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Research Article

Exploration of Some Novel Integral Inequalities Pertaining to the New Class of (k, ρ)-Conformable Fractional Integrals

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Abstract: Conformable integrals and derivatives have received more attention in recent years as a means of determining different kinds of inequalities. In the research work, we define a novel class of (k, ρ)-conformable fractional integrals ((k, ρ)-CFI). Also, we establish the refinement of the reverse Minkowski inequality incorporating the (k, ρ)-conformable fractional integral operators. The proposed (k, ρ)-conformable fractional integral operators are used to present the two new theorems that correlate with this inequality, along with declarations and verifications of other inequalities. The inequalities presented in this work are more general as compared to the existing literature. The special cases of our main findings are given in the paper.

Keywords: conformable fractional derivative, conformable fractional integral, inequalities, fractional integrals

MSC: 26A33, 26A51

1. Introduction

In general, integral and derivative operators are generalized in the calculus of non-integer order, also known as fractional calculus. The literature contains a wide variety of definitions for fractional integral operators, including the Hadamard, Weyl, Erdélyi-Kober, Riemann-Liouville, and Katugampola fractional integrals [1–4]. Local fractional conformable derivative as well as integral operators are a novel family of fractional operators that were recently introduced by Khalil et al. [5] and Adeljawad [6]. By adding new parameters to such fractional integral operators, one can generalize fractional operators and derive the associated integral inequalities. Among them are Hadamard, Hermite-Hadamard, Opial, Grüss, and Ostrowski [7–13]. Katugampola [14] proposed a generalized fractional integral operator. Jarad et al. [15] presented the conformable fractional operators using the conventional fractional calculus iteration method. These advancements promote further study to present new ideas for combining fractional operators and obtaining integral inequalities for the extended fractional operators.

Utilizing integral inequalities is significant in several scientific domains, including engineering, physics, and mathematics. We specifically suggest impulse equations, integral differential equations, linear conversion stability and initial-value problems [16, 17]. The readers are also suggested to the work presented by [9, 18, 19]. Inequalities pertaining

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to the fractional integral operators have several practical applications in various scientific domains. Furthermore, the concept of fractional calculus plays a vital role in resolving many other unique function concerns as well as differential and integral equations. Consequently, the new integral inequalities result has been made achievable. It is now possible to obtain new results involving integral inequalities; therefore, certain applications have been made [16, 17]. We mention a few of them, namely Jensen, Hermite-Hadamard, Hardy, Holder, and Minkowski inequalities [20–26].

The aim of this paper is to present how the new fractional conformable operators are generalized and extends the reverse Minkowski inequality and certain other related inequalities [4, 27–30] by employing the generalized $(k,\ \rho)$-conformable fractional integrals.

This paper is classified as follows. Section 2 presents the fundamental definitions and annotations of fractional integrals along with our recently established generalized $(k,\ \rho)$-conformable fractional integrals. We establish suitable spaces for such operators as well as the theorems pertaining to the reverse Minkowski inequality. Section 3 presents our fundamental outcomes, which comprise the reverse Minkowski integral inequality involving the generalized $(k,\ \rho)$-fractional conformable integral. We demonstrate some other type of integral inequalities by utilizing the $(k,\ \rho)$-conformable fractional integrals in Section 4. The conclusion is presented in Section 5.

2. Notations and preliminaries

In this section, we consider the following well known results. Also, we defined the new generalized $(k,\ \rho)$-conformable fractional integrals.

Definition 1 A function $\lambda(z)$ is said to be in the space $L_p[a,\ b]$, if

$$\left(\int_a^b |\lambda(z)|^p\, dz\right)^{\frac{1}{p}} < \infty,\ 1 \leq p < \infty.$$

Theorem 1 [21] Suppose that the two functions $\lambda_1,\ \lambda_2 \in L_p[a,\ b]$ be positive, with $1 \leq p < \infty,\ 0 < \int_a^b \lambda_1^p(\tau)d\tau < \infty$, and $0 < \int_a^b \lambda_2^p(\tau)d\tau < \infty$. If $0 < n \leq \frac{\lambda_1(\tau)}{\lambda_2(\tau)} \leq m$ for $n,\ m \in \mathbb{R}^+$ and $\forall\ \tau \in [a,\ b]$, then

$$\left(\int_a^b \lambda_1^p(\tau)d\tau\right)^{\frac{1}{p}} + \left(\int_a^b \lambda_2^p(\tau)d\tau\right)^{\frac{1}{p}} \leq c_1 \left(\int_a^b \lambda_1^p + \lambda_2^p(\tau)d\tau\right)^{\frac{1}{p}}$$

with $c_1 = \dfrac{m(n+1)+(m+1)}{(n+1)(m+1)}$.

Theorem 2 [22] Suppose that the two functions $\lambda_1,\ \lambda_2 \in L_p[a,\ b]$ are positive, and with $1 \leq p < \infty, 0 < \int_a^b \lambda_1^p(\tau)d\tau < \infty$, and $0 < \int_a^b \lambda_2^p(\tau)d\tau < \infty$, if $0 < n \leq \frac{\lambda_1(\tau)}{\lambda_2(\tau)} \leq m$ for $n,\ m \in \mathbb{R}^+$ and $\forall\ \tau\ \in [a,\ b]$, then

$$\left(\int_a^b \lambda_1^p(\tau)d\tau\right)^{\frac{2}{p}} + \left(\int_a^b \lambda_2^p(\tau)d\tau\right)^{\frac{2}{p}} \geq c_2 \left(\int_a^b \lambda_1^p(\tau)d\tau\right)^{\frac{1}{p}} + \left(\int_a^b \lambda_2^p(\tau)d\tau\right)^{\frac{2}{p}}$$

with $c_2 = \dfrac{(n+1)(m+1)}{m} - 2$.

Definition 2 A function $\lambda(z)$ is said to be in the space $L_{p,\ \omega}\ [a,\ b]$, if

$$\left(\int_a^b |\lambda(z)|^p z^{\omega} dz\right)^{\frac{1}{p}} < \infty,\ 1 \le p < \infty,\ \omega \ge 0.$$

Definition 3 Let $X_{c,\ p}(a,\ b)$ is the space for $c \in\ R,\ a < b$ and $1 \le p < \infty$ includes those complex valued Lebesgue measurable functions with f on $(a,\ b)$ with $\|f\|_{X_{c,\ p}}$, where

$$\|f\|_{X_{c,\ p}} = \left(\int_a^b |z^c \lambda(z)|^p \frac{dz}{z}\right)^{\frac{1}{p}}\ (1 \le p < \infty),$$

and for $p = \infty$,

$$\|f\|_{X_{c,\ \infty}} = ess\ \sup\ _{z \in (a,\ b)} [z^c |f(z)|].$$

In particular, the function space $X_{c,\ p}\ (a,\ b)$ coincides with the space $L_p(a,\ b)$ for $c = \dfrac{1}{p}$ see [1].

Definition 4 [1, 3] The right Riemann-Liouville fractional integral of order $\eta > 0$ is given by:

$$({}^{\eta}\mathscr{I}_b\lambda)(\tau) = \frac{1}{\Gamma(\eta)} \int_{\tau}^{b} (\Upsilon - \tau)^{\eta - 1}\ \lambda(\Upsilon) d\Upsilon.$$

Definition 5 [1, 3] The left Riemann-Liouville fractional integral of order $\eta > 0$, for $\eta \in\ C$ and $\Re(\eta) > 0$, is defined by

$$({}_{a}^{\eta}\mathscr{I}\ \lambda)(\tau) = \frac{1}{\Gamma(\eta)} \int_{a}^{\tau} (\tau - \Upsilon)^{\eta - 1}\ \lambda(\Upsilon) d\Upsilon. \tag{1}$$

Definition 6 [15] The left Riemann-Liouville fractional derivative of order $\eta > 0$ is defined for $\eta \in\ C$ and $\Re(\eta) > 0$ by

$$({}_{a}\mathscr{D}^{\eta}\lambda)\ (\tau) = \left(\frac{d}{d\tau}\right)^{n} ({}_{a}\mathscr{I}^{n-\eta}\ \lambda)(\tau),\ n = [\eta] + 1$$

The right Riemann-Liouville fractional derivative [15] of order $\eta > 0$ is defined as

$$(\mathscr{D}_b^{\nabla}\lambda)(\tau) = \left(-\frac{d}{d\tau}\right)^{n} \left({}^{n-\eta}\mathscr{I}_b\lambda\right)(\tau).$$

Definition 7 The left Caputo fractional derivative [15] of order $\eta,\ \Re(\eta) > 0$ is given by

$$\left({}_{a}^{\mathtt{C}}\mathscr{D}^{\nabla}\lambda\right)(\tau)=({}_{a}\mathscr{I}^{n-\eta}\,\lambda^{(n)})(\tau),\; n=[\eta]+1$$

while the left Caputo fractional derivative [15] having order $\eta>0$ is given by

$$\left({}^{\mathtt{C}}\mathscr{D}_{b}^{\eta}\lambda\right)(\tau)=\left({}^{n-\eta}\mathscr{I}_{b}(-1)^{n}\lambda^{(n)}\right)(\tau).$$

Definition 8 The left Hadamard fractional integral [15] of the order $\eta>0$ is defined by

$$({}_{a}\mathscr{F}^{\,\eta}\lambda)(\tau)=\frac{1}{\Gamma(\eta)}\int_{a}^{\tau}(ln(\tau)-ln(\Upsilon))^{\eta-1}\lambda(\Upsilon)\frac{d\Upsilon}{\Upsilon} \tag{2}$$

and the right Hadamard fractional integral [15] of order $\eta>0$ is given by

$$(\mathscr{F}_{b}^{\eta}\lambda)(\tau)=\frac{1}{\Gamma(\eta)}\int_{\tau}^{b}((\ln(y)-\ln(\tau)))^{\eta-1}\lambda(\Upsilon)\frac{d\Upsilon}{\Upsilon}.$$

Definition 9 [15] The left Hadamard fractional derivative of the order $\eta>0$ is defined by

$$({}_{a}\mathscr{G}^{\,\eta}\lambda)(\tau)=\left(\tau\frac{d}{d\tau}\right)^{n}({}_{a}\mathscr{F}^{n-\eta}\,\lambda)(\tau),\; n=[\eta]+1$$

and the right Hadamard fractional derivative of order $\eta>0$ is defined by

$$(\mathscr{G}_{b}^{\eta}\lambda)(\tau)=\left(-\tau\frac{d}{d\tau}\right)^{n}\left({}^{n-\eta}\mathscr{F}_{b}\lambda\right)(\tau).$$

Definition 10 [28] For a real function $\lambda\in X_{c,\,p}(a,\,b)$, the left and right Katugampola fractional integrals having order $\eta>0$, $\rho>0$ and $\Re(\eta)>0$ is defined by

$$({}_{a}\mathscr{K}^{\eta,\,\rho}\lambda)(\tau)=\frac{(\rho)^{1-\eta}}{\Gamma(\eta)}\int_{a}^{\tau}(\tau^{\rho}-\Upsilon^{\rho})^{\eta-1}\lambda(\Upsilon)\frac{d\Upsilon}{\Upsilon^{1-\rho}} \tag{3}$$

and

$$(\mathscr{K}_{b}^{\eta,\,\rho})(\tau)=\frac{(\rho)^{1-\eta}}{\Gamma(\eta)}\int_{\tau}^{b}(\Upsilon^{\rho}-\tau^{\rho})^{\eta-1}\lambda(\Upsilon)d\Upsilon,$$

respectively.

