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Research Article

Complexity and Entropy of Sequence of Some Families of Graphs Created by a Triangle that Have the Same Average Degree

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Abstract: In physics, complex circuits that need multiple mathematical operations to analyze can be reduced to simpler equivalent circuits using equivalent transformations. These modifications can also be used to find the number of spanning trees in specific graph families. In the current study, we calculate the explicit formulae for the number of spanning trees of sequences of new families of graphs formed by a triangle with the same average degree using our understanding of difference equations, electrically equivalent transformations, and weighted generating function rules. We conclude by comparing our graphs' entropy to similar graphs with an average degree of four.

***Keywords*:** number of spanning trees, electrically equivalent transformations, entropy

MSC: 05C30, 05C50, 05C63

1. Introduction

There has been a lot of interest in the topic of finding closed-form formulations for the complexity (Number of spanning trees) in various graph types. Enumerating chemical isomers [1, 2], extending network analysis methods in psychological networks [3], counting Eulerian circuits [4, 5], and resolving unsolvable issues like the traveling salesman and Steiner tree problems [6] are all important applications of this study area. Additionally, examining various graph types can help find the most complicated graphs, which has applications for network resilience [7, 8].

The number of spanning trees $\tau(G)$ of a finite connected undirected graph G is an acyclic $(n-1)$-edge spanning subgraph. This number can be found in a variety of ways. Kirchhoff [9] gave the famous matrix tree theorem: if D is the diagonal matrix of the degrees of G and A denote the adjacency matrix of G, Kirchhoff matrix $L = D - A$ has all its cofactors equal to $\tau(G)$.

Another way to determine a graph's complexity is to use its Laplacian eigenvalues. Consider a graph with k vertices that is linked. The following formula was obtained by Kelmans and Chelnoknov [10]:

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$$\tau(G) = \frac{1}{k}\prod_{i=1}^{k-1}\mu_i. \tag{1}$$

where $k = \mu_1 \geq \mu_2 \geq \ldots \geq \mu_k = 0$ are the eigenvalues of the Kirchhoff matrix L.

Degenerating the graph through successive elimination of contraction of its edges represent the core of another way to compute the complexity of a graph [11–13]. If $G = (V, E)$ is a multigraph with $e \in E(G)$, then $G - e$ denotes the graph obtained by deleting an arbitrary edge e and $G.e$ is the graph obtained from G by contracting the degree until its endpoints are a single vertex. The formula for computing the number of spanning trees of a multigraph G is given by:

$$\tau(G) = \tau(G - e) + \tau(G.e). \tag{2}$$

This formula is beautiful but not practically useful (grows exponentially with the size of the graph-may be as many as $2^{|E(G)|}$ terms. For a summary of further techniques and methods for calculating number of spanning trees of graphs, see [14–17].

1.1 *Materials and methods*

An edge-weighted graph, whose weights represent the conductance of the corresponding edges, may be thought of as an electrical network, which is why Kirchhoff was motivated to research electrical networks. The quotient of the (weighted) number of spanning trees and the (weighted) number of so-called thickets-that is, spanning forests with exactly two components and the characteristic that each component contains precisely one of the vertices $u,\ v$ can be used to express the effect conductance between two vertices $u,\ v$ [18–21]. The impact of a few basic modifications on the quantity of spanning trees is listed below. The weighted number of spanning trees G is indicated by $\tau(G)$ and let G be an edge weighted graph and $G^{'}$ be the associated electrically equivalent graph.

• Parallel edges: When two parallel edges in G, each with conductances u and v, are merged into a single edge in $G^{'}$ with a conductance of $u+v$, the count of spanning trees, $\tau(G^{'})$, remains unchanged compared to $\tau(G)$.

• Serial edges: If two serial edges in G, with conductances u and v, are combined into a single edge in $G^{'}$ with a conductance of $uv/(u+v)$, then $\tau(G^{'})$ can be calculated as $(1/(u+v)$ multiplied by $\tau(G)$.

• Δ-Y Transformation: When a triangle in G, with conductances $u,\ v$ and w is transformed into an electrically equivalent star graph in $G^{'}$ with conductances $x = (uv+vw+wu)/u,\ y = (uv+vw+wu)/v$, and $z = (uv+vw+wu)/w$, the count of spanning trees in $G^{'},\ \tau(G^{'})$, can be determined as $(uv+vw+wu)^2/uvw$ multiplied by $\tau(G)$.

• Y-Δ Transformation: If a star graph in G, with conductances $u,\ v$ and w, is converted into an electrically equivalent triangle in $G^{'}$ with conductances $x = vw/(u+v+w),\ y = uw/(u+v+w)$ and $z = uv/(u+v+w)$, then $\tau(G^{'})$ is given by $1/(u+v+w)$ multiplied by $\tau(G)$.

2. Results

In mathematics, it is common to derive new structures from existing ones. This principle extends to graphs, where numerous new graphs can be generated from a given set. In this study, we determine the complexity for four novel types of graphs of the same average degree we named it $\mathscr{M}_n,\ \mathscr{N}_n,\ \mathscr{Q}_n$ and $\mathscr{R}_n$ respectively.

2.1 *Number of spanning trees in the sequences of the graph $\mathscr{M}_n$*

The Figure 1 $\mathscr{M}_n,\ n = 3$ is obtained by replacing the central triangle in the graph $\mathscr{M}_2$ by a copy of $\mathscr{M}_2$. In general, the graph $\mathscr{M}_n$ is obtained by replacing the central triangle in $\mathscr{M}_{n-1}$ with $\mathscr{M}_2$. According to this construction, the number of total vertices $|V(\mathscr{M}_n)|$ and edges $|E(\mathscr{M}_n)|$ are $|V(\mathscr{M}_n)| = 9n-6$ and $|E(\mathscr{M}_n)| = 21n-18$, $n = 1, 2, \ldots$. The average degree of the graph $\mathscr{M}_n$ in the large n limit is $\frac{14}{3}$.

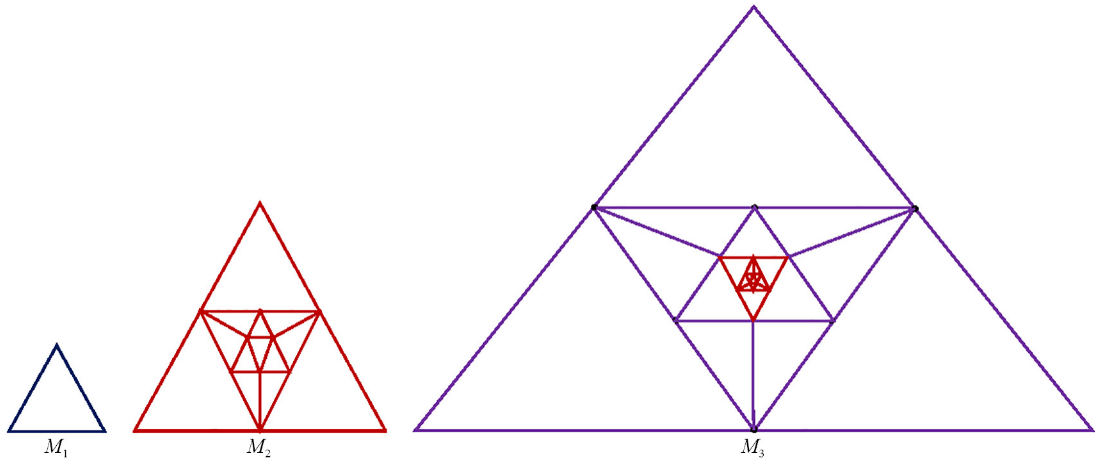
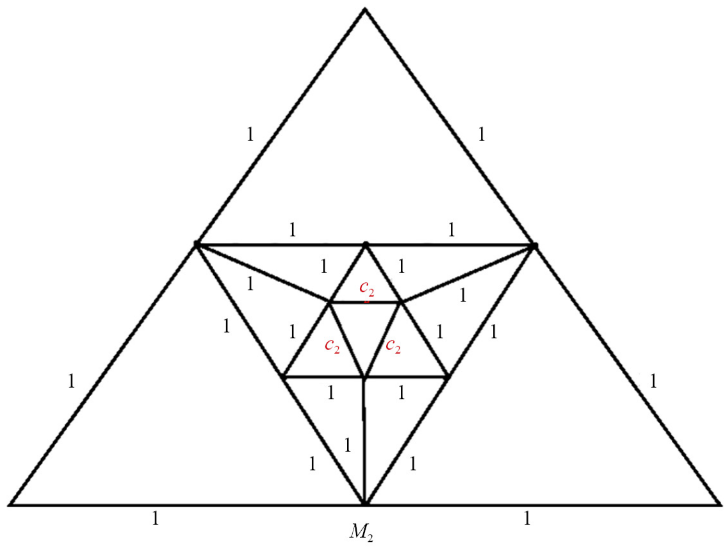


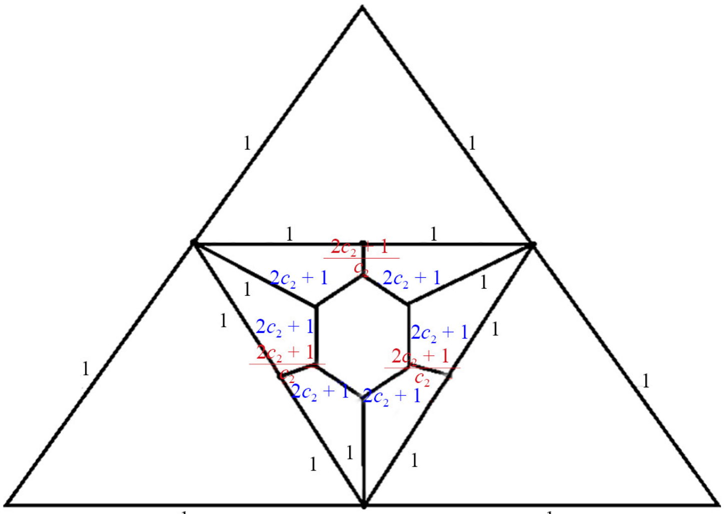
Figure 1. Some sequences of graph $\mathscr{M}_n$

Theorem 1 For $n \geq 1$, the number of spanning trees in the sequence of the graph $\mathscr{M}_n$ is given by

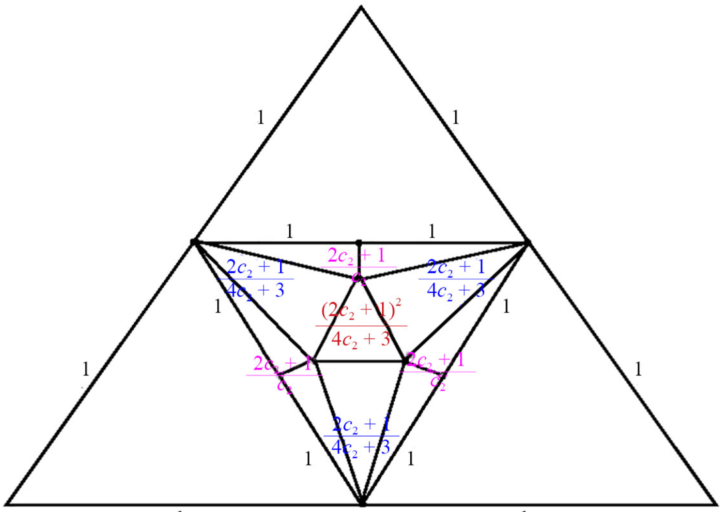
$$\frac{3\times 4^{2n-3}\left(\left(5\left(8+3\sqrt{7}\right)\right)^n\left(-23+9\sqrt{7}\right)-\left(40-15\sqrt{7}\right)^n\left(23+9\sqrt{7}\right)\right)^2\left(2+5\sqrt{7}+\left(23+10\sqrt{7}\right)\left(127+48\sqrt{7}\right)^{n-1}\right)^2}{625\left(-19+\left(44+15\sqrt{7}\right)\left(127+48\sqrt{7}\right)^{n-1}\right)^2}$$

Proof. We convert $\mathscr{M}_i$ to $\mathscr{M}_{i-1}$ via the electrically equivalent transformation. The conversion procedure from $\mathscr{M}_2$ to $\mathscr{M}_1$ is shown in Figure 2.

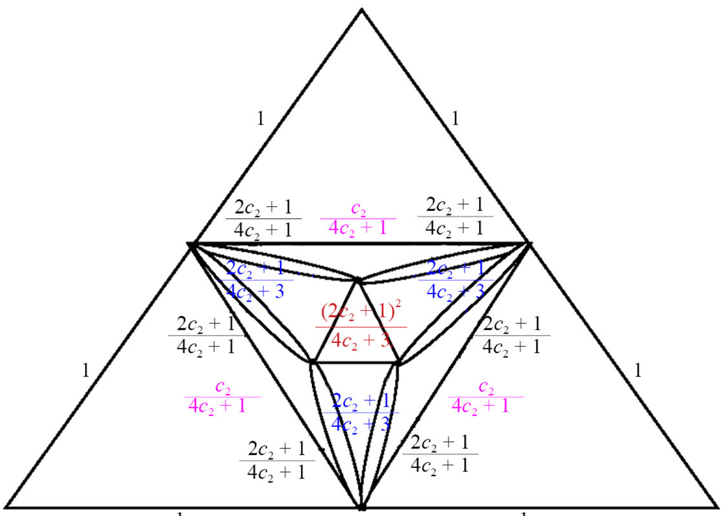



1
2c2 + 1
c2
2c2 + 1
2c2 + 1

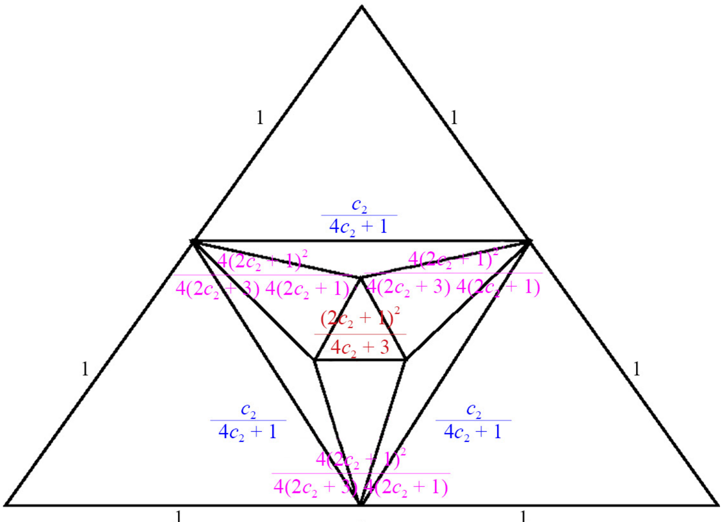
G_1


1
2c2 + 1
4c2 + 3
2c2 + 1
c2
(2c2 + 1)2
4c2 + 3

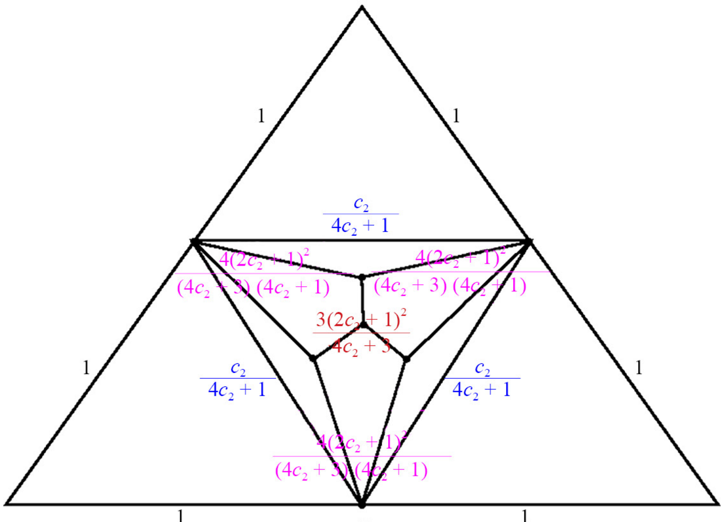
G_2


1
2c2 + 1
4c2 + 1
c2
4c2 + 1
2c2 + 1
4c2 + 3
(2c2 + 1)2
4c2 + 3

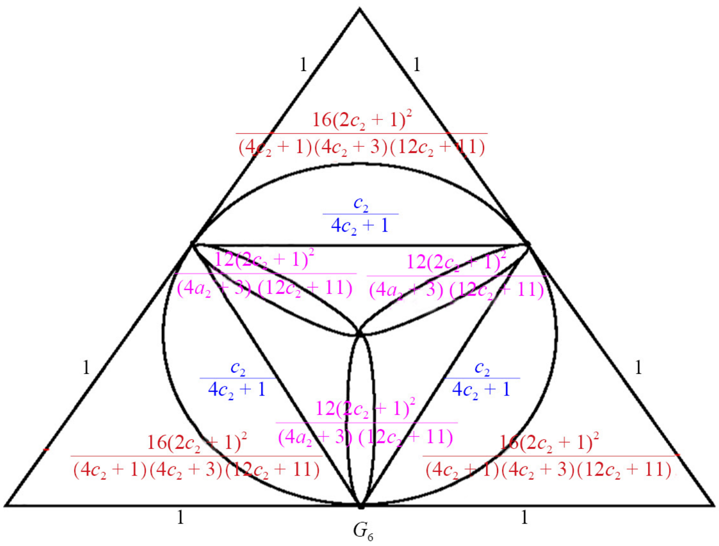
G_3


1
1
$\frac{c_2}{4c_2+1}$
$\frac{4(2c_2+1)^2}{4(2c_2+3)\,4(2c_2+1)}$
$\frac{4(2c_2+1)^2}{4(2c_2+3)\,4(2c_2+1)}$
$\frac{(2c_2+1)^2}{4c_2+3}$
1
1
$\frac{c_2}{4c_2+1}$
$\frac{c_2}{4c_2+1}$
$\frac{4(2c_2+1)^2}{4(2c_2+3)\,4(2c_2+1)}$
1
1

