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Research Article

On Solvability of Group Equations Over Torsion-Free Groups

Mairaj Bibi[1*], Daniyal Asif[1], Muhammad Shoaib Arif[2], Kamaleldin Abodayeh[2]

[1] Department of Mathematics, COMSATS University Islamabad, Park Road, Islamabad, 45550, Pakistan
[2] Department of Mathematics and Sciences, College of Humanities and Sciences, Prince Sultan University, Riyadh, 11586, Saudi Arabia
E-mail: marif@psu.edu.sa

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Abstract: Let A be a group, $\langle t \rangle$ the free group generated by t, and let $r(t) \in A * \langle t \rangle$. The group equation $r(t) = 1$ is said to have a solution over A if a solution exists in some group containing A. In other words, this means that the canonical map $A \longrightarrow \left\langle \dfrac{A * \langle t \rangle}{r(t)} \right\rangle$ is injective. A group is said to be torsion-free if the order of every non-identity element is infinite. There is a conjecture (attributed to F. Levin) that every group equation over an arbitrary torsion-free group is solvable. It has been proved that this conjecture holds true for group equations of length at most seven. This study examines the solvability of group equations of length eight over a torsion-free group.

***Keywords*:** group equations, torsion-free groups, combinatorial group theory, relative presentation

MSC: 20E06, 20F05, 20F70, 20K15

1. Introduction

Let A be a non-trivial group, and t be an element distinct from A. A *group equation* in t is an expression

$$r(t) = a_1 t^{\xi_1} a_2 t^{\xi_2} a_3 t^{\xi_3} \ \dots \ a_n t^{\xi_n} = 1 \quad (a_j \in A, \ \xi_j = \pm 1),$$

where $a_{j+1} \neq 1$ if $\xi_j = -\xi_{j+1}$ for $1 \leq j \leq n-1$ (subscripts modulo n). The constants or coefficients a_j are associated with the terms, and n is referred to as the *length* of the group equation $r(t)$. A solution for the group equation $r(t)$ exists if there is an embedding ϕ of A into a group B, and an element $b \in B$ such that $\phi\ (a_1) b^{\xi_1}\ \phi\ (a_2) b^{\xi_2} \ \dots \ \phi\ (a_n) b^{\xi_n} = 1$ in B. It is a standard result in combinatorial group theory that $r(t) = 1$ has a solution if and only if the canonical map $\eta : A \longrightarrow \dfrac{A * \langle t \rangle}{C}$ is injective, where C is the conjugate closure of $r(t) = 1$ in the free product $A * \langle t \rangle$. The group equation $r(t) = 1$ is classified as follows: *Singular* if $\sum_{j=1}^{n} \xi_j = 0$; *Unimodular* if $\sum_{j=1}^{n} \xi_j = 1$; *Regular* if $\sum_{j=1}^{n} \xi_j \neq 0$; and *Positive* if $\sum_{j=1}^{n} \xi_j = n$. See [1] for more information about group equations.

A group is called *torsion* if all its elements have finite order, *torsion* $-$ *free* if the order of every non-identity element is infinite, and *mixed* if it contains elements of both finite and infinite order. Examples of torsion groups include the Klein

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four-group (K_4), the symmetric group S_3, and the multiplicative group $\{\pm 1, \pm \iota\}$. Torsion-free groups include free groups and $(\mathbb{Z}, +)$. Mixed groups include the multiplicative group of all non-zero complex numbers $(\mathbb{C}\setminus\{0\}, \times)$ and the group of all non-zero real numbers $(\mathbb{R}\setminus\{0\}, \times)$.

Combinatorial group theory involves the analysis of groups in terms of presentation, where a set of generators and defining relations are considered. In 1882, Dyck [2] introduced presentation as a new approach in mathematical research. The study of group equations in combinatorial group theory was initiated by Neumann [3] in 1943. Higman et al. [4] discussed the embedding theorem for groups in 1949 and proved the solvability of the group equation $ata^{'}t^{-1} = 1$, where $a, a^{'} \in A$. Levin [5] explored the solvability of arbitrary positive group equations, drawing motivation from the solvability of algebraic equations for fields.

The concept of solvability has been extensively explored across various mathematical domains, including fractional differential equations and nonlinear systems [6–8]. However, in the context of groups, the notion of solvability takes on a different form. It is crucial to note that not all group equations are solvable. For instance, suppose the equation $r(t) = a^{-1}ta't^{-1} = 1$ over a group A, where $a, a' \in A$ have different orders. If a solution exists in some overgroup A' of A, then there exists a monomorphism $\varphi : A \to A'$ and an element $t \in A'$ such that $\varphi(a^{-1})t\varphi(a')t^{-1} = 1$, which implies $\varphi(a') = t^{-1}\varphi(a)t$, i.e., $\varphi(a')$ is conjugate to $\varphi(a)$. Since conjugate elements have the same order and φ preserves order, we get $\mathrm{ord}(a) = \mathrm{ord}(a')$, contradicting the assumption. Hence, the equation is not solvable. This illustrates that solvability of group equations can depend heavily on the structure of the group and the properties of the equation. Two significant conjectures addressing this are the *Kervaire-Laudenbach (K-L) conjecture* [9], which asserts that every regular group equation is solvable, and the *Levin conjecture* [5], which posits that all group equations are solvable over torsion-free groups.