Definition 11 [4] The left and right Katugampola fractional derivatives has order $\eta>0$ and for $\rho>0$ and $\Re(\eta)>0$ are respectively defined by

$$({}_a\mathscr{L}^{\eta,\,\rho}\lambda)(\tau) = \partial^n({}_a\mathscr{K}^{n-\eta,\,\rho}\lambda)(\tau) = \frac{\partial^n\left(\tau^\rho - \Upsilon^\rho\right)^{1-n+\eta}}{\Gamma(n-\eta)} \int_a^\tau \left(\tau^\rho - \Upsilon^\rho\right)^{n-\eta-1} \lambda(\Upsilon)\frac{d\Upsilon}{\Upsilon^{1-\rho}}$$

and

$$(\mathscr{L}_b^{\eta,\,\rho}\lambda)(\tau) = (-\partial)^n(\mathscr{K}_b^{n-\eta,\,\rho})(\tau) = \frac{(-\partial)^n\left(\Upsilon^\rho - \tau^\rho\right)^{1-n+\eta}}{\Gamma(n-\eta)} \int_\tau^b \left(\Upsilon^\rho - \tau^\rho\right)^{n-\eta-1} \lambda(\Upsilon)\frac{d\Upsilon}{\Upsilon^{1-\rho}}$$

where $\partial = \tau^{1-\rho}\dfrac{d}{d\tau}$.

Dahmani [29] presented the following two Theorem associated with reverse Minkowski integral inequality by using Riemann-Liouville fractional integrals.

Theorem 3 [29] For $\eta > 0$ and $p \geq 1$. Let $\lambda_1,\ \lambda_2 \in\ L_{1,\,\omega}[a,\ \tau]$ be the two positive functions in $[0,\ \infty)$ such that, $\forall\ \tau > a,\ {}_a\mathscr{I}_k^{\eta,\,\omega}\lambda_1^p(\tau) < \infty$ and ${}_a\mathscr{I}_k^{\eta,\,\omega}\lambda_2^p(\tau) < \infty$. If $0 < n \leq \dfrac{\lambda_1(\Upsilon)}{\lambda_2(\Upsilon)} \leq m$ for $n,\ m \in\ \mathbb{R}^+$ and $\forall\ \Upsilon \in [a,\ \tau]$, then

$$\left({}_a\mathscr{I}_k^{\eta,\,\omega}\lambda_1^p(\tau)\right)^{\frac{1}{p}} + \left({}_a\mathscr{I}_k^{\eta,\,\omega}\lambda_2^p(\tau)\right)^{\frac{1}{p}} \leq c_1\left({}_a\mathscr{I}_k^{\eta,\,\omega}\left(\lambda_1+\lambda_2\right)^p(\tau)\right)^{\frac{1}{p}}$$

with $c_1 = \dfrac{m(n+1)+(m+1)}{(n+1)(m+1)}$.

Theorem 4 [29] For $\eta > 0$ and $p \geq 1$. Let $\lambda_1,\ \lambda_2 \in\ L_{1,\,\omega}[a,\ \tau]$ are two positive functions in $[0,\ \infty)$ such that, $\forall\ \tau > a,\ {}_a\mathscr{I}_k^{\eta,\,\omega}\lambda_1^p(\tau) < \infty$ and ${}_a\mathscr{I}_k^{\eta,\,\omega}\lambda_2^p(\tau) < \infty$. If $0 < n \leq \dfrac{\lambda_1(\Upsilon)}{\lambda_2(\Upsilon)} \leq m$ for $n,\ m \in\ \mathbb{R}^+$ and $\forall\ \Upsilon \in [a,\ \tau]$, then

$$\left({}_a\mathscr{I}_k^{\eta,\,\omega}\lambda_1^p(\tau)\right)^{\frac{2}{p}} + \left({}_a\mathscr{I}_k^{\eta,\,\omega}\lambda_2^p(\tau)\right)^{\frac{2}{p}} \geq c_2\left({}_a\mathscr{I}_k^{\eta,\,\omega}\lambda_1{}^p(\tau)\right)^{\frac{1}{p}}\left({}_a\mathscr{I}_k^{\eta,\,\omega}\lambda_2{}^p(\tau)\right)^{\frac{1}{p}},$$

with $c_2 = \dfrac{(n+1)(m+1)}{m} - 2$.

Using the Hadamard fractional integral operator, Chinchane et al. [30] and Taf et al. [31] proved the subsequent pair of theorems for the reverse Minkowski inequality.

Theorem 5 For $\eta > 0$ and $p \geq 1$. Let $\lambda_1,\ \lambda_2 \in\ L_{1,\,\omega}[a,\ \tau]$ be the two positive functions in $[0,\ \infty)$ such that, $\forall\ \tau > a,\ {}_a\mathscr{F}_k^{\eta,\,\omega}\lambda_1^p(\tau) < \infty$ and ${}_a\mathscr{F}_k^{\eta,\,\omega}\lambda_2^p(\tau) < \infty$. If $0 < n \leq \dfrac{\lambda_1(\Upsilon)}{\lambda_2(\Upsilon)} \leq m$ for $n,\ m \in\ \mathbb{R}^+$ and $\forall\ \Upsilon \in [a,\ \tau]$, then

$$\left({}_a\mathscr{F}_k^{\eta,\,\omega}\lambda_1^p(\tau)\right)^{\frac{1}{p}} + \left({}_a\mathscr{F}_k^{\eta,\,\omega}\lambda_2^p(\tau)\right)^{\frac{1}{p}} \leq c_1\left({}_a\mathscr{F}_k^{\eta,\,\omega}\left(\lambda_1+\lambda_2\right)^p(\tau)\right)^{\frac{1}{p}}$$

with $c_1 = \dfrac{m(n+1)+(m+1)}{(n+1)(m+1)}$.

Theorem 6 For $\eta > 0$ and $p \geq 1$. Let $\lambda_1,\ \lambda_2 \in\ L_{1,\,\omega}[a,\ \tau]$ are two positive functions in $[0,\ \infty)$ such that, $\forall\ \tau > a,\ {}_a\mathscr{F}_k^{\eta,\,\omega}\lambda_1^p(\tau) < \infty$ and ${}_a\mathscr{F}_k^{\eta,\,\omega}\lambda_2^p(\tau) < \infty$. If $0 < n \leq \dfrac{\lambda_1(\Upsilon)}{\lambda_2(\Upsilon)} \leq m$ for $n,\ m \in \mathbb{R}^+$ and $\forall\ \Upsilon \in [a,\ \tau]$, then

$$\left({}_a\mathscr{F}_k^{\eta,\,\omega}\lambda_1^p(\tau)\right)^{\frac{2}{p}} + \left({}_a\mathscr{F}_k^{\eta,\,\omega}\lambda_2^p(\tau)\right)^{\frac{2}{p}} \geq c_2\left({}_a\mathscr{F}_k^{\eta,\,\omega}\lambda_1{}^p(\tau)\right)^{\frac{1}{p}}\left({}_a\mathscr{F}_k^{\eta,\,\omega}\lambda_2{}^p(\tau)\right)^{\frac{1}{p}},$$

with $c_2 = \dfrac{(n+1)(m+1)}{m} - 2$.

Using the fractional integral of Saigo, Chinchane et al. [32] established the reverse Minkowski inequality, as well as lately, Chinchane [33] demonstrated the classical inequality using the k-fractional integral. A novel generalized fractional integrals was presented by Jarad et al. [15] in 2017. Mubeen et al. [34] presented the generalized k-conformable fractional integral operators and provided a generalization of the reverse Minkowski integral inequalities.

Definition 12 [6] If $\lambda \in L_1[a,\ b]$, then the left conformable fractional integral of order $\eta > 0$, is defined by

$$ {}_a\mathbf{I}^{\eta}\lambda(\Upsilon) = \int_a^{\Upsilon} \lambda(\tau)(\tau - a)^{\eta-1} d\tau;\ 0 \le a < \Upsilon < b \le \infty,\ 0 < \ \eta < 1. $$

Definition 13 [6] If $\lambda \in L_1[a,\ b]$, then the right conformable fractional integral of order $\eta > 0$ is defined by

$$ \mathbf{I}_b^{\eta}\lambda(\Upsilon) = \int_{\Upsilon}^{b} \lambda(\tau)(b - \tau)^{\eta-1} d\tau;\ 0 \le a < \Upsilon < b \le \infty,\ 0 < \ \eta < 1. $$

Definition 14 If $\lambda \in L_{1,\ \omega}[a,\ b]$, then the generalized left conformable fractional integral ${}_a^{\eta}\mathscr{J}^{\omega}$ having order $\eta \in \mathbb{C},\ \Re(\eta) > 0$ and $\omega > 0$, defined by Jared et al. [15], is follows:

$$ {}_a^{\eta}\mathscr{J}^{\omega}\lambda(\tau) = \frac{(\omega)^{1-\eta}}{\Gamma(\eta)} \int_a^{\tau} \left((\tau - a)^{\omega} - (\Upsilon - a)^{\omega}\right)^{\eta-1} (\Upsilon - a)^{\omega-1} \lambda(\Upsilon) d\Upsilon;\ 0 \le a < \tau < b \le \infty \tag{4} $$

where the Euler gamma function is denoted by Γ.

Definition 15 If $\lambda \in\ L_{1,\ \omega}[a,\ b]$, the right fractional conformable integral ${}^{\eta}\mathscr{J}_b^{\omega}$ having order $\eta \in\ \mathbb{C},\ \Re(\eta) > 0$ and $\omega > 0$, defined by Jared et al. [15], is follows:

$$ {}^{\eta}\mathscr{J}_b^{\omega}\lambda(\tau) = \frac{(\omega)^{1-\eta}}{\Gamma(\eta)} \int_{\tau}^{b} \left((b - \tau)^{\omega} - (b - \Upsilon)^{\omega}\right)^{\eta-1} (b - \Upsilon)^{\omega-1}\ \lambda(\Upsilon) d\Upsilon;\ 0 \le a < \tau < b \le \infty, $$

where the Euler gamma function is denoted by Γ.