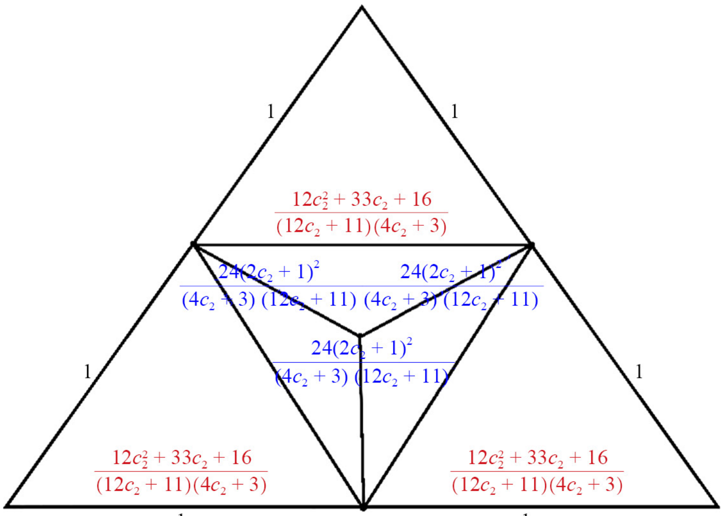
G_4


1
1
$\frac{c_2}{4c_2+1}$
$\frac{4(2c_2+1)^2}{(4c_2+3)\,(4c_2+1)}$
$\frac{4(2c_2+1)^2}{(4c_2+3)\,(4c_2+1)}$
$\frac{3(2c_2+1)^2}{4c_2+3}$
1
1
$\frac{c_2}{4c_2+1}$
$\frac{c_2}{4c_2+1}$
$\frac{4(2c_2+1)^2}{(4c_2+3)\,(4c_2+1)}$
1
1

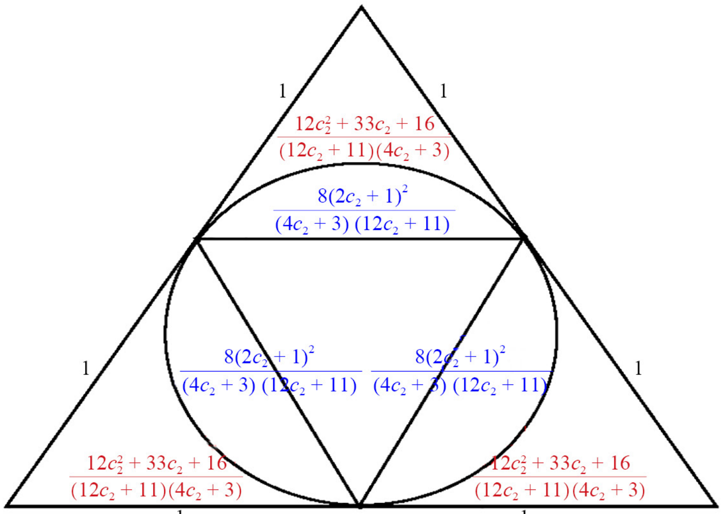
G_5


1
1
$\frac{16(2c_2+1)^2}{(4c_2+1)(4c_2+3)(12c_2+11)}$
$\frac{c_2}{4c_2+1}$
$\frac{12(2c_2+1)^2}{(4a_2+3)\,(12c_2+11)}$
$\frac{12(2c_2+1)^2}{(4a_2+3)\,(12c_2+11)}$
1
1
$\frac{c_2}{4c_2+1}$
$\frac{c_2}{4c_2+1}$
$\frac{12(2c_2+1)^2}{(4a_2+3)\,(12c_2+11)}$
$\frac{16(2c_2+1)^2}{(4c_2+1)(4c_2+3)(12c_2+11)}$
$\frac{16(2c_2+1)^2}{(4c_2+1)(4c_2+3)(12c_2+11)}$
1
1

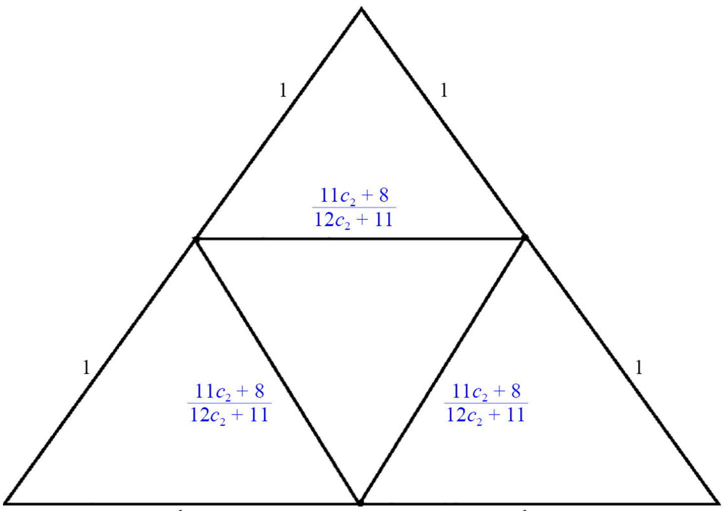
G_6


1
1
$\frac{12c_2^2 + 33c_2 + 16}{(12c_2 + 11)(4c_2 + 3)}$
$\frac{24(2c_2 + 1)^2}{(4c_2 + 3)(12c_2 + 11)}$
$\frac{24(2c_2 + 1)^2}{(4c_2 + 3)(12c_2 + 11)}$
1
$\frac{24(2c_2 + 1)^2}{(4c_2 + 3)(12c_2 + 11)}$
1
$\frac{12c_2^2 + 33c_2 + 16}{(12c_2 + 11)(4c_2 + 3)}$
$\frac{12c_2^2 + 33c_2 + 16}{(12c_2 + 11)(4c_2 + 3)}$
1
1

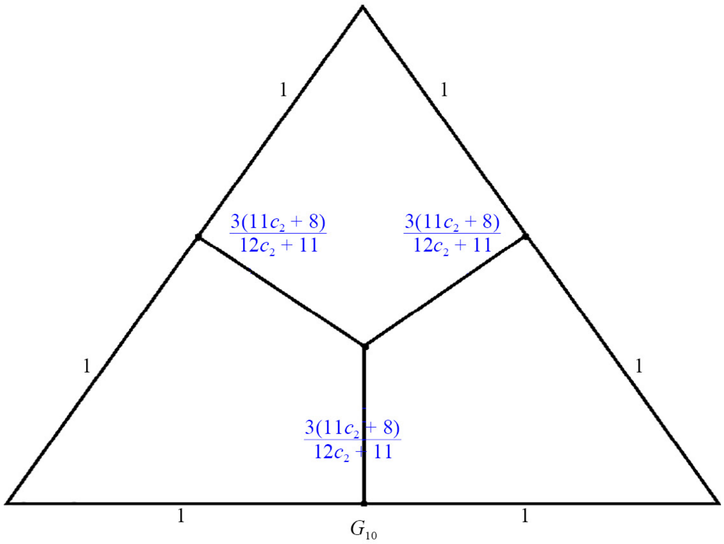
G_7


1
1
$\frac{12c_2^2 + 33c_2 + 16}{(12c_2 + 11)(4c_2 + 3)}$
$\frac{8(2c_2 + 1)^2}{(4c_2 + 3)(12c_2 + 11)}$
1
$\frac{8(2c_2 + 1)^2}{(4c_2 + 3)(12c_2 + 11)}$
$\frac{8(2c_2 + 1)^2}{(4c_2 + 3)(12c_2 + 11)}$
1
$\frac{12c_2^2 + 33c_2 + 16}{(12c_2 + 11)(4c_2 + 3)}$
$\frac{12c_2^2 + 33c_2 + 16}{(12c_2 + 11)(4c_2 + 3)}$
1
1

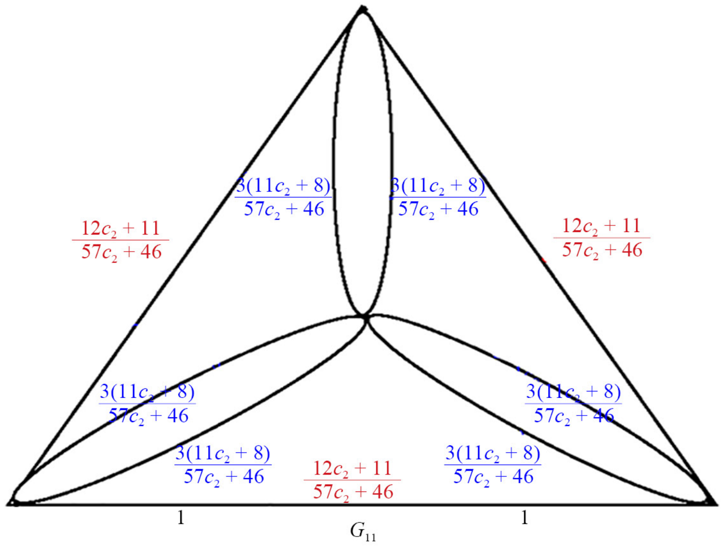
G_8


1
1
$\frac{11c_2 + 8}{12c_2 + 11}$
1
$\frac{11c_2 + 8}{12c_2 + 11}$
$\frac{11c_2 + 8}{12c_2 + 11}$
1
1
1

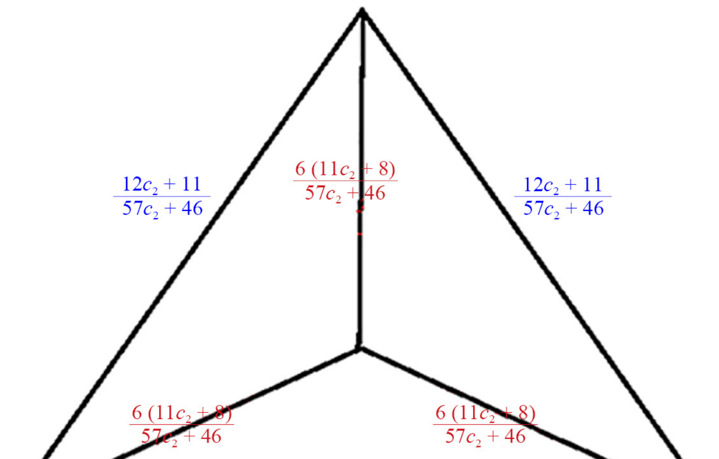
G_9


1
1
$\frac{3(11c_2+8)}{12c_2+11}$
$\frac{3(11c_2+8)}{12c_2+11}$
1
1
$\frac{3(11c_2+8)}{12c_2+11}$
1
1

G_{10}


$\frac{3(11c_2+8)}{57c_2+46}$
$\frac{3(11c_2+8)}{57c_2+46}$
$\frac{12c_2+11}{57c_2+46}$
$\frac{12c_2+11}{57c_2+46}$
$\frac{3(11c_2+8)}{57c_2+46}$
$\frac{3(11c_2+8)}{57c_2+46}$
$\frac{3(11c_2+8)}{57c_2+46}$
$\frac{12c_2+11}{57c_2+46}$
$\frac{3(11c_2+8)}{57c_2+46}$
1
1

G_{11}


$\frac{12c_2+11}{57c_2+46}$
$\frac{6(11c_2+8)}{57c_2+46}$
$\frac{12c_2+11}{57c_2+46}$
$\frac{6(11c_2+8)}{57c_2+46}$
$\frac{6(11c_2+8)}{57c_2+46}$
$\frac{12c_2+11}{57c_2+46}$

G_{12}

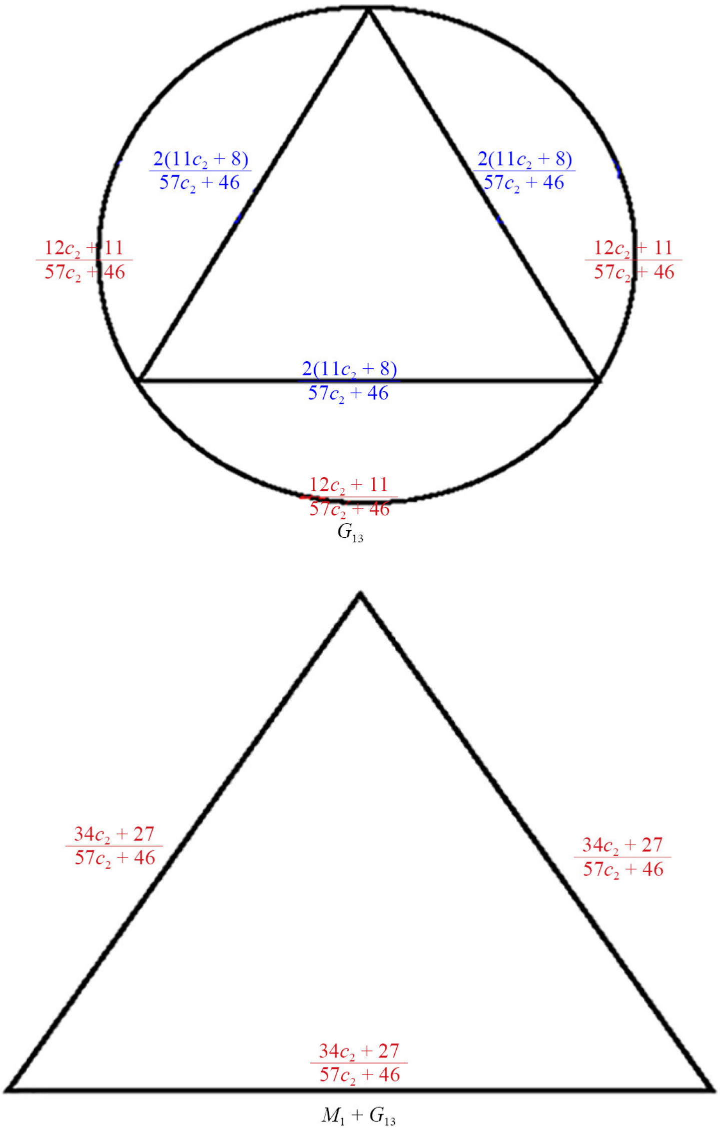


Figure 2. The transformations from $\mathscr{M}_2$ to $\mathscr{M}_1$

The following transformations result from using the attributes listed in Section 2:

$$\tau(G_1)=\left[\frac{(2c_2+1)^2}{c_2}\right]^3\tau(\mathscr{M}_2),\ \tau(G_2)=\left[\frac{1}{4c_2+3}\right]^3\tau(G_1),$$

$$\tau(G_3)=\left[\frac{c_2}{4c_2+1}\right]^3\tau(G_2),\ \tau(G_4)=\tau(G_3),\ \tau(G_5)=\frac{9\,(2c_2+1)^2}{4c_{2+}3}\tau(G_4),$$

$$\tau(G_6)=\left[\frac{(4c_2+3)(4c_2+1)}{(2c_2+1)^2(12c_2+11)}\right]^3\tau(G_5),\ \tau(G_7)=\tau(G_6),$$

$$\tau(G_8)=\frac{(4c_{2+}3)\,(12c_{2+}11)}{72\,(2c_2+1)^2}\tau(G_7),\ \tau(G_9)=\tau\,(G_8)\,,\ \tau\,(G_{10})=\frac{9\,(11c_2+8)}{(12c_2+11)}\tau\,(G_9)\,,$$

$$\tau\,(G_{11})=\left[\frac{12c_2+11}{57c_2+46}\right]^3\tau\,(G_{10})\,,\ \tau(G_{12})=\tau(G_{11}),\ \tau\,(G_{13})=\frac{57c_2+46}{18\,(11c_2+8)}\tau\,(G_{12})$$

and $\tau\,(\mathscr{M}_1)=\tau\,(G_{13})$.

