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Research Article

# Multiple Timewise Coefficient Determination Problem From Heat Moment Observations

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**Abstract:** Consider the inverse problem of determining multiple timewise coeffcient and the solution function satisfying the parabolic equation with the direct initial and Dirichlet boundary conditions from the heat moment observations. This formulation ensures the unique solvability of the inverse problem. However, the problem still suffers from ill-posedness. Since small errors in the input data cause large errors in the output solution. The finite difference method is developed as a direct solver, whilst the inverse problem solver is reformulated as nonlinear least-squares minimization. The optimization problem was solved numerically using the *lsqnonlin* routine from the MATLAB toolbox. The exact and noisy input data are inverted numerically. Numerical results are presented and discussed to illustrate the performance of the inversion for timewise coefficients.

***Keywords***: inverse problem, parabolic equation, heat moment observations, nonlinear optimization

**MSC:** 35R30, 35K10, 35A09, 35A01, 35A02

## 1. Introduction

Many mathematical models for physical problems in engineering and science need stable is the solutions for indirect (inverse) problem. Inverse problems are usually posed as nonclassical problems for some partial differential equation. They often belong to the class of ill-posed problems in which the solution either does not exist or is not unique nor is the solution unstable. Therefore, a reliable and efficient method should be applied in order to gain a stable solution.

The problem of coefficient identification in heat parabolic equation was investigated theoretically and numerically by many authors, see [1–3] for one-dimensional and one coefficient. In [4–6] for two coefficients whilst for three or more in [7], just to mention only a few. In all these papers the author(s) investigate the numerical retrieval of unknown coefficients from several types of overdetermination conditions that ensure unique solvability, for example, Cauchy data, heat flux, mass/energy specification, and heat moments. Huntul et al. [8–14] investigate an inverse problem numerically to recover the timewise term in the rectangular domain with the additional temperature measurement as an overdetermination condition while Isgendarov et al. [15] investigated an inverse problem theoretically with the first kind integral condition and proved the uniqueness of the solution using the principle of contraction mappings. Tekin [16] considered an inverse problem to determine a time-dependent potential in a pseudo-hyperbolic equation with an over-determination condition.


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The inverse problem investigated in this paper has already proven to be uniquely solvable, but no numerical determination has been attempted so far, and the current research aims to undertake the numerical solution of this problem. In this study, the novelty consists in the development of a numerical optimization method for solving this nonlinear inverse coefficient problem. Numerically, the implementation is realized using the MATLAB subroutine *lsqnonlin*.

The structure of the paper is as follows. In Section 2, the mathematical formulation of the inverse problem is presented. In Section 3, the numerical solution of the direct problem is based on the finite difference method with the Crank-Nicolson scheme. In Section 4, the initial value for the unknown coefficients is derived from input and overdetermination conditions. In Section 5, the minimization algorithm to solve the inverse problem is presented. The numerical results are discussed in Section 6. Finally, conclusions are highlighted in Section 7.

## 2. Mathematical formulation

In the fixed domain $Q_T := \{(x, t)|\ 0 < x < l,\ 0 < t < T\}$, we consider the inverse problem given by the parabolic equation

$$\frac{\partial u}{\partial t}(x, t) = a(t)\frac{\partial^2 u}{\partial x^2}(x, t) + \Big(\alpha(t)x^2 + \beta(t)x + \gamma(t)\Big)\frac{\partial u}{\partial x}(x, t) + f(x, t),\ (x, t) \in Q_T, \tag{1}$$

with known thermal conductivity $a(t) > 0$, $t \in [0, T]$ and known heat source $f(x, t)$, unknown temperature $u(x, t)$ and unknown timewise coefficients $\alpha(t)$, $\beta(t)$ and $\gamma(t)$, respectively, subject to the initial condition

$$u(x, 0) = \varphi(x),\ x \in [0, l], \tag{2}$$

the Dirichlet boundary conditions

$$u(0, t) = \mu_1(t),\ u(l, t) = \mu_2(t), \quad t \in [0, T], \tag{3}$$

and the heat moment observations

$$\int_0^l u(x, t)dx = \nu_1(t),\ \int_0^l xu(x, t)dx = \nu_2(t),\ \int_0^l x^2u(x, t)dx = \nu_3(t),\ t \in [0, T]. \tag{4}$$

The existence and uniqueness of th solution of the inverse problem (1)-(4) were established in [4] and reads as stated in the following two theorems.

**Theorem 1** (Existence of the solution)

Assume the following conditions are satisfied:

**(A)** $\mu_i(t) \in C^1[0, T],\ i = 1, 2,\ \nu_i(t) \in C^1[0, T],\ i = 1, 2, 3,\ \varphi(x) \in C^1[0, l],\ f(x, t) \in C^{1,0}(\overline{Q_T})$;

**(B)** $\varphi(x) > 0,\ x \in [0, l],\ \mu_1(t) > 0,\ \mu_2(t) > 0,\ t \in [0, T],\ f(x, t) \geq 0,\ (x, t) \in (\overline{Q_T})$;

**(C)** $R_1(t) \equiv \mu_2(t)\Big(12\nu_2^2(t) + 6\mu_1(t)\nu_3(t) + 9\nu_3^2(t) + \nu_1^2(t) - 6\nu_1(t)\nu_3(t) - 12\nu_2(t)\nu_3(t)\Big) - \mu_2(t)\Big(4\nu_1(t)\nu_2(t) + 2\mu_1(t)\nu_2(t)\Big) + 12\nu_1(t)\nu_2(t)\nu_3(t) - 8\nu_2^3(t) - 9\mu_1(t)\nu_3^2(t) > 0, R_2(t) \equiv 4\nu_1^2(t) - 8\mu_2(t)\nu_1(t) + 8\mu_2(t)\nu_2(t) + 4\mu_2(t)\mu_1(t) + 8\mu_1(t)\nu_2(t) > 0,\ a(t) > 0$;

**(D)** $\mu_1(0) = \varphi(0),\ \mu_2(0) = \varphi(l),\ \int_0^l \varphi(x)dx = \nu_1(0),\ \int_0^l x\varphi(x)dx = \nu_2(0),\ \int_0^l x^2\varphi(x)dx = \nu_3(0)$.

Then, there exists a number, $t_0,\ 0 \le t_0 \le T$, which is determined by input data, such that the problem (1)-(4) has a solution $(\alpha(t),\ \beta(t),\ \gamma(t),\ u(x,\ t)) \in C[0,\ t_0] \times C[0,\ t_0] \times C[0,\ t_0] \times (C^{2,\ 1}(Q_{t_0}) \cap C(\overline{Q_{t_0}}))$.

**Theorem 2** (Uniqueness of the solution) Let the assumptions **(B)** and **(C)** hold.

Then, the inverse problem given by equation (1)-(4) admits a unique solution $(\alpha(t),\ \beta(t),\ \gamma(t),\ u(x,\ t)) \in C[0,\ T] \times C[0,\ T] \times C[0,\ T] \times (C^{2,\ 1}(Q_T) \cap C(\overline{Q_T}))$.

## 3. Numerical solution of direct problem

In this section, we consider the direct initial boundary value problem given by equations (1)-(3). We use the finite-difference method (FDM) with a Crank-Nicholson scheme [17], which is unconditionally stable and second-order accurate in space and time. The discrete form of the direct problem is as follows. We denote $u(x_i,\ t_j) = u_{i,\ j}$, $a(t_j) = a_j$, $\alpha(t_j) = \alpha_j$, $\beta(t_j) = \beta_j$, $\gamma(t_j) = \gamma_j$, and $f(x_i,\ t_j) = f_{i,\ j}$, where $x_i = i\Delta x$, $t_j = j\Delta t$ for $i = \overline{0,\ M}$, $j = \overline{0,\ N}$, and $\Delta x = \frac{l}{M}$, $\Delta t = \frac{T}{N}$. Then the problem (1)-(3) can be discretised as

$$-A_{i,\ j+1}u_{i-1,\ j+1} + (1 + B_{j+1})u_{i,\ j+1} - C_{i,\ j+1}u_{i+1,\ j+1}$$

$$= A_{i,\ j}u_{i-1,\ j} + (1 - B_j)u_{i,\ j} + C_{i,\ j}u_{i+1,\ j} + \frac{\Delta t}{2}(f_{i,\ j} + f_{i,\ j+1}),$$

$$i = \overline{1,\ (M-1)},\quad j = \overline{0,\ N}, \tag{5}$$

$$u_{i,\ 0} = \varphi(x_i),\ i = \overline{0,\ M}, \tag{6}$$

$$u_{0,\ j} = \mu_1(t_j),\ u_{M,\ j} = \mu_2(t_j),\ j = \overline{0,\ N}, \tag{7}$$

where

$$A_{i,\ j} = \frac{(\Delta t)a_j}{2(\Delta x)^2} - \frac{(\Delta t)\alpha_j x_i^2}{4(\Delta x)} - \frac{(\Delta t)\beta_j x_i}{4(\Delta x)} - \frac{(\Delta t)\gamma_j}{4(\Delta x)},\ B_j = \frac{(\Delta t)a_j}{(\Delta x)^2},$$

$$C_{i,\ j} = \frac{(\Delta t)a_j}{2(\Delta x)^2} + \frac{(\Delta t)\alpha_j x_i^2}{4(\Delta x)} + \frac{(\Delta t)\beta_j x_i}{4(\Delta x)} + \frac{(\Delta t)\gamma_j}{4(\Delta x)}. \tag{8}$$

