

Research Article

Solvability of a Doubly Singular Boundary Value Problem Arising in Front Propagation for Reaction-Diffusion Equations

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Abstract: The paper deals with the solvability of the following doubly singular boundary value problem

$$\begin{cases} \dot{z} = cg(u) - f(u) - \frac{h(u)}{z^\alpha} \\ z(0^+) = 0, z(1^-) = 0, z(u) > 0 \text{ in } (0, 1) \end{cases}$$

naturally arising in the study of the existence and properties of travelling waves for reaction-diffusion-convection equations governed by the p -Laplacian operator. Here c , α are real parameters, with $\alpha > 0$, and f , g , h are continuous functions in $[0, 1]$, with

$$h(0) = h(1), h(u) > 0 \text{ in } (0, 1).$$

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1. Introduction

One of the most popular topics in the study of diffusion equations, in their various possible variants (reaction-diffusion, convection-diffusion, aggregation-reaction-diffusion, etc.) is the existence and the properties of travelling fronts. These particular solutions v , satisfying $v(\tau, x) = u(x - c\tau)$ for some one-variable function u , have a relevant role in understanding asymptotic behavior of the other solutions of the PDE (see, e.g., [1–3]).

When searching for travelling fronts, the original PDE reduces to an autonomous ordinary differential equation and if the fronts are monotone this can be further reduced to a first-order singular ordinary equation or, equivalently, to a singular integral equation. For instance, the travelling fronts of the general reaction-diffusion-convection equation with accumulation term

$$f(v)v_x + g(v)v_\tau = (\mathcal{D}(v)v_x)_x + \rho(v)$$

are solutions of the following autonomous second-order equation

$$(D(u)u')' + (cg(u) - f(u))u' + \rho(u) = 0$$

where c is the wave speed and $'$ means the derivative with respect to the wave variable t . Therefore, the existence of monotone fronts is equivalent to the solvability of the following first-order equation

$$\dot{z} = cg(u) - f(u) - \frac{D(u)\rho(u)}{z}.$$

Hence, when the reaction term ρ vanishes at 0 and 1 (and it is positive elsewhere), then the study of the existence of monotone travelling fronts, connecting the equilibria 0 and 1, involves the following doubly singular boundary value problem:

$$\begin{cases} \dot{z} = cg(u) - f(u) - \frac{h(u)}{z} \\ z(0^+) = z(1^-) = 0 \\ z(u) > 0 \text{ in } (0, 1) \end{cases} \quad (1)$$

where we have put $h(u) := D(u)\rho(u)$ (see [4]).

Problem (1) admits an equivalent formulation as a singular Volterra-type integral equation:

$$\begin{cases} z(u) = \int_0^u \left(cg(u) - f(u) - \frac{h(u)}{z} \right) du \\ z(1^-) = 0 \\ z(u) > 0 \text{ in } (0, 1) \end{cases}$$

(see [5] in the case of constant g).

In the last years, diffusion equations governed by the p -Laplacian operator, that is

$$f(v)v_x + g(v)v_\tau = \Delta_p(v) + \rho(v)$$

have been the subject of growing interest (see, e.g. [6–10]). In this case, problem (1) becomes

$$\begin{cases} \dot{z} = cg(u) - f(u) - \frac{h(u)}{z^\alpha} \\ z(0^+) = z(1^-) = 0 \\ z(u) > 0 \text{ in } (0, 1). \end{cases} \quad (2)$$

for a suitable $\alpha > 0$ depending on p .

Despite a very extensive literature on the subject, a complete study about the solvability of problem (2) seems not known, at least in such a general form (see [11] for a recent result without the convection and accumulation terms f, g).

In this paper we provide an existence and non-existence result for problem (2), given in terms of admissible wave speeds c , related to the value of the parameter α . More in detail, the main result is the following:

Theorem 1 Let $f, g, h \in C([0, 1])$ be such that

$$h(0) = h(1) = 0 \text{ and } h(u) > 0 \text{ for every } u \in (0, 1). \quad (3)$$

Assume that $g(0) > 0$ and $\int_0^u g(s) ds > 0$ for every $u \in [0, 1]$. Moreover, assume that

$$\text{there exists the limit } h_{\alpha, 0} := \lim_{u \rightarrow 0^+} \frac{h(u)}{u^\alpha} \in [0, +\infty]. \quad (4)$$

Then, if $h_{\alpha, 0} = +\infty$ problem (2) does not admit solutions for any $c \in \mathbb{R}$.