When these fourteen transformations are combined, we obtain

$$\tau(\mathscr{M}_2)=16(57c_2+46)^2\tau(\mathscr{M}_1). \tag{3}$$

Further

$$\tau(\mathscr{M}_n)=\prod_{i=2}^{n}16(57c_2+46)^2\tau(\mathscr{M}_1)=3\times(16)^{n-1}c_1^2\left[\prod_{i=2}^{n}(57c_i+46)\right]^2 \tag{4}$$

where $c_{i-1}=\dfrac{34c_i+27}{57c_i+46}$, $i=2,\ 3,\ \ldots,\ n$. Its characteristic equation is $57\alpha^2+12\alpha-27=0$, which have two roots $\alpha_1=\dfrac{-2-5\sqrt{7}}{19}$ and $\alpha_2=\dfrac{-2+5\sqrt{7}}{19}$. Subtracting both roots from each side of $c_{i-1}=\dfrac{34c_i+27}{57c_i+46}$, we get

$$c_{i-1}-\frac{-2-5\sqrt{7}}{19}=\frac{34c_i+27}{57a_i+46}+\frac{2+5\sqrt{7}}{19}=5\left(8+3\sqrt{7}\right).\frac{c_i+\dfrac{2+5\sqrt{7}}{19}}{(57c_i+46)}; \tag{5}$$

$$c_{i-1}-\frac{-2+5\sqrt{7}}{19}=\frac{34c_i+27}{57a_i+46}+\frac{2-5\sqrt{7}}{19}=5\left(8-3\sqrt{7}\right).\frac{c_i+\dfrac{2-5\sqrt{7}}{19}}{(57c_i+46)}. \tag{6}$$

Let $d_i=\dfrac{c_i+\dfrac{2+5\sqrt{7}}{19}}{c_i+\dfrac{2-5\sqrt{7}}{19}}$. Then by Equations (5) and (6), we get $d_{i-1}=\left(127+48\sqrt{7}\right)d_i$ and $d_i=(127+48\sqrt{7})^{n-i}d_n$. Therefore

$$c_i=\frac{(127+48\sqrt{7})^{n-i}\left(\dfrac{-2+5\sqrt{7}}{19}\right)d_n+\dfrac{2+5\sqrt{7}}{19}}{(127+48\sqrt{7})^{n-i}\,d_n-1}.$$

Thus

$$c_1 = \frac{(127+48\sqrt{7})^{n-1}\left(23+10\sqrt{7}\right)+(2+5\sqrt{7})}{(127+48\sqrt{7})^{n-1}\left(44+15\sqrt{7}\right)\ -\ 19}. \tag{7}$$

Using the formula $c_{n-1} = \dfrac{34c_n+27}{57c_n+46}$ and designating the coefficients of $34c_n+27$ and $57c_n+46$ as a_n and b_n we have

$$57c_n+46 = a_0\,(34c_n+27)+b_0\,(57c_n+46),$$

$$57c_{n-1}+46 = \frac{a_1\,(34c_n+27)+b_1\,(57c_n+46)}{a_0\,(34c_n+27)+b_0\,(57c_n+46)},$$

$$57c_{n-2}+46 = \frac{a_2\,(34c_n+27)+b_2\,(57c_n+46)}{a_1\,(34c_n+27)+b_1\,(57c_n+46)},$$

$$\vdots$$

$$57c_{n-i}+46 = \frac{a_i\,(34c_n+27)+b_i\,(57c_n+46)}{a_{i-1}\,(34c_n+27)+b_{i-1}\,(57c_n+46)}, \tag{8}$$

$$57c_{n-(i+1)}+46 = \frac{a_{i+1}\,(34c_n+27)+b_{i+1}\,(57c_n+46)}{a_i\,(34c_n+27)+b_i(57+46)}, \tag{9}$$

$$\vdots$$

$$57c_2+46 = \frac{a_{n-2}\,(34c_n+27)+b_{n-2}\,(57c_n+46)}{a_{n-3}\,(34c_n+27)+b_{n-3}\,(57c_n+46)},$$

When Equation (8) is substituted into Equation (4), we get

$$\tau(\mathscr{M}_n) = 3\times(16)^{n-1}c_1^2[a_{n-2}(34c_n+27)+b_{n-2}(57c_n+46)]^2 \tag{10}$$

where $a_0=0,\ b_0=1$ and $a_1=57,\ b_1=46$.

By the expression $c_{n-1} = \dfrac{34c_n+27}{57c_n+46}$ and Equations (8) and (9), we have

$$a_{i+1} = 80a_i - 25a_{i-1};\ b_{i+1} = 80b_i - 25b_{i-1}. \tag{11}$$

Equation (11) has the characteristic equation $\beta^2-80\beta+25=0$. Its roots are $\beta_1 = 40+15\sqrt{7}$ and $\beta_2 = 40-15\sqrt{7}$. The general solutions of Equation (11) are $a_i = h_1\beta_1^i+h_2\beta_2^i;\ b_i = k_1\beta_1^i+d_2\beta_2^i$.

Given the initial conditions $a_0 = 0,\ b_0 = 1$ and $a_1 = 57,\ b_1 = 46$, we have

$$a_i = \frac{19\sqrt{7}}{70}(40+15\sqrt{7})^i - \frac{19\sqrt{7}}{70}(40-15\sqrt{7})^i;$$
$$b_i = \left(\frac{35+2\sqrt{7}}{70}\right)(40+15\sqrt{7})^i + \left(\frac{35-2\sqrt{7}}{70}\right)(40-15\sqrt{7})^i. \tag{12}$$

There is no electrically similar transition for $\mathscr{M}_n$ if $c_n = 1$. When Equation (12) is inserted into Equation (10), we obtain

$$\tau(\mathscr{M}_n) = 3\times(16)^{n-1}c_1^2\left[\left(\frac{103+39\sqrt{7}}{2}\right)(40+15\sqrt{7})^{n-2} + \left(\frac{103-39\sqrt{7}}{2}\right)(40-15\sqrt{7})^{n-2}\right]^2,\ n\geq 2. \tag{13}$$

Equation (13) is satisfied for $n = 1$ and $\tau(\mathscr{M}_1) = 3$. Thus, the number of spanning trees in the sequence of the graph $\mathscr{M}_n$ is determined by

$$\tau(\mathscr{M}_n) = 3\times(16)^{n-1}c_1^2\left[\left(\frac{103+39\sqrt{7}}{2}\right)(40+15\sqrt{7})^{n-2} + \left(\frac{103-39\sqrt{7}}{2}\right)(40-15\sqrt{7})^{n-2}\right]^2,\ n\geq 1. \tag{14}$$

where

$$c_1 = \frac{(127+48\sqrt{7})^{n-1}\left(23+10\sqrt{7}\right)+(2+5\sqrt{7})}{(127+48\sqrt{7})^{n-1}\left(44+15\sqrt{7}\right)\ -\ 19},\ n\geq 1. \tag{15}$$

The result is obtained by inserting Equation (15) into Equation (14). □

2.2 *Number of spanning trees in the sequences of the graph* $\mathscr{N}_n$

The graph $\mathscr{N}_n$ is defined recursively using the graphs $\mathscr{N}_1$ (triangle or K_3) and $\mathscr{N}_2$ as shown in Figure 3. The graph $\mathscr{N}_n,\ n = 3$ is obtained by replacing the central triangle in the graph $\mathscr{N}_2$ by a copy of $\mathscr{N}_2$. In general, the graph $\mathscr{N}_n$ is obtained by replacing the central triangle in $\mathscr{N}_{n-1}$ with $\mathscr{N}_2$. According to this construction, the number of total vertices $|V(\mathscr{N}_n)|$ and edges $|E(\mathscr{N}_n)|$ are $|V(\mathscr{N}_n)| = 9n - 6$ and $|E(\mathscr{N}_n)| = 21n - 18,\ n = 1,\ 2,\ \ldots$. The average degree of the graph $\mathscr{N}_n$ in the large n limit is $\frac{14}{3}$.

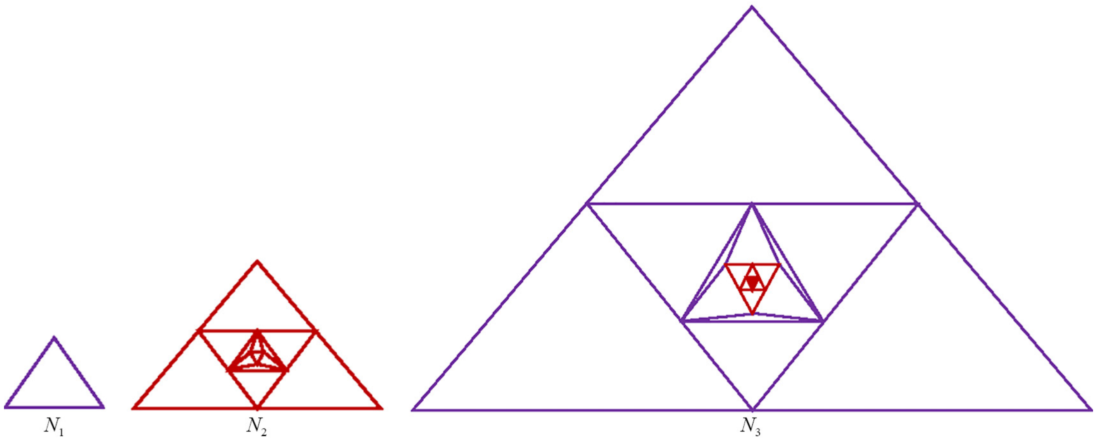
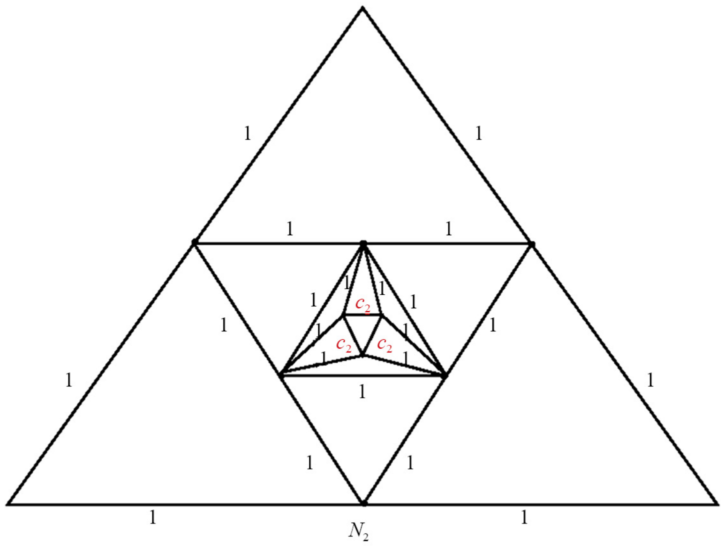


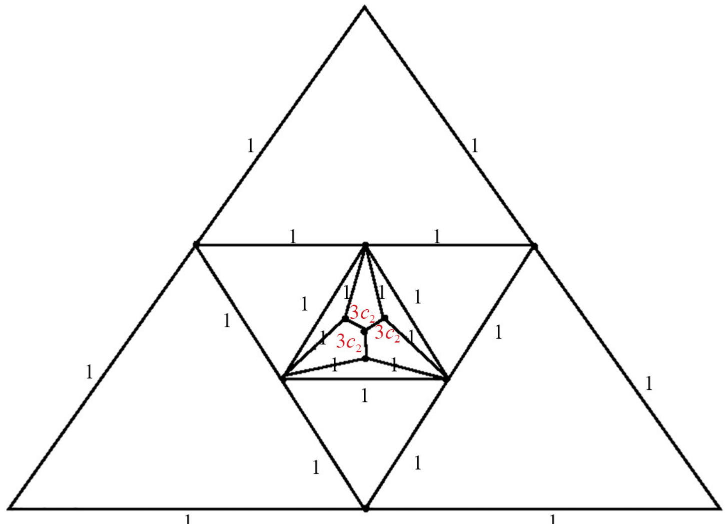
Figure 3. Some sequences of the graph $\mathscr{N}_n$

Theorem 1 For $n \geq 1$, the number of spanning trees in the sequence of the graph $\mathscr{N}_n$ is given by

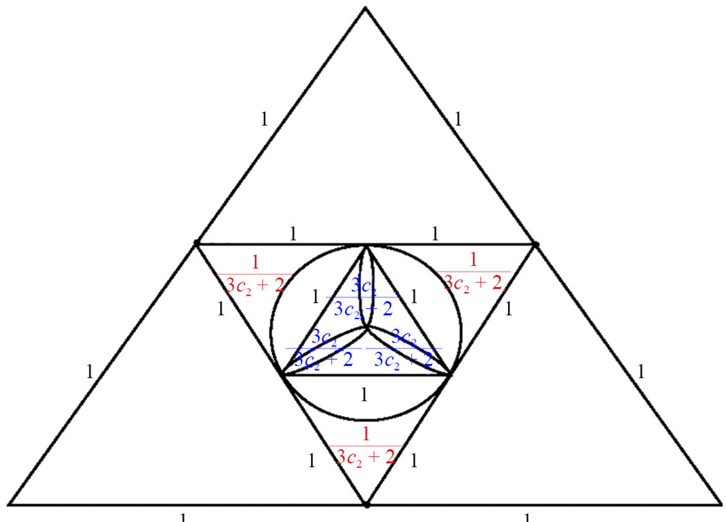
$$\frac{3\times 2^{3n-5}\left(1+\sqrt{1045}+2^{n}\left(9407-291\sqrt{1045}\right)^{1-n}\left(28+\sqrt{1045}\right)\right)^{2}\left(\left(-17-10\sqrt{\frac{55}{19}}\right)\left(\frac{1}{2}\left(97-3\sqrt{1045}\right)\right)^{n}+\frac{1}{19}\times 2^{1-n}\left(97+3\sqrt{1045}\right)^{n-2}\left(2489+77\sqrt{1045}\right)\right)^{2}}{\left(27-3\times 2^{-n}\left(37+\sqrt{1045}\right)\left(9407+291\sqrt{1045}\right)^{n-1}\right)^{2}}$$

Proof. We convert $\mathscr{N}_i$ to $\mathscr{N}_{i-1}$ via the electrically equivalent transformation. The conversion procedure from $\mathscr{N}_2$ to $\mathscr{N}_1$ is shown in Figure (4).