At each time step $t_{j+1}$, for $j = \overline{0,\ (N-1)}$, using the Dirichlet boundary conditions (7), the above difference equation can be reformulated as a $(M-1) \times (M-1)$ system of linear equations of the form,

$$H\mathbf{u_{j+1}} = G\mathbf{u_j} + \mathbf{k}, \tag{9}$$

where $\mathbf{u_{j+1}} = (u_{1,\ j+1},\ u_{2,\ j+1},\ \ldots,\ u_{M-2,\ j+1},\ u_{M-1,\ j+1})^{\text{tr}}$,

$$H=\begin{pmatrix}(1+B_{j+1}) & -C_{1,\ j+1} & 0 & \dots & 0 & 0 & 0\\ -A_{2,\ j+1} & (1+B_{j+1}) & -C_{2,\ j+1} & \dots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & -A_{M-2,\ j+1} & (1+B_{j+1}) & -C_{M-2,\ j+1}\\ 0 & 0 & 0 & \dots & 0 & -A_{M-1,\ j+1} & (1+B_{j+1})\end{pmatrix},$$

$$G=\begin{pmatrix}(1-B_{j}) & C_{1,\ j} & 0 & \dots & 0 & 0 & 0\\ A_{2,\ j} & (1-B_{j}) & C_{2,\ j} & \dots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & A_{M-2,\ j} & (1-B_{j}) & C_{M-2,\ j}\\ 0 & 0 & 0 & \dots & 0 & A_{M-1,\ j} & (1-B_{j})\end{pmatrix},$$

and

$$\mathbf{k}=\begin{pmatrix}\frac{\Delta t}{2}(f_{1,\ j}+f_{1,\ j+1})+A_{1,\ j}\mu_1(t_j)+C_{1,\ j+1}\mu_1(t_{j+1})\\ \frac{\Delta t}{2}(f_{2,\ j}+f_{2,\ j+1})\\ \vdots\\ \frac{\Delta t}{2}(f_{M-2,\ j}+f_{M-2,\ j+1})\\ \frac{\Delta t}{2}(f_{M-1,\ j}+f_{M-1,\ j+1})+A_{M-1,\ j}\mu_2(t_j)+C_{M-1,\ j+1}\mu_2(t_{j+1})\end{pmatrix}.$$

The numerical solutions for heat moment in equation (4) on the interval $t\in[0,\ T]$ have been calculated using the following $O((\Delta x)^2)$ finite difference approximation and trapezoidal rule formulas:

$$\nu_{n+1}(t_j)=\int_0^l x^n u(x,\ t_j)dx=\frac{l}{2N}\left(x_0^n u_{0,\ j}+x_M^n u_{M,\ j}+2\sum_{i=1}^{M-1}x_i^n u_{i,\ j}\right),\ n=0,\ 1,\ 2,\ j=\overline{0,\ N},\tag{10}$$

with the convention that $x_0^0=1$.

## 4. Initial value for the unknown coefficients

Apart from the numerical reconstruction of the unknown coefficients, it is crucial to find the initial value for the unknowns $\alpha(0)$, $\beta(0)$ and $\gamma(0)$ using the input data equations (2)-(4). It can drive a formula for the initial values by multiplying the equation (1) by $x$ or $x^2$ and integrating the resulting equation with respect to $x$ over the interval $[0,\ l]$, we arrive at the following systems:

$$\alpha(t)(\ell^2\mu_2(t)-2\nu_2(t))+\beta(t)(\ell\mu_2(t)-\nu_1(t))+\gamma(t)(\mu_2(t)-\mu_1(t))=:L_1(t),\tag{11}$$

$$\alpha(t)(\ell^3\mu_2(t)-3\nu_3(t))+\beta(t)(\ell^2\mu_2(t)-2\nu_2(t))+\gamma(t)(\ell\mu_2(t)-\nu_1(t)) =: L_2(t), \tag{12}$$

$$\alpha(t)(\ell^4\mu_2(t)-4\int_0^\ell x^3u(x,\ t)dx)+\beta(t)(\ell^3\mu_2(t)-3\nu_3(t))$$
$$+\gamma(t)(\ell^2\mu_2(t)-2\nu_2(t)) =: L_3(t), \tag{13}$$

where the right hand side functions $L_i(t)$, $i=1,\ 2,\ 3$ are as follows:

$$L_1(t)=\nu_1'(t)-a(t)(\ell u_x(\ell,\ t)-u_x(0,\ t))+\int_0^\ell f(x,\ t)dx, \tag{14}$$

$$L_2(t)=\nu_2'(t)-a(t)(\ell u_x(\ell,\ t)-\mu_2(t)+\mu_1(t))+\int_0^\ell xf(x,\ t)dx, \tag{15}$$

$$L_3(t)=\nu_3'(t)-a(t)(\ell^2u_x(\ell,\ t)-2\ell\mu_2(t)+2\nu_1(t))+\int_0^\ell x^2f(x,\ t)dx, \tag{16}$$

evaluating the system of equations (11)-(13) at the initial time we have the following systems:

$$\alpha(0)(\ell^2\mu_2(0)-2\nu_2(0))+\beta(0)(\ell\mu_2(0)-\nu_1(0))+\gamma(0)(\mu_2(0)-\mu_1(0)) =: L_1(0), \tag{17}$$

$$\alpha(0)(\ell^3\mu_2(0)-3\nu_3(0))+\beta(0)(\ell^2\mu_2(0)-2\nu_2(0))+\gamma(0)(\ell\mu_2(0)-\nu_1(0)) =: L_2(0), \tag{18}$$

$$\alpha(0)(\ell^4\mu_2(0)-4\int_0^\ell x^3\phi(x)dx)+\beta(0)(\ell^3\mu_2(0)-3\nu_3(0))$$
$$+\gamma(0)(\ell^2\mu_2(0)-2\nu_2(0)) =: L_3(0). \tag{19}$$

Solving the above system for the unknown $\alpha(0)$, $\beta(0)$ and $\gamma(0)$ we obtain the following closed forms:

$$\begin{aligned}\alpha(0)= & -\left((\mu_2-\mu_1)\left(l^2\mu_2-2\nu_2\right)-(l\mu_2-\nu_1)^2\right)\left(L_1\left(l^2\mu_2-2\nu_2\right)-(\mu_2-\mu_1)L_3\right)\\ & -\left((\mu_2-\mu_1)\left(l^3\mu_2-3\nu_3\right)-(l\mu_2-\nu_1)\left(l^2\mu_2-2\nu_2\right)\right)\left(L_1\left(l\mu_2-\nu_1\right)-(\mu_2-\mu_1)L_2\right)\\ & \Big/\left((\mu_2-\mu_1)\left(l^2\mu_2-2\nu_2\right)-(l\mu_2-\nu_1)^2\right)\left((\mu_2-\mu_1)\left(l^4\mu_2-4\mathrm{D}\right)-\left(l^2\mu_2-2\nu_2\right)^2\right)\\ & -\left((\mu_2-\mu_1)\left(l^3\mu_2-3\nu_3\right)-(l\mu_2-\nu_1)\left(l^2\mu_2-2\nu_2\right)\right)^2,\end{aligned} \tag{20}$$

$$\beta(0) = -\Big( -4\mathrm{D}\mu_2 l L_1 - 4\mathrm{D}\mu_1 L_2 + 4\mathrm{D}\mu_2 L_2 + 4\mathrm{D}L_1 \nu_1 - \mu_2 l^4 L_1 \nu_1 + \mu_1\mu_2 l^4 L_2$$

$$+ 2\mu_2 l^3 L_1 \nu_2 - \mu_1\mu_2 l^3 L_3 + \mu_2 l^2 L_3 \nu_1 - 4\mu_2 l^2 L_2 \nu_2 + 3\mu_2 l^2 L_1 \nu_3 + 2\mu_2 l L_3 \nu_2 + 3\mu_1 L_3 \nu_3$$

$$- 3\mu_2 L_3 \nu_3 + 4L_2 \nu_2^2 - 2L_3 \nu_1 \nu_2 - 6L_1 \nu_2 \nu_3$$

$$\Big/ 4\mathrm{D}\mu_1\mu_2 l^2 - 8\mathrm{D}\mu_2 l \nu_1 - 8\mathrm{D}\mu_1 \nu_2 + 8\mathrm{D}\mu_2 \nu_2 + 4\mathrm{D}\nu_1^2 - \mu_2 l^4 \nu_1^2 + 2\mu_1\mu_2 l^4 \nu_2$$

$$+ 4\mu_2 l^3 \nu_1 \nu_2 - 6\mu_1\mu_2 l^3 \nu_3 - 12\mu_2 l^2 \nu_2^2 + 6\mu_2 l^2 \nu_1 \nu_3$$

$$+ 12\mu_2 l \nu_2 \nu_3 + 9\mu_1 \nu_3^2 - 9\mu_2 \nu_3^2 + 8\nu_2^3 - 12\nu_1 \nu_2 \nu_3\Big), \tag{21}$$

$$\gamma(0) = -\Big( -4\mathrm{D}\mu_2 l^2 L_1 + 4\mathrm{D}\mu_2 l L_2 - 4\mathrm{D}L_2 \nu_1 + 8\mathrm{D}L_1 \nu_2 + \mu_2 l^4 L_2 \nu_1$$

$$- 2\mu_2 l^4 L_1 \nu_2 - \mu_2 l^3 L_3 \nu_1 - 2\mu_2 l^3 L_2 \nu_2 + 6\mu_2 l^3 L_1 \nu_3 + 4\mu_2 l^2 L_3 \nu_2 - 3\mu_2 l^2 L_2 \nu_3$$

$$- 3\mu_2 l L_3 \nu_3 - 4L_3 \nu_2^2 - 9L_1 \nu_3^2 + 3L_3 \nu_1 \nu_3 + 6L_2 \nu_2 \nu_3 \Big/ \big( -4\mathrm{D}\mu_1\mu_2 l^2 + 8\mathrm{D}\mu_2 l \nu_1 + 8\mathrm{D}\mu_1 \nu_2$$

$$- 8\mathrm{D}\mu_2 \nu_2 - 4\mathrm{D}\nu_1^2 + \mu_2 l^4 \nu_1^2 - 2\mu_1\mu_2 l^4 \nu_2 - 4\mu_2 l^3 \nu_1 \nu_2 + 6\mu_1\mu_2 l^3 \nu_3 + 12\mu_2 l^2 \nu_2^2 - 6\mu_2 l^2 \nu_1 \nu_3$$

$$- 12\mu_2 l \nu_2 \nu_3 - 9\mu_1 \nu_3^2 + 9\mu_2 \nu_3^2 - 8\nu_2^3 + 12\nu_1 \nu_2 \nu_3\big)\Big), \tag{22}$$

where the symbol $\mathrm{D} := \int_0^\ell x^3 \phi(x) dx$, also all the values of the functions in the above equations are evaluated at $t = 0$.