Otherwise, if $h_{\alpha, 0} < +\infty$, there exists a threshold value c^* such that problem (2) admits solution if and only if $c \geq c^*$. Moreover, put

$$G_0 := \inf_{u \in (0, 1)} \int_0^u g(s) ds, F_0 := \sup_{u \in (0, 1)} \int_0^u f(s) ds, H_0 := \sup_{u \in (0, 1)} \int_0^u \frac{h(s)}{s^\alpha} ds, \quad (5)$$

(where \int stands for the mean value) we have

$$\frac{f(0)}{g(0)} + \frac{\alpha + 1}{g(0)} \left(\frac{h_{\alpha, 0}}{\alpha^\alpha} \right)^{\frac{1}{\alpha+1}} \leq c^* \leq \frac{F_0}{G_0} + \frac{\alpha + 1}{G_0} \left(\frac{H_0}{\alpha^\alpha} \right)^{\frac{1}{\alpha+1}}. \quad (6)$$

Finally, for every $c \geq c^*$ the solution is unique.

As a consequence of the previous theorem, we have that problem (2) admits a solution for some $c \in \mathbb{R}$ if and only if the value $h_{\alpha, 0}$ is finite. This places an upper bound on the admissible values of α ; namely, if $h(u) \sim Cu^q$ as $u \rightarrow 0^+$, then $h_{\alpha, 0} < +\infty$ if and only if $\alpha \leq q$.

Finally, note that when g is constant, say $g(u) \equiv 1$, then (6) reduces to

$$f(0) + (\alpha + 1) \left(\frac{h_{\alpha, 0}}{\alpha^\alpha} \right)^{\frac{1}{\alpha+1}} \leq c^* \leq \sup_{u \in (0, 1)} \int_0^u f(s) ds + (\alpha + 1) \left(\frac{1}{\alpha^\alpha} \sup_{u \in (0, 1)} \int_0^u \frac{h(s)}{s^\alpha} ds \right)^{\frac{1}{\alpha+1}}.$$

When, in addition, $\alpha = 1$ (the standard case) we obtain

$$f(0) + 2\sqrt{h(0)} \leq c^* \leq \sup_{u \in (0, 1)} \int_0^u f(s) ds + 2\sqrt{\sup_{u \in (0, 1)} \int_0^u \frac{h(s)}{s} ds}$$

that is the estimate proved in [12].

2. Preliminary results

In this section we establish some preliminary results concerning the solutions of the equation

$$\dot{z} = cg(u) - f(u) - \frac{h(u)}{z^\alpha} \quad (7)$$

for given $c \in \mathbb{R}$, $\alpha > 0$, and continuous functions f, g, h in $(0, 1)$ satisfying (3).

The first Lemma concerns the maximal existence interval for the solutions of (7) and the behavior at the extrema of the interval $(0, 1)$.

Lemma 1 Let $z \in C^1(a, b)$ be a positive solution of equation (7), where $(a, b) \subset (0, 1)$ is its maximal existence interval. Then, $a = 0$ and both the limits $z(0^+)$ and $z(b^-)$ exist and are finite.

Moreover, if (4) holds, then $h_{\alpha, 0} < +\infty$, and there exists $\dot{z}(0)$. Furthermore, $\dot{z}(0)$ is a zero of the function $\eta_0(t) := |t|^{\alpha+1} - (cg(0) - f(0))|t|^\alpha + h_{\alpha, 0}$.

Similarly, if $b = 1$ and

$$\text{there exists the limit } h_{\alpha, 1} := \lim_{u \rightarrow 1^-} \frac{-h(u)}{(1-u)^\alpha} \in [-\infty, 0],$$

then $h_{\alpha, 1} > -\infty$ and there exists $\dot{z}(1)$. Moreover, $\dot{z}(1)$ is a zero of the function $\eta_1(t) := |t|^{\alpha+1} + (cg(1) - f(1))|t|^\alpha + h_{\alpha, 1}$.