Definition 16 [34] The left and right $(k,\ \omega)$-conformable fractional integrals having order $\eta \in \mathbb{C},\ \Re(\eta) > 0$ of a continuous function $\lambda(\Upsilon)$ on $[0,\ \infty)$, are respectively given by

$$ {}_a^{\eta}\tau_k^{\omega}\lambda(\tau) = \frac{(\omega)^{1-\frac{\eta}{k}}}{k\Gamma_k(\eta)} \int_a^{\tau} \left((\tau - a)^{\omega} - (\Upsilon - a)^{\omega}\right)^{\frac{\eta}{k}-1} (\Upsilon - a)^{\omega-1}\ \lambda(\Upsilon) d\Upsilon;\ 0 \le a < \tau < b \le \infty \tag{5} $$

and

$$ {}^{\eta}\tau_{b,\ k}^{\omega}\ \lambda(\tau) = \frac{(\omega)^{1-\frac{\eta}{k}}}{k\Gamma_k(\eta)} \int_{\tau}^{b} \left((b - \tau)^{\omega} - (b - \Upsilon)^{\omega}\right)^{\frac{\eta}{k}-1} (b - \Upsilon)^{\omega-1}\ \lambda(\Upsilon) d\Upsilon;\ 0 \le a < \tau < b \le \infty, $$

if these integrals exist, for $k > 0,\ \omega \in\ \mathbb{R} \setminus \{0\}$.

Definition 17 [35] Let $\eta \in C,\ \Re(\eta) > 0$. The left and right ρ-CFI are respectively given by

$$
{}_{a}^{\eta}\tau_{\omega}^{\rho}\lambda(\tau) = \frac{1}{\Gamma(\eta)} \int_{a}^{\tau} \left(\frac{(\rho(\sqcup) - \rho(a))^{\omega} - (\rho(\varsigma) - \rho(a))^{\omega}}{\omega} \right)^{\eta - 1} \frac{\lambda(\varsigma)\rho'(\varsigma)d\varsigma}{(\rho(\varsigma) - \rho(a))^{1-\omega}} \tag{6}
$$

and

$$
{}^{\eta}\tau_{b}^{\omega,\,\rho}\lambda(\sqcup) = \frac{1}{\Gamma(\eta)} \int_{\tau}^{b} \left(\frac{(\rho(b) - \rho(\tau))^{\omega} - (\rho(b) - \rho(\varsigma))^{\omega}}{\omega} \right)^{\eta - 1} \frac{\lambda(\varsigma)\rho'(\varsigma)d\varsigma}{(\rho(b) - \rho(\varsigma))^{1-\omega}} \tag{7}
$$

Next, we define the generalized $(k,\ \rho)$ conformable fractional integrals ($(k,\ \rho)-$CFI) as follows:

Definition 18 Let $\eta \in \mathbb{C}$, $\Re(\eta) > 0$. The left $(k,\ \rho)$-CFI is defined by

$$
{}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}\lambda(\tau) = \frac{(\omega)^{1-\frac{\eta}{k}}}{k\Gamma_k(\eta)} \int_{a}^{\tau} \left((\rho(\tau) - \rho(a))^{\omega} - (\rho(\Upsilon) - \rho(a)^{\omega}\right)^{\frac{\eta}{k} - 1} (\rho(\Upsilon) - \rho(a))^{\omega - 1} \lambda(\Upsilon)\rho'(\Upsilon)d\Upsilon;
$$

$$
0 \le a < \tau < b \le \infty \tag{8}
$$

and the right $(k,\ \rho)$-CFI is defined by

$$
{}^{\eta}\mathfrak{G}_{b,\,k}^{\omega,\,\rho}\lambda(\tau) = \frac{(\omega)^{1-\frac{\eta}{k}}}{k\Gamma_k(\eta)} \int_{\tau}^{b} \left((\rho(b) - \rho(\tau))^{\omega} - (\rho(b) - \rho(\Upsilon)^{\omega}\right)^{\frac{\eta}{k} - 1} (\rho(b) - \rho(\Upsilon))^{\omega - 1} \lambda(\Upsilon)\rho'(\Upsilon)d\Upsilon;
$$

$$
0 \le a < \tau < b \le \infty \tag{9}
$$

Remark 1 (1) Let us choose $k = 1$ in (8), we get (7).
(2) Let us choose $\rho(\tau) = \tau$ in (8), we get (5).
(3) Let us choose $k = 1$ and $\rho(\tau) = \tau$ in (5), we get (4).
(4) Let us choose $\rho(\tau) = \tau,\ \omega = 1$ and $k = 1$, we get (1).
(5) If we take $a = 0,\ \eta \to 0$, then (8) reduces to (2).
(6) If we take $a = 0$ in (8), we get (3).

Theorem 7 Let $\lambda \in L_1[a,\ b],\ \omega \in \mathbb{R}\backslash\{0\}$ and $k > 0$. Then the left and right $(k,\ \rho)$-CFI, i.e., ${}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}\lambda(\Upsilon)$ and ${}^{\eta}\mathfrak{G}_{b,\,k}^{\omega,\,\rho}\lambda(\Upsilon)$ exist for any $\Upsilon \in [a,\ b]$, $\Re(\eta) > 0$.

Proof. Let $\nabla' := [a,\ b] \times [a,\ b]$ and $P : \nabla' \to \mathbb{R}$ such that

$$
P((\rho(\Upsilon),\ \rho(\tau)) = \left((\rho(\Upsilon) - \rho(a))^{\omega} - (\rho(\tau) - \rho(a)^{\omega}\right)^{\frac{\eta}{k} - 1} (\rho(\tau) - \rho(a))^{\omega - 1} \lambda(\tau)\rho'(\tau)
$$

It is obviously clear that,

$$
P = P_{+} + P_{-}
$$

where

$$P_{+}\left((\rho(\Upsilon),\ \rho(\tau)\right)=\begin{cases}\left((\rho(\Upsilon)-\rho(a))^{\omega}-(\rho(\tau)-\rho(a))^{\omega}\right)^{\frac{\eta}{k}-1}(\rho(\tau)-\rho(a))^{\omega-1}\rho^{'}(\tau);\ a\leq\tau\leq\Upsilon\leq b,\\ 0;\ \Upsilon\leq\tau\leq b\end{cases}$$

and

$$P_{-}\left((\rho(\Upsilon),\ \rho(\tau)\right)=\begin{cases}\left((\rho(\tau)-\rho(a))^{\omega}-(\rho(\Upsilon)-\rho(a)^{\omega}\right)^{\frac{\eta}{k}-1}(\rho(\Upsilon)-\rho(a))^{\omega-1}\rho^{'}(\Upsilon);\ a\leq\tau\leq\Upsilon\leq b,\\ 0;\ a\leq\Upsilon\leq\tau\leq b.\end{cases}$$

Since $\mathscr{P}'$ is measurable on ∇', then it might be expressed as

$$\begin{aligned}\int_a^b P\left((\rho(\Upsilon),\ \rho(\tau)\right)d\tau &= \int_a^{\Upsilon} P\left((\rho(\Upsilon),\ \rho(\tau)\right)d\tau\\ &= \int_a^{\Upsilon}\left((\rho(\Upsilon)-\rho(a))^{\omega}-(\rho(\tau)-\rho(a)^{\omega}\right)^{\frac{\eta}{k}-1}(\rho(\tau)-\rho(a))^{\omega-1}\rho^{'}(\tau)d\tau.\end{aligned}$$

By assumption

$$r=(\rho(\Upsilon)-\rho(a))^{\omega}-(\rho(\tau)-\rho(a))^{\omega}$$

$$dr=-\omega\left(\rho(\tau)-\rho(a)\right)^{\omega-1}\rho^{'}(\tau)d\tau$$

$$\int_a^b P\left((\rho(\Upsilon),\ \rho(\tau)\right)d\tau=-\frac{1}{\omega}\int_a^{\Upsilon}(r)^{\frac{\eta}{k}-1}dr=\frac{k}{\omega\eta}(\rho(\Upsilon)-\rho(a))^{\frac{\omega\eta}{k}}$$

Utilizing the double integral, it yields

$$\begin{aligned}\int_a^b\left(\int_a^b P(\rho(\Upsilon),\ \rho(\tau))\,|\lambda\rho(\Upsilon)|\,d\tau\right)d\Upsilon &= \int_a^b|\lambda\rho(\Upsilon)|\left(\int_a^b P(\rho(\Upsilon),\ \rho(\tau))\,d\tau\right)d\Upsilon\\ &= \frac{k}{\omega\eta}\int_a^b(\rho(\Upsilon)-\rho(a))^{\frac{\omega\eta}{k}}\,|\lambda\rho(\Upsilon)|\,d\Upsilon\\ &\leq \frac{k}{\omega\eta}(\rho(b)-\rho(a))^{\frac{\omega\eta}{k}}\int_a^b|\lambda\rho(\Upsilon)|\,d\Upsilon.\end{aligned}$$

It follows that

$$\int_a^b \left(\int_a^b P(\rho(\Upsilon),\, \rho(\tau))\, |\lambda\rho(\Upsilon)|\, d\tau\right) d\Upsilon = \int_a^b |\lambda\rho(\Upsilon)| \left(\int_a^b P(\rho(\Upsilon),\, \rho(\tau))\, d\tau\right) d\Upsilon$$

$$\le \frac{k}{\omega\eta}(\rho(b)-\rho(a))^{\frac{\omega\eta}{k}} \|\lambda\rho(\Upsilon)\|_{L_1[a,\, b]} < \infty.$$

As by Tonelli's theorem, the function $\mathscr{Q}: \nabla' \to \mathbb{R}$ such that $\mathscr{Q}(\rho(\Upsilon),\, \rho(\tau)) := P(\rho(\Upsilon),\, \rho(\tau))\, \lambda(\rho(\Upsilon))$ can be integrated over ∇'. Thus by Fubini's theorem $\int_a^b P(\rho(\Upsilon),\, \rho(\tau))\, \lambda(\rho(\Upsilon))\, d\Upsilon$ is an integrable function over $[a,\, b]$ as a function of $\Upsilon \in [a,\, b]$, that is, ${}^{\eta}_{a}\mathfrak{G}^{\omega,\,\rho}_{k}\lambda(\Upsilon)$ exists. The existence of the right $(k,\, \rho)$-CFI ${}^{\eta}\mathfrak{G}^{\omega,\,\rho}_{b,\,k}\lambda(\Upsilon)$ may be demonstrated in a similar way. □

3. Reverse minkowski inequality involving generalized (k, ρ)-CFI operators

Our major contribution to prove the reverse Minkowski integral inequalities via the generalized conformable fractional integrals in this section.

Theorem 8 For $\omega \in \mathbb{R}\backslash\{0\}$, $\eta > 0$, $p \ge 1$ and $k > 0$. Let $\lambda_1,\, \lambda_2 \in L_{1,\,\omega}[a,\, \tau]$ be the two positive functions in $[0,\, \infty)$ such that, $\forall\, \tau > a$, ${}^{\eta}_{a}\mathfrak{G}^{\omega,\,\rho}_{k}\lambda_1^p(\tau) < \infty$ and ${}^{\eta}_{a}\mathfrak{G}^{\omega,\,\rho}_{k}\lambda_2^p(\tau) < \infty$. If $0 < n \le \dfrac{\lambda_1(\Upsilon)}{\lambda_2(\Upsilon)} \le m$ for $n,\, m \in \mathbb{R}^+$ and $\forall\, \Upsilon \in [a,\, \tau]$ then,

$$\left({}^{\eta}_{a}\mathfrak{G}^{\omega,\,\rho}_{k}\lambda_1^p(\tau)\right)^{\frac{1}{p}} + \left({}^{\eta}_{a}\mathfrak{G}^{\omega,\,\rho}_{k}\lambda_2^p(\tau)\right)^{\frac{1}{p}} \le c_1 \left({}^{\eta}_{a}\mathfrak{G}^{\omega,\,\rho}_{k}(\lambda_1+\lambda_2)^p(\tau)\right)^{\frac{1}{p}},$$

with $c_1 = \dfrac{m(n+1)+(m+1)}{(n+1)(m+1)}$.