## 5. Numerical approach to solve the inverse problem

The nonlinear inverse problem (1)-(4) can be formulated as a nonlinear minimization of the least-squares objective function

$$F(\alpha,\ \beta,\ \gamma) := \left\| \int_0^l u(x,\ t)dx - \nu_1(t) \right\|^2 + \left\| \int_0^l xu(x,\ t)dx - \nu_2(t) \right\|^2$$

$$+ \left\| \int_0^l x^2 u(x,\ t)dx - \nu_3(t) \right\|^2 + \lambda \left( \left\| \alpha(t) \right\|^2 + \left\| \beta(t) \right\|^2 + \left\| \gamma(t) \right\|^2 \right), \tag{23}$$

where $u(x,\ t)$ solves (1)-(3) for given $(\alpha,\ \beta,\ \gamma)$, respectively, $\lambda \geq 0$ is regularization parameter and the norm is the $L^2[0,\ T]$-norm. The discretizations of (23) is

$$F(\alpha,\ \beta,\ \gamma) = \sum_{j=1}^{N} \left[ \int_0^l u(x,\ t_j)dx - \nu_1(t_j) \right]^2 + \sum_{j=1}^{N} \left[ \int_0^l xu(x,\ t_j)dx - \nu_2(t_j) \right]^2$$

$$+ \sum_{j=1}^{N} \left[ \int_0^l x^2 u(x,\ t_j)dx - \nu_3(t_j) \right]^2 + \lambda \left( \sum_{j=1}^{N} \alpha_j^2 + \sum_{j=1}^{N} \beta_j^2 + \sum_{j=1}^{N} \gamma_j^2 \right). \tag{24}$$

The unregularized case, i.e., $\lambda = 0$, yields the ordinary nonlinear least-squares method which is usually producing unstable solutions when noisy data are inverted. The minimization of the objective function (24) is carried out utilizing the MATLAB subroutine *lsqnonlin* [18]. This routine attempts to find the minimum of a sum of squares by starting from the initial guesses $\alpha^{(0)}$, $\beta^{(0)}$, $\gamma^{(0)}$ computed from equations (20)-(22), respectivley. Simple bounds on the variable are allowed and the explicit calculation (analytical or numerical) of the gradient is not required to be supplied by the user. Furthermore, within *lsqnonlin*, we use the Trust Region Reflective algorithm [19], which is based on the interior-reflective Newton method. The sensitivity, robustness, and performance of Trust-region algorithms can be found in [20, 21]. More information about optimization techniques can be found in [22–25]. We have compiled this routine with the following specifications:

- Number of variables $M = N$.
- Maximum number of iterations, (MaxIter)= 400.
- Maximum number of objective function evaluations, (MaxFunEvals) $= 10^2 \times$ (number of variables).
- Termination tolerance on the function value, (TolFun) $= 10^{-15}$.
- $x$ Tolerance, ($x$Tol) $= 10^{-15}$.

## 6. Numerical results and discussion

In this section, we present a few test examples in order to test the accuracy and stability of the numerical method introduced in Section 5. The root mean square error (rmse) is used to evaluate the accuracy of the numerical results as follows:

$$rmse(\alpha(t)) = \left[ \frac{1}{N} \sum_{j=1}^{N} \left( \alpha^{Numerical}(t_j) - \alpha^{Exact}(t_j) \right)^2 \right]^{1/2}, \tag{25}$$

$$rmse(\beta(t)) = \left[ \frac{1}{N} \sum_{j=1}^{N} \left( \beta^{Numerical}(t_j) - \beta^{Exact}(t_j) \right)^2 \right]^{1/2}, \tag{26}$$

$$rmse(\gamma(t)) = \left[ \frac{1}{N} \sum_{j=1}^{N} \left( \gamma^{Numerical}(t_j) - \gamma^{Exact}(t_j) \right)^2 \right]^{1/2}. \tag{27}$$

The inverse problem (1)-(4) is solved subject to both exact and noisy heat moment measurements (4). The noisy data are numerically simulated as:

$$v_1^{\varepsilon 1}(t_j) = v_1(t_j) + \varepsilon 1_j, \quad v_2^{\varepsilon 2}(t_j) = v_2(t_j) + \varepsilon 2_j, \quad v_3^{\varepsilon 3}(t_j) = v_3(t_j) + \varepsilon 3_j, \quad j = \overline{1,\, N}, \tag{28}$$

where $\varepsilon 1_j$, $\varepsilon 2_j$, $\varepsilon 3_j$ are random variables generated from a Gaussian normal distribution with mean zero and standard deviations $\sigma_1$, $\sigma_2$, $\sigma_3$ given as follows:

$$\sigma_1 = p \times \max_{t \in [0,\, T]} |v_1(t)|, \quad \sigma_2 = p \times \max_{t \in [0,\, T]} |v_2(t)|, \quad \sigma_3 = p \times \max_{t \in [0,\, T]} |v_3(t)|, \tag{29}$$

where $p$ represents the percentage of noise. We use the MATLAB function *normrnd* to generate the random variables $\boldsymbol{\varepsilon_1} = (\varepsilon 1_j)_{j=\overline{1,\, N}}$, $\boldsymbol{\varepsilon_2} = (\varepsilon 2_j)_{j=\overline{1,\, N}}$, $\boldsymbol{\varepsilon_3} = (\varepsilon 3_j)_{j=\overline{1,\, N}}$ as follows:

$$\boldsymbol{\varepsilon_1} = normrnd(0,\, \sigma_1,\, N), \quad \boldsymbol{\varepsilon_2} = normrnd(0,\, \sigma_2,\, N), \quad \boldsymbol{\varepsilon_3} = normrnd(0,\, \sigma_3,\, N). \tag{30}$$

For simplicity, in all the numerical results presented below we take $l = T = 1$.

## 6.1 *Example 1*

In this test example, we consider the inverse problem given by (1)-(4) and the input data

$$\varphi(x) = u(x,\ 0) = 8 - x + x^2 + x^3, \ \ \mu_1(t) = u(0,\ t) = 8e^t, \ \ \mu_2(t) = u(1,\ t) = 1 + 8e^t,$$

$$a(t) = 1 + t, \ \ f(x,\ t) = 3x(-2 + t(-2 + x + x^2 + x^3))$$

$$+ e^t(6 - x + x^2 + t(-3 + x + x^2 + 2x^3)), \tag{31}$$

$$v_1(t) = 1/4 + (47e^t)/6, \quad v_2(t) = 1/5 + (47e^t)/12, \quad v_3(t) = 1/6 + (157e^t)/60. \tag{32}$$

It can be easily checked by direct substitution that the analytical solution of the inverse problem (1)-(4) with the input data (31) and (32) is given by

$$u(x,\ t) = x^3 + e^t(8 - x + x^2), \tag{33}$$

and

$$\alpha(t) = -t, \quad \beta(t) = -t, \quad \gamma(t) = -t. \tag{34}$$

First, we assess the convergence and accuracy of the FDM solver of Section 3 for solving the direct problem (1)-(3) with the input data (31), when $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ are known and given by (34). The numerical results for the interior temperature $u(x, t)$ have been obtained in excellent agreement with the analytical solution (33) and the absolute error between them is shown in Figure 1 with various mesh sizes $M = N \in \{10, 20, 40\}$. Apart from the interior temperature, another output of interest is the data (4), which analytically is given by (32). Table 1 shows that the analytical and numerical solutions for this quantity is repeated twice with various mesh sizes $M = N \in \{10, 20, 40\}$ are in very good agreement. Also, the root mean square errors *rmse* defined by

$$rmse(\nu_i(t)) = \left[\frac{1}{N}\sum_{j=1}^{N}\left(\nu_i^{Numerical}(t_j) - \nu_i^{Exact}(t_j)\right)^2\right]^{1/2}, \ i = 1, 2, 3, \tag{35}$$

indicated in Table 1, show more clearly the convergence of the numerical FDM solution to the analytical solution (32).