Proof. Note that $\dot{z}(u) > cg(u) - f(u)$ in (a, b) , so \dot{z} is bounded below, hence, we deduce that both the limits $z(a^+) := \lim_{u \rightarrow a^+} z(u)$ and $z(b^-) := \lim_{u \rightarrow b^-} z(u)$ exist (finite or infinite). If $z(a^+) = +\infty$, then equation (7) implies that $\lim_{u \rightarrow a^+} \dot{z}(u) = cg(a) - f(a) \in \mathbb{R}$, a contradiction. An analogous reasoning works for $z(b^-)$. Therefore, if $a > 0$, since (a, b) is the maximal existence interval of z , we infer $z(a^+) = 0$, implying $\lim_{u \rightarrow a^+} (cg(u) - f(u))z^\alpha(u) - h(u) = -h(a) < 0$. Hence, given a real $0 < \varepsilon < h(a)$, we have $\dot{z}(u) = cg(u) - f(u) - \frac{h(u)}{z^\alpha(u)} < -\frac{\varepsilon}{z^\alpha(u)} < 0$ in a right neighbourhood of a , which is a contradiction. This implies $a = 0$.

Assume now (4) and put

$$\zeta(u) := \frac{z(u)}{u}, \quad L := \limsup_{u \rightarrow 0^+} \zeta(u), \quad \ell := \liminf_{u \rightarrow 0^+} \zeta(u). \quad (8)$$

Suppose, by contradiction, $\ell < L$. So, for every $k \in (\ell, L)$ there exist decreasing sequences $(u_n)_n, (v_n)_n$, converging to 0, such that

$$\zeta(u_n) = \zeta(v_n) = k, \quad \dot{\zeta}(u_n) \geq 0, \quad \dot{\zeta}(v_n) \leq 0 \quad \text{for every } n \geq 1.$$

Since $\dot{\zeta}(u) = \frac{1}{u}(\dot{z}(u) - \zeta(u))$, we get $\dot{z}(u_n) \geq \zeta(u_n) = k$, so

$$k \leq \dot{z}(u_n) = cg(u_n) - f(u_n) - \frac{h(u_n)}{z^\alpha(u_n)} = cg(u_n) - f(u_n) - \frac{h(u_n)}{(ku_n)^\alpha}.$$

Passing to the limit as $n \rightarrow +\infty$, we have that $h_{0, \alpha}$ is finite and $k \leq cg(0) - f(0) - \frac{h_{\alpha, 0}}{k^\alpha}$, that is

$$k^{\alpha+1} - (cg(0) - f(0))k^\alpha + h_{\alpha, 0} \leq 0.$$

By means of a similar argument, by using the sequence $(v_n)_n$ instead of $(u_n)_n$, it is possible to show that $k^{\alpha+1} - (cg(0) - f(0))k^\alpha + h_{\alpha, 0} \geq 0$, hence we conclude

$$k^{\alpha+1} - (cg(0) - f(0))k^\alpha + h_{\alpha, 0} = 0 \text{ for all } k \in (\ell, L),$$

a contradiction. Then, $\ell = L$ and the limit $\lambda := \lim_{u \rightarrow 0^+} \frac{z(u)}{u} \geq 0$ exists.

Finally, since

$$\dot{z}(u) = cg(u) - f(u) - \frac{h(u)}{u^\alpha} \cdot \frac{u^\alpha}{z^\alpha(u)}, \quad (9)$$

then $\lambda < +\infty$, otherwise $\dot{z}(u) \rightarrow cg(0) - f(0)$ as $u \rightarrow 0^+$, a contradiction; so $\lambda \in \mathbb{R}$.

If $\lambda > 0$ then by (9) we deduce the existence of the limit $\dot{z}(0^+) := \lim_{u \rightarrow 0^+} \dot{z}(u)$ too; moreover, its value necessarily coincides with λ . So, passing again to the limit as $u \rightarrow 0^+$ in (9), we infer

$$\lambda^{\alpha+1} = (cg(0) - f(0))\lambda^\alpha - h_{\alpha, 0}$$

that is, $\dot{z}(0)$ is a zero of the function $\eta_0(t)$. Whereas, if $\lambda = 0$, then $h_{\alpha, 0} = 0$ too, otherwise $\dot{z}(u) \rightarrow +\infty$, a contradiction. Then, even if $\lambda = 0$, it is a zero of the function $\eta_0(t)$.