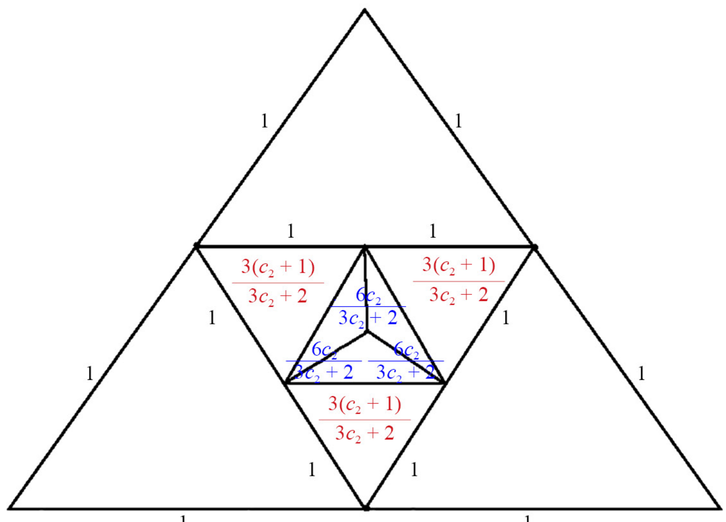




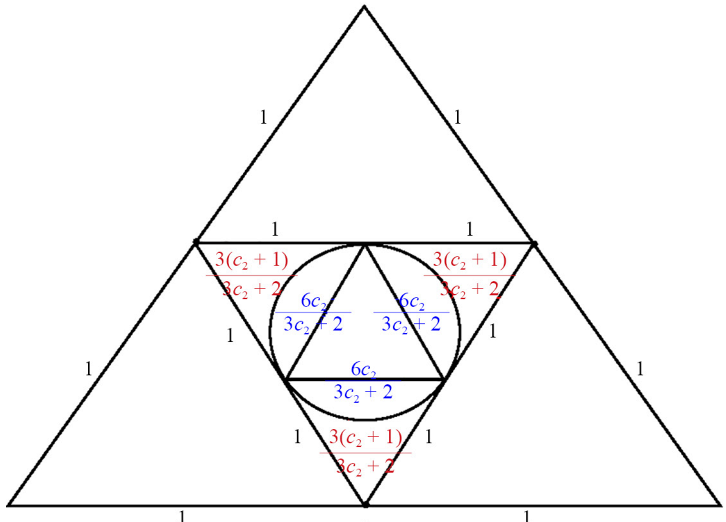
G_1



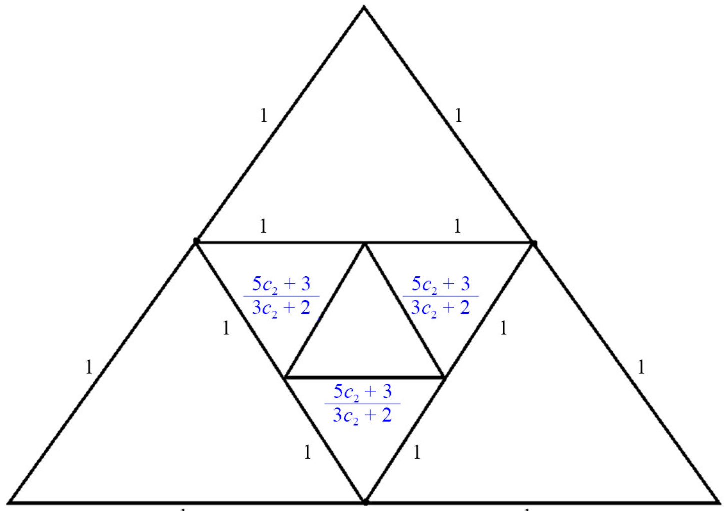
G_2



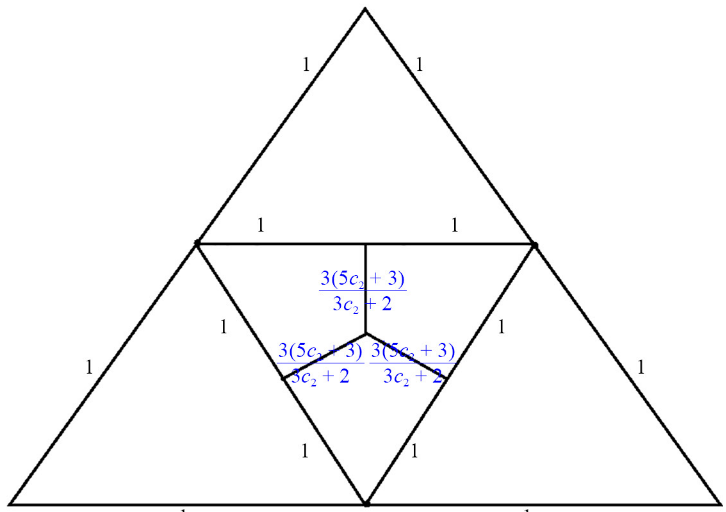
G_3


1
1
1
1
3(c2 + 1)
3c2 + 2
6c2
3c2 + 2
1
1
1
1

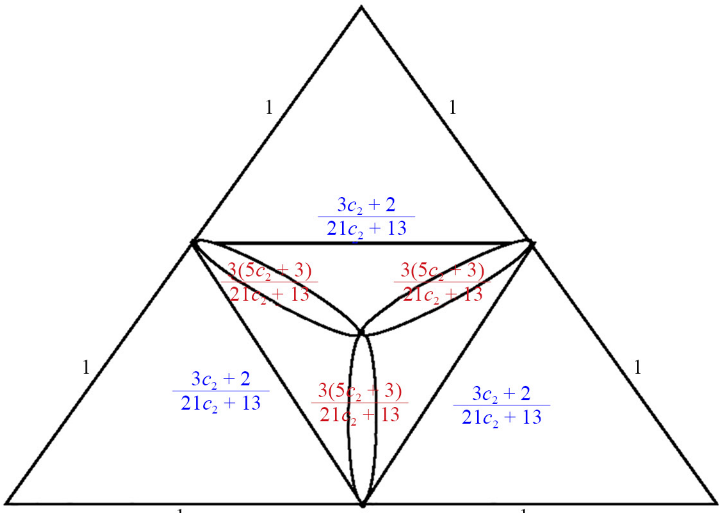
G_4


1
1
1
1
5c2 + 3
3c2 + 2
1
1
1
1

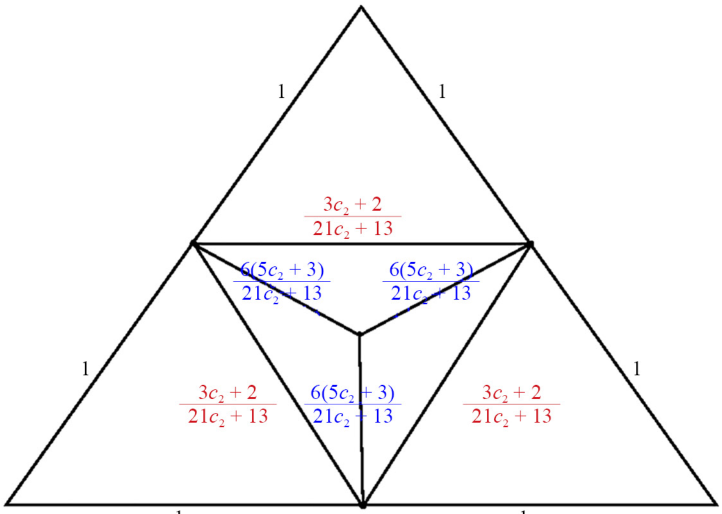
G_5


1
1
1
1
3(5c2 + 3)
3c2 + 2
1
1
1
1

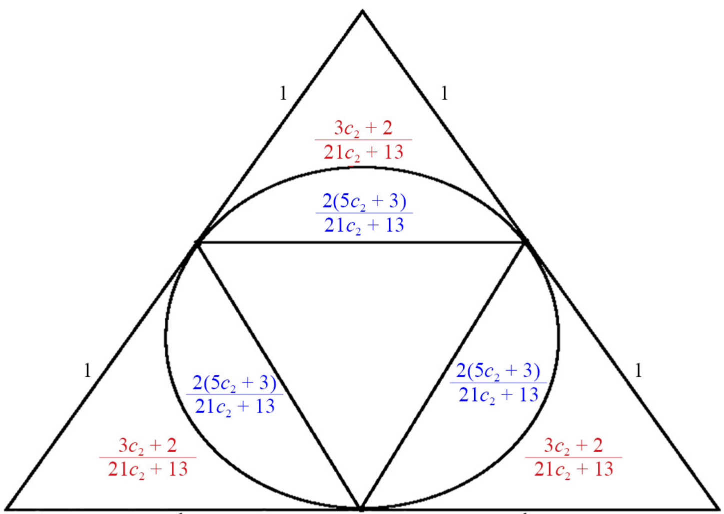
G_6


1
1
$\frac{3c_2+2}{21c_2+13}$
$\frac{3(5c_2+3)}{21c_2+13}$
$\frac{3(5c_2+3)}{21c_2+13}$
1
$\frac{3c_2+2}{21c_2+13}$
$\frac{3(5c_2+3)}{21c_2+13}$
$\frac{3c_2+2}{21c_2+13}$
1
1
1

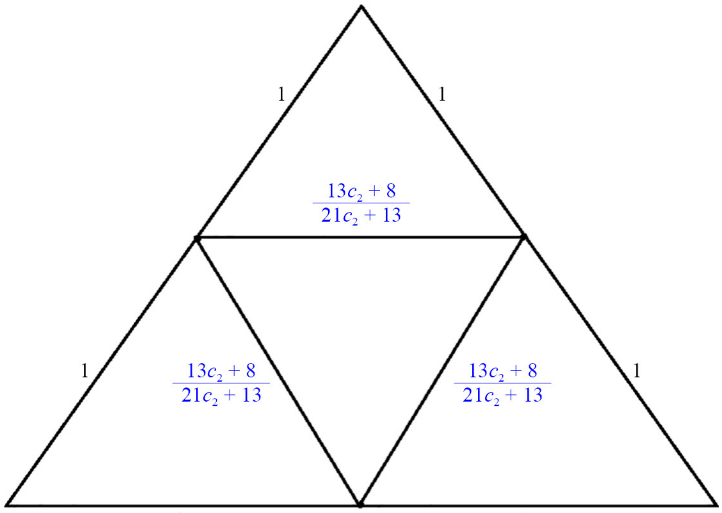
G_7


1
1
$\frac{3c_2+2}{21c_2+13}$
$\frac{6(5c_2+3)}{21c_2+13}$
$\frac{6(5c_2+3)}{21c_2+13}$
1
$\frac{3c_2+2}{21c_2+13}$
$\frac{6(5c_2+3)}{21c_2+13}$
$\frac{3c_2+2}{21c_2+13}$
1
1
1

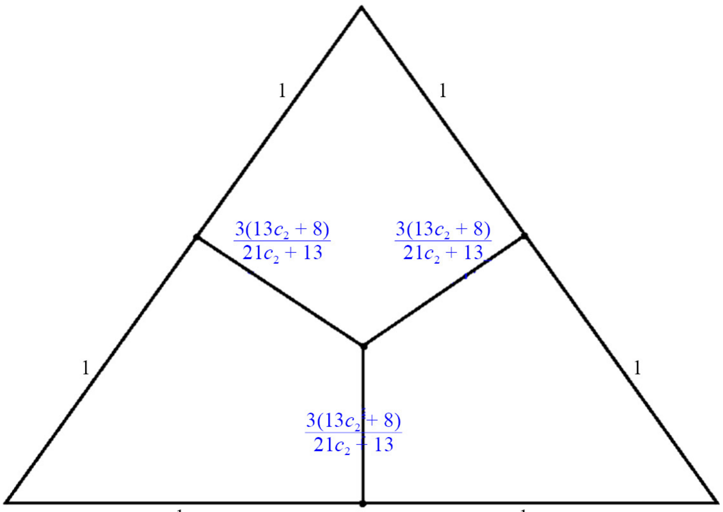
G_8


1
1
$\frac{3c_2+2}{21c_2+13}$
$\frac{2(5c_2+3)}{21c_2+13}$
1
$\frac{2(5c_2+3)}{21c_2+13}$
$\frac{2(5c_2+3)}{21c_2+13}$
1
$\frac{3c_2+2}{21c_2+13}$
$\frac{3c_2+2}{21c_2+13}$
1
1

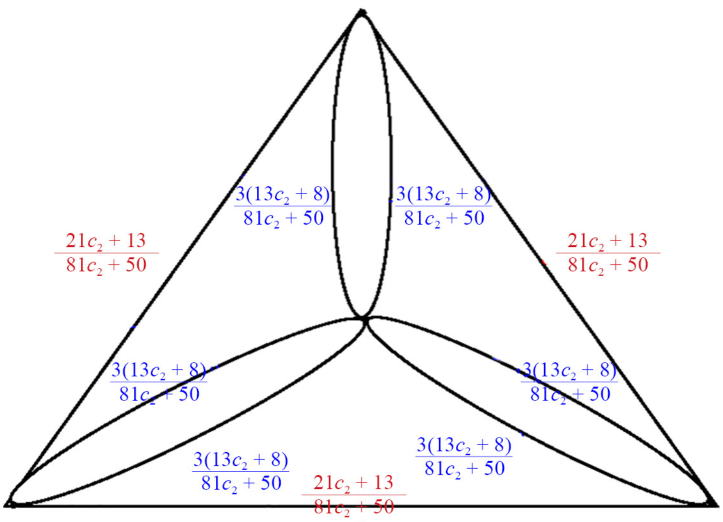
G_9


1
1
13c2 + 8 / 21c2 + 13
1
13c2 + 8 / 21c2 + 13
13c2 + 8 / 21c2 + 13
1
1
1

G_{10}


1
1
3(13c2 + 8) / 21c2 + 13
3(13c2 + 8) / 21c2 + 13
1
1
3(13c2 + 8) / 21c2 + 13
1
1

G_{11}


3(13c2 + 8) / 81c2 + 50
3(13c2 + 8) / 81c2 + 50
21c2 + 13 / 81c2 + 50
21c2 + 13 / 81c2 + 50
3(13c2 + 8) / 81c2 + 50
3(13c2 + 8) / 81c2 + 50
3(13c2 + 8) / 81c2 + 50
3(13c2 + 8) / 81c2 + 50
21c2 + 13 / 81c2 + 50

G_{12}

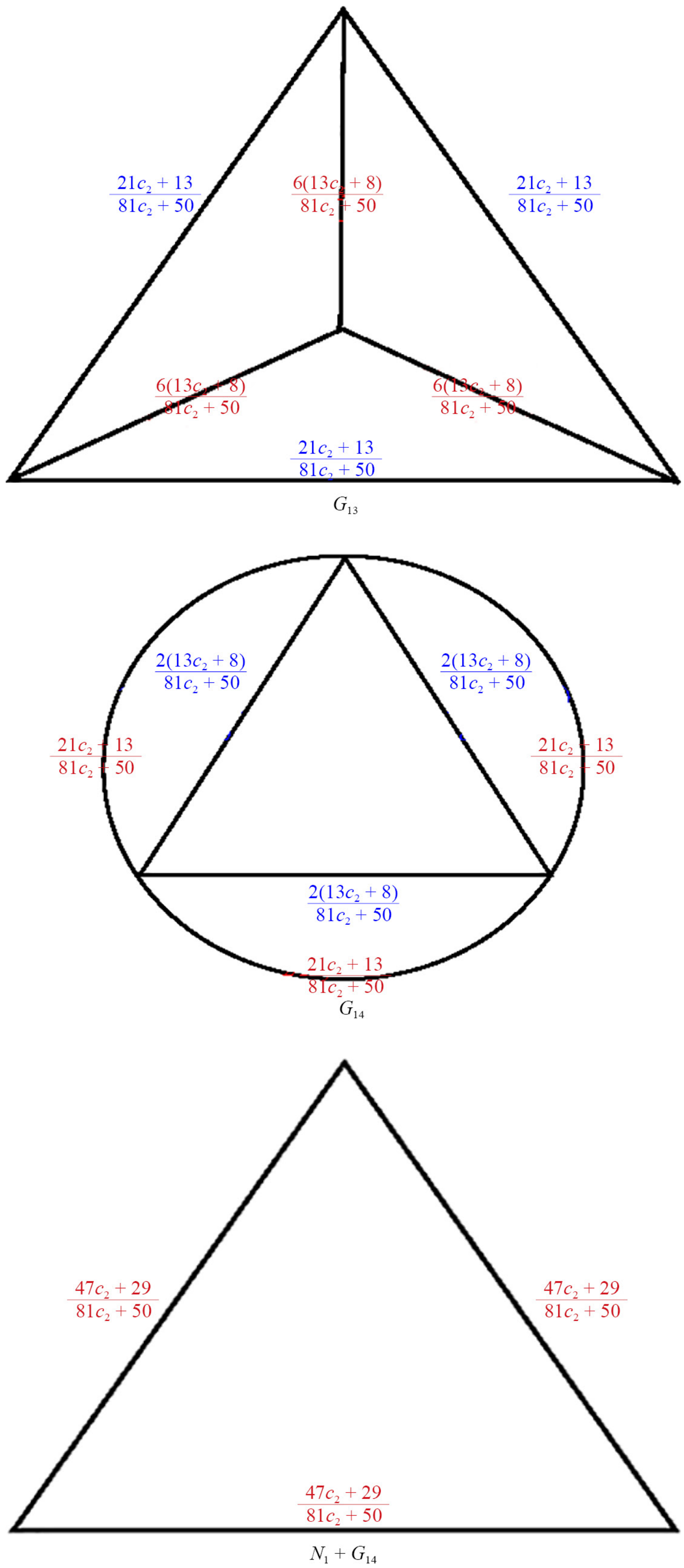


Figure 4. The transformations from $\mathcal{N}_2$ to $\mathcal{N}_1$

The following transformations result from using the attributes listed in Section 2:

$$\tau(G_1) = 9c_2\tau(\mathscr{N}_2),\ \tau(G_2) = \left[\frac{1}{3c_2+2}\right]^3 \tau(G_1),$$

$$\tau(G_3) = \tau(G_2),\ \tau(G_4) = \frac{3c_2+2}{18c_2}\tau(G_3),$$

$$\tau(G_5) = \tau(G_4),\ \tau(G_6) = \frac{9(5c_2+3)}{3c_2+2}\tau(G_5)$$

$$\tau(G_7) = \left[\frac{3c_2+2}{21c_2+13}\right]^3 \tau(G_6),\ \ \tau(G_8) = \tau(G_7),$$

$$\tau(G_9) = \frac{21c_2+13}{18(5c_2+3)}\tau(G_8),\ \tau(G_{10}) = \tau(G_9),$$

$$\tau(G_{11}) = \frac{9(13c_2+8)}{21c_2+13}\tau(G_{10}),\ \tau(G_{12}) = \left[\frac{21c_2+13}{81c_2+50}\right]^3 \tau(G_{11}),$$

$$\tau(G_{13}) = \tau(G_{12}),\ \tau(G_{14}) = \frac{81c_2+50}{18(13c_2+8)}\tau(G_{13}),$$

and $\tau(\mathscr{N}_1) = \tau(G_{14})$.