Proof. By given hypothesis $\dfrac{\lambda_1(\Upsilon)}{\lambda_2(\Upsilon)} \le m$, $a \le \Upsilon \le \tau$, it may be expressed as

$$\lambda_1(\Upsilon) \le m\lambda_2(\Upsilon)$$

$$\lambda_1(\Upsilon) \le m\lambda_2(\Upsilon) - m\lambda_1(\Upsilon) + m\lambda_1(\Upsilon)$$

$$\lambda_1(\Upsilon) + m\lambda_1(\Upsilon) \le m\lambda_2(\Upsilon) + m\lambda_1(\Upsilon)$$

$$\lambda_1(\Upsilon) + m\lambda_1(\Upsilon) \le m(\lambda_1(\Upsilon) + \lambda_2(\Upsilon))$$

$$[m+1]\lambda_1(\Upsilon) \le m(\lambda_1(\Upsilon) + \lambda_2(\Upsilon))$$

$$[m+1]^p {\lambda_1}^p(\Upsilon) \le m^p(\lambda_1(\Upsilon) + \lambda_2(\Upsilon))^p. \tag{10}$$

Both sides of Equation (10) are multiplied by

$$\frac{(\omega)^{1-\frac{\eta}{k}}\left((\rho(\tau)-\rho(a))^{\omega}-(\rho(\Upsilon)-\rho(a))^{\omega}\right)^{\frac{\eta}{k}-1}(\rho(\Upsilon)-\rho(a))^{\omega-1}\rho'(\Upsilon)}{k\Gamma_k(\eta)}$$

and then integrating from a to τ with respect to the variable Υ, we get

$$\frac{[m+1]^p(\omega)^{1-\frac{\eta}{k}}}{k\Gamma_k(\eta)}\int_a^{\tau}\left((\rho(\tau)-\rho(a))^{\omega}-(\rho(\Upsilon)-\rho(a))^{\omega}\right)^{\frac{\eta}{k}-1}(\rho(\Upsilon)-\rho(a))^{\omega-1}\lambda_1{}^p(\Upsilon)\rho'(\Upsilon)d\Upsilon$$

$$\leq\frac{m^p(\omega)^{1-\frac{\eta}{k}}}{k\Gamma_k(\eta)}\int_a^{\tau}\left((\rho(\tau)-\rho(a))^{\omega}-(\rho(\Upsilon)-\rho(a)^{\omega}\right)^{\frac{\eta}{k}-1}(\rho(\Upsilon)-\rho(a))^{\omega-1}(\lambda_1+\lambda_2)^p(\Upsilon)\rho'(\Upsilon)d\Upsilon. \tag{11}$$

Thus, it may be expressed as

$$[m+1]^p\left({}_a^{\eta}\mathfrak{G}_k^{\omega,\,\rho}\lambda_1^p(\tau)\right)^p\leq m^p\left({}_a^{\eta}\mathfrak{G}_k^{\omega,\,\rho}(\lambda_1+\lambda_2)^p(\tau)\right)^p$$

$$\left({}_a^{\eta}\mathfrak{G}_k^{\omega,\,\rho}\lambda_1^p(\tau)\right)^{\frac{1}{p}}\leq\frac{m}{m+1}\left({}_a^{\eta}\mathfrak{G}_k^{\omega,\,\rho}(\lambda_1+\lambda_2)^p(\tau)\right)^{\frac{1}{p}}. \tag{12}$$

On the other hand, we have $n\leq\dfrac{\lambda_1(\Upsilon)}{\lambda_2(\Upsilon)}\Rightarrow n\lambda_2(\Upsilon)\leq\lambda_1(\Upsilon)$, it follows

$$n\lambda_2(\Upsilon)\leq\lambda_1(\Upsilon)+\lambda_2(\Upsilon)-\lambda_2(\Upsilon)$$

$$n\lambda_2(\Upsilon)+\lambda_2(\Upsilon)\leq\lambda_1(\Upsilon)+\lambda_2(\Upsilon)$$

$$n+1\,[\lambda_2(\Upsilon)]\leq\lambda_1(\Upsilon)+\lambda_2(\Upsilon)$$

$$\frac{n+1}{n}[\lambda_2(\Upsilon)]\leq\frac{1}{n}\lambda_1(\Upsilon)+\lambda_2(\Upsilon)$$

$$\left(1+\frac{1}{n}\right)^p\lambda_2^p(\Upsilon)\leq\left(\frac{1}{n}\right)^p(\lambda_1(\Upsilon)+\lambda_2(\Upsilon))^p. \tag{13}$$

Again, by multiplying both sides of (13) with

$$\frac{(\omega)^{1-\frac{\eta}{k}}\left((\rho(\tau)-\rho(a))^{\omega}-(\rho(\Upsilon)-\rho(a))^{\omega}\right)^{\frac{\eta}{k}-1}(\rho(\Upsilon)-\rho(a))^{\omega-1}\rho'(\Upsilon)}{k\Gamma_k(\eta)}$$

and then integrating from a to τ with respect to the variable Υ, we obtain

$$\frac{\left(1+\frac{1}{n}\right)^{p}(\omega)^{1-\frac{\eta}{k}}}{k\Gamma_k(\eta)}\int_a^{\tau}\left((\rho(\tau)-\rho(a))^{\omega}-(\rho(\Upsilon)-\rho(a))^{\omega}\right)^{\frac{\eta}{k}-1}(\rho(\Upsilon)-\rho(a))^{\omega-1}\lambda_1{}^{p}(\Upsilon)\rho'(\Upsilon)d\Upsilon$$

$$\leq\frac{\left(\frac{1}{n}\right)^{p}(\omega)^{1-\frac{\eta}{k}}}{k\Gamma_k(\eta)}\int_a^{\tau}\left((\rho(\tau)-\rho(a))^{\omega}-(\rho(\Upsilon)-\rho(a))^{\omega}\right)^{\frac{\eta}{k}-1}(\rho(\Upsilon)-\rho(a))^{\omega-1}(\lambda_1+\lambda_2)^{p}(\Upsilon)\rho'(\Upsilon)d\Upsilon.$$

Thus, it may be expressed as

$$\left(1+\frac{1}{n}\right)^{p}\left({}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}\lambda_1^{p}(\tau)\right)^{p}\leq\left(\frac{1}{n}\right)^{p}\left({}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}(\lambda_1+\lambda_2)^{p}(\tau)\right)^{p}$$

$$\left({}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}\lambda_1^{p}(\tau)\right)^{\frac{1}{p}}\leq\frac{1}{n+1}\left({}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}(\lambda_1+\lambda_2)^{p}(\tau)\right)^{\frac{1}{p}}. \tag{14}$$

The desired Theorem 8 is deduced from (12) and (14).

The reverse Minkowski inequality involving generalized $(k,\,\rho)$-CFI is given in Theorem 8. □

Theorem 9 For $k>0,\ \omega\in\mathbb{R}\backslash\{0\}$, $\eta>0$ and $p\geq 1$. Let $\lambda_1,\ \lambda_2\in L_{1,\ \omega}[a,\ \tau]$ be the two positive functions in $[0,\ \infty)$ such that, $\forall\ \tau>a,\ {}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}\lambda_1^{p}(\tau)<\infty$ and ${}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}\lambda_2^{p}(\tau)<\infty$. If $0<n\leq\dfrac{\lambda_1(\Upsilon)}{\lambda_2(\Upsilon)}\leq m$ for $n,\ m\in\ \mathbb{R}^{+}$ and $\forall\ \Upsilon\in[a,\ \tau]$, then we have

$$\left({}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}\lambda_1^{p}(\tau)\right)^{\frac{2}{p}}+\left({}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}\lambda_2^{p}(\tau)\right)^{\frac{2}{p}}\geq c_2\left({}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}\lambda_1^{p}(\tau)\right)^{\frac{1}{p}}\left({}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}\lambda_2^{p}(\tau)\right)^{\frac{1}{p}} \tag{15}$$

with $c_2=\dfrac{(n+1)(m+1)}{m}-2.$

Remark 2 The product of (12) and (14) yields

$$\left({}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}\lambda_1^{p}(\tau)\right)^{\frac{1}{p}}\left({}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}\lambda_1^{p}(\tau)\right)^{\frac{1}{p}}\leq\frac{m}{((m+1)}\frac{1}{(n+1)}\left({}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}(\lambda_1+\lambda_2)^{p}(\tau)\right)^{\frac{1}{p}}\left({}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}(\lambda_1+\lambda_2)^{p}(\tau)\right)^{\frac{1}{p}}$$

It follows that

$$\frac{(n+1)(m+1)}{m}\left({}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}\lambda_1^{p}(\tau)\right)^{\frac{1}{p}}\left({}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}\lambda_1^{p}(\tau)\right)^{\frac{1}{p}}\leq\left({}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}(\lambda_1+\lambda_2)^{p}(\tau)\right)^{\frac{2}{p}} \tag{16}$$

On the right side of (16), involving the Minkowski inequality, we obtain

$$\frac{(n+1)(m+1)}{m}\left({}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}\lambda_1^{p}(\tau)\right)^{\frac{1}{p}}\left({}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}\lambda_1^{p}(\tau)\right)^{\frac{1}{p}}\leq\left(\left({}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}\lambda_1^{p}(\tau)\right)^{\frac{1}{p}}+\left({}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}\lambda_1^{p}(\tau)\right)^{\frac{1}{p}}\right)^{2} \tag{17}$$

It can be obtained from (17) that

$$\left({}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}\lambda_{1}^{p}(\tau)\right)^{\frac{2}{p}}+\left({}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}\lambda_{1}^{p}(\tau)\right)^{\frac{2}{p}}\geq\left(\frac{(n+1)(m+1)}{m}-2\right)\left({}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}\lambda_{1}^{p}(\tau)\right)^{\frac{1}{p}}\left({}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}\lambda_{1}^{p}(\tau)\right)^{\frac{1}{p}}$$

which completes the desired inequality (15).

Remark 3 i. If we choose $k = 1$, then from Theorems 8 and 9, we get certain new results for the fractional operator recently defined by [35].

ii. If we choose $\rho(\tau) = \tau$, then Theorems 8 and 9 reduce to the work presented by Mubeen et al. [34].

iii. Similarly, Theorems 8 and 9 will reduce to the work presented by [29–31] by applying certain conditions given in Remark 1.