The local analysis near the point 1 can be made by the same way, putting in (8)

$$\zeta(u) = \frac{z(u)}{1-u}, \quad L := \limsup_{u \rightarrow 1^-} \zeta(u), \quad \ell := \liminf_{u \rightarrow 1^-} \zeta(u).$$

□

In what follows we will make use of the method of lower and upper-solutions.

Recall that a function $z \in C^1(I)$, with $I \subset (0, 1)$, is a lower-solution [upper-solution] for equation (7) if

$$\dot{z} \leq [\geq] cg(u) - f(u) - \frac{h(u)}{z^\alpha} \text{ for all } u \in I.$$

For the reader's convenience, we recall a classic comparison result (see e.g. [13, Theorems 9.5-9.6])

Lemma 2 Let z be a solution of (7) in $I \subset (0, 1)$ and let $u_0 \in I$ be fixed.

Let y be an upper-solution of (7) in the same interval, then

- (1) if $z(u_0) \leq y(u_0)$, we have $z(u) \leq y(u)$ for all $u \geq u_0$.
- (2) if $z(u_0) \geq y(u_0)$, we have $z(u) \geq y(u)$ for all $u \leq u_0$.

Whereas, if y is a lower-solution, then

- (1) if $z(u_0) \geq y(u_0)$, then $z(u) \geq y(u)$ for all $u \geq u_0$.

(2) if $z(u_0) \leq y(u_0)$, then $z(u) \leq y(u)$ for all $u \leq u_0$.

The following result provides the uniform boundedness by below of the solutions of (7) on the compact subintervals of $(0, 1)$.

Lemma 3 Let us fix $c_0 \in \mathbb{R}$ and $\alpha > 0$. For every $r \in \left(0, \frac{1}{2}\right)$ there exists $\delta = \delta_r > 0$ such that for every $c \in \mathbb{R}$ with $|c - c_0| < \delta$ and for positive solution z_c of equation (7), defined in $(0, 1)$, we have

$$z_c(u) \geq \delta \text{ for every } u \in [r, 1 - r].$$

Proof. Let us fix $r \in \left(0, \frac{1}{2}\right)$ and put $m := \min\{h(u) : u \in [r, 1 - r]\} > 0$. Let us consider the map $(c, u, z) \mapsto (cg(u) - f(u))z^\alpha - h(u)$, which is uniformly continuous in the compact set $[c_0 - 1, c_0 + 1] \times [0, 1] \times [0, m]$. Hence, there exists $\delta = \delta_r < (mr)^{\frac{1}{\alpha+1}}$, such that if $u_0 \in [r, 1 - r]$, then we have

$$(cg(u) - f(u))z^\alpha - h(u) < -h(u_0) + \frac{m\alpha}{\alpha + 1} \leq -\frac{m}{\alpha + 1} \text{ when } |u - u_0|, |z|, |c - c_0| < \delta. \quad (10)$$

For every fixed $u_0 \in [r, 1 - r]$, define the function

$$\psi(u) := (\delta^{\alpha+1} - m(u - u_0))^{\frac{1}{\alpha+1}} \text{ for } u_0 \leq u \leq u_0 + \delta^{\alpha+1}/m,$$

where the extremum $u^* := u_0 + \delta^{\alpha+1}/m < 1$, by the choice of δ .

Observe that $\psi(u) \leq \delta$ for every $u \in (u_0, u^*)$, so by (10) we can deduce that for every c with $|c - c_0| < \delta$ we have

$$\psi(u) > cg(u) - f(u) - \frac{h(u)}{\psi^\alpha(u)}, \text{ for every } u \in (u_0, u^*)$$

that is ψ is an upper-solution for equation (7) in such an interval, with $\psi(u_0) = \delta$ and $\psi(u^*) = 0$.

Therefore, if z_c is a solution of problem (2) defined in $(0, 1)$, then $z_c(u_0) < \delta = \psi(u_0)$. Indeed, since ψ is an upper-solution for equation (7) in the interval (u_0, u^*) , then necessarily $z_c(u) \leq \psi(u)$ in the same interval, implying that $z_c(u^*) = 0$, which is a contradiction since z_c is defined and positive on the whole interval $(0, 1)$. Therefore, we conclude that $z_c(u_0) \geq \delta$.