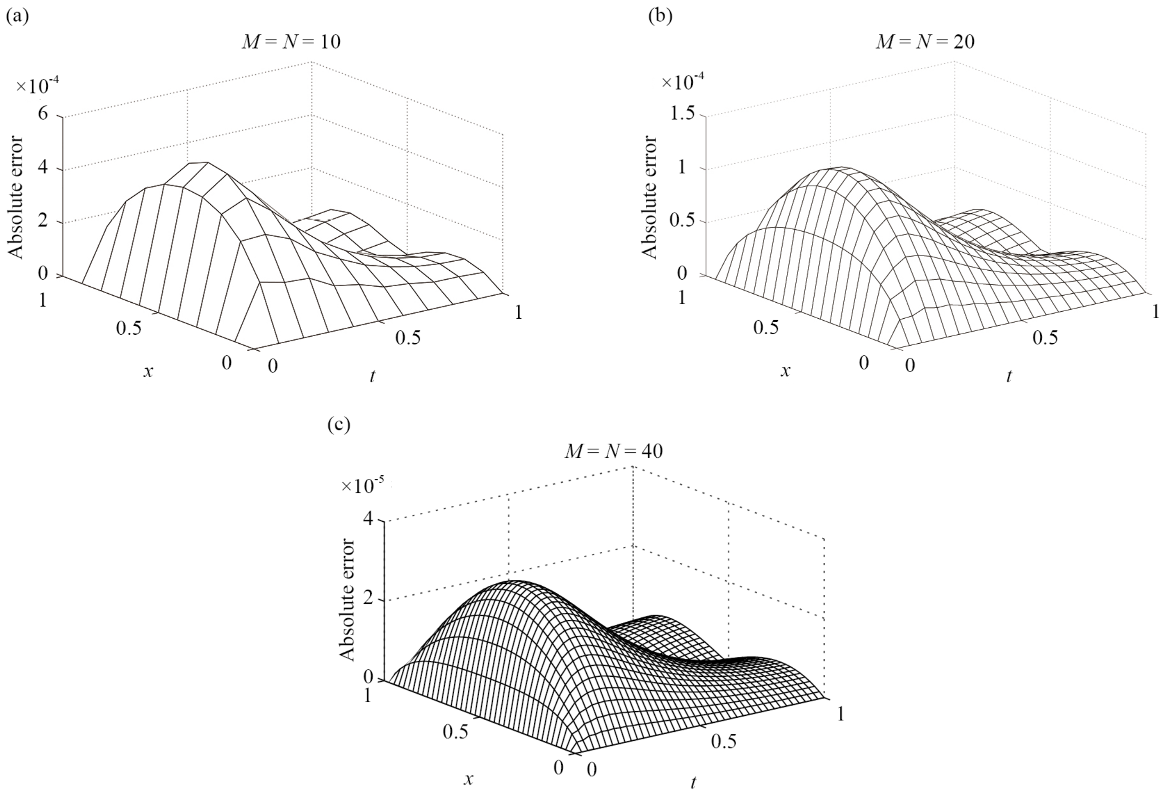



**Figure 1.** The absolute error between the analytical (33) and numerical temperatures $u(x, t)$, with $M = N \in \{10, 20, 40\}$, for direct problem

**Table 1.** The exact (32) and numerical solutions for $v_1(t)$, $v_2(t)$ and $v_3(t)$, with various $M = N \in \{10,\ 20,\ 40\}$, for direct problem. The *rmse* values (35) are also included

| $t$ | 0.1 | 0.2 | 0.3 | ... | 0.9 | | *rmse* |
|---|---|---|---|---|---|---|---|
| $v_1(t)$ | 8.9118 | 9.8225 | 10.8289 | ... | 19.5235 | $M = N = 10$ | 0.0057 |
| | 8.9083 | 9.8189 | 10.8252 | ... | 19.5185 | $M = N = 20$ | 0.0014 |
| | 8.9075 | 9.8180 | 10.8242 | ... | 19.5173 | $M = N = 40$ | 0.0003 |
| | 8.9072 | 9.8177 | 10.8238 | ... | 19.5169 | exact | 0 |
| $v_2(t)$ | 4.5330 | 4.9883 | 5.4915 | ... | 9.8388 | $M = N = 10$ | 0.0049 |
| | 4.5297 | 4.9850 | 5.4881 | ... | 9.8348 | $M = N = 20$ | 0.0012 |
| | 4.5289 | 4.9841 | 5.4872 | ... | 9.8338 | $M = N = 40$ | 0.0003 |
| | 4.5286 | 4.9838 | 5.4869 | ... | 9.8334 | exact | 0 |
| $v_3(t)$ | 3.0784 | 3.3842 | 3.7222 | ... | 6.6416 | $M = N = 10$ | 0.0306 |
| | 3.0635 | 3.3681 | 3.7046 | ... | 6.6124 | $M = N = 20$ | 0.0075 |
| | 3.0598 | 3.3640 | 3.7003 | ... | 6.6051 | $M = N = 40$ | 0.0019 |
| | 3.0585 | 3.3627 | 3.6988 | ... | 6.6026 | exact | 0 |

Next, we investigate the inverse problem. We fix $M = N = 40$ and start the investigation for determining the unknown time-dependent coefficients $\alpha(t)$, $\beta(t)$, $\gamma(t)$ and the temperature $u(x,\ t)$, when there is no noise in the input data (4), i.e. $p = 0$ in (29). The graph of the function $R_1(t)$ and $R_2(t)$ given by Theorem 2 condition **(C)** is shown in Figure 2. From this figure it can be seen that this functions never vanishes over the time interval $t \in [0,\ 1]$ and hence condition **(C)** is satisfied. Consequently, according to Theorem 2, a solution to the inverse problem given by equations (1)-(4) with data (31) and (32) is unique. The unregularized objective function $F$ (24), as a function of the number of iterations, and the analytical (34) and numerical curves for $\alpha(t)$, $\beta(t)$ and $\gamma(t)$, with no regularization parameter, i.e. $\lambda = 0$ are shown in Figure 3. It is observed that the numerical outcomes are not accurate with $rmse(\alpha)$ =24.0891, $rmse(\beta)$ =17.7692 and $rmse(\gamma)$ =1.9385. The regularized objective function $F$ (24), and the analytical (34) and numerical curves for $\alpha(t)$, $\beta(t)$ and $\gamma(t)$, with regularization parameter $\lambda > 0$ are shown in Figure 4. It is determined from all chosen $\lambda$ that $\lambda \in \{10^{-5},\ 10^{-4},\ 10^{-3}\}$ provides an acceptable and stable accurate estimate for the timewise coefficient $\alpha(t)$, $\beta(t)$ and $\gamma(t)$, yielding $rmse(\alpha) \in \{0.1506,\ 0.1347,\ 0.2123\}$, $rmse(\beta) \in \{0.0464,\ 0.0452,\ 0.1224\}$ and $rmse(\gamma) \in \{0.1557,\ 0.0897,\ 0.0941\}$.

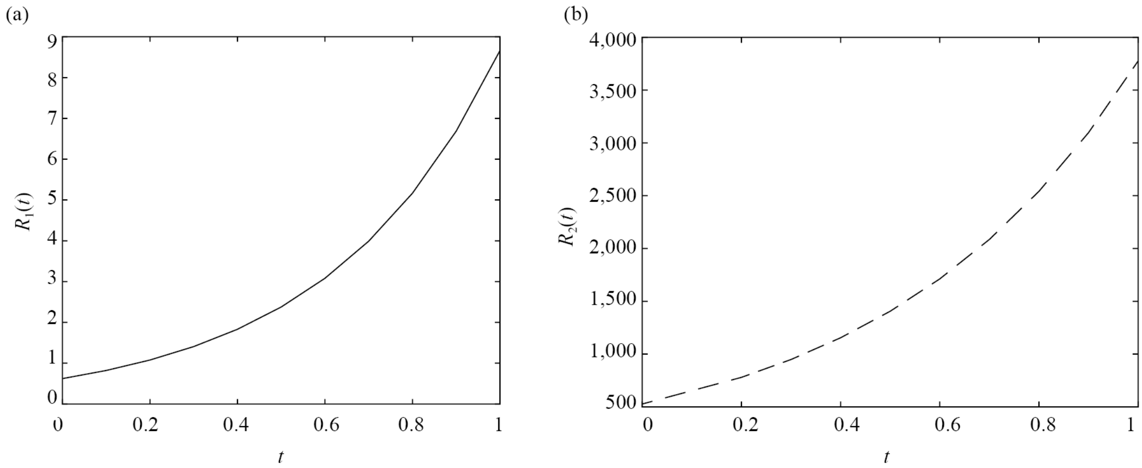



**Figure 2.** The graph of the functions: (a) $R_1(t)$ and (b) $R_2(t)$, as a function of time $t$, given by Theorem 2 condition **(C)**, for Examples 1 and 2

Now, we associate $p = 0.1\%$ noise with the simulated data (28), as in equation (29). It is significant to note that the inverse problem is not well posed therefore, we anticipate that the cost function needs to be regularized for the sake of stability and accuracy in results. Figure 5 shows the unregularized objective function $F$ (24), and the analytical (34) and numerical curves for $\alpha(t)$, $\beta(t)$, and $\gamma(t)$, with no regularization parameter. From Figures 5(b), 5(c) and 5(d) it is clear that, as expected, for $\lambda = 0$ we obtain inaccurate and unstable solutions with $rmse(\alpha) = 49.9903$, $rmse(\beta) = 54.0524$ and $rmse(\gamma) = 40.5789$, as the problem is noise-sensitive and ill-posed. Hence, regularization process is crucial for stable solutions. For this, the regularization parameter $\lambda > 0$ is chosen to be $10^{-4}$, $10^{-3}$ (see Figures 6(b), 6(c), and 6(d) obtaining $rmse(\alpha) \in \{0.3900,\ 0.2174\}$, $rmse(\beta) \in \{0.4790,\ 0.1672\}$ and $rmse(\gamma) \in \{0.4846,\ 0.1615\}$, which provide stable and comparatively accurate approximations for the timewise functions $\alpha(t)$, $\beta(t)$ and $\gamma(t)$. Other details about the *rmse* values (25)-(27), the number of iterations, and the computational time, with and without regularization are listed in Table 2. Eventually, from Figures 4, 6, and Table 2 (for Example 1), it is observed that the MATLAB simulation results are fairly stable and accurate.