When these fourteen transformations are combined, we obtain

$$\tau(\mathscr{N}_2) = 8(81c_2+50)^2\tau(\mathscr{N}_1). \tag{16}$$

Further

$$\tau(\mathscr{N}_n) = \prod_{i=2}^{n} 8(81c_2+50)^2\tau(\mathscr{N}_1) = 3\times(8)^{n-1}c_1^2\left[\prod_{i=2}^{n}(81c_i+50)\right]^2 \tag{17}$$

where $c_{i-1} = \frac{47c_i+29}{81c_i+50}$, $i = 2, 3, \ldots, n$. Its characteristic equation is $81\alpha^2+3\alpha-29 = 0$, which have two roots $\alpha_1 = \frac{-1-\sqrt{1045}}{54}$ and $\alpha_2 = \frac{-1+\sqrt{1045}}{54}$. Subtracting both roots from each side of $c_{i-1} = \frac{47c_i+29}{81c_i+50}$, we get

$$c_{i-1} - \frac{-1-\sqrt{1045}}{54} = \frac{47c_i+29}{81c_i+50} + \frac{1+\sqrt{1045}}{54} = \left(97+3\sqrt{1045}\right).\frac{c_i+\dfrac{1+\sqrt{1045}}{54}}{2(81c_i+50)}; \tag{18}$$

$$c_{i-1}-\frac{-1+\sqrt{1045}}{54}=\frac{47c_i+29}{81c_i+50}+\frac{1-\sqrt{1045}}{54}=\left(97-3\sqrt{1045}\right).\frac{c_i+\dfrac{1-\sqrt{1045}}{54}}{2\left(81c_i+50\right)}. \tag{19}$$

Let $d_i=\frac{c_i+\dfrac{1+\sqrt{1045}}{54}}{c_i+\dfrac{1-\sqrt{1045}}{54}}$. Then by Equations (18) and (19), we get $d_{i-1}=\left(\frac{9407+291\sqrt{1045}}{2}\right)d_i$ and $d_i=\left(\frac{9407+291\sqrt{1045}}{2}\right)^{n-i}d_n$.

Therefore,

$$c_i=\frac{\left(\dfrac{9407+291\sqrt{1045}}{2}\right)^{n-i}\left(\dfrac{-1+\sqrt{1045}}{54}\right)d_n+\dfrac{1+\sqrt{1045}}{54}}{\left(\dfrac{9407+291\sqrt{1045}}{2}\right)^{n-i}d_n-1}.$$

Thus

$$c_1=\frac{\left(\dfrac{9407+291\sqrt{1045}}{2}\right)^{n-1}\left(28+\sqrt{1045}\right)+\dfrac{1+\sqrt{1045}}{2}}{3\left(\dfrac{9407+291\sqrt{1045}}{2}\right)^{n-1}\left(\dfrac{37+\sqrt{1045}}{2}\right)-27}. \tag{20}$$

Using the formula $c_{n-1}=\frac{47c_n+29}{81c_n+50}$ and designating the coefficients of $47c_n+29$ and $81c_n+50$ as a_n and b_n we have

$$81c_n+50=a_0\left(47c_n+29\right)+b_0\left(81c_n+50\right),$$

$$81c_{n-1}+50=\frac{a_1\left(47c_n+29\right)+b_1\left(81c_n+50\right)}{a_0\left(47c_n+29\right)+b_0\left(81c_n+50\right)},$$

$$81c_{n-2}+50=\frac{a_2\left(47c_n+29\right)+b_2\left(81c_n+50\right)}{a_1\left(47c_n+29\right)+b_1\left(81c_n+50\right)},$$

$$\vdots$$

$$81c_{n-i}+50=\frac{a_i\left(47c_n+29\right)+b_i\left(81c_n+50\right)}{a_{i-1}\left(47c_n+29\right)+b_{i-1}\left(81c_n+50\right)}, \tag{21}$$

$$81c_{n-(i+1)}+50=\frac{a_{i+1}(47c_n+29)+b_{i+1}(81c_n+50)}{a_i(47c_n+29)+b_i(81c_n+50)}, \tag{22}$$

$$\vdots$$

$$81c_2+50=\frac{a_{n-2}(47c_n+29)+b_{n-2}(81c_n+50)}{a_{n-3}(47c_n+29)+b_{n-3}(81c_n+50)}.$$

When Equation (21) is substituted into Equation (17), we get

$$\tau(\mathscr{N}_n)=3\times(8)^{n-1}c_1^2[a_{n-2}(47c_n+29)+b_{n-2}(81c_n+50)]^2 \tag{23}$$

where $a_0=0,\ b_0=1$ and $a_1=81,\ b_1=50$.

By the expression $c_{n-1}=\dfrac{47c_n+29}{81c_n+50}$ and Equations (21) and (22), we have

$$a_{i+1}=97a_i-a_{i-1};\ b_{i+1}=97b_i-b_{i-1}. \tag{24}$$

Equation (24) has the characteristic equation $\beta^2-97\beta+1=0$. Its roots are $\beta_1=\dfrac{97+3\sqrt{1045}}{2}$ and $\beta_2=\dfrac{97-3\sqrt{1045}}{2}$. The general solutions of Equations (24) are $a_i=h_1\beta_1^i+h_2\beta_2^i;\ b_i=k_1\beta_1^i+d_2\beta_2^i$.

Given the initial conditions $a_0=0,\ b_0=1$ and $a_1=81,\ b_1=50$, we obtain

$$a_i=\frac{27\sqrt{1045}}{1045}\left(\frac{97+3\sqrt{1045}}{2}\right)^i-\frac{27\sqrt{1045}}{1045}\left(\frac{97-3\sqrt{1045}}{2}\right)^i;$$

$$b_i=\left(\frac{1045+\sqrt{1045}}{2090}\right)\left(\frac{97+3\sqrt{1045}}{2}\right)^i+\left(\frac{1045-\sqrt{1045}}{2090}\right)\left(\frac{97-3\sqrt{1045}}{2}\right)^i. \tag{25}$$

There is no electrically similar transition for $\mathscr{N}_n$ if $c_n=1$. When Equation (25) is inserted into Equation (23), we obtain

$$\tau(\mathcal{N}_n) = 3 \times (8)^{n-1} c_1^2 \left[\left(\frac{131 + 77\sqrt{\frac{55}{19}}}{2} \right) \left(\frac{97 + 3\sqrt{1045}}{2} \right)^{n-2} + \left(\frac{131 - 77\sqrt{\frac{55}{19}}}{2} \right) \left(\frac{97 - 3\sqrt{1045}}{2} \right)^{n-2} \right]^2, \tag{26}$$

$n \geq 2.$

Equation (26) is satisfied for $n = 1$ and $\tau(\mathcal{N}_1) = 3$. Thus, the number of spanning trees in the sequence of the graph $\mathcal{N}_n$ is determined by

$$\tau(\mathcal{N}_n) = 3 \times (8)^{n-1} c_1^2 \left[\left(\frac{131 + 77\sqrt{\frac{55}{19}}}{2} \right) \left(\frac{97 + 3\sqrt{1045}}{2} \right)^{n-2} + \left(\frac{131 - 77\sqrt{\frac{55}{19}}}{2} \right) \left(\frac{97 - 3\sqrt{1045}}{2} \right)^{n-2} \right]^2, \tag{27}$$

$n \geq 1.$

where

$$c_1 = \frac{\left(\frac{9407 + 291\sqrt{1045}}{2} \right)^{n-1} \left(28 + \sqrt{1045} \right) + \frac{1 + \sqrt{1045}}{2}}{3 \left(\frac{9407 + 291\sqrt{1045}}{2} \right)^{n-1} \left(\frac{37 + \sqrt{1045}}{2} \right) - 27}, \; n \geq 1. \tag{28}$$

The result is obtained by inserting Equation (28) into Equation (27). □

2.3 *Number of spanning trees in the sequences of the graph* $\mathcal{Q}_n$

The graph Q_n is defined recursively using the graphs Q_1 (triangle or K_3) and Q_2 as shown in Figure 5. The graph Q_n, $n = 3$ is obtained by replacing the central triangle in the graph Q_2 by a copy of Q_2. In general, the graph Q_n is obtained by replacing the central triangle in Q_{n-1} with Q_2. According to this construction, the number of total vertices $|V(\mathcal{Q}_n)|$ and edges $|E(\mathcal{Q}_n)|$ are $|V(\mathcal{Q}_n)| = 9n - 6$ and $|E(\mathcal{Q}_n)| = 21n - 18$, $n = 1, 2, \ldots$. The average degree of the graph $\mathcal{Q}_n$ in the large n limit is $\frac{14}{3}$.

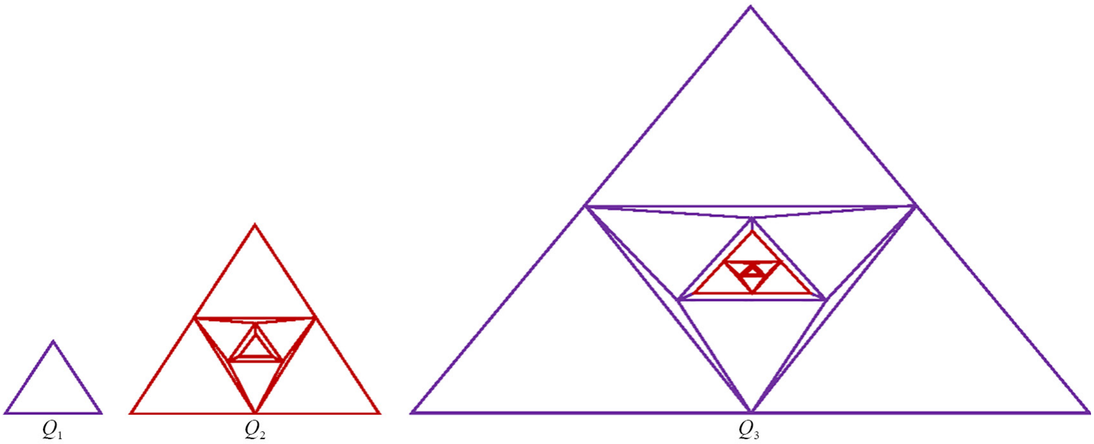
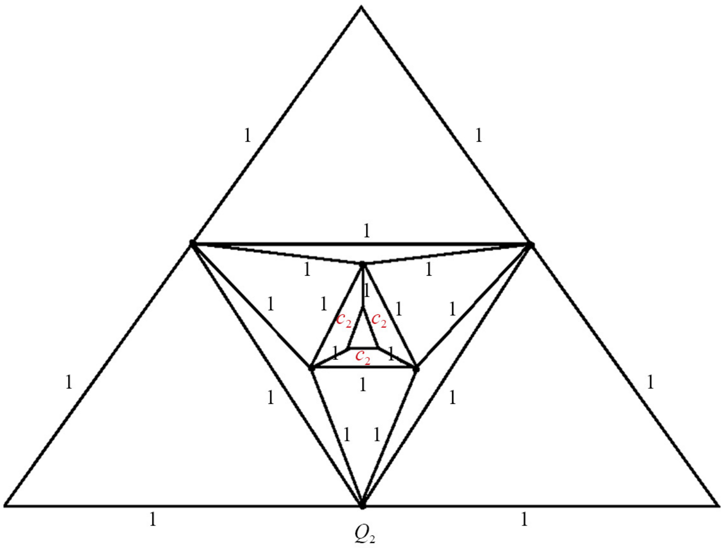


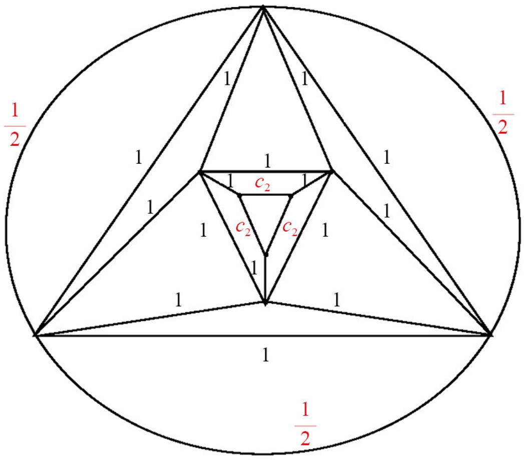
Figure 5. Some sequences of the graph $\mathscr{Q}_n$

Proof. For $n \geq 1$, the number of spanning trees in the sequence of the graph $\mathscr{Q}_n$ is given by

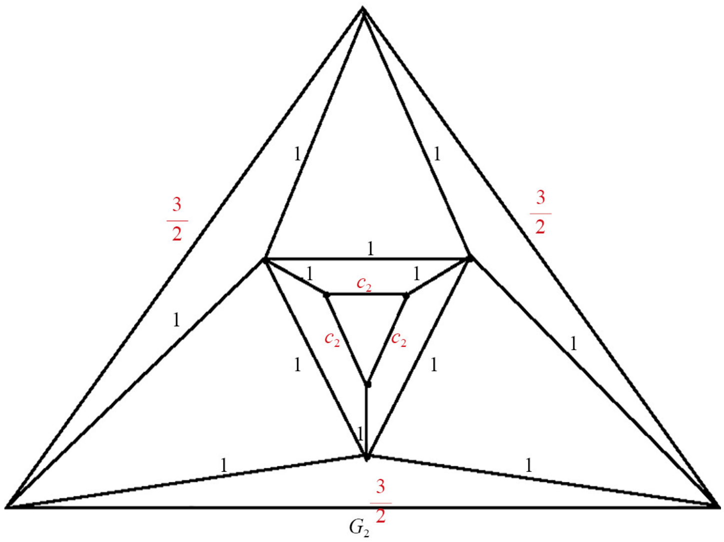
$$\frac{\begin{array}{c} 3\times 4^{n-3}\left(-17\times 2^{n+1}\left(1532+107\sqrt{205}\right)\right. \\ \left.+\left(223+19\sqrt{205}\right)\left(1847+129\sqrt{205}\right)^{n}\right)^{2}\left(\left(2050-143\sqrt{205}\right)\left(43+3\sqrt{205}\right)^{n}\right. \\ \left.+\left(43-3\sqrt{205}\right)^{n}\left(2050+143\sqrt{205}\right)\right)^{2} \end{array}}{42025\left(51\times 2^{n}\left(1847+129\sqrt{205}\right)+\left(103+\sqrt{205}\right)\left(1847+129\sqrt{205}\right)^{n}\right)^{2}}$$

We convert $\mathscr{Q}_i$ to $\mathscr{Q}_{i-1}$ via the electrically equivalent transformation. The conversion procedure from $\mathscr{Q}_2$ to $\mathscr{Q}_1$ is shown in Figure 6.