The assertion follows from the arbitrariness of u_0 . □

Lemma 4 Let $(c_n)_n$ be a decreasing (not necessarily strictly decreasing) sequence converging to c_0 and let $(z_n(u))_n$ be a sequence of positive solutions of equation (7) in $(0, 1)$ for $c = c_n$, pointwise convergent to a function $z_0(u)$ in $(0, 1)$. Then we have $z_0(u) > 0$ for every $u \in (0, 1)$ and z_0 is a solution of equation (7) in $(0, 1)$ for $c = c_0$.

Proof. As a consequence of Lemma 3, by the arbitrariness of $r > 0$ we have $z_0(u) > 0$ for every $u \in (0, 1)$. Moreover, again by Lemma 3, for every $r \in \left(0, \frac{1}{2}\right)$ we have

$$c_n g(u) - f(u) \leq c_n g(u) - f(u) - \frac{h(u)}{z_n^\alpha(u)} \leq c_n g(u) - f(u) - \frac{h(u)}{\delta^\alpha} \text{ for every } u \in [r, 1 - r],$$

for every $n \in \mathbb{N}$. So, by virtue of the Dominated Convergence Theorem we get that, for every $u, u^* \in [r, 1-r]$, we have

$$\begin{aligned} z_0(u) - z_0(u^*) &= \lim_{n \rightarrow +\infty} (z_n(u) - z_n(u^*)) = \lim_{n \rightarrow +\infty} \int_{u^*}^u \left(f(s) - c_n g(s) - \frac{h(s)}{z_n^\alpha(s)} \right) ds \\ &= \int_{u^*}^u \left(f(s) - c_0 g(s) - \frac{h(s)}{z_0^\alpha(s)} \right) ds. \end{aligned}$$

Hence, $z_0(u) = f(u) - c_0 g(u) - \frac{h(u)}{z_0^\alpha(u)}$ for every $u \in [r, 1-r]$. Since $r > 0$ is arbitrary, we get that z_0 solves equation (7) in the whole interval $(0, 1)$. \square

Remark 1 In light of the proof of the previous Lemma, the assertion holds even if the sequence $(z_n)_n$ is defined in a subinterval $(a, b) \subset (0, 1)$.

In the following result a comparison criterium is proved, in order to establish the existence and non-existence of the solutions of problem (2).

Proposition 1 Suppose (3) and assume that there exists a lower-solution φ for equation (7) in the whole interval $(0, 1)$, such that $\varphi(0^+) = 0$ and $\varphi(u) > 0$ for every $u \in (0, 1)$.

Then, there exists a C^1 -function $z : (0, 1) \rightarrow \mathbb{R}$, solution of the singular boundary value problem (2), such that $0 < z(u) < \varphi(u)$ for every $u \in (0, 1)$.

Proof. Notice that $\dot{\varphi}(u) \leq c g(u) - f(u)$, so φ is bounded above, implying the existence of the limit $\varphi(1^-) < +\infty$.

Let us divide the proof into two cases.

Case 1: $\varphi(1^-) > 0$.

For each $n \in \mathbb{N}$ let z_n be the solution of equation (7), satisfying the condition $z_n(1) = \varphi(1^-)/n$. By applying Lemma 1, we derive that z_n is defined in the whole interval $(0, 1]$. Furthermore, by virtue of Lemma 2 we get $z_n(u) \leq \varphi(u)$ for every $u \in (0, 1]$. Since (7) has a unique solution passing through a given point, we have $\varphi(u) \geq z_n(u) \geq z_{n+1}(u) > 0$ for each $n \in \mathbb{N}$ and every $u \in (0, 1)$. Put $\zeta(u) := \lim_{n \in \mathbb{N}} z_n(u)$, in force of Lemma 4, we get $\zeta(u) > 0$ in $(0, 1)$ and it is a solution of equation (7). Moreover, since $z_n(0^+) = 0$ for every $n \in \mathbb{N}$, we have $\zeta(0^+) = 0$. Finally, by definition, $z_n(1) \rightarrow 0 = \zeta(1^-)$.

Case 2: $\varphi(1^-) = 0$.