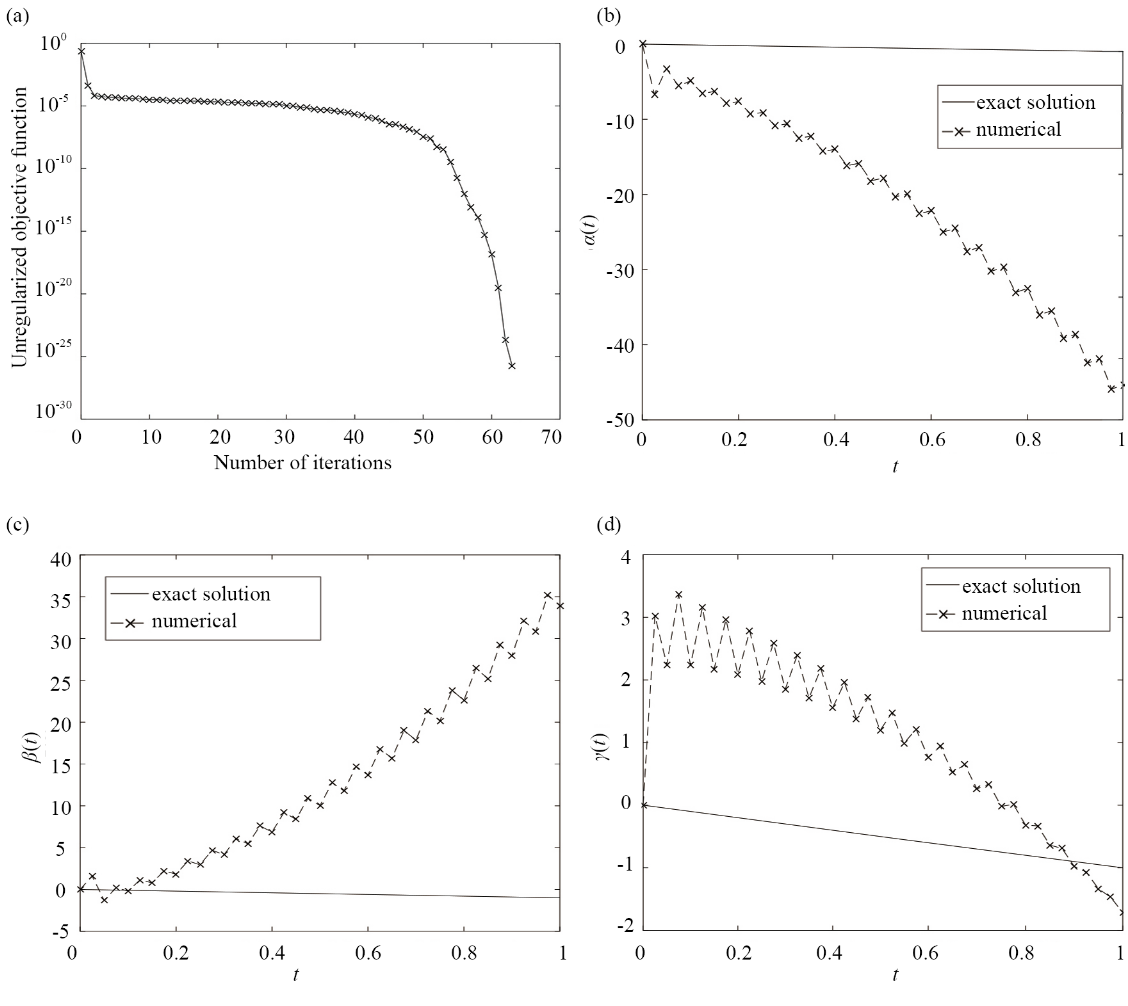



**Figure 3.** (a) The unregularized objective function $F$ (24), as a function of the number of iterations, and the analytical (34) and numerical curves for: (b) $\alpha(t)$, (c) $\beta(t)$ and (d) $\gamma(t)$, with no noise, i.e. $p = 0$ and no regularization parameter, i.e. $\lambda = 0$, for Example 1

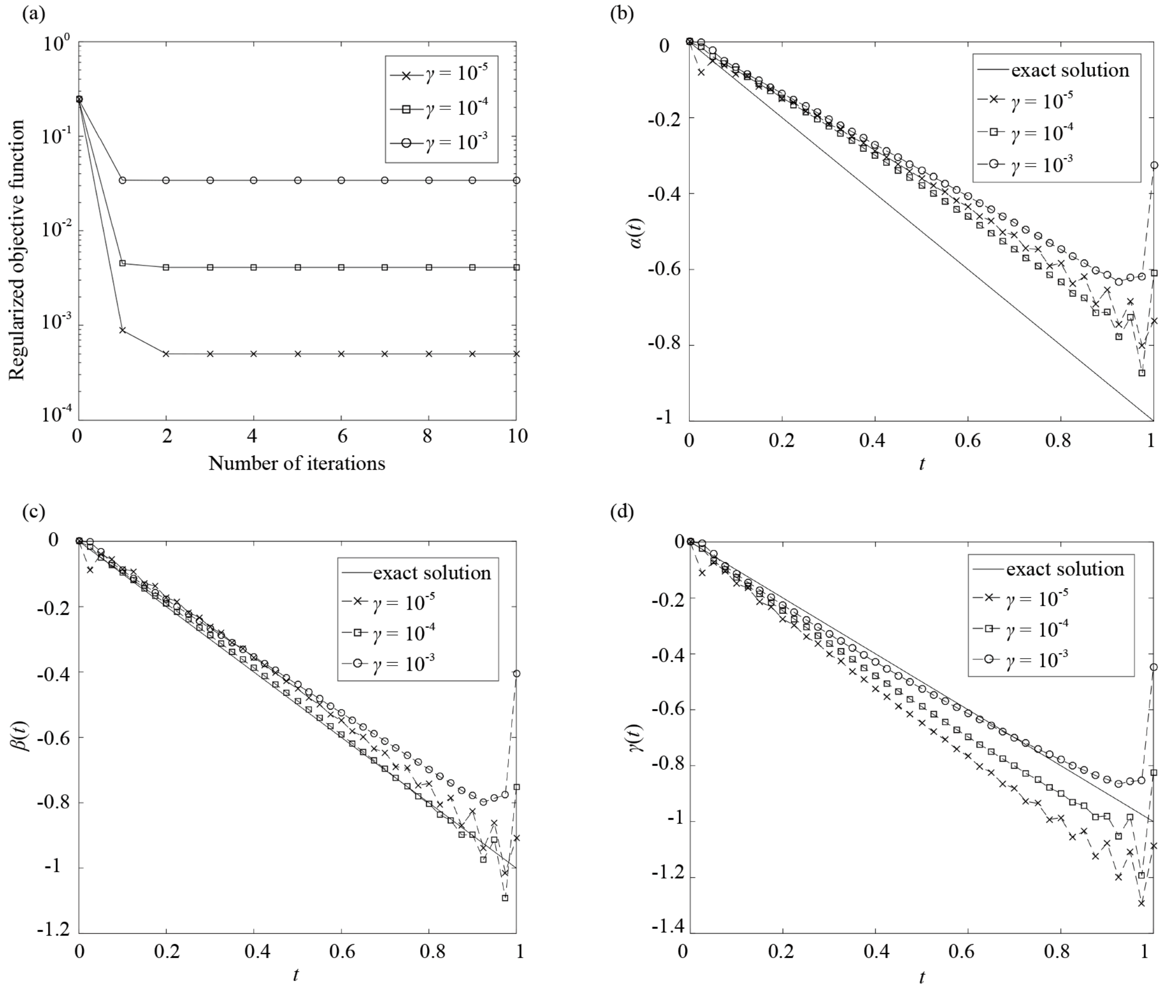
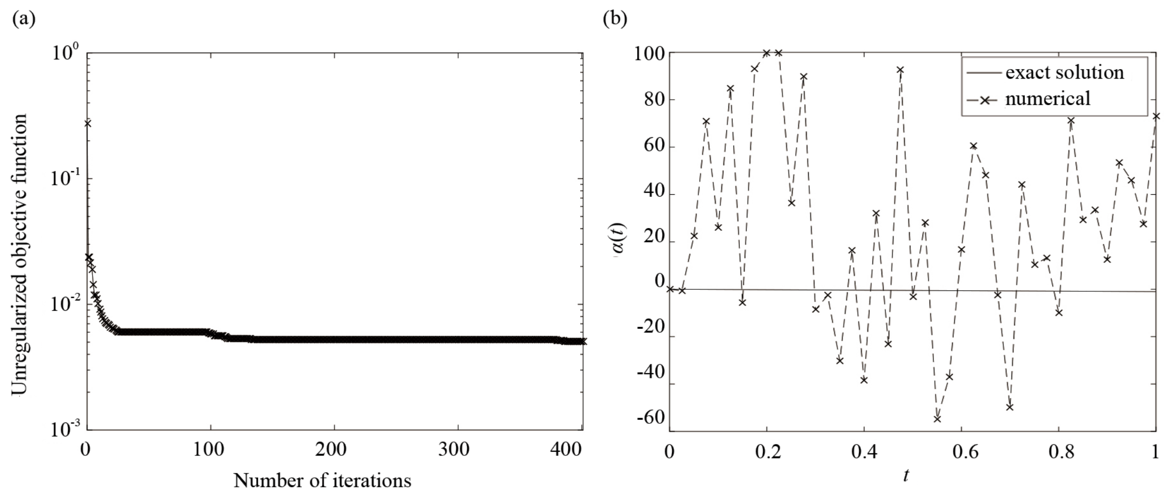



**Figure 4.** (a) The regularized objective function $F$ (24), and the analytical (34) and numerical curves for: (b) $\alpha(t)$, (c) $\beta(t)$ and (d) $\gamma(t)$, with no noise, i.e. $p = 0$ and with regularization parameter $\lambda \in \{10^{-5},\ 10^{-4},\ 10^{-3}\}$, for Example 1