4. Associated fractional integral inequalities

In this section, we present the generalization of certain other types of the related integral inequalities which were provided by the researchers for distinct operators cited in literature. These inequalities are generalized by using the suggested generalized $(k,\ \rho)$-CFI operator defined in (8).

Theorem 10 For $k > 0,\ \omega \in \mathbb{R}\backslash\{0\},\ \eta > 0$ and $p,\ q \geq 1$ and $\dfrac{1}{p}+\dfrac{1}{q}=1$. Let $\lambda_1,\ \lambda_2 \in L_{1,\,\omega}[a,\ \tau]$ be the two positive functions in $[0,\ \infty)$ such that, $\forall\ \tau > a,\ {}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}\lambda_{1}^{p}(\tau) < \infty$ and ${}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}\lambda_{2}^{p}(\tau) < \infty$. If $0 < n \leq \dfrac{\lambda_1(\Upsilon)}{\lambda_2(\Upsilon)} \leq m$ for $n,\ m \in \mathbb{R}^{+}$ and $\forall\,\Upsilon \in [a,\ \tau]$, then

$$\left({}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}\lambda_{1}^{p}(\tau)\right)^{\frac{1}{p}}+\left({}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}\lambda_{2}^{p}(\tau)\right)^{\frac{1}{q}}\leq\left(\frac{m}{n}\right)^{\frac{1}{pq}}\left({}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}\lambda_{1}^{\frac{1}{p}}(\tau)\lambda_{2}^{\frac{1}{q}}(\tau)\right) \tag{18}$$

Proof. Under the given conditions $\dfrac{\lambda_1(\Upsilon)}{\lambda_2(\Upsilon)} \leq m,\ a \leq \Upsilon \leq \tau$, it may be expressed as

$$\lambda_1(\Upsilon) \leq m\lambda_2(\Upsilon)$$

$$\lambda_1^{\frac{1}{q}}(\Upsilon)\ \leq (m)^{\frac{1}{q}}\ \lambda_2^{\frac{1}{q}}(\Upsilon)$$

$$\lambda_2^{\frac{1}{q}}(\Upsilon) \geq (m)^{-\frac{1}{q}}\ \lambda_1^{\frac{1}{q}}(\Upsilon) \tag{19}$$

Multiplying (19) by $\lambda^{\frac{1}{p}}(\Upsilon)$, we get

$$\lambda_1^{\frac{1}{p}}(\Upsilon)\lambda_2^{\frac{1}{q}}(\Upsilon) \geq (m)^{-\frac{1}{q}}\ \lambda_1(\Upsilon) \tag{20}$$

Multiplying (20) with

$$\frac{(\omega)^{1-\frac{\eta}{k}}\left((\rho(\tau)-\rho(a))^{\omega}-(\rho(\Upsilon)-\rho(a))^{\omega}\right)^{\frac{\eta}{k}-1}(\rho(\Upsilon)-\rho(a))^{\omega-1}\rho'(\Upsilon)}{k\Gamma_k(\eta)}$$

and then integrating from a to τ with respect to the variable Υ, we obtain

$$\frac{(m)^{-\frac{1}{\Vert}}(\omega)^{1-\frac{\eta}{k}}}{k\Gamma_k(\eta)}\int_a^{\tau}\left((\rho(\tau)-\rho(a))^{\omega}-(\rho(\Upsilon)-\rho(a))^{\omega}\right)^{\frac{\eta}{k}-1}(\rho(\Upsilon)-\rho(a))^{\omega-1}\lambda_1(\Upsilon)\rho'(\Upsilon)d\Upsilon$$

$$\leq\frac{(\omega)^{1-\frac{\eta}{k}}}{k\Gamma_k(\eta)}\int_a^{\tau}\left((\rho(\tau)-\rho(a))^{\omega}-(\rho(\Upsilon)-\rho(a))^{\omega}\right)^{\frac{\eta}{k}-1}(\rho(\Upsilon)-\rho(a))^{\omega-1}\lambda_1^{\frac{1}{p}}(\Upsilon)\lambda_2^{\frac{1}{q}}(\Upsilon)\rho'(\Upsilon)d\Upsilon.$$

Thus, it may be expressed as

$$(m)^{-\frac{1}{pq}}\left({}_a^{\eta}\mathfrak{G}_k^{\omega,\,\rho}\lambda_1^{p}(\tau)\right)^{\frac{1}{p}}\leq\left({}_a^{\eta}\mathfrak{G}_k^{\omega,\,\rho}\lambda_1^{\frac{1}{p}}(\tau)\lambda_2^{\frac{1}{q}}(\tau)\right)^{\frac{1}{p}} \tag{21}$$

On the other hand, as $n\leq\dfrac{\lambda_1(\Upsilon)}{\lambda_2(\Upsilon)}\Rightarrow n\lambda_2(\Upsilon)\leq\lambda_1(\Upsilon)$, it follows

$$n^{\frac{1}{p}}\lambda_2{}^{\frac{1}{p}}(\Upsilon)\leq\lambda_1{}^{\frac{1}{p}}(\Upsilon). \tag{22}$$

By multiplying (22) with $\lambda_2^{\frac{1}{q}}(\Upsilon)$ and using $\dfrac{1}{p}+\dfrac{1}{q}=1$, we obtain

$$n^{\frac{1}{p}}\lambda_2(\Upsilon)\leq\lambda_1{}^{\frac{1}{p}}(\Upsilon)\lambda_2^{\frac{1}{q}}(\Upsilon) \tag{23}$$

Again, multiplying (23) with $\dfrac{(\omega)^{1-\frac{\eta}{k}}\left((\rho(\tau)-\rho(a))^{\omega}-(\rho(\Upsilon)-\rho(a))^{\omega}\right)^{\frac{\eta}{k}-1}(\rho(\Upsilon)-\rho(a))^{\omega-1}\rho'(\Upsilon)}{k\Gamma_k(\eta)}$ and then integrating from a to τ with regard to the variable Υ, we obtain

$$\frac{n^{\frac{1}{p}}(\omega)^{1-\frac{\eta}{k}}}{k\Gamma_k(\eta)}\int_a^{\tau}\left((\rho(\tau)-\rho(a))^{\omega}-(\rho(\Upsilon)-\rho(a))^{\omega}\right)^{\frac{\eta}{k}-1}(\rho(\Upsilon)-\rho(a))^{\omega-1}\lambda_2(\Upsilon)\rho'(\Upsilon)d\Upsilon$$

$$\leq\frac{(\omega)^{1-\frac{\eta}{k}}}{k\Gamma_k(\eta)}\int_a^{\tau}\left((\rho(\tau)-\rho(a))^{\omega}-(\rho(\Upsilon)-\rho(a))^{\omega}\right)^{\frac{\eta}{k}-1}(\rho(\Upsilon)-\rho(a))^{\omega-1}\lambda_1{}^{\frac{1}{p}}(\Upsilon)\lambda_2^{\frac{1}{q}}(\Upsilon)\rho'(\Upsilon)d\Upsilon$$

Thus, it may be written as

$$n^{\frac{1}{p}}\left({}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}\lambda_2(\tau)\right)\leq {}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}\lambda_1{}^{\frac{1}{p}}(\tau)\lambda_2^{\frac{1}{q}}(\tau)$$

$$n^{\frac{1}{pq}}\left({}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}\lambda_2(\tau)\right)^{\frac{1}{q}}\leq\left({}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}\lambda_1{}^{\frac{1}{p}}(\tau)\lambda_2^{\frac{1}{q}}(\tau)\right)^{\frac{1}{q}} \tag{24}$$

The product (21) and (24) gives

$$\left({}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}\lambda_1^{p}(\tau)\right)^{\frac{1}{p}}\left({}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}\lambda_2(\tau)\right)^{\frac{1}{q}}\leq\left(\frac{m}{n}\right)^{\frac{1}{pq}}\left({}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}\lambda_1^{\frac{1}{p}}(\tau)\lambda_2^{\frac{1}{q}}(\tau)\right)$$

the desired result (18) is thus completed. □

Theorem 11 For $\omega\in\mathbb{R}\backslash\{0\}$, $\eta>0,\ k>0$ and $p,\ q\geq 1$ and $\dfrac{1}{p}+\dfrac{1}{q}=1$. Let $\lambda_1,\ \lambda_2\in L_{1,\,\omega}[a,\ \tau]$ be the two positive functions in $[0,\ \infty)$ such that, $\forall\ \tau>a$, ${}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}\lambda_1^{p}(\tau)<\infty$ and ${}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}\lambda_2^{p}(\tau)<\infty$. If $0<n\leq\dfrac{\lambda_1(\Upsilon)}{\lambda_2(\Upsilon)}\leq m$ for $n,\ m\in\mathbb{R}^{+}$ and $\forall\ \Upsilon\in[a,\ \tau]$, then

$${}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}\lambda_1(\tau)\lambda_2(\tau)\leq c_3\left({}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}(\lambda_1^{p}+\lambda_2^{p})(\tau)\right)+c_4\left({}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}(\lambda_1^{q}+\lambda_2^{q})(\tau)\right) \tag{25}$$

with $c_3=\dfrac{2^{p-1}m^{p}}{p(m+1)^{p}}$ and $c_4=\dfrac{2^{q-1}}{q(n+1)^{q}}$.

Proof. The following identity is obtained by using the given hypothesis:

$$(m+1)^{p}\lambda_1^{p}(\Upsilon)\leq m^{p}\left(\lambda_1+\lambda_2\right)^{p}(\Upsilon) \tag{26}$$

Multiplying (26) with $\dfrac{(\omega)^{1-\frac{\eta}{k}}\left((\rho(\tau)-\rho(a))^{\omega}-(\rho(\Upsilon)-\rho(a))^{\omega}\right)^{\frac{\eta}{k}-1}(\rho(\Upsilon)-\rho(a))^{\omega-1}\rho'(\Upsilon)}{k\Gamma_k(\eta)}$ and then integrating from a to τ with regard to the variable Υ, we obtain

$$\frac{(m+1)^{p}(\omega)^{1-\frac{\eta}{k}}}{k\Gamma_k(\eta)}\int_a^{\tau}\left((\rho(\tau)-\rho(a))^{\omega}-(\rho(\Upsilon)-\rho(a))^{\omega}\right)^{\frac{\eta}{k}-1}(\rho(\Upsilon)-\rho(a))^{\omega-1}\lambda_1^{p}(\Upsilon)\rho'(\Upsilon)d\Upsilon$$

$$\leq\frac{m^{p}(\omega)^{1-\frac{\eta}{k}}}{k\Gamma_k(\eta)}\int_a^{\tau}\left((\rho(\tau)-\rho(a))^{\omega}-(\rho(\Upsilon)-\rho(a))^{\omega}\right)^{\frac{\eta}{k}-1}(\rho(\Upsilon)-\rho(a))^{\omega-1}\left(\lambda_1+\lambda_2\right)^{p}(\Upsilon)\rho'(\Upsilon)d\Upsilon.$$