For each $n \in \mathbb{N}$ let z_n be the solution of equation (7), satisfying the condition $z_n\left(\frac{n}{n+1}\right) = \varphi\left(\frac{n}{n+1}\right)$, defined in its maximal existence interval (a_n, b_n) . By Lemma 1 we have $a_n = 0$ for every $n \in \mathbb{N}$; moreover, by Lemma 2 we have $z_n(u) \leq \varphi(u)$ for every $u \in \left(0, \frac{n}{n+1}\right)$ and $z_n(u) \geq \varphi(u)$ for every $u \in \left(\frac{n}{n+1}, b_n\right)$. Hence, since by Lemma 1 there exists $z_n(b_n^-)$ and it is finite, we also infer $b_n = 1$ for every $n \in \mathbb{N}$.

Furthermore, since $z_n\left(\frac{n}{n+1}\right) = \varphi\left(\frac{n}{n+1}\right) \geq z_{n+1}\left(\frac{n}{n+1}\right)$, again by the uniqueness of the solution of equation (7) passing through a point, we also derive that $z_n(u) = z_{n+1}(u)$ for each $u \in (0, 1)$ or $z_n(u) > z_{n+1}(u)$ for each $u \in (0, 1)$. Therefore, the sequence $(z_n)_n$ is decreasing. Set $\zeta(u) = \lim_{n \rightarrow +\infty} z_n(u)$; by applying Lemma 4 (with $c_n = c$ for every n) we have $\zeta(u) > 0$ for all $u \in (0, 1)$ and ζ is a solution of equation (7). Finally, we have $\zeta(u) \leq \varphi(u)$ in $(0, 1)$, hence $\zeta(0^+) = \varphi(0^+) = 0$ and $\zeta(1^-) = \varphi(1^-) = 0$. \square

Corollary 1 Let (3) be satisfied. Assume that there exists a continuous positive function $\psi : (0, 1) \rightarrow \mathbb{R}$ such that $\psi(0^+) = 0$ and

$$\psi(u) \leq \int_0^u \left(c g(s) - f(s) - \frac{h(s)}{\psi^\alpha(s)} \right) ds \quad \text{for every } u \in (0, 1). \quad (11)$$

Then, the singular boundary value problem (2) admits a solution $z \in C^1(0, 1)$.

Proof. Put $\varphi(u) := \int_0^u \left(cg(s) - f(s) - \frac{h(s)}{\psi^\alpha(s)} \right) ds$. Since ψ is positive, by (11) also φ is positive. Moreover, φ is differentiable, with

$$\varphi(u) = cg(u) - f(u) - \frac{h(u)}{\psi(u)} \leq cg(u) - f(u) - \frac{h(u)}{\varphi(u)} \text{ for every } u \in (0, 1).$$

Finally, since $\varphi(0^+) = 0$, then φ satisfies all the assumptions of Proposition 1. So, there exists a solution z of problem (2) satisfying $0 < z(u) \leq \varphi(u)$ for every $u \in (0, 1)$. \square

We conclude the section by stating a result concerning the behavior of a suitable function, which will be used in the proof of Theorem 1. We omit the proof since it is trivial.

Lemma 5 Let β, γ be fixed with $\gamma \geq 0$, and let

$$M_{\beta, \gamma}(t) := t^{\alpha+1} - \beta t^\alpha + \gamma, \quad t \geq 0.$$

Then, $M_{\beta, \gamma}$ has minimum $\mu_{\alpha, \beta}$ which has the same sign of the value $(\alpha + 1) \left(\frac{\gamma}{\alpha^\alpha} \right)^{\frac{1}{\alpha+1}} - \beta$.

3. Proof of Theorem 1 and some examples

Proof of Theorem 1.

First of all, let us prove that if problem (2) admits a solution for some c , then it is unique. To this aim, assume by contradiction that for a fixed c , problem (2) admits two different solutions z_1, z_2 . Since the differential equation in (2) admits a unique solution passing through a given point, we get $z_1(u) \neq z_2(u)$ for every $u \in (0, 1)$. So, we have $z_1(u) < z_2(u)$ for every $u \in (0, 1)$ (or vice versa) and then

$$0 = z_1(1^-) - z_1(0^+) = \int_0^1 \left(f(u) - cg(u) - \frac{h(u)}{z_1^\alpha(u)} \right) du < \int_0^1 \left(f(u) - cg(u) - \frac{h(u)}{z_2^\alpha(u)} \right) du = z_2(1^-) - z_2(0^+) = 0,$$

a contradiction.