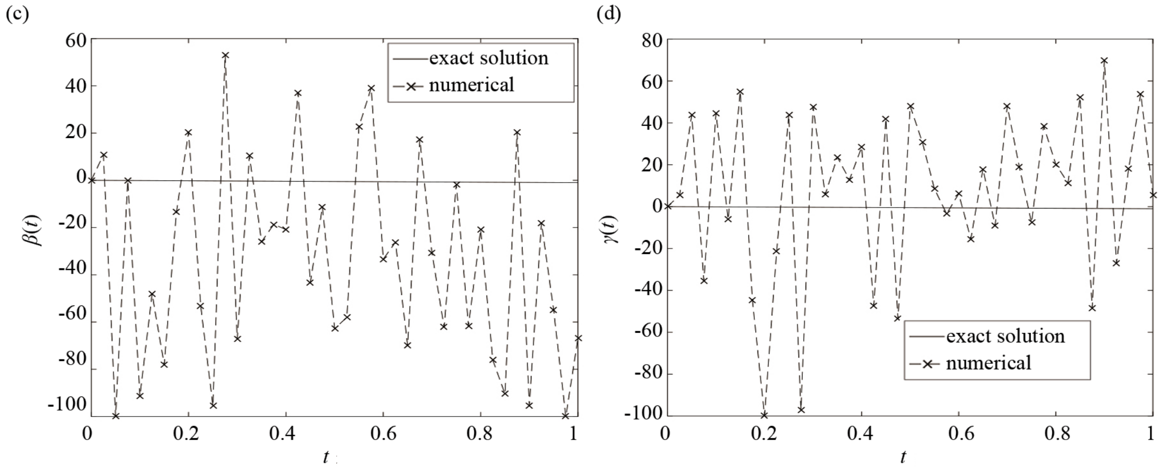



**Figure 5.** (a) The unregularized objective function $F$ (24), and the analytical (34) and numerical curves for: (b) $\alpha(t)$, (c) $\beta(t)$ and (d) $\gamma(t)$, with $p = 0.1\%$ noise and with no regularization parameter, i.e. $\lambda = 0$, for Example 1

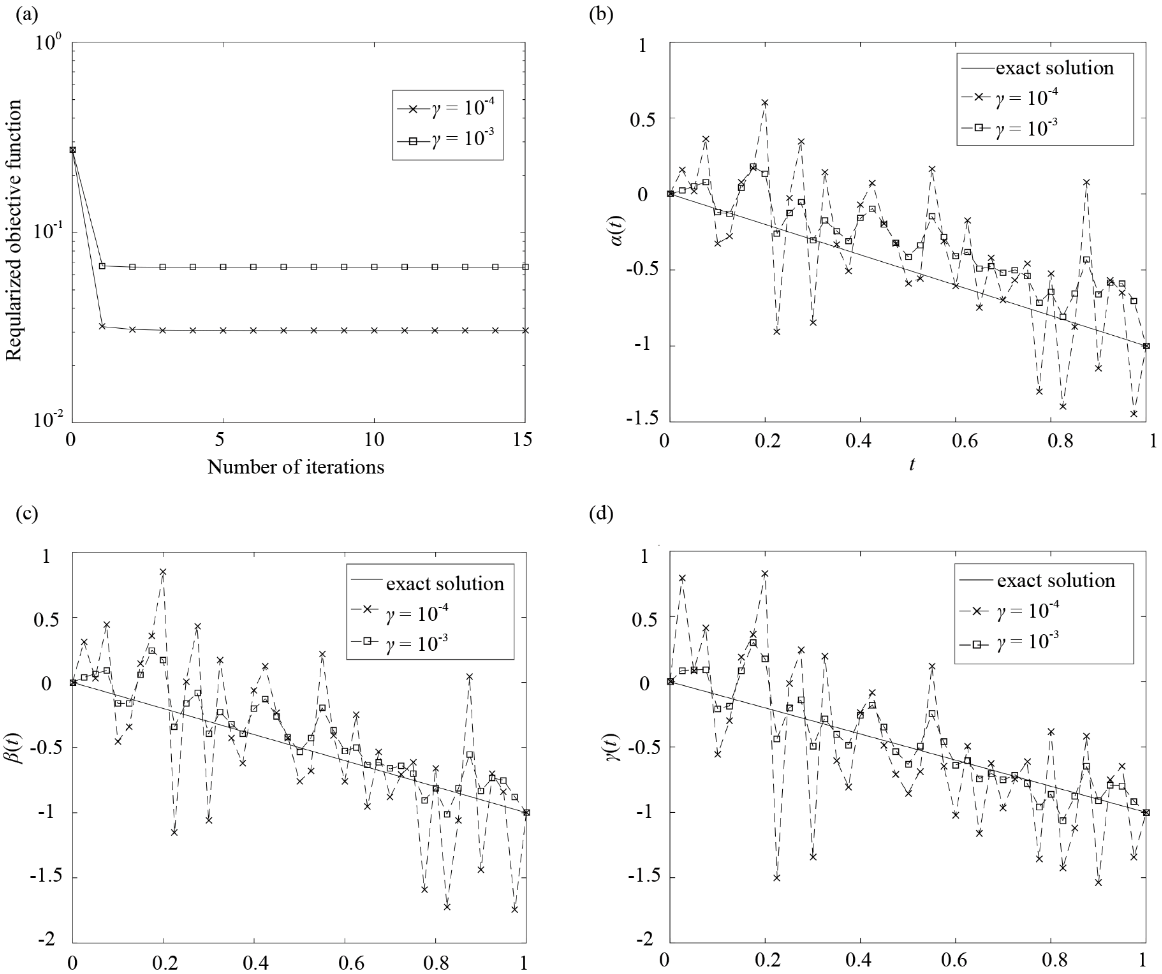



**Figure 6.** (a) The regularized objective function $F$ (24), and the analytical (34) and numerical curves for: (b) $\alpha(t)$, (c) $\beta(t)$ and (d) $\gamma(t)$, with $p = 0.1\%$ noise and with regularization parameter $\lambda \in \{10^{-4},\ 10^{-3}\}$, for Example 1

**Table 2.** The *rmse* values in (25)-(27), number of iterations and computational time for $p \in \{0, 0.1\%\}$ noise, with and without regularization parameter $\lambda$, for Examples 1 and 2

| Example 1 | | | | | | |
|---|---|---|---|---|---|---|
| Noise | Regularization | $rmse(\alpha)$ | $rmse(\beta)$ | $rmse(\gamma)$ | Iter | Time |
| $p = 0$ | $\lambda = 0$ | 24.0891 | 17.7692 | 1.9385 | 63 | 32 mins |
| | $\lambda = 10^{-5}$ | 0.1506 | 0.0464 | 0.1557 | 10 | 5 mins |
| | $\lambda = 10^{-4}$ | 0.1347 | 0.0452 | 0.0897 | 10 | 5 mins |
| | $\lambda = 10^{-3}$ | 0.2123 | 0.1224 | 0.0941 | 10 | 5 mins |
| $p = 0.1\%$ | $\lambda = 0$ | 49.9903 | 54.0524 | 40.5789 | 401 | 4 hours |
| | $\lambda = 10^{-5}$ | 1.0169 | 1.2915 | 1.1515 | 15 | 8 mins |
| | $\lambda = 10^{-4}$ | 0.3900 | 0.4790 | 0.4846 | 15 | 8 mins |
| | $\lambda = 10^{-3}$ | 0.2174 | 0.1672 | 0.1615 | 15 | 8 mins |
| | $\lambda = 10^{-2}$ | 0.4333 | 0.3881 | 0.3516 | 15 | 8 mins |
| | $\lambda = 10^{-1}$ | 0.5642 | 0.5566 | 0.5493 | 15 | 8 mins |
| Example 2 | | | | | | |
| Noise | Regularization | $rmse(\alpha)$ | $rmse(\beta)$ | $rmse(\gamma)$ | Iter | Time |
| $p = 0$ | $\lambda = 0$ | 23.8992 | 17.5273 | 1.8597 | 52 | 26 mins |
| | $\lambda = 10^{-5}$ | 0.1874 | 0.1022 | 0.1696 | 15 | 15 mins |
| | $\lambda = 10^{-4}$ | 0.2113 | 0.1495 | 0.1630 | 15 | 15 mins |
| | $\lambda = 10^{-3}$ | 0.2860 | 0.2173 | 0.1989 | 15 | 15 mins |
| $p = 0.1\%$ | $\lambda = 0$ | 50.6334 | 53.5710 | 40.1817 | 401 | 4 hours |
| | $\lambda = 10^{-5}$ | 1.0136 | 1.3021 | 1.1643 | 20 | 10 mins |
| | $\lambda = 10^{-4}$ | 0.4311 | 0.5061 | 0.5263 | 20 | 10 mins |
| | $\lambda = 10^{-3}$ | 0.2880 | 0.2310 | 0.2103 | 20 | 10 mins |
| | $\lambda = 10^{-2}$ | 0.4744 | 0.4339 | 0.3999 | 20 | 10 mins |
| | $\lambda = 10^{-1}$ | 0.5904 | 0.5834 | 0.5762 | 20 | 10 mins |

## 6.2 *Example 2*

The previous example has recovered the smooth timewise coefficients $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ given by equation (34). In this example, we assess the numerical scheme for reconstructing the non-smooth coefficients given by

$$\alpha(t) = -\cos^2(\pi t), \quad \beta(t) = -\cos^2(\pi t), \quad \gamma(t) = -\cos^2(\pi t). \tag{36}$$

The exact solution for $u(x, t)$ is given by (33) and the input data is kept the same as it was used in Example 1, and the source function $f(x, t)$ for this example is given as:

$$f(x, t) = -2(1+t)(e^t + 3x) + e^t(8 - x + x^2)$$
$$+ (1 + x + x^2)(3x^2 + e^t(-1 + 2x))\cos^2(\pi t). \tag{37}$$

With the above data, the assumptions of Theorems 1 and 2 can also be verified to affirm that we have a unique solution to the problem. The initial approximation for $\underline{\alpha}$, $\underline{\beta}$, and $\underline{\gamma}$ are supposed to be

$$\alpha^0(t_j) = \alpha(0) = -1, \;\; \beta^0(t_j) = \beta(0) = -1, \;\; \gamma^0(t_j) = \gamma(0) = -1, \;\; j = \overline{0,\, N}. \tag{38}$$

Note that the values of $\alpha(0)$, $\beta(0)$ and $\gamma(0)$ are available from (20)-(22). First, we use $M = N = 40$ and $p = 0$ to retrieve the unknown coefficients $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ for the exact input data. In Figure 7(a), the unregularized function $F$ (24), i.e. $\lambda = 0$ has been plotted versus a number of iterations. It can be noted that the tolerance of $O(10^{-25})$ is achieved. The analytical (36) and numerical solutions to the functions $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ are portrayed in Figures 7(b), 7(c) and 7(d). It is observed that the numerical outcomes are inaccurate with $rmse(\alpha) = 23.8992$, $rmse(\beta) = 17.5273$ and $rmse(\gamma) = 1.8597$. The regularized objective function $F$ (24), and the analytical (36) and numerical curves for $\alpha(t)$, $\beta(t)$ and $\gamma(t)$, with regularization parameter $\lambda > 0$ are showen in Figure 8. It is observed that from all chosen $\lambda$ that $\lambda \in \{10^{-5},\, 10^{-4},\, 10^{-3}\}$ provides an acceptable and stable estimate for the timewise coefficients $\alpha(t)$, $\beta(t)$ and $\gamma(t)$, obtaining $rmse(\alpha) \in \{0.1874,\, 0.2113,\, 0.2860\}$, $rmse(\beta) \in \{0.1022,\, 0.1495,\, 0.2173\}$ and $rmse(\gamma) \in \{0.1696,\, 0.1630,\, 0.1989\}$.