Thus, it follows that

$$\left({}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}\lambda_1^{p}(\tau)\right)\leq\frac{m^{p}}{(m+1)^{p}}\,{}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}\left(\lambda_1+\lambda_2\right)^{p}(\tau) \tag{27}$$

On the other hand, as $0 < n \leq \dfrac{\lambda_1(\Upsilon)}{\lambda_2(\Upsilon)}$, $a < \Upsilon < \tau$, it follows

$$n\lambda_2(\Upsilon) \leq \lambda_1(\Upsilon)$$

$$n\lambda_2(\Upsilon) + \lambda_2(\Upsilon) \leq \lambda_1(\Upsilon) + \lambda_2(\Upsilon)$$

$$\lambda_2(\Upsilon)[n+1] \leq \lambda_1(\Upsilon) + \lambda_2(\Upsilon)$$

$$\lambda_2{}^q(\Upsilon)[n+1]^q \leq (\lambda_1 + \lambda_2)^q(\Upsilon) \tag{28}$$

Next, multiplying (28) with $\dfrac{(\omega)^{1-\frac{\eta}{k}}\left((\rho(\tau)-\rho(a))^{\omega}-(\rho(\Upsilon)-\rho(a))^{\omega}\right)^{\frac{\eta}{k}-1}(\rho(\Upsilon)-\rho(a))^{\omega-1}\rho'(\Upsilon)}{k\Gamma_k(\eta)}$ and then integrating from a to τ with regard to the variable Υ, we obtain

$$\frac{[n+1]^q(\omega)^{1-\frac{\eta}{k}}}{k\Gamma_k(\eta)}\int_a^{\tau}\left((\rho(\tau)-\rho(a))^{\omega}-(\rho(\Upsilon)-\rho(a))^{\omega}\right)^{\frac{\eta}{k}-1}(\rho(\Upsilon)-\rho(a))^{\omega-1}\lambda_2{}^q(\Upsilon)\rho'(\Upsilon)d\Upsilon$$

$$\leq \frac{(\omega)^{1-\frac{\eta}{k}}}{k\Gamma_k(\eta)}\int_a^{\tau}\left((\rho(\tau)-\rho(a))^{\omega}-(\rho(\Upsilon)-\rho(a))^{\omega}\right)^{\frac{\eta}{k}-1}(\rho(\Upsilon)-\rho(a))^{\omega-1}(\lambda_1+\lambda_2)^q(\Upsilon)\rho'(\Upsilon)d\Upsilon$$

It follows that

$$[n+1]^q\left({}_a^{\eta}\mathfrak{G}_k^{\omega,\,\rho}\lambda_2{}^q(\tau)\right) \leq {}_a^{\eta}\mathfrak{G}_k^{\omega,\,\rho}(\lambda_1+\lambda_2)^q(\tau)$$

$$\left({}_a^{\eta}\mathfrak{G}_k^{\omega,\,\rho}\lambda_2{}^q(\tau)\right) \leq \frac{1}{[n+1]^q}{}_a^{\eta}\mathfrak{G}_k^{\omega,\,\rho}(\lambda_1+\lambda_2)^q(\tau) \tag{29}$$

Considering the Young's inequality

$$\lambda_1(\Upsilon) + \lambda_2(\Upsilon) \leq \frac{\lambda_1{}^p(\Upsilon)}{p} + \frac{\lambda_2{}^q(\Upsilon)}{q} \tag{30}$$

Now, multiplying (30) with $\dfrac{(\omega)^{1-\frac{\eta}{k}}\left((\rho(\tau)-\rho(a))^{\omega}-(\rho(\Upsilon)-\rho(a))^{\omega}\right)^{\frac{\eta}{k}-1}(\rho(\Upsilon)-\rho(a))^{\omega-1}\rho'(\Upsilon)}{k\Gamma_k(\eta)}$ and then integrating from a to τ with regard to the variable Υ, we obtain

$$\frac{(\omega)^{1-\frac{\eta}{k}}}{k\Gamma_k(\eta)}\int_a^\tau \left((\rho(\tau)-\rho(a))^\omega-(\rho(\Upsilon)-\rho(a))^\omega\right)^{\frac{\eta}{k}-1}(\rho(\Upsilon)-\rho(a))^{\omega-1}\lambda_1(\Upsilon)+\lambda_2(\Upsilon)\rho'(\Upsilon)d\Upsilon$$

$$\leq \frac{(\omega)^{1-\frac{\eta}{k}}}{pk\Gamma_k(\eta)}\int_a^\tau \left((\rho(\tau)-\rho(a))^\omega-(\rho(\Upsilon)-\rho(a))^\omega\right)^{\frac{\eta}{k}-1}(\rho(\Upsilon)-\rho(a))^{\omega-1}\lambda_1{}^p(\Upsilon)\rho'(\Upsilon)d\Upsilon$$

$$+\frac{(\omega)^{1-\frac{\eta}{k}}}{q k\Gamma_k(\eta)}\int_a^\tau \left((\rho(\tau)-\rho(a))^\omega-(\rho(\Upsilon)-\rho(a))^\omega\right)^{\frac{\eta}{k}-1}(\rho(\Upsilon)-\rho(a))^{\omega-1}\lambda_2{}^q(\Upsilon)\rho'(\Upsilon)d\Upsilon$$

It follows that

$$ {}_a^\eta\mathfrak{G}_k^{\omega,\,\rho}(\lambda_1\lambda_2)(\tau)\leq\frac{1}{p}\left({}_a^\eta\mathfrak{G}_k^{\omega,\,\rho}\lambda_1{}^p(\tau)\right)+\frac{1}{q}\left({}_a^\eta\mathfrak{G}_k^{\omega,\,\rho}\lambda_2{}^q(\tau)\right) \tag{31}$$

The substitution of (27) and (29) in (31) yields

$$ {}_a^\eta\mathfrak{G}_k^{\omega,\,\rho}(\lambda_1\lambda_2)(\tau)\leq\frac{m^p}{p(m+1)^p}\left({}_a^\eta\mathfrak{G}_k^{\omega,\,\rho}(\lambda_1+\lambda_2)^p(\tau)\right)+\frac{1}{q[n+1]^q}\left({}_a^\eta\mathfrak{G}_k^{\omega,\,\rho}(\lambda_1+\lambda_2)^q(\tau)\right) \tag{32}$$

Making use of the inequality $(\curvearrowleft+\Upsilon)^\omega\leq 2^{\omega-1}(\curvearrowleft^\omega+\Upsilon^\omega)$, $\omega>1$, $\curvearrowleft,\ \Upsilon>0$, we get

$$ {}_a^\eta\mathfrak{G}_k^{\omega,\,\rho}(\lambda_1+\lambda_2)^p(\tau)\leq 2^{p-1}\,{}_a^\eta\mathfrak{G}_k^{\omega,\,\rho}(\lambda_1{}^p+\lambda_2{}^p)(\tau) \tag{33}$$

and

$$ {}_a^\eta\mathfrak{G}_k^{\omega,\,\rho}(\lambda_1+\lambda_2)^q(\tau)\leq 2^{q-1}\,{}_a^\eta\mathfrak{G}_k^{\omega,\,\rho}(\lambda_1{}^q+\lambda_2{}^q)(\tau) \tag{34}$$

Thus, by substituting (33) and (34) in (32), we get the desired inequality (26). □

Theorem 12 For $k>0$, $\omega\in\mathbb{R}\setminus\{0\}$, $\eta>0$ and $p\geq 1$. Let $\lambda_1,\ \lambda_2\in L_{1,\ \omega}[a,\ \tau]$ be the two positive functions in $[0,\ \infty)$ such that, $\forall\ \tau>a$, ${}_a^\eta\mathfrak{G}_k^{\omega,\,\rho}\lambda_1^p(\tau)<\infty$ and ${}_a^\eta\mathfrak{G}_k^{\omega,\,\rho}\lambda_2^p(\tau)<\infty$. If $0<c<n\leq\dfrac{\lambda_1(\Upsilon)}{\lambda_2(\Upsilon)}\leq m$ for $n,\ m\in\ \mathbb{R}^+$ and $\forall\ \Upsilon\in[a,\ \tau]$ then,

$$\frac{m+1}{m-c}\left({}_a^\eta\mathfrak{G}_k^{\omega,\,\rho}\lambda_1(\tau)-c\lambda_2(\tau)\right)\leq\left({}_a^\eta\mathfrak{G}_k^{\omega,\,\rho}\lambda_1^p(\tau)\right)^{\frac{1}{p}}+\left({}_a^\eta\mathfrak{G}_k^{\omega,\,\rho}\lambda_2^p(\tau)\right)^{\frac{1}{p}}$$

$$\leq\frac{n+1}{n-c}\left({}_a^\eta\mathfrak{G}_k^{\omega,\,\rho}\lambda_1(\tau)-c\lambda_2(\tau)\right)^{\frac{1}{p}} \tag{35}$$

Proof. By means of the given supposition $0 < c < n \leq m$, we obtain

$$nc \leq mc \Rightarrow nc + n \leq nc + m \leq mc + m$$

$$\Rightarrow (m+1)(n-c) \leq (n+1)(m-c).$$

It follows that

$$\frac{(m+1)}{(m-c)} \leq \frac{(n+1)}{(n-c)}.$$

Moreover, we have

$$n - c \leq \frac{\lambda_1(\Upsilon) - c\lambda_2(\Upsilon)}{\lambda_2(\Upsilon)} \leq m - c.$$

It follows that

$$\frac{(\lambda_1(\Upsilon) - c\lambda_2(\Upsilon))^p}{(m-c)^p} \leq \lambda_2^p(\Upsilon) \leq \frac{(\lambda_1(\Upsilon) - c\lambda_2(\Upsilon))^p}{(n-c)^p} \tag{36}$$

Again, we can write

$$\frac{1}{m} \leq \frac{\lambda_2(\Upsilon)}{\lambda_1(\Upsilon)} \leq \frac{1}{n} \Rightarrow \frac{n-c}{cn} \leq \frac{\lambda_1(\Upsilon) - c\lambda_2(\Upsilon)}{c\lambda_1(\Upsilon)} \leq \frac{m-c}{cm}$$

It becomes

$$\left(\frac{m}{m-c}\right)^p (\lambda_1(\Upsilon) - c\lambda_2(\Upsilon))^p \leq \lambda_1{}^p(\Upsilon) \leq \left(\frac{n}{n-c}\right)^p (\lambda_1(\Upsilon) - c\lambda_2(\Upsilon))^p \tag{37}$$