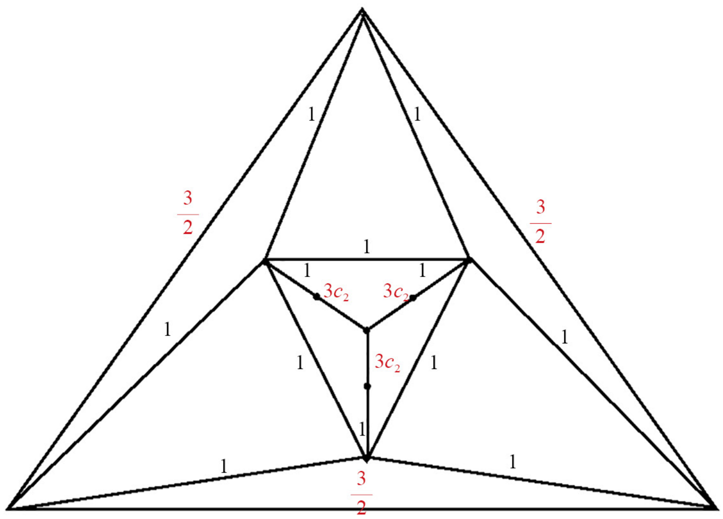



1
1
$\frac{1}{2}$
$\frac{1}{2}$
1
1
1
1
c_2
1
1
1
c_2
c_2
1
1
1
1
1
$\frac{1}{2}$

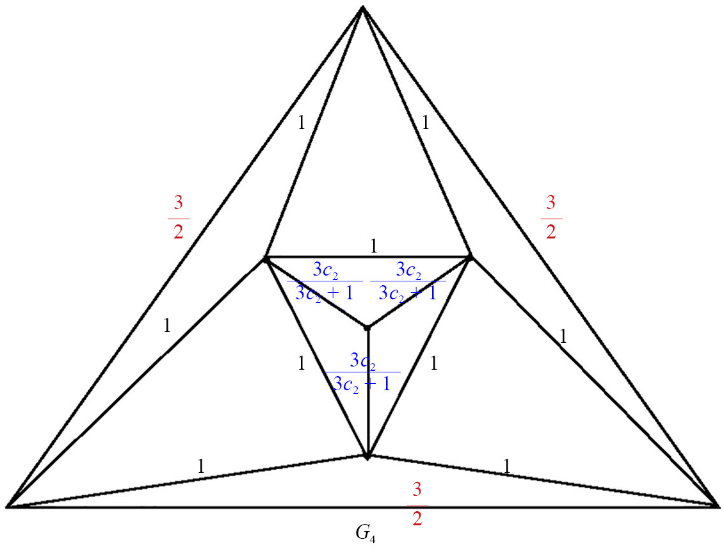
G_1


1
1
$\frac{3}{2}$
$\frac{3}{2}$
1
1
c_2
1
1
c_2
c_2
1
1
1
1
1
1
$\frac{3}{2}$

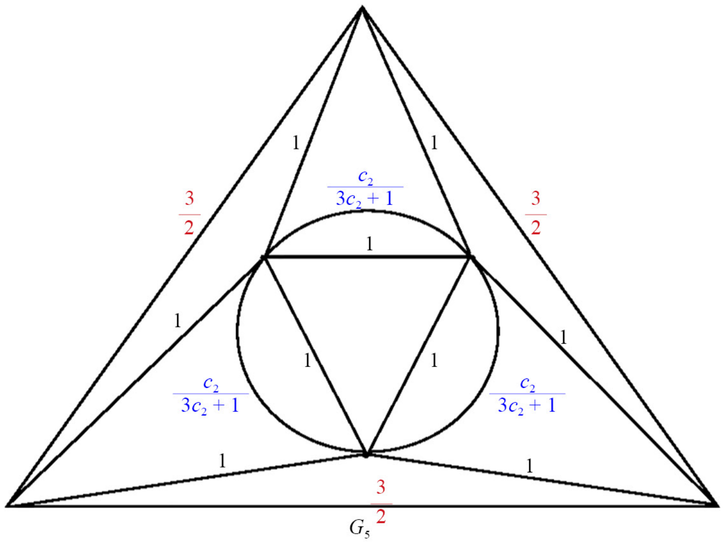
G_2


1
1
$\frac{3}{2}$
$\frac{3}{2}$
1
1
1
$3c_2$
$3c_2$
1
1
$3c_2$
1
1
1
1
1
$\frac{3}{2}$

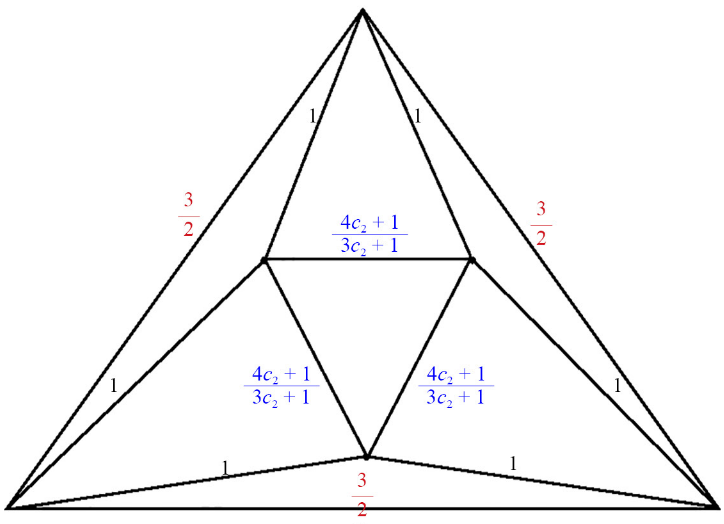
G_3


1
1
3/2
3/2
1
3c2/(3c2 + 1)
3c2/(3c2 + 1)
1
1
3c2/(3c2 + 1)
1
1
1
1
3/2

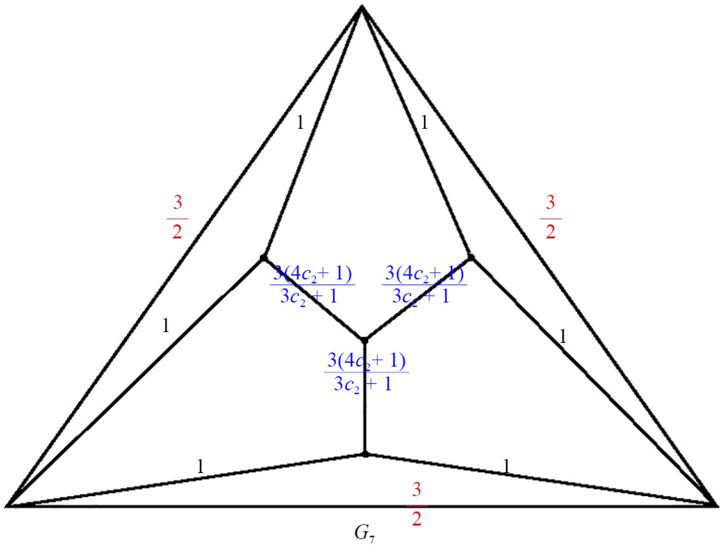
G_4


1
1
c2/(3c2 + 1)
3/2
3/2
1
1
1
1
1
1
c2/(3c2 + 1)
c2/(3c2 + 1)
1
1
3/2

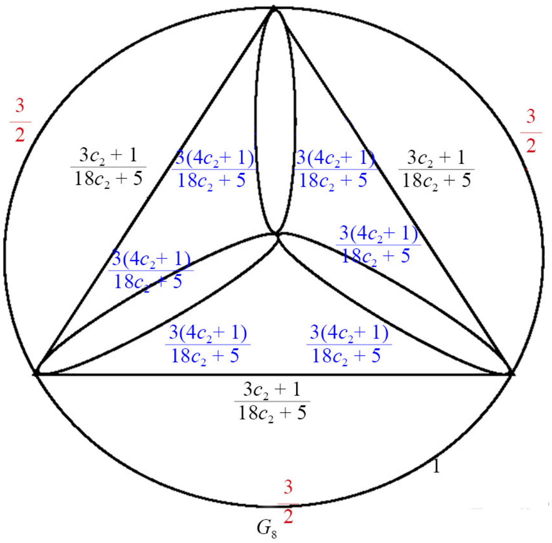
G_5


1
1
3/2
3/2
(4c2 + 1)/(3c2 + 1)
1
(4c2 + 1)/(3c2 + 1)
(4c2 + 1)/(3c2 + 1)
1
1
1
3/2

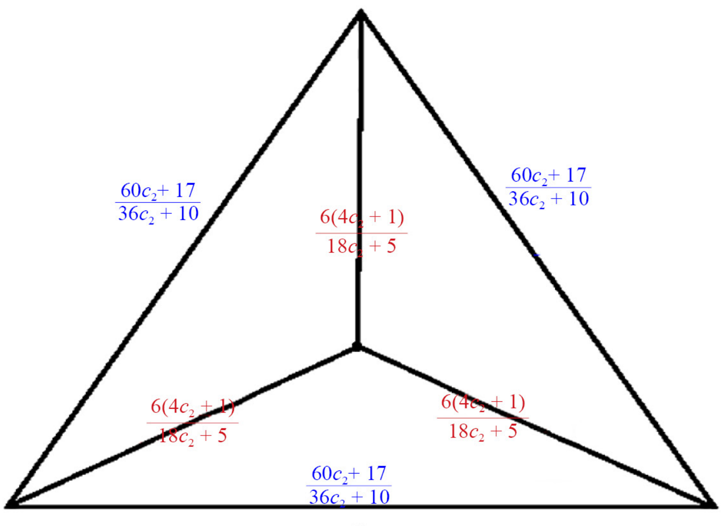
G_6


1
1
3/2
3/2
3(4c2+ 1)/3c2 + 1
3(4c2+ 1)/3c2 + 1
1
1
3(4c2+ 1)/3c2 + 1
1
1
3/2

G_7


3/2
3/2
3c2 + 1/18c2 + 5
3(4c2+ 1)/18c2 + 5
3(4c2+ 1)/18c2 + 5
3c2 + 1/18c2 + 5
3(4c2+ 1)/18c2 + 5
3(4c2+ 1)/18c2 + 5
3(4c2+ 1)/18c2 + 5
3(4c2+ 1)/18c2 + 5
3c2 + 1/18c2 + 5
1
3/2

G_8


60c2+ 17/36c2 + 10
60c2+ 17/36c2 + 10
6(4c2 + 1)/18c2 + 5
6(4c2 + 1)/18c2 + 5
6(4c2 + 1)/18c2 + 5
60c2+ 17/36c2 + 10

G_9

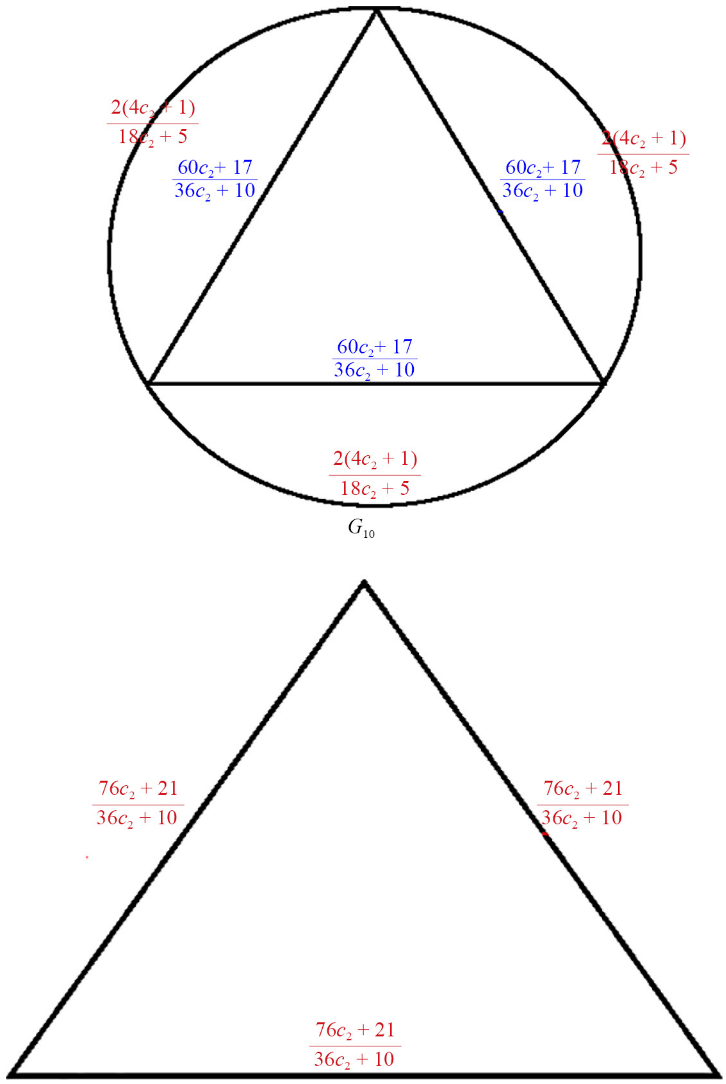


Figure 6. The transformations from $\mathscr{Q}_2$ to $\mathscr{Q}_1$

The following transformations result from using the attributes listed in Section 2:

$$\tau(G_1) = \left[\frac{1}{2}\right]^3 \tau(\mathscr{Q}_2),\ \tau(G_2) = \tau(G_1),\ \tau(G_3) = 9c_2\tau(G_2),$$

$$\tau(G_4) = \left[\frac{1}{3c_2+1}\right]^3 \tau(G_3),\ \tau(G_5) = \frac{3c_2+1}{9c_2}\tau(G_4),\ \tau(G_6) = \tau(G_5),$$

$$\tau(G_7) = \frac{9(4c_2+1)}{(3c_2+1)}\tau(G_6),\ \tau(G_8) = \left[\frac{(3c_2+1)}{(18c_2+5)}\right]^3 \tau(G_7),\ \tau(G_9) = \tau(G_8),$$

$$\tau(G_{10}) = \frac{18c_{2+}5}{18(4c_{2+}1)}\tau(G_9) \text{ and } \tau(\mathscr{Q}_1) = \tau(G_{10}).$$

When these eleven transformations are combined, we obtain

$$\tau(\mathscr{Q}_2) = 4(36c_2+10)^2\tau(\mathscr{Q}_1). \tag{29}$$

Further

$$\tau(\mathscr{Q}_n) = \prod_{i=2}^{n} 4(36c_2+10)^2\tau(\mathscr{Q}_1) = 3\times(4)^{n-1}c_1^2\left[\prod_{i=2}^{n}(36c_i+10)\right]^2 \tag{30}$$

where $c_{i-1} = \frac{76c_i+21}{36c_i+10}$, $i = 2,\ 3,\ \ldots,\ n$. Its characteristic equation is $36\alpha^2 - 66\alpha - 21 = 0$, which have two roots $\alpha_1 = \frac{11-\sqrt{205}}{12}$ and $\alpha_2 = \frac{11+\sqrt{205}}{12}$. Subtracting both roots from each side of $c_{i-1} = \frac{76c_i+21}{36c_i+10}$, we get

$$c_{i-1} - \frac{11-\sqrt{205}}{12} = \frac{76c_i+21}{36c_i+10} - \frac{11-\sqrt{205}}{12} = \left(43+3\sqrt{205}\right).\frac{c_i - \frac{11-\sqrt{205}}{12}}{(36c_i+10)}; \tag{31}$$

$$c_{i-1} - \frac{11+\sqrt{205}}{12} = \frac{76c_i+21}{36c_i+10} - \frac{11+\sqrt{205}}{12} = \left(43-3\sqrt{205}\right).\frac{c_i - \frac{11+\sqrt{205}}{12}}{(36c_i+10)}. \tag{32}$$

Let $d_i = \frac{c_i - \frac{11-\sqrt{205}}{12}}{c_i - \frac{11+\sqrt{205}}{12}}$. Then by Equations (31) and (32), we get $d_{i-1} = \left(\frac{1847+129\sqrt{205}}{2}\right)d_i$ and $d_i = \left(\frac{1847+129\sqrt{205}}{2}\right)^{n-i} d_n$.

Therefore

$$c_i = \frac{\left(\frac{1847+129\sqrt{205}}{2}\right)^{n-i}\left(\frac{11+\sqrt{205}}{12}\right)d_n + \frac{11-\sqrt{205}}{12}}{\left(\frac{1847+129\sqrt{205}}{2}\right)^{n-i}d_n - 1}.$$

Thus

$$c_1 = \frac{\left(\frac{1847+129\sqrt{205}}{2}\right)^{n-1}\left(223+19\sqrt{205}\right)+17(11-\sqrt{205})}{2\left(\frac{1847+129\sqrt{205}}{2}\right)^{n-1}\left(103+\sqrt{205}\right)+204}. \tag{33}$$

Using the formula $c_{n-1} = \frac{76c_n+21}{36c_n+10}$ and designating the coefficients of $76c_n+21$ and $36c_n+10$ as a_n and b_n we have

$$36c_n+10 = a_0\left(76c_n+21\right)+b_0\left(36c_n+10\right),$$

$$36c_{n-1}+10 = \frac{a_1\left(76c_n+21\right)+b_1\left(36c_n+10\right)}{a_0\left(76c_n+21\right)+b_0\left(36c_n+10\right)},$$

$$36c_{n-2}+10 = \frac{a_2\left(76c_n+21\right)+b_2\left(36c_n+10\right)}{a_1\left(76c_n+21\right)+b_1\left(36c_n+10\right)},$$

$$\vdots$$

$$36c_{n-i}+10 = \frac{a_i\left(76c_n+21\right)+b_i\left(36c_n+10\right)}{a_{i-1}\left(76c_n+21\right)+b_{i-1}\left(36c_n+10\right)}, \tag{34}$$

$$36c_{n-(i+1)}+10 = \frac{a_{i+1}\left(76c_n+21\right)+b_{i+1}\left(36c_n+10\right)}{a_i\left(76c_n+21\right)+b_i\left(36c_n+10\right)}, \tag{35}$$

$$\vdots$$

$$51c_2+40 = \frac{a_{n-2}\left(76c_n+21\right)+b_{n-2}\left(36c_n+10\right)}{a_{n-3}\left(76c_n+21\right)+b_{n-3}\left(36c_n+10\right)}.$$

When Equation (34) is substituted into Equation (40), we get

$$\tau(\mathscr{Q}_n) = 3\times(4)^{n-1}c_1^2[a_{n-2}(76c_n+21)+b_{n-2}(36c_n+10)]^2 \tag{36}$$

where $a_0 = 0,\ b_0 = 1$ and $a_1 = 36,\ b_1 = 10$.