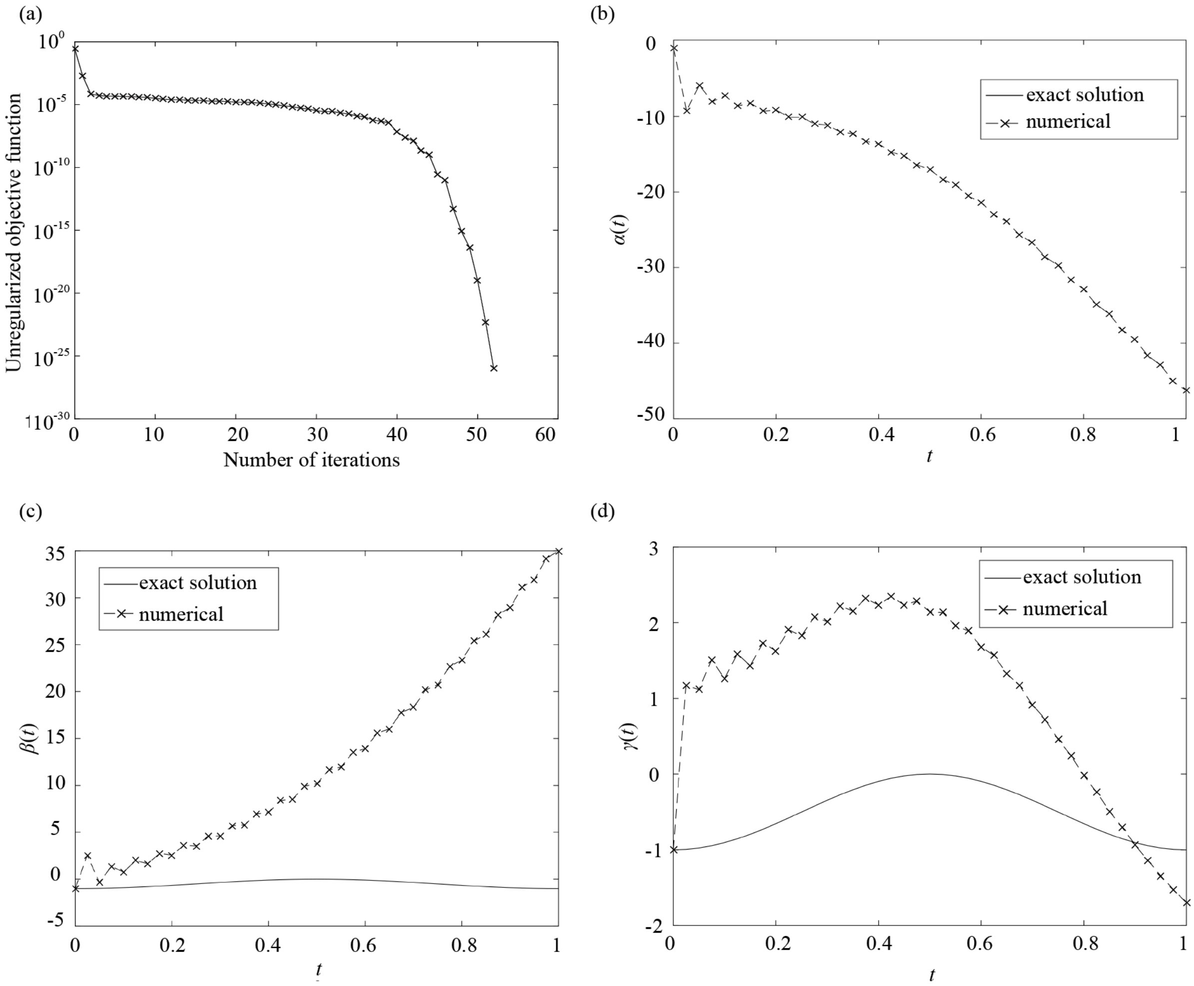



**Figure 7.** (a) The unregularized objective function $F$ (24), and the analytical (36) and numerical curves for: (b) $\alpha(t)$, (c) $\beta(t)$ and (d) $\gamma(t)$, with no noise, i.e. $p = 0$ and no regularization parameter, i.e. $\lambda = 0$, for Example 2

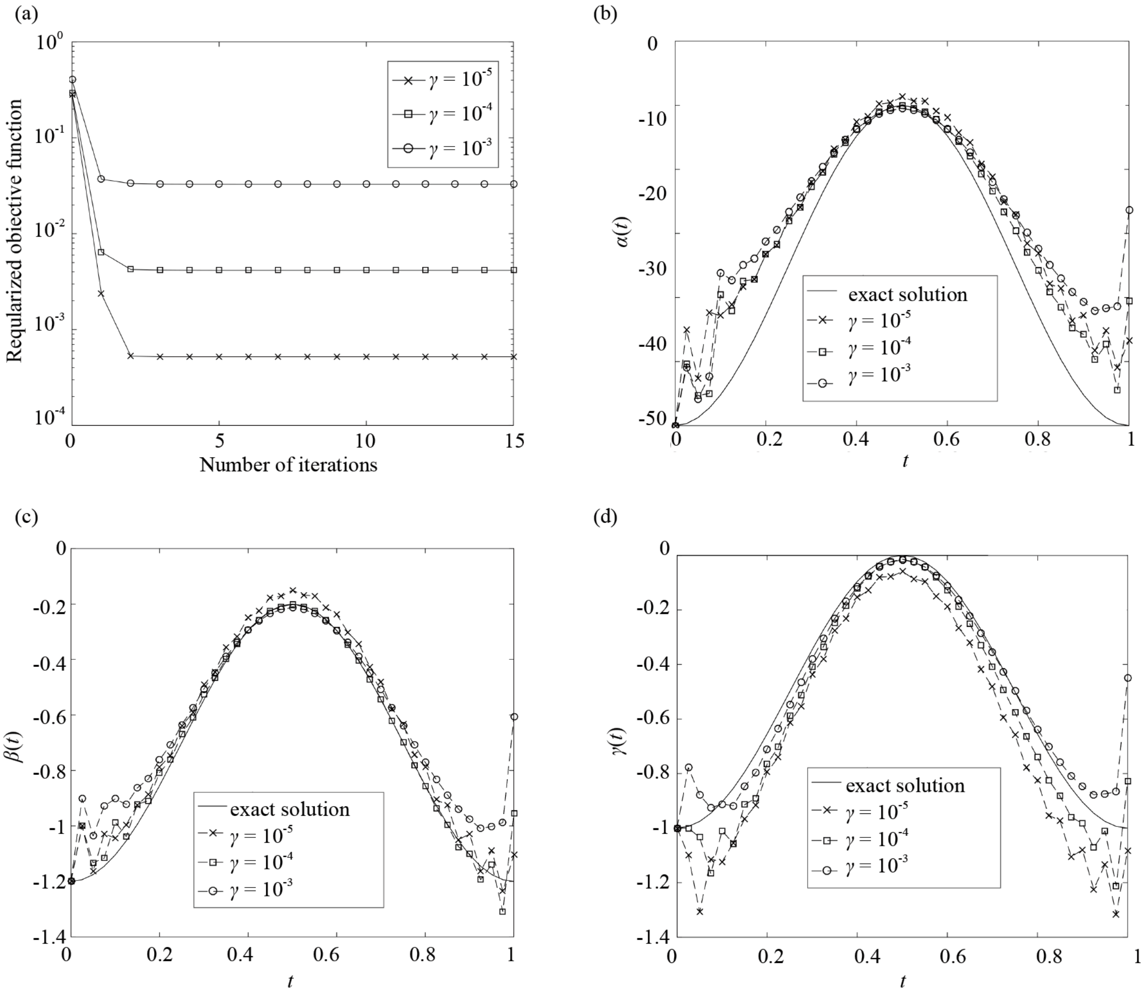



**Figure 8.** (a) The regularized objective function $F$ (24), and the analytical (36) and numerical curves for: (b) $\alpha(t)$, (c) $\beta(t)$ and (d) $\gamma(t)$, with no noise, i.e. $p = 0$ and with regularization parameter $\lambda \in \{10^{-5},\ 10^{-4},\ 10^{-3}\}$, for Example 2

Next, we discuss the case of noisy input data with $p = 0.1\%$ by including Gaussian random noise in $v_1(t)$, $v_2(t)$ and $v_3(t)$. As anticipated earlier, without regularization, i.e., $\lambda = 0$, the least-squares minimization returns an unstable solution. Hence, for restoration of the stability, we need to utilize the Tikhonov regularization approach by entering the stabilizer term in $F$ (24). The analytical (36) and approximate solutions for $\alpha(t)$, $\beta(t)$ and $\gamma(t)$, with and without regularization are depicted in Figures 9 and 10. From Figures 9(b), 9(c) and 9(d), it can be noticed that the unstable (highly oscillatory) and inaccurate results are obtained for $\alpha(t)$, $\beta(t)$ and $\gamma(t)$, if no regularization is installed with $rmse(\alpha) = 50.6334$, $rmse(\beta) = 53.5710$ and $rmse(\gamma) = 40.1817$. In order to stabilize the timewise coefficients $\alpha$, $\beta$ and $\gamma$, we employed regularization with $\lambda \in \{10^{-4},\ 10^{-3}\}$ (see From Figures 10(b), 10(c) and 10(d)), obtaining $rmse(\alpha) \in \{0.4311,\ 0.2880\}$, $rmse(\beta) \in \{0.5061,\ 0.2310\}$ and $rmse(\gamma) \in \{0.5263,\ 0.2103\}$. Overall, the computational outcomes produced by the FDM approach together with Tikhonov's regularization advocates that accurate and stable approximate solutions can be achieved for the ill-posed problem.