By multiplying (36) with $\dfrac{(\omega)^{1-\frac{\eta}{k}} \left((\rho(\tau) - \rho(a))^\omega - (\rho(\Upsilon) - \rho(a))^\omega\right)^{\frac{\eta}{k}-1} (\rho(\Upsilon) - \rho(a))^{\omega-1} \rho'(\Upsilon)}{k\Gamma_k(\eta)}$ and then integrating from a to τ with regard to the variable Υ, we obtain

$$\frac{(\omega)^{1-\frac{\eta}{k}}}{(m-c)^p k\Gamma_k(\eta)}\int_a^\tau \left((\rho(\tau)-\rho(a))^\omega-(\rho(\Upsilon)-\rho(a))^\omega\right)^{\frac{\eta}{k}-1}(\rho(\Upsilon)-\rho(a))^{\omega-1}(\lambda_1(\Upsilon)-\lambda_2(\Upsilon))^p\rho^{'}(\Upsilon)d\Upsilon$$

$$\leq \frac{(\omega)^{1-\frac{\eta}{k}}}{k\Gamma_k(\eta)}\int_a^\tau \left((\rho(\tau)-\rho(a))^\omega-(\rho(\Upsilon)-\rho(a))^\omega\right)^{\frac{\eta}{k}-1}(\rho(\Upsilon)-\rho(a))^{\omega-1}\lambda_2{}^p(\Upsilon)\rho^{'}(\Upsilon)d\Upsilon$$

$$\leq \frac{(\omega)^{1-\frac{\eta}{k}}}{(n-c)^p k\Gamma_k(\eta)}\int_a^\tau \left((\rho(\tau)-\rho(a))^\omega-(\rho(\Upsilon)-\rho(a))^\omega\right)^{\frac{\eta}{k}-1}(\rho(\Upsilon)-\rho(a))^{\omega-1}(\lambda_1(\Upsilon)-c\lambda_2(\Upsilon))^p\rho^{'}(\Upsilon)d\Upsilon$$

Thus, it follows that

$$\frac{1}{m-c}\left({}_a^\eta\mathfrak{G}_k^{\omega,\,\rho}(\lambda_1(\tau)-c\lambda_2(\tau))^p\right)^{\frac{1}{p}}\leq\left({}_a^\eta\mathfrak{G}_k^{\omega,\,\rho}\lambda_2{}^p(\tau)\right)^{\frac{1}{p}}\leq\frac{1}{n-c}\left({}_a^\eta\mathfrak{G}_k^{\omega,\,\rho}(\lambda_1(\tau)-c\lambda_2(\tau))^p\right)^{\frac{1}{p}}\tag{38}$$

Similarly, by performing the same procedure with (37), we get

$$\frac{m}{m-c}\left({}_a^\eta\mathfrak{G}_k^{\omega,\,\rho}(\lambda_1(\tau)-c\lambda_2(\tau))^p\right)^{\frac{1}{p}}\leq\left({}_a^\eta\mathfrak{G}_k^{\omega,\,\rho}\lambda_1{}^p(\tau)\right)^{\frac{1}{p}}\leq\frac{n}{n-c}\left({}_a^\eta\mathfrak{G}_k^{\omega,\,\rho}(\lambda_1(\tau)-c\lambda_2(\tau))^p\right)^{\frac{1}{p}}\tag{39}$$

Thus, by adding (38) and (39), we obtain the proof of (35). □

Theorem 13 For $k>0,\ \omega\in\ \mathbb{R}\backslash\{0\},\ \eta>0$ and $p\geq1$. Let $\lambda_1,\ \lambda_2\in\ L_{1,\,\omega}[a,\ \tau]$ be the two positive functions in $[0,\ \infty)$ such that, $\forall\ \tau>a,\ {}_a^\eta\mathfrak{G}_k^{\omega,\,\rho}\lambda_1^p(\tau)<\infty$ and ${}_a^\eta\mathfrak{G}_k^{\omega,\,\rho}\lambda_2^p(\tau)<\infty$. If $0\leq a\leq\lambda_1(\Upsilon)\leq A$ and $0\leq b\leq\lambda_2(\Upsilon)\leq B$ for $n,\ m\in\ \mathbb{R}^+\ \forall\ \Upsilon\in[a,\ \tau]$, then

$$\left({}_a^\eta\mathfrak{G}_k^{\omega,\,\rho}\lambda_1^p(\tau)\right)^{\frac{1}{p}}+\left({}_a^\eta\mathfrak{G}_k^{\omega,\,\rho}\lambda_2^p(\tau)\right)^{\frac{1}{p}}\leq c_5\left({}_a^\eta\mathfrak{G}_k^{\omega,\,\rho}(\lambda_1+\lambda_2)^p(\tau)\right)^{\frac{1}{p}}\tag{40}$$

with $c_5=\dfrac{A(a+B)+B(A+b)}{(A+b)(a+B)}$.

Proof. By the given conditions, we have

$$\frac{1}{B}\leq\frac{1}{\lambda_2(\tau)}\leq\frac{1}{b}\tag{41}$$

Multiplying (41) with $0\leq a\leq\lambda_1(\Upsilon)\leq A$, we obtain

$$\frac{a}{B}\leq\frac{\lambda_1(\tau)}{\lambda_2(\tau)}\leq\frac{A}{b}\tag{42}$$

From (42), we have

$$\lambda_1^p(\Upsilon) \leq \left(\frac{A}{b+A}\right)^p (\lambda_1(\Upsilon)+\lambda_2(\Upsilon))^p \tag{43}$$

and

$$\lambda_2^p(\Upsilon) \leq \left(\frac{B}{a+B}\right)^p (\lambda_1(\Upsilon)+\lambda_2(\Upsilon))^p \tag{44}$$

Now, by conducting the product of (43) with

$$\frac{(\omega)^{1-\frac{\eta}{k}}\left((\rho(\tau)-\rho(a))^\omega-(\rho(\Upsilon)-\rho(a))^\omega\right)^{\frac{\eta}{k}-1}(\rho(\Upsilon)-\rho(a))^{\omega-1}\rho'(\Upsilon)}{k\Gamma_k(\eta)}$$

and then integrating from a to τ with regard to the variable Υ, we obtain

$$\frac{(\omega)^{1-\frac{\eta}{k}}}{k\Gamma_k(\eta)}\int_a^\tau \left((\rho(\tau)-\rho(a))^\omega-(\rho(\Upsilon)-\rho(a))^\omega\right)^{\frac{\eta}{k}-1}(\rho(\Upsilon)-\rho(a))^{\omega-1}\lambda_1^p(\Upsilon)\rho'(\Upsilon)d\Upsilon$$

$$\leq \frac{A^p(\omega)^{1-\frac{\eta}{k}}}{(b+A)^p k\Gamma_k(\eta)}\int_a^\tau \left((\rho(\tau)-\rho(a))^\omega-(\rho(\Upsilon)-\rho(a))^\omega\right)^{\frac{\eta}{k}-1}(\rho(\Upsilon)-\rho(a))^{\omega-1}(\lambda_1(\Upsilon)+\lambda_2(\Upsilon))^p\rho'(\Upsilon)d\Upsilon$$

Thus, it follows that

$$\left({}_a^\eta\mathfrak{G}_k^{\omega,\,\rho}\lambda_1^p(\tau)\right)^{\frac{1}{p}} \leq \frac{A}{b+A}\left({}_a^\eta\mathfrak{G}_k^{\omega,\,\rho}(\lambda_1+\lambda_2)^p(\tau)\right)^{\frac{1}{p}} \tag{45}$$

Similarly, by conducting the product of (44) with

$$\frac{(\omega)^{1-\frac{\eta}{k}}\left((\rho(\tau)-\rho(a))^\omega-(\rho(\Upsilon)-\rho(a))^\omega\right)^{\frac{\eta}{k}-1}(\rho(\Upsilon)-\rho(a))^{\omega-1}\rho'(\Upsilon)}{k\Gamma_k(\eta)}$$

and then integrating from a to τ with regard to the variable Υ, we obtain

$$\frac{(\omega)^{1-\frac{\eta}{k}}}{k\Gamma_k(\eta)}\int_a^\tau \left((\rho(\tau)-\rho(a))^\omega-(\rho(\Upsilon)-\rho(a))^\omega\right)^{\frac{\eta}{k}-1}(\rho(\Upsilon)-\rho(a))^{\omega-1}\lambda_2^p(\Upsilon)\rho'(\Upsilon)d\Upsilon$$

$$\leq \frac{B^p(\omega)^{1-\frac{\eta}{k}}}{(a+B)^p k\Gamma_k(\eta)}\int_a^\tau \left((\rho(\tau)-\rho(a))^\omega-(\rho(\Upsilon)-\rho(a))^\omega\right)^{\frac{\eta}{k}-1}(\rho(\Upsilon)-\rho(a))^{\omega-1}\times(\lambda_1(\Upsilon)+\lambda_2(\Upsilon))^p\rho'(\Upsilon)d\Upsilon$$

Thus, it can be written as

$$\left({}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}\lambda_2^p(\tau)\right)^{\frac{1}{p}} \leq \frac{B}{a+B}\left({}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}(\lambda_1+\lambda_2)^p(\tau)\right)^{\frac{1}{p}} \tag{46}$$

The desired proof of (40) is thus obtained by adding (45) and (46). □

Theorem 14 For $k>0,\ \omega\in\mathbb{R}\backslash\{0\},\ \eta>0$. Let $\lambda_1,\ \lambda_2\in L_{1,\,\omega}[a,\ \tau]$ be the two positive functions in $[0,\ \infty)$ such that, $\forall\ \tau>a,\ {}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}\lambda_1^p(\tau)<\infty$ and ${}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}\lambda_2^p(\tau)<\infty$. If $0<n\leq\dfrac{\lambda_1(\Upsilon)}{\lambda_2(\Upsilon)}\leq m$ for $n,\ m\in\mathbb{R}^+$ and $\forall\ \Upsilon\in[a,\ \tau]$, then we have

$$\frac{1}{m}\left({}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}\lambda_1(\tau)+\lambda_2(\tau)\right)\ \leq\frac{1}{(n+1)(m+1)}\left({}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}(\lambda_1+\lambda_2)^2(\tau)\right)\leq\frac{1}{n}\left({}_{a}^{\eta}\mathfrak{G}_{k}^{\omega,\,\rho}\lambda_1(\tau)\lambda_2(\tau)\right) \tag{47}$$

Proof. By making the use of $0<n\leq\dfrac{\lambda_1(\Upsilon)}{\lambda_2(\Upsilon)}\leq m$, we have

$$n\lambda_2(\Upsilon)\leq\lambda_1(\Upsilon)\leq m\lambda_2(\Upsilon)$$

$$\lambda_2(\Upsilon)(n+1)\leq\lambda_2(\Upsilon)+\lambda_1(\Upsilon)\leq\lambda_2(\Upsilon)(m+1) \tag{48}$$

Moreover, it follows that $\dfrac{1}{m}\leq\dfrac{\lambda_2(\Upsilon)}{\lambda_1(\Upsilon)}\leq\dfrac{1}{n}$, which gives

$$\lambda_1(\Upsilon)\left(\frac{m+1}{m}\right)\leq\lambda_2(\Upsilon)+\lambda_1(\Upsilon)\leq\left(\frac{n+1}{n}\right) \tag{49}$$

The multiplication of (48) and (49) yields,

$$\frac{\lambda_1(\Upsilon)\lambda_2(\Upsilon)}{m}\leq\frac{(\lambda_1(\Upsilon)+\lambda_2(\Upsilon))^2}{(n+1)(m+1)}\leq\frac{\lambda_1(\Upsilon)\lambda_2(\Upsilon)}{n} \tag{50}$$

