_1)+(x_2-t)^{\varrho}\hbar(x_2)}{H(\zeta)\Gamma(\varrho+1)} - \frac{2(1-\zeta)\hbar(t)}{H(\zeta)} \right|$$

$$\leq \frac{\zeta(t-x_1)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)} \left(\frac{1}{\varrho p+1}\right)^{\frac{1}{p}} \left| \hbar'\left(\frac{t+r}{2}\right) \right| + \frac{(x_2-t)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)} \left(\frac{1}{\varrho p+1}\right)^{\frac{1}{p}} \left| \hbar'\left(\frac{x_2+t}{2}\right) \right|,$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $q > 1$, $t \in [x_1, x_2]$, $\zeta \in (0, 1]$ and $H(\zeta) > 0$.

Proof. By utilizing Lemma 3.1, we have

$$\left| \Im^{\zeta,\varrho}_{x_1,t}\hbar(t) + \Im^{\zeta,\varrho}_{x_2,t}\hbar(t) - \zeta \frac{(t-x_1)^{\varrho}\hbar(x_1)+(x_2-t)^{\varrho}\hbar(x_2)}{H(\zeta)\Gamma(\varrho+1)} - \frac{2(1-\zeta)\hbar(t)}{H(\zeta)} \right|$$

$$\leq \frac{\zeta(t-x_1)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)} \int_0^1 (1-\eta)^{\varrho} \mid \hbar'(\eta t + (1-\eta)x_1) \mid d\eta$$

$$+ \frac{\zeta(x_2-t)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)} \int_0^1 \eta^{\varrho} \mid \hbar'(\eta x_2 + (1-\eta)t) \mid d\eta.$$

By employing the Hölder integral inequality, we obtain

$$\left|\Im_{x_1,t}^{\zeta,\varrho}\hbar(t)+\Im_{x_2,t}^{\zeta,\varrho}\hbar(t)-\zeta\frac{(t-x_1)^{\varrho}\hbar(x_1)+(x_2-t)^{\varrho}\hbar(x_2)}{H(\zeta)\Gamma(\varrho+1)}-\frac{2(1-\zeta)\hbar(t)}{H(\zeta)}\right|$$

$$\leq\frac{\zeta(t-x_1)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)}\left(\int_0^1(1-\eta)^{\varrho p}d\eta\right)^{\frac{1}{p}}\left(\int_0^1|\hbar'(\eta t+(1-\eta)x_1)|^q\,d\eta\right)^{\frac{1}{q}}$$

$$+\frac{\zeta(x_2-t)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)}\left(\int_0^1\eta^{\varrho p}d\eta\right)^{\frac{1}{p}}\left(\int_0^1|\hbar'(\eta x_2+(1-\eta)t)|^q\,d\eta\right)^{\frac{1}{q}}.$$

Now, by employing the concavity of $|\hbar'|^q$ and Jensen integral inequality, we obtain

$$\left|\Im_{x_1,t}^{\zeta,\varrho}\hbar(t)+\Im_{x_2,t}^{\zeta,\varrho}\hbar(t)-\zeta\frac{(t-x_1)^{\varrho}\hbar(x_1)+(x_2-t)^{\varrho}\hbar(x_2)}{H(\zeta)\Gamma(\varrho+1)}-\frac{2(1-\zeta)\hbar(t)}{H(\zeta)}\right|$$

$$\leq\frac{\zeta(t-x_1)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)}\left(\int_0^1(1-\eta)^{\varrho p}d\eta\right)^{\frac{1}{p}}\left(\int_0^1|\hbar'(\eta t+(1-\eta)x_1)|^q\,d\eta\right)^{\frac{1}{q}}$$

$$+\frac{\zeta(x_2-t)^{\varrho+1}}{H(\zeta)\Gamma(\varrho+1)}\left(\int_0^1\eta^{\varrho p}d\eta\right)^{\frac{1}{p}}\left(\int_0^1|\hbar'(\eta x_2+(1-\eta)t)|^q\,d\eta\right)^{\frac{1}{q}}. \tag{11}$$

Since

$$\int_0^1|\hbar'(\eta t+(1-\eta)x_1)|^q\,d\eta\leq\int_0^1\eta^0\,|\hbar'(\eta t+(1-\eta)x_1)|^q\,d\eta$$

$$\leq\left(\int_0^1\eta^0d\eta\right)\left|\hbar'\left(\frac{1}{\int_0^1\eta^0d\eta}\int_0^1(\eta t+(1-\eta)x_1)d\eta\right)\right|^q$$

$$\leq\left|\hbar'\left(\frac{x_1+t}{2}\right)\right|^q. \tag{12}$$

Similarly

$$\int_0^1|\hbar'(\eta x_2+(1-\eta)t)|^q\,d\eta\leq\left|\hbar'\left(\frac{x_2+t}{2}\right)\right|^q. \tag{13}$$

By substituting (13) and (12) in (11) and then solving the integrals, we get the desire assertion. □

Corollary 3.29 Applying Theorem 3.28 for $t=\dfrac{x_1+x_2}{2}$, we get the following inequality

$$\left|\mathfrak{I}^{\zeta,\varrho}_{x_1,\frac{x_1+x_2}{2}}\hbar\left(\frac{x_1+x_2}{2}\right)+\mathfrak{I}^{\zeta,\varrho}_{x_2,\frac{x_1+x_2}{2}}\hbar\left(\frac{x_1+x_2}{2}\right)-\zeta\frac{(x_2-x_1)^{\varrho}}{2^{\varrho}H(\zeta)\Gamma(\varrho+1)}[\hbar(x_1)+\hbar(x_2)]-\frac{2(1-\zeta)\hbar\left(\frac{x_1+x_2}{2}\right)}{H(\zeta)}\right|$$

$$\leq\frac{\zeta(x_2-x_1)^{\varrho+1}}{2^{\varrho+1}H(\zeta)\Gamma(\varrho+1)}\left(\frac{1}{\varrho p+1}\right)^{\frac{1}{p}}\left[\left|\hbar'\left(\frac{3x_1+x_2}{4}\right)\right|+\left|\hbar'\left(\frac{3x_2+x_1}{4}\right)\right|\right].$$

Remark 3.30 Applying Theorem 3.28 for $\zeta=\varrho$, we get Theorem 2.16 proved by Set et al. [31].
Remark 3.31 Applying Theorem 3.28 for $\zeta=\varrho=1$, we get Theorem 6 given in [32].

4. Concluding remarks

In this study, we have established Hermite-Hadamard type integral inequalities for convex functions by employing Hattaf-fractional integral operators. These inequalities for fractional integral operators are obtained by using the applications of Jensen integral inequality, Young's inequality, power-mean inequality, and Hölder inequality. The inequalities proved in this study are more general as compared to the previous work. The inequalities in term of Atangana-Baleanu fractional integral operators will be restored if we put $\zeta=\varrho$ (see for example [31]). Also, the classical inequalities will be restored if we put $\zeta=\varrho=1$ (see for example [32]). These inequalities will promote future research in the field of mathematical inequities. One can obtain certain other type inequalities by using Hattaf fractional integral operators such as Grüss, Chebyshev, Reverse Minkowski's and Ostrowski type integral inequalities.

Authors' contributions

The authors have worked equally when writing this paper. All authors read and approved the final manuscript.

Acknowledgement

The authors A. Aloqaily and N. Mlaiki would like to thank Prince Sultan University for paying the publication fees for this work through TAS LAB.

Conflict of interest

The authors declare that they have no competing interests.

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