Let us now assume (4). Notice that if $h_0, \alpha = +\infty$ then by Lemma 1 problem (2) does not admit solutions for any $c \in \mathbb{R}$. So, from now on we assume $h_0, \alpha < +\infty$.

Let us fix a value

$$c > \frac{F_0}{G_0} + \frac{\alpha + 1}{G_0} \left(\frac{H_0}{\alpha^\alpha} \right)^{\frac{1}{\alpha+1}}$$

(see (5)). Then, if we consider the function $M_{\beta, \gamma}$, defined in Lemma 5, for $\beta := cG_0 - F_0$ and $\gamma := H_0$, we have that the minimum of the function $M_{\beta, \gamma}$ is negative. Let $L > 0$ be such that $M_{\beta, \gamma}(L) < 0$. Hence $L^{\alpha+1} < (cG_0 - F_0)L^\alpha - H_0$, implying

$$L < cG_0 - F_0 - \frac{H_0}{L^\alpha} \leq c \int_0^u g(s) ds - \int_0^u f(s) ds - \int_0^u \frac{h(s)}{(Ls)^\alpha} ds \text{ for every } u \in (0, 1].$$

Therefore, put $\psi(u) := Lu$ we have

$$\psi(u) < \int_0^u \left(cg(s) - f(s) + \frac{h(s)}{\psi^\alpha(s)} \right) ds \text{ for every } u \in (0, 1]$$

and from Corollary 1 we deduce that problem (2) admits a solution.

Let us consider now a value c satisfying

$$c < \frac{f(0)}{g(0)} + \frac{\alpha + 1}{g(0)} \left(\frac{h_0, \alpha}{\alpha^\alpha} \right)^{\frac{1}{\alpha+1}}.$$

Put $\beta := cg(0) - f(0)$ and $\gamma := h_0, \alpha$, in this case by Lemma 5 we have that $M_{\beta, \gamma}(t) > 0$ for every $t \geq 0$. On the other hand, if a solution z of problem (2) exists, then by Lemma 1 $\dot{z}(0)$ should be a zero of the function $M_{\beta, \gamma}$, a contradiction. So, in this case problem (2) does not admit solutions.

Finally, put $\Gamma := \{c : \text{problem (2) admits solution}\}$ and let $c^* := \inf \Gamma$. In view of what we have just observed, the value c^* satisfies estimate (6).

In order to show that problem (2) admits a solution for $c = c^*$, consider a decreasing sequence $(c_n)_n$ in Γ , converging to c^* and let z_n be the solution of problem (2) for $c = c_n$. Put

$$M := \max_{(u, c) \in [0, 1] \times [c^*, c_1]} (cg(u) - f(u)), \quad m := \min_{(u, c) \in [0, 1] \times [c^*, c_1]} (cg(u) - f(u)). \quad (12)$$

Let $(r_n)_n$ be a decreasing sequence converging to 0, with $r_1 < \frac{1}{2}$, and put $I_k := [r_k, 1 - r_k]$. Of course, $I_k \subset I_{k+1}$ and $(0, 1) = \cup_{k \geq 1} I_k$.

By Lemma 3, there exists a value $\delta_{r_1} > 0$ such that for every $n \geq 1$ we have

$$c_n g(u) - f(u) \geq c_n g(u) - f(u) - \frac{h(u)}{z_n^\alpha(u)} \geq c_n g(u) - f(u) - \frac{h(u)}{\delta_{r_1}^\alpha}, \text{ for every } u \in I_1.$$

Therefore, by (12) we have

$$m - \frac{h(u)}{\delta_{r_1}^\alpha} \leq \dot{z}_n(u) \leq M, \text{ for every } u \in I_1$$

from which we deduce the equicontinuity of the sequence of functions $(z_n)_n$ in I_1 . Moreover, since $0 < z_n(u) \leq M$ in I_1 , then the sequence $(z_n)_n$ is equibounded in I_1 . Then, we can apply the Ascoli-Arzelà Theorem to derive that there exists a subsequence $(z_n^{(1)})_n$ which uniformly converges to a certain function $z_0^{(1)}$ in I_1 .

Let us consider now the interval $I_2 \supset I_1$. By means of the same reasoning, we infer the existence of a further subsequence $(z_n^{(2)})_n$ which uniformly converges to a function $z_0^{(2)}$ in I_2 . Moreover, we have $z_0^{(2)}(u) = z_0^{(1)}(u)$ for every $u \in I_1$.