Conducting the product of (50) with $\dfrac{(\omega)^{1-\frac{\eta}{k}}\left((\rho(\tau)-\rho(a))^{\omega}-(\rho(\Upsilon)-\rho(a))^{\omega}\right)^{\frac{\eta}{k}-1}(\rho(\Upsilon)-\rho(a))^{\omega-1}\rho'(\Upsilon)}{k\Gamma_k(\eta)}$ and then integrating from a to τ with regard to the variable Υ, we obtain

$$\frac{(\omega)^{1-\frac{\eta}{k}}}{mk\Gamma_k(\eta)}\int_a^{\tau}\left((\rho(\tau)-\rho(a))^{\omega}-(\rho(\Upsilon)-\rho(a))^{\omega}\right)^{\frac{\eta}{k}-1}(\rho(\Upsilon)-\rho(a))^{\omega-1}\lambda_1(\Upsilon)\lambda_2(\Upsilon)\rho'(\Upsilon)d\Upsilon$$

$$\leq c_6\frac{(\omega)^{1-\frac{\eta}{k}}}{k\Gamma_k(\eta)}\int_a^{\tau}\left((\rho(\tau)-\rho(a))^{\omega}-(\rho(\Upsilon)-\rho(a))^{\omega}\right)^{\frac{\eta}{k}-1}(\rho(\Upsilon)-\rho(a))^{\omega-1}(\lambda_1(\Upsilon)+\lambda_2(\Upsilon))^2\rho'(\Upsilon)d\Upsilon$$

$$\leq \frac{(\omega)^{1-\frac{\eta}{k}}}{nk\Gamma_k(\eta)}\int_a^{\tau}\left((\rho(\tau)-\rho(a))^{\omega}-(\rho(\Upsilon)-\rho(a))^{\omega}\right)^{\frac{\eta}{k}-1}(\rho(\Upsilon)-\rho(a))^{\omega-1}\lambda_1(\Upsilon)\lambda_2(\Upsilon)\rho'(\Upsilon)d\Upsilon$$

with $c_6=\dfrac{1}{(n+1)(m+1)}$.

Thus, the desired inequality (47) can be obtained. □

Theorem 15 For $k>0$, $\omega\in\mathbb{R}\backslash\{0\}$, $\eta>0$ and $p\geq 1$. Let $\lambda_1,\ \lambda_2\in L_{1,\ \omega}[a,\ \tau]$ be the two positive functions in $[0,\ \infty)$ such that, $\forall\ \tau>a$, ${}_a^{\eta}\mathfrak{G}_k^{\omega,\ \rho}\lambda_1^p(\tau)<\infty$ and ${}_a^{\eta}\mathfrak{G}_k^{\omega,\ \rho}\lambda_2^p(\tau)<\infty$. If $0<n\leq\dfrac{\lambda_1(\Upsilon)}{\lambda_2(\Upsilon)}\leq m$ for $n,\ m\in\mathbb{R}^+$ and $\forall\,\Upsilon\in[a,\ \tau]$, then

$$\left({}_a^{\eta}\mathfrak{G}_k^{\omega,\ \rho}\lambda_1^p(\tau)\right)^{\frac{1}{p}}+\left({}_a^{\eta}\mathfrak{G}_k^{\omega,\ \rho}\lambda_2^p(\tau)\right)^{\frac{1}{p}}\leq 2\left({}_a^{\eta}\mathfrak{G}_k^{\omega,\ \rho}h^p\left(\lambda_1(\tau),\ \lambda_2(\tau)\right)\right)^{\frac{1}{p}} \tag{51}$$

where

$$h(\lambda_1(\Upsilon),\ \lambda_2(\Upsilon))=max\left\{m\left[\left(\frac{m}{n}+1\right)\lambda_1(\tau)-m\lambda_2(\tau)\right],\ \frac{(n+m)\lambda_2(\tau)-\lambda_1(\tau)}{n}\right\}$$

Proof. In the circumstances specified $0<n\leq\dfrac{\lambda_1(\Upsilon)}{\lambda_2(\Upsilon)}\leq m,\ a\leq\Upsilon\leq\tau$. It may be expressed as

$$0<n\leq m+n-\frac{\lambda_1(\Upsilon)}{\lambda_2(\Upsilon)} \tag{52}$$

and

$$m+n-\frac{\lambda_1(\Upsilon)}{\lambda_2(\Upsilon)}\leq m. \tag{53}$$

Using (52) and (55), we get

$$\lambda_2(\Upsilon)<\frac{(m+n)\lambda_2(\Upsilon)-\lambda_1(\Upsilon)}{n}\leq h(\lambda_1(\Upsilon),\ \lambda_2(\Upsilon)) \tag{54}$$

where

$$h(\lambda_1(\Upsilon),\ \lambda_2(\Upsilon)) = \max\left\{ m\left[\left(\frac{m}{n}+1\right)\lambda_1(\tau) - m\lambda_2(\tau)\right],\ \frac{(n+m)\lambda_2(\tau)-\lambda_1(\tau)}{n} \right\}$$

This also implies, based on hypothesis that $0 < \frac{1}{m} \leq \frac{\lambda_2(\Upsilon)}{\lambda_1(\Upsilon)} \leq \frac{1}{n}$, which produces

$$\frac{1}{m} \leq \frac{1}{m} + \frac{1}{n} - \frac{\lambda_2(\Upsilon)}{\lambda_1(\Upsilon)} \tag{55}$$

and

$$\frac{1}{m} + \frac{1}{n} - \frac{\lambda_2(\Upsilon)}{\lambda_1(\Upsilon)} \leq \frac{1}{n} \tag{56}$$

We obtain from (55) and (56)

$$\frac{1}{m} \leq \frac{\left(\frac{1}{m}+\frac{1}{n}\right)\lambda_1(\Upsilon)-\lambda_2(\Upsilon)}{\lambda_1(\Upsilon)} \leq \frac{1}{n} \tag{57}$$

This can also be written as

$$\begin{aligned}
\lambda_1(\Upsilon) &\leq m\left(\frac{1}{m}+\frac{1}{n}\right)\lambda_1(\Upsilon) - m\lambda_2(\Upsilon) \\
&= \frac{m(m+n)\lambda_1(\Upsilon) - m^2 n\lambda_2(\Upsilon)}{nm} \\
&= \left(\frac{m}{n}+1\right)\lambda_1(\Upsilon) - m\lambda_2(\Upsilon) \\
&\leq m\left[\left(\frac{m}{n}+1\right)\lambda_1(\Upsilon) - m\lambda_2(\Upsilon)\right] \\
&\leq h(\lambda_1(\Upsilon),\lambda_2(\Upsilon))
\end{aligned} \tag{58}$$

From (54) and (58), we have

$$\lambda_1{}^p(\Upsilon) \leq h^p(\lambda_1(\Upsilon),\lambda_2(\Upsilon)) \tag{59}$$

and

$$\lambda_2{}^p(\Upsilon) \leq h^p\left(\lambda_1(\Upsilon), \lambda_2(\Upsilon)\right) \tag{60}$$

Now, conducting the product of (59) with $\dfrac{(\omega)^{1-\frac{\eta}{k}}\left((\rho(\tau)-\rho(a))^{\omega}-(\rho(\Upsilon)-\rho(a))^{\omega}\right)^{\frac{\eta}{k}-1}(\rho(\Upsilon)-\rho(a))^{\omega-1}\rho'(\Upsilon)}{k\Gamma_k(\eta)}$ and then integrating from a to τ with regard to the variable Υ, we obtain

$$\frac{(\omega)^{1-\frac{\eta}{k}}}{k\Gamma_k(\eta)}\int_a^{\tau}\left((\rho(\tau)-\rho(a))^{\omega}-(\rho(\Upsilon)-\rho(a))^{\omega}\right)^{\frac{\eta}{k}-1}(\rho(\Upsilon)-\rho(a))^{\omega-1}\lambda_1^p(\Upsilon)\rho'(\Upsilon)d\Upsilon$$

$$\leq \frac{(\omega)^{1-\frac{\eta}{k}}}{k\Gamma_k(\eta)}\int_a^{\tau}\left((\rho(\tau)-\rho(a))^{\omega}-(\rho(\Upsilon)-\rho(a))^{\omega}\right)^{\frac{\eta}{k}-1}(\rho(\Upsilon)-\rho(a))^{\omega-1}h^p\left(\lambda_1(\Upsilon), \lambda_2(\Upsilon)\right)\rho'(\Upsilon)d\Upsilon$$

Thus, it may be expressed as

$$\left({}_a^{\eta}\mathfrak{G}_k^{\omega,\,\rho}\lambda_1^p(\tau)\right)^{\frac{1}{p}} \leq \left({}_a^{\eta}\mathfrak{G}_k^{\omega,\,\rho}h^p\left(\lambda_1(\Upsilon),\, \lambda_2(\Upsilon)\right)(\tau)\right)^{\frac{1}{p}} \tag{61}$$

Similarly, from (60), we obtain

$$\left({}_a^{\eta}\mathfrak{G}_k^{\omega,\,\rho}\lambda_2^p(\tau)\right)^{\frac{1}{p}} \leq \left({}_a^{\eta}\mathfrak{G}_k^{\omega,\,\rho}h^p\left(\lambda_1(\Upsilon),\, \lambda_2(\Upsilon)\right)(\tau)\right)^{\frac{1}{p}} \tag{62}$$

From (61) and (62), the desired outcome (53) is obtained.

The results presented in this paper can be easily reduced to classical inequalities under suitable parameter values involving distinct classical fractional integrals cited in literature. □

Remark 4 i. If we choose $k = 1$, then from Theorems 10-15, we get certain new results for the fractional operator recently defined by [35].

ii. If we choose $\rho(\tau) = \tau$, then Theorems 10-15 reduce to the work presented by Mubeen et al. [34].

iii. Similarly, Theorems 10-15 will be reduced to the work presented by [29–31] by applying certain conditions given in Remark 1.

5. Conclusions

The paper begins with a summary of fractional integrals as defined by Hadamard, Katugampola, and Riemann-Liouville, along with newly defined generalized fractional integral operators defined earlier by [15, 34, 35]. We proved the existence and formulation of $(k,\ \rho)$-CFI operators. Then $(k,\ \rho)$-CFI operators are used to generalize the reverse Minkowski inequality. The inequalities involving fractional integrals such as Hadamard, Katugampola, and Riemann-can easily be restored by applying the conditions given in Remark 1. We also presented certain other types of integral inequalities via the $(k,\ \rho)$-CFI. With the application of the recently proposed fractional integral operators, several inequalities can be generalized. The integral inequalities presented in this paper can also be reduced to the inequalities involving the generalized fractional operators recently defined by [35]. One can present the Grüss-type inequality, Chebyshev inequality, and Chebyshev-Grüss type inequality for the new class of proposed operators.

Author contributions

The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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Competing interests

The authors declare that they have no competing interests.

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