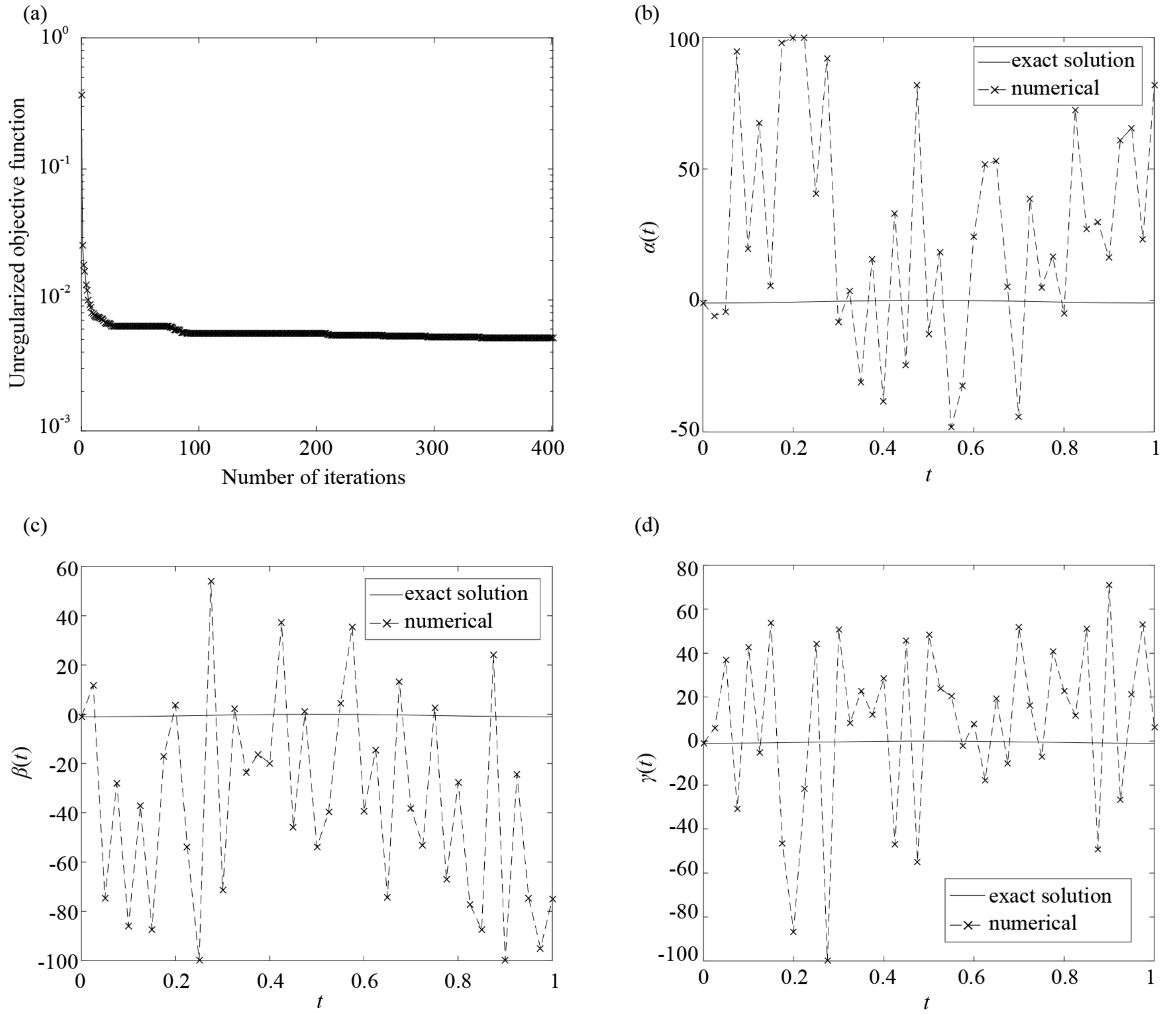
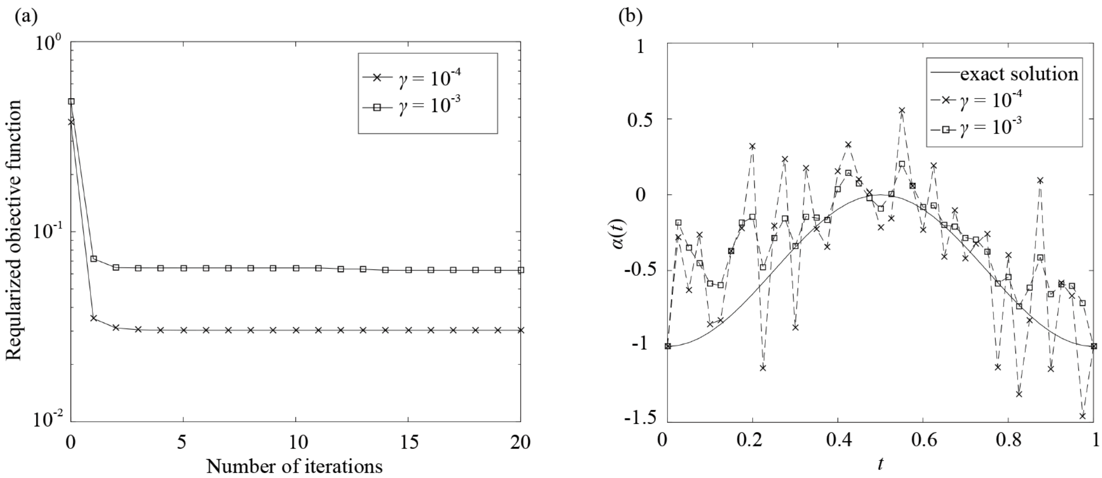



**Figure 9.** (a) The unregularized objective function $F$ (24), and the analytical (36) and numerical curves for: (b) $\alpha(t)$, (c) $\beta(t)$ and (d) $\gamma(t)$, with $p = 0.1\%$ noise and with no regularization parameter, i.e. $\lambda = 0$, for Example 2



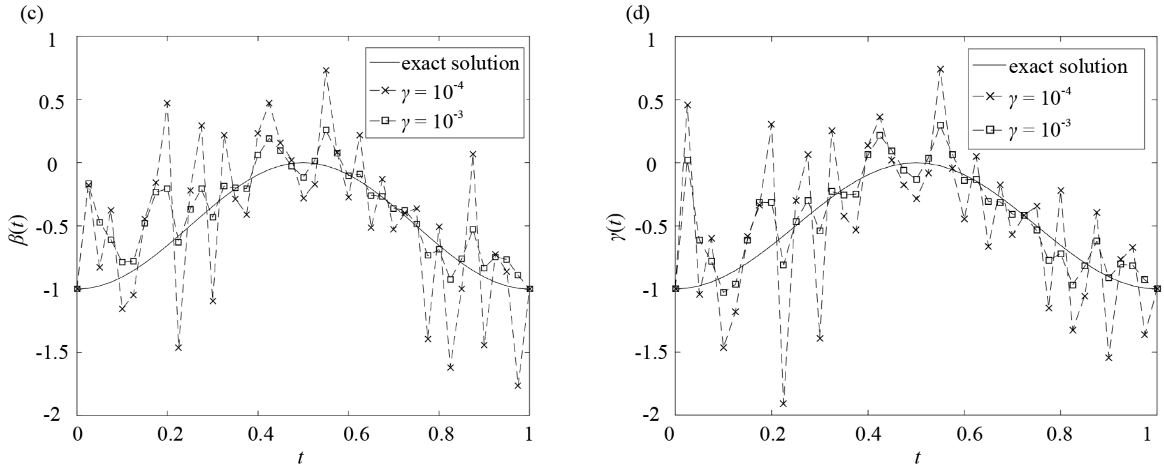



**Figure 10.** (a) The regularized objective function $F$ (24), and the analytical (36) and numerical curves for: (b) $\alpha(t)$, (c) $\beta(t)$ and (d) $\gamma(t)$, with $p = 0.1\%$ noise and with regularization parameter $\lambda \in \{10^{-4}, 10^{-3}\}$, for Example 2

# 7. Conclusions

The determination of multiple timewise coefficient $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ along with the solution function $u(x, t)$ from heat moment observations in the one-dimensional parabolic equation has been investigated. The resulting non-linear the optimization problem was solved computationally by means of the MATLAB subroutine *lsqnonlin*. Since the problem under consideration was ill-posed, therefore, the Tikhonov regularization was utilized in order to tackle the stability. The *rmse* values for various noise levels $p$ without and with regularization were compared. Throughout the paper numerical results have been compared with their analytical solutions. The main difficulty in regularization when we solve ill-posed problem is how to choose an appropriate regularization parameter $\lambda$ which compromises between accuracy and stability. However, one can use techniques such as the L-curve method [26] or Morozov's discrepancy principle [27] to find such a parameter, but in our work, we have used trial and error. The numerical results for the problem show that stable and accurate approximate results have been obtained.

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# Conflict of interest

The author declares no competing financial interest.

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