By the expression $c_{n-1} = \frac{76c_n+21}{36c_n+10}$ and Equations (34) and (35), we have

$$a_{i+1} = 86a_i - 4a_{i-1};\ b_{i+1} = 86b_i - 4b_{i-1}. \tag{37}$$

Equation (37) has the characteristic equation $\beta^2 - 86\beta + 4 = 0$. Its roots are $\beta_1 = 43 + 3\sqrt{205}$ and $\beta_2 = 43 - 3\sqrt{205}$. The general solutions of Equation (37) are $a_i = h_1\beta_1^i + h_2\beta_2^i;\ b_i = k_1\beta_1^i + d_2\beta_2^i$.
Given the initial conditions $a_0 = 0,\ b_0 = 1$ and $a_1 = 36,\ b_1 = 10$, we have

$$a_i = \frac{6\sqrt{205}}{205}(43+3\sqrt{205})^i - \frac{6\sqrt{205}}{205}(43-3\sqrt{205})^i;$$

$$b_i = \left(\frac{205-11\sqrt{205}}{410}\right)(43+3\sqrt{205})^i + \left(\frac{205+11\sqrt{205}}{410}\right)(43-3\sqrt{205})^i. \tag{38}$$

There is no electrically similar transition for $\mathscr{Q}_n$ if $c_n = 1$. When Equation (38) is inserted into Equation (36), we obtain

$$\tau(\mathscr{Q}_n) = 3\times(4)^{n-1}c_1^2\left[\left(\frac{4715+329\sqrt{205}}{205}\right)(43+3\sqrt{205})^{n-2} + \left(\frac{4715-329\sqrt{205}}{205}\right)(43-3\sqrt{205})^{n-2}\right]^2, \tag{39}$$

$n \geq 2$.

Equation (39) is satisfied for $n = 1$ and $\tau(\mathscr{Q}_1) = 3$. Thus, the number of spanning trees in the sequence of the graph $\mathscr{Q}_n$ is determined by

$$\tau(\mathscr{Q}_n) = 3\times(4)^{n-1}c_1^2\left[\left(\frac{4715+329\sqrt{205}}{205}\right)(43+3\sqrt{205})^{n-2} + \left(\frac{4715-329\sqrt{205}}{205}\right)(43-3\sqrt{205})^{n-2}\right]^2, \tag{40}$$

$n \geq 1$.

where

$$c_1 = \frac{\left(\frac{1847+129\sqrt{205}}{2}\right)^{n-1}\left(223+19\sqrt{205}\right)+17(11-\sqrt{205})}{2\left(\frac{1847+129\sqrt{205}}{2}\right)^{n-1}\left(103+\sqrt{205}\right)+204},\ n \geq 1. \tag{41}$$

The result is obtained by inserting Equation (41) into Equation (40). □

2.4 *Number of spanning trees in the sequences of the graph $\mathscr{R}_n$*

The graph $\mathscr{R}_n$ is defined recursively using the graphs $\mathscr{R}_1$ (triangle or K_3) and $\mathscr{R}_2$ as shown in Figure 7. The graph $\mathscr{R}_n,\ n = 3$ is obtained by replacing the central triangle in the graph $\mathscr{R}_2$ by a copy of $\mathscr{R}_2$. In general, the graph $\mathscr{R}_n$ is obtained by replacing the central triangle in $\mathscr{R}_{n-1}$ with $\mathscr{R}_2$. According to this construction, the number of total vertices $|V(\mathscr{R}_n)|$ and edges $|E(\mathscr{R}_n)|$ are $|V(\mathscr{R}_n)| = 9n - 6$ and $|E(\mathscr{R}_n)| = 21n - 18$, $n = 1, 2, \ldots$. The average degree of $\mathscr{R}_n$ in the large n limit is $\frac{14}{3}$.

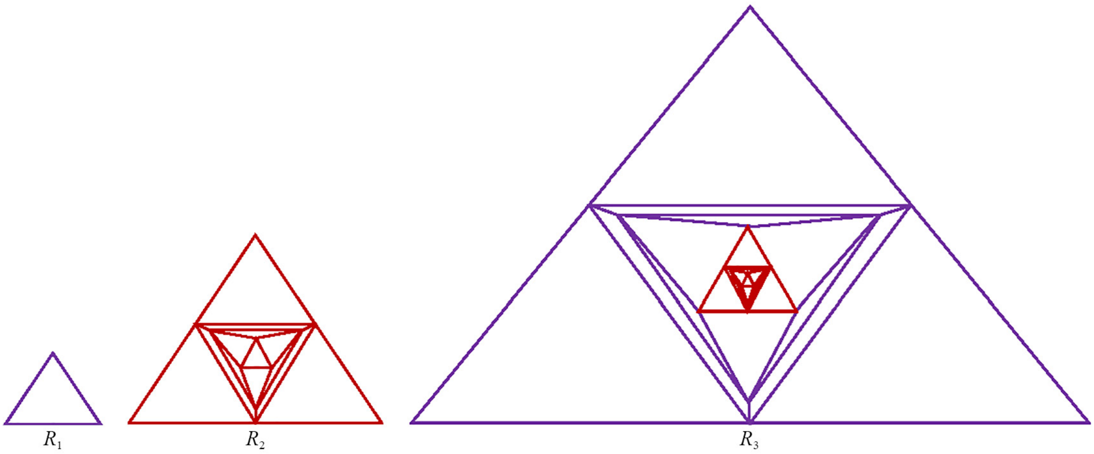
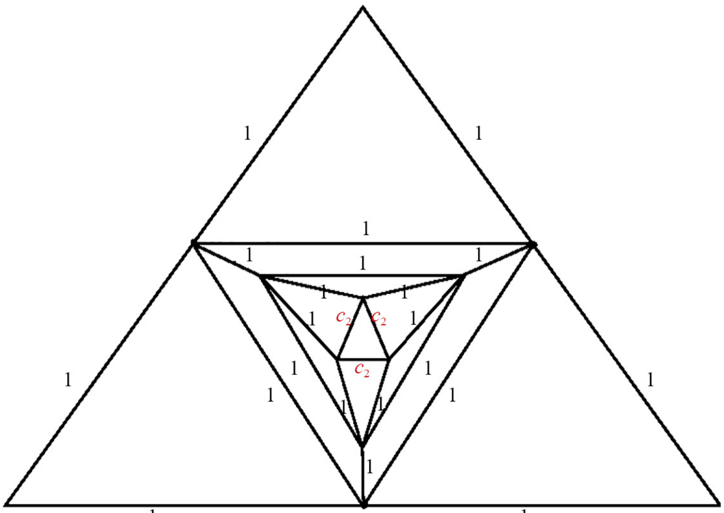


Figure 7. Some sequences of the graph $\mathscr{R}_n$

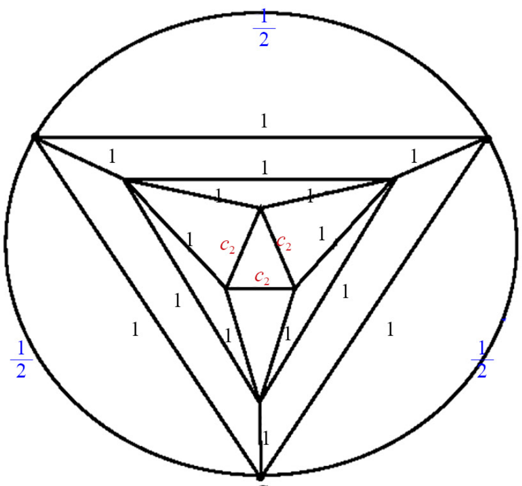
For $n \geq 1$, the number of spanning trees in the sequence of the graph $\mathscr{R}_n$ is given by

$$\frac{3\times 4^{n-4}\left(\left(287-20\sqrt{205}\right)\left(43+3\sqrt{205}\right)^{n}+\left(43-3\sqrt{205}\right)^{n}\left(287+20\sqrt{205}\right)^{2}\left(-3\left(-7+\sqrt{205}\right)+\left(\frac{1}{2}\left(1847+129\sqrt{205}\right)\right)^{n}\left(-9779+683\sqrt{205}\right)^{2}}{1681\left(9+2^{-n}\left(23+\sqrt{205}\right)\left(1847+129\sqrt{205}\right)^{n-1}\right)^{2}}.$$

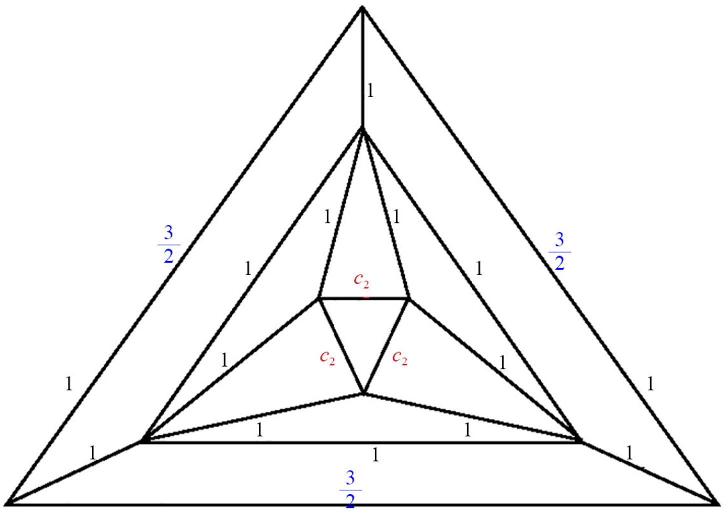
Proof. We convert $\mathscr{R}_i$ to $\mathscr{R}_{i-1}$ via the electrically equivalent transformation. The conversion procedure from $\mathscr{R}_2$ to $\mathscr{R}_1$ is shown in Figure 8.


1
1
1
1
1
1
1
1
1
c_2
c_2
1
c_2
1
1
1
1
1
1
1
1
1
1

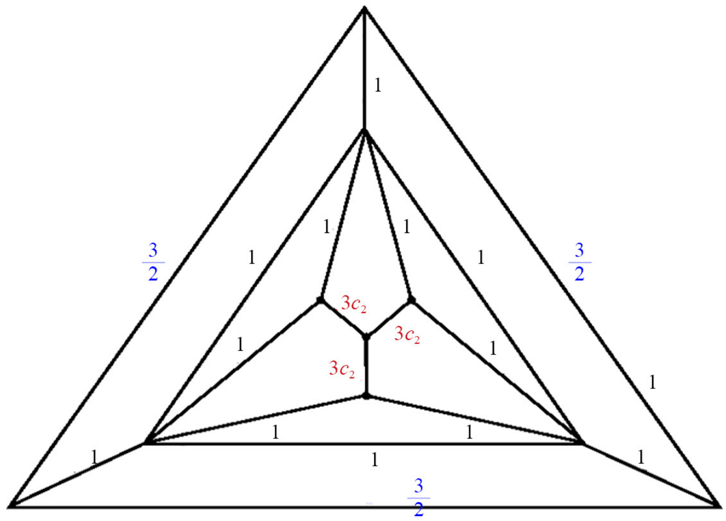
R_2


$\frac{1}{2}$
1
1
1
1
1
1
1
c_2
c_2
1
c_2
1
1
1
1
1
1
$\frac{1}{2}$
$\frac{1}{2}$
1

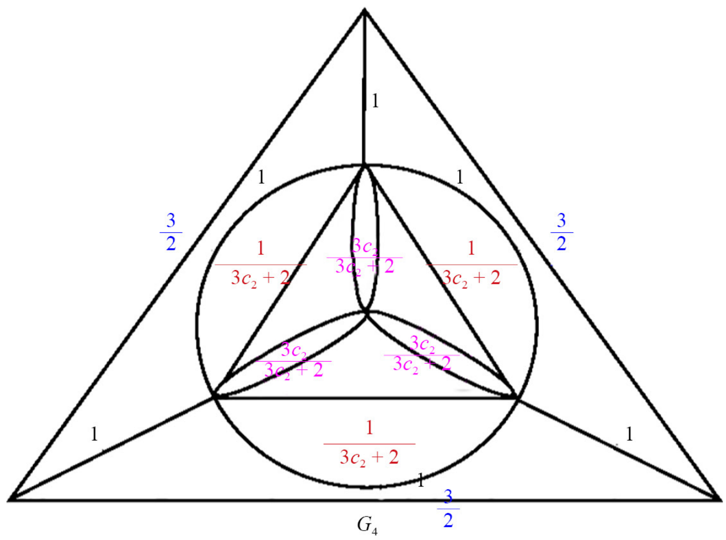
G_1


1
1
1
$\frac{3}{2}$
$\frac{3}{2}$
1
1
c_2
1
c_2
c_2
1
1
1
1
1
1
1
1
$\frac{3}{2}$

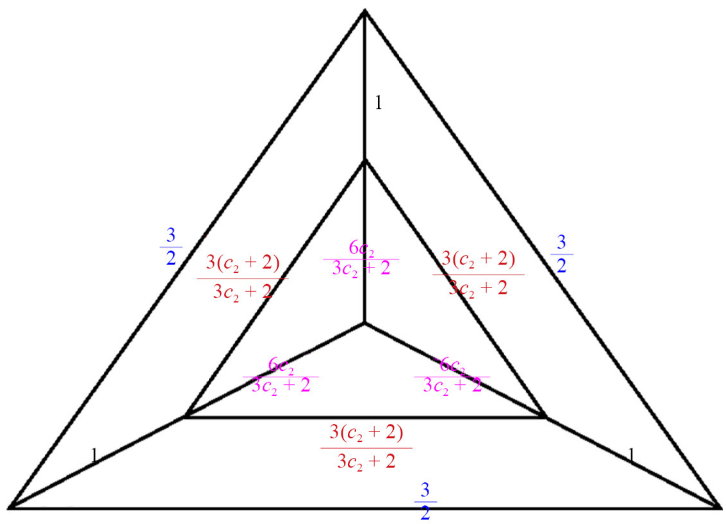
G_2


1
1
1
1
1
1
1
1
1
1
1
1
3/2
3/2
3/2
3c2
3c2
3c2

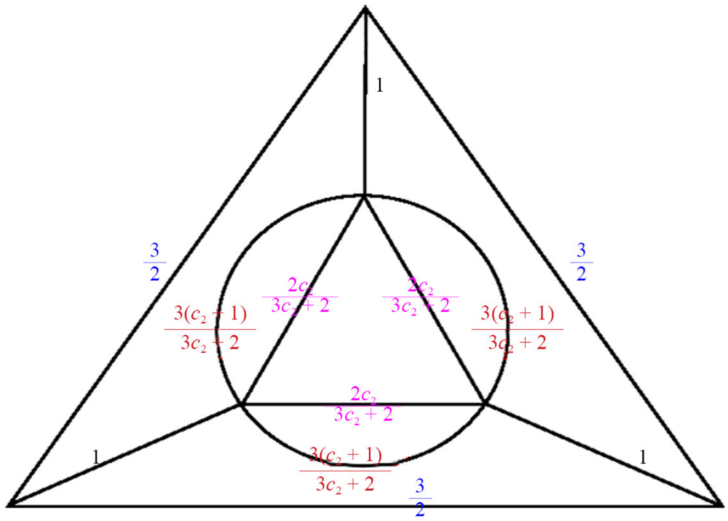
G_3


1
1
1
1
1
1
3/2
3/2
3/2
1/(3c2 + 2)
1/(3c2 + 2)
1/(3c2 + 2)
3c2/(3c2 + 2)
3c2/(3c2 + 2)
3c2/(3c2 + 2)

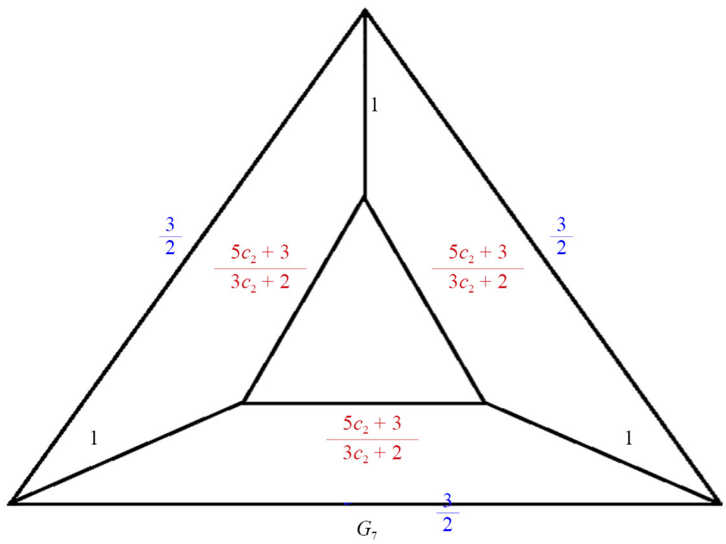
G_4


1
1
1
3/2
3/2
3/2
3(c2 + 2)/(3c2 + 2)
3(c2 + 2)/(3c2 + 2)
3(c2 + 2)/(3c2 + 2)
6c2/(3c2 + 2)
6c2/(3c2 + 2)
6c2/(3c2 + 2)