Proceeding by this diagonal argument, one shows that for all $k \in \mathbb{N}$ the sequence $(z_n^{(k)})_n$ admits a subsequence $(z_n^{(k+1)})_n$ which uniformly converges to a function $z_0^{(k+1)}$ in I_{k+1} , and $z_0^{(k+1)} = z_0^{(k)}(u)$ for all $u \in I_k$.

Finally, let us define $\zeta_0 : (0, 1) \rightarrow \mathbb{R}$, as $\zeta_0(u) = z_0^{(k)}(u)$ if $u \in I_k$. By what we have just observed, ζ_0 is a well-defined function. Moreover, in each interval I_k the function ζ_0 is uniform limit of the sequence of solutions $(z_n^{(k)})_n$ of problem (2) for $c = c_n$. So, by applying Lemma 4 we deduce that ζ_0 is a solution of problem (2) for $c = c^*$ in the interval I_k . By the arbitrariness of k we conclude that that ζ_0 is a positive solution on the whole interval $(0, 1)$.

Observe now that by the monotonicity of the integral function of g we get

$$z_n(u) \leq \int_0^u (c_n g(s) - f(s)) ds \leq \int_\alpha^u (c_1 g(s) - f(s)) ds$$

for all $n \in \mathbb{N}$ and $u \in (0, 1)$. So, also $\zeta_0(u) \leq \int_0^u (c_1 g(s) - f(s)) ds$, for every $u \in (0, 1)$, implying that $\zeta_0(0^+) = 0$.

Furthermore, by virtue of Lemma 1, we deduce that there exists the limit $\zeta_0(1^-) \in [0, +\infty)$. So, all the assumptions of Proposition 1 are satisfied by the function ζ_0 and we can conclude that problem (2) admits a solution for $c = c^*$.

Finally, let us now show that problem (2) admits solution for every $c > c^*$. To this aim, let z_* be the solution of problem (2) for $c = c^*$. Again by the monotonicity of the integral function of g , for every $c > c^*$, we have

$$z_*(u) = \int_\alpha^u \left(c^* g(s) - f(s) - \frac{h(s)}{z_*^\alpha(s)} \right) ds \leq \int_\alpha^u \left(c g(s) - f(s) - \frac{h(s)}{z_*^\alpha(u)} \right) ds$$

So, z^* satisfies condition (11) of Corollary 1 and we can infer that problem (2) admits solution also for every $c > c^*$. \square

Example 1 Let us consider equation (7) with

$$f(u) \equiv 0, \quad g(u) = u + 1, \quad h(u) = u^2(1 - u).$$

In this case we have $h_{0, \alpha} < +\infty$ if and only if $\alpha \leq 2$, so equation (7) admits solution if and only if $\alpha \leq 2$. Moreover, since $F_0 = 0$ and $g(0) = G_0 = 1$, when $\alpha = 2$ we have $h_{0, \alpha} = H_0 = 1$, so estimate (6) becomes $3\sqrt[3]{\frac{1}{4}} \leq c^* \leq 3\sqrt[3]{\frac{1}{4}}$, that is $c^* = \frac{3}{\sqrt[3]{4}}$.

Example 2 Let us consider equation (7) with

$$f(u) = u, \quad g(u) = 1 - u, \quad h(u) = u(1 - u).$$

In this case we have $h_{0, \alpha} < +\infty$ if and only if $\alpha \leq 1$, so equation (7) admits solution if and only if $\alpha \leq 1$. Moreover, we have $F_0 = G_0 = \frac{1}{2}$ and $g(0) = 1$. Furthermore, when $\alpha = 1$ we have $h_{0, \alpha} = H_0 = 1$, so estimate (6) for $\alpha = 1$ becomes $2 \leq c^* \leq 5$.

4. Conclusions

In this paper we have established an existence result for problem (2), stating that there exist a solution if and only if the value $h_{0, \alpha}$ is finite (see (4)). In this case, there are infinitely many admissible wave speeds, whose minimum value satisfies estimate (6).

The present result can be used in the study of travelling waves for reaction-diffusion-advection equations governed by the p-Laplacian operator.

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Conflict of interest

The author declares no competing financial interest.

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