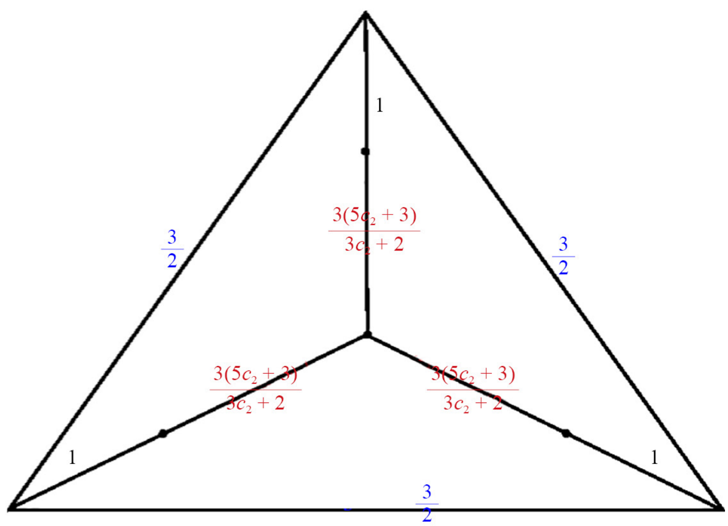
G_5


1
$\frac{3}{2}$
$\frac{3}{2}$
$\frac{2c_2}{3c_2+2}$
$\frac{2c_2}{3c_2+2}$
$\frac{3(c_2+1)}{3c_2+2}$
$\frac{3(c_2+1)}{3c_2+2}$
$\frac{2c_2}{3c_2+2}$
1
1
$\frac{3(c_2+1)}{3c_2+2}$
$\frac{3}{2}$

G_6


1
$\frac{3}{2}$
$\frac{3}{2}$
$\frac{5c_2+3}{3c_2+2}$
$\frac{5c_2+3}{3c_2+2}$
1
$\frac{5c_2+3}{3c_2+2}$
1
$\frac{3}{2}$

G_7


1
$\frac{3(5c_2+3)}{3c_2+2}$
$\frac{3}{2}$
$\frac{3}{2}$
$\frac{3(5c_2+3)}{3c_2+2}$
$\frac{3(5c_2+3)}{3c_2+2}$
1
1
$\frac{3}{2}$

G_8

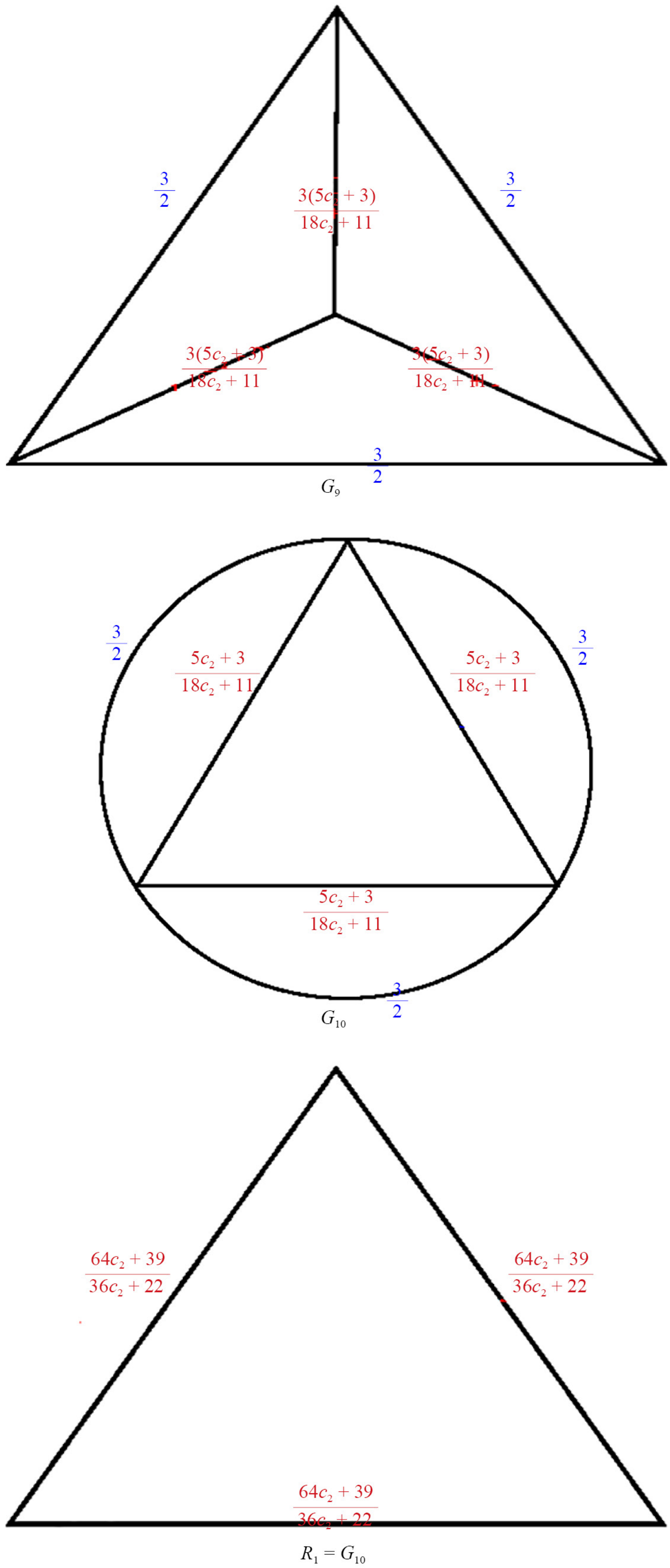


Figure 8. The transformations from $\mathscr{R}_2$ to $\mathscr{R}_1$

The following transformations result from using the attributes listed in Section 2:

$$\tau(G_1) = \left[\frac{1}{2}\right]^3 \tau(\mathscr{R}_2),\ \tau(G_2) = \tau(G_1),\ \tau(G_3) = 9c_2\tau(G_2),$$

$$\tau(G_4) = \left[\frac{1}{3c_{2+}1}\right]^3 \tau(G_3),\ \tau(G_5) = \tau(G_4),\ \tau(G_6) = \frac{3c_{2+}2}{18c_2}\tau(G_5),$$

$$\tau(G_7) = \tau(G_6),\ \tau(G_8) = \frac{9\,(5c_{2+}3)}{(3c_2+2)}\tau(G_7),\ \tau(G_9) = \left[\frac{3c_{2+}2}{18c_{2+}11}\right]^3 \tau(G_8),$$

$$\tau(G_{10}) = \frac{18c_{2+}11}{9(5c_{2+}3)}\tau(G_9) \text{ and } \tau(\mathscr{R}_1) = \tau(G_{10}).$$

When these eleven transformations are combined, we obtain

$$\tau(\mathscr{R}_2) = 4(36c_2+22)^2\tau(\mathscr{R}_1). \tag{42}$$

Further

$$\tau(\mathscr{R}_n) = \prod_{i=2}^{n} 4(36c_2+22)^2\tau(\mathscr{R}_1) = 3\times(4)^{n-1}c_1^2\left[\prod_{i=2}^{n}(36c_i+22)\right]^2 \tag{43}$$

where $c_{i-1} = \dfrac{64c_i+39}{36c_i+22}$, $i = 2,\ 3,\ \ldots,\ n$. Its characteristic equation is $36\alpha^2 - 42\alpha - 39 = 0$, which have two roots $\alpha_1 = \dfrac{7-\sqrt{205}}{12}$ and $\alpha_2 = \dfrac{7+\sqrt{205}}{12}$. Subtracting both roots from each side of $c_{i-1} = \dfrac{64c_i+39}{36c_i+22}$, we get

$$c_{i-1} - \frac{7-\sqrt{205}}{12} = \frac{64c_i+39}{36c_i+22} - \frac{7-\sqrt{205}}{12} = \left(43+3\sqrt{205}\right).\frac{c_i - \dfrac{7-\sqrt{205}}{12}}{(36c_i+22)}; \tag{44}$$

$$c_{i-1} - \frac{7+\sqrt{205}}{12} = \frac{64c_i+39}{36c_i+22} - \frac{7+\sqrt{205}}{12} = \left(43-3\sqrt{205}\right).\frac{c_i - \dfrac{7+\sqrt{205}}{12}}{(36c_i+22)}. \tag{45}$$

Let $d_i = \dfrac{c_i - \dfrac{7-\sqrt{205}}{12}}{c_i - \dfrac{7+\sqrt{205}}{12}}$. Then by Equations (44) and (45), we get $d_{i-1} = \left(\dfrac{1847+129\sqrt{205}}{2}\right) d_i$ and $d_i = \left(\dfrac{1847+129\sqrt{205}}{2}\right)^{n-i} d_n$.

Therefore

$$c_i = \frac{\left(\dfrac{1847+129\sqrt{205}}{2}\right)^{n-i} \left(\dfrac{7+\sqrt{205}}{12}\right) d_n + \dfrac{7-\sqrt{205}}{12}}{\left(\dfrac{1847+129\sqrt{205}}{2}\right)^{n-i} d_n - 1}.$$

Thus

$$c_1 = \frac{\left(\dfrac{1847+129\sqrt{205}}{2}\right)^{n-1} \left(61+5\sqrt{205}\right) + 3(7-\sqrt{205})}{2\left(\dfrac{1847+129\sqrt{205}}{2}\right)^{n-1} \left(23+\sqrt{205}\right) + 36}. \tag{46}$$

Using the formula $c_{n-1} = \dfrac{64c_n+39}{36c_n+22}$ and designating the coefficients of $64c_n+39$ and $36c_n+22$ as a_n and b_n we have

$$36c_n + 22 = a_0\left(64c_n+39\right) + b_0\left(36c_n+22\right),$$

$$36c_{n-1} + 22 = \frac{a_1\left(64c_n+39\right) + b_1\left(36c_n+22\right)}{a_0\left(64c_n+39\right) + b_0\left(36c_n+22\right)},$$

$$36c_{n-2} + 22 = \frac{a_2\left(64c_n+39\right) + b_2\left(36c_n+22\right)}{a_1\left(64c_n+39\right) + b_1\left(36c_n+22\right)},$$

$$\vdots$$

$$36c_{n-i} + 22 = \frac{a_i\left(64c_n+39\right) + b_i\left(36c_n+22\right)}{a_{i-1}\left(64c_n+39\right) + b_{i-1}\left(36c_n+22\right)}, \tag{47}$$

$$36c_{n-(i+1)} + 22 = \frac{a_{i+1}\left(64c_n+39\right) + b_{i+1}\left(36c_n+22\right)}{a_i\left(64c_n+39\right) + b_i\left(36c_n+22\right)}, \tag{48}$$

$$\vdots$$

$$36c_2+40=\frac{a_{n-2}\,(64c_n+39)+b_{n-2}\,(36c_n+22)}{a_{n-3}\,(64c_n+39)+b_{n-3}\,(36c_n+22)},$$

when Equation (47) is substituted into Equation (43), we get

$$\tau(\mathscr{R}_n)=3\times(4)^{n-1}c_1^2[a_{n-2}(64c_n+39)+b_{n-2}(36c_n+22)]^2 \tag{49}$$

where $a_0=0,\ b_0=1$ and $a_1=36,\ b_1=22$.

By the expression $c_{n-1}=\dfrac{64c_n+39}{36c_n+22}$ and Equations (47) and (48), we have

$$a_{i+1}=86a_i-4a_{i-1};\ b_{i+1}=86b_i-4b_{i-1}. \tag{50}$$

Equation (50) has the characteristic equation $\beta^2-86\beta+4=0$. Its roots are $\beta_1=43+3\sqrt{205}$ and $\beta_2=43-3\sqrt{205}$. The general solutions of Equation (50) are $a_i=h_1\beta_1^i+h_2\beta_2^i;\ b_i=k_1\beta_1^i+d_2\beta_2^i$.

Given the initial conditions $a_0=0,\ b_0=1$ and $a_1=36,\ b_1=22$, we obtain

$$a_i=\frac{6\sqrt{205}}{205}(43+3\sqrt{205})^i-\frac{6\sqrt{205}}{205}(43-3\sqrt{205})^i;$$

$$b_i=\left(\frac{205-7\sqrt{205}}{410}\right)(43+3\sqrt{205})^i+\left(\frac{205+7\sqrt{205}}{410}\right)(43-3\sqrt{205})^i. \tag{51}$$

There is no electrically similar transition for $\mathscr{R}_n$ if $c_n=1$. When Equation (51) is entered into Equation (49), we obtain

$$\tau(\mathscr{R}_n)=3\times(4)^{n-1}c_1^2\left[\left(29+83\sqrt{\frac{5}{41}}\right)(43+3\sqrt{205})^{n-2}+\left(\left(29-83\sqrt{\frac{5}{41}}\right)\right)(43-3\sqrt{205})^{n-2}\right]^2, \tag{52}$$

$n\geq 2$.

Equation (52) is satisfied for $n=1$ and $\tau(\mathscr{R}_1)=3$. Thus, the number of spanning trees in the sequence of the graph $\mathscr{R}_n$ is determined by

$$\tau(\mathscr{R}_n)=3\times(4)^{n-1}c_1^2\left(29+83\sqrt{\frac{5}{41}}\right)(43+3\sqrt{205})^{n-2}+\left(\left(29-83\sqrt{\frac{5}{41}}\right)\right)(43-3\sqrt{205})^{n-22},\ n\geq 1. \tag{53}$$

where

$$c_1 = \frac{\left(\frac{1847+129\sqrt{205}}{2}\right)^{n-1}\left(61+5\sqrt{205}\right)+3(7-\sqrt{205})}{2\left(\frac{1847+129\sqrt{205}}{2}\right)^{n-1}\left(23+\sqrt{205}\right)+36}, \; n \geq 1. \tag{54}$$

The result is obtained by inserting Equation (54) into Equation (53). □

2.5 *Numerical results*

The values of the number of spanning trees in the graphs $\mathscr{M}_n$, $\mathscr{N}_n$, $\mathscr{Q}_n$ and $\mathscr{R}_n$ are shown in the following Table 1:

Table 1. Illustrates some of the values of the number of spanning trees in the graphs $\mathscr{M}_n$, $\mathscr{N}_n$, $\mathscr{Q}_n$ and $\mathscr{R}_n$

n	$\tau(\mathscr{M}_n)$	$\tau(\mathscr{N}_n)$	$\tau(\mathscr{Q}_n)$	$\tau(\mathscr{R}_n)$
1	3	3	3	3
2	178608	138624	112908	127308
3	18102547200	10431675072	3337067712	3762879168
4	1839172800000000	785046125446656	98617002700800	111200583364608
5	186857053105152000000	59079430548947140608	2914328809629990912	3286198676466352128

3. Spanning tree entropy

After having explicit Formulas for the number of spanning trees of the sequence of the three families of graphs $\mathscr{M}_n$, $\mathscr{N}_n$, $\mathscr{Q}_n$ and $\mathscr{R}_n$, we can calculate its spanning tree entropy Z which is a finite number and a very interesting quantity characterizing the network structure, defined as in [22, 23] as: For a graph G,

$$Z(G) = \lim_{n\to\infty} \frac{\ln \tau(G)}{|V(G)|}. \tag{55}$$

$$Z(\mathscr{M}_n) = \frac{2}{9}\left(\ln\ 20\left[8+3\ \sqrt{7}\right]\right) = 1.280975924;$$

$$Z(\mathscr{N}_n) = \frac{1}{9}\left(\ln[32] - 2ln\left[97-3\sqrt{1045}\right]\right) = 1.247627878;$$

$$Z(\mathscr{Q}_n) = \frac{1}{9}\left(\ln[4] + 2ln\left[43+3\sqrt{205}\right]\right) = 1.143767379;$$

$$Z(\mathscr{R}_n) = \frac{2}{9}\left(\ln\left[86+6\sqrt{205}\right]\right) = 1.143767379.$$

Now we compare the value of entropy in our graphs with other graphs. The entropy of the graph $\mathscr{M}_n$ is larger than the entropy of the graph $\mathscr{N}_n$ and the graphs $\mathscr{Q}_n$ *and* $\mathscr{R}_n$ have the same entropy. In addition the entropy of the families $\mathscr{M}_n$ and $\mathscr{N}_n$ which have average degree $\frac{14}{3}$ is larger than the entropy of fractal scale free lattice [24] which has the entropy 1.040 and 3-prism graph of average degree 4 which has entropy 1.0445 [25] and two dimensional Sierpinski gasket [26] which has the entropy 1.166 of the same average degree 4 but the entropy of the families $\mathscr{Q}_n$ and $\mathscr{R}_n$ which have average degree $\frac{14}{3}$ is smaller than the entropy of two dimensional Sierpinski gasket.

4. Conclusions

In this work, we enumerate the number of spanning trees in the sequences of three sequences of graphs of average degree $\frac{14}{3}$ using electrically equivalent transformations. An advantage of this method lies in the avoidance of laborious computation of Laplacian spectra that is needed for a generic method for determining spanning trees.

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Conflict of interest

There is no conflict of interest for this study.

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