Significant progress has been made in the literature to verify these conjectures. Howie [10] proved these conjectures for group equations of length 3. In 1987, Stallings [11] proved that if ξ_j periodically changes sign exactly twice, then a solution exists for $r(t) = 1$ over a torsion-free group. Edjvet and Howie [12] verified the solvability of group equations of length 4. In 1993, Klyachko [13] proved the existence of a solution for unimodular group equations over a torsion-free group. Furthermore, Klyachko's approach was extended by Rourke and Fenn [14, 15]. Clifford and Goldstein [16–18], along with Clark [19], discussed the existence of solutions over torsion-free groups by using the equation's pattern. In [20], Evangelidou proved the solvability of regular group equations of length 5. Ivanov and Klyachko in [21] verified that group equations for torsion-free groups of length 6 are solvable. Recently, in [22], Bibi and Edjvet proved the Levin conjecture for length 7. In [23], the authors have made progress in establishing the Levin conjecture for a specific type of equation. In this article, we investigate the solvability of group equations of length 8. There are eighteen distinct group equations of length 8 up to inversion cyclic permutation and $t \leftrightarrow t^{-1}$. The authors in [24] solve one of these group equations. In this work, we consider eleven of the remaining seventeen group equations and examine their solvability over torsion-free groups. In what follows, we will prove the following theorem:

Theorem 1 Any of the following group equations is solvable over a torsion-free group:

- $r_1(t) = atbtctdtetftgtht = 1$
- $r_2(t) = atbtctdtetftgtht^{-1} = 1$
- $r_3(t) = atbtctdtetftgt^{-1}ht^{-1} = 1$
- $r_4(t) = atbtctdtetft^{-1}gt^{-1}ht^{-1} = 1$
- $r_5(t) = atbtctdtet^{-1}ft^{-1}gt^{-1}ht^{-1} = 1$
- $r_6(t) = atbtctdtetft^{-1}gtht^{-1} = 1$
- $r_7(t) = atbtctdt^{-1}etft^{-1}gtht^{-1} = 1$
- $r_8(t) = atbtctdtet^{-1}ftgt^{-1}ht^{-1} = 1$
- $r_9(t) = atbtctdt^{-1}etftgt^{-1}ht^{-1} = 1$
- $r_{10}(t) = atbtct^{-1}dtetft^{-1}gtht^{-1} = 1$
- $r_{11}(t) = atbtctdt^{-1}etft^{-1}gt^{-1}ht^{-1} = 1$

The rest of the paper is organized as follows: In Section 2, we discuss the methods used in the proof of the main theorem. This includes some preliminaries, definitions, and known results, followed by an overview of the techniques used. In Section 3, we present the proof of the theorem. Finally, Section 4 contains the conclusion.

2. A proof methodology

In this research, our objective is to investigate the solvability of the group equation $r(t) = 1$. The equation $r(t) = 1$ is said to be solvable over a group A if the canonical map

$$A \longrightarrow \left\langle \frac{A * \langle t \rangle}{r(t)} \right\rangle$$

is injective, where $\langle t \rangle$ is the free group generated by t, and $r(t) \in A * \langle t \rangle$. To prove the solvability, some definitions and methods are required.

2.1 *Preliminaries*

In this section, we present some essential definitions and known results that will be used throughout the paper.

Definition 1 [17] A pattern P is a singly connected network embedded in $\mathbb{R}^2$ with a single node, called the "centre", that is connected to all edges. If a corner in P is bounded by a pair of outward edges, it is called a "source". If it is bounded by inward edges, it is called a "sink", and otherwise it is "neutral". Every group equation $r(t) = 1$ has a corresponding pattern denoted by $P_{r(t)}$. The derivative pattern, denoted by $P'_{r(t)}$, is obtained by adding outward edges at each of the source corners and inward edges at each of the sink corners, followed by the removal of all initial edges of $P_{r(t)}$ (see Figure 1).

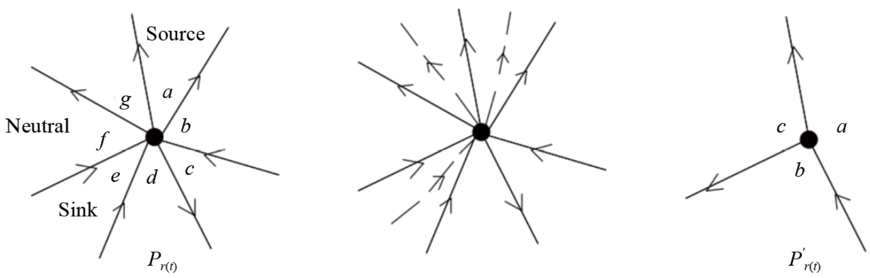

Figure 1. Pattern $P_{r(t)}$ and its derivative $P'_{r(t)}$

Theorem 2 [17] Any group equation has a solution over a torsion-free group if the group equation (derivative) up to cyclic permutation has the form

$$r(t) = \left(\prod_{m=1}^{p} a_m t^{-1} b_m t \right) \left(\prod_{n=1}^{q-1} c_n t \right) = 1,$$

where $a_m,\ b_m,\ c_n \in A$, $p \geq 1$, and $q \geq 2$.

Corollary 1 [17] If some derivative of $s(t)$=1 has the form discussed in Theorem 2, then the equation $s(t) = 1$ is solvable over the given torsion free group.

Definition 2 A pattern $P_{r(t)}$ is said to be *stable* if $P'_{r(t)} = P_{r(t)}$, and *pre-stable* if $P''_{r(t)} = P'_{r(t)}$. A pattern is stable only if it has v outgoing (or incoming) edges, where $v \in \mathbb{N}$ and $v \geq 2$.

Note that every stable pattern is also pre-stable. If not stable, a pattern $P_{r(t)}$ is pre-stable only if it falls into one of the following types:

Type 1: A pattern with a single outgoing or incoming edge.

Type 2: A pattern consisting only of alternating edges.

Type 3: A pattern of the form $(\sigma_1, \beta_1, \ldots, \sigma_n, \beta_n)$, where $n \geq 1$, $\sigma_j \geq 1$, and $\beta_j \geq 1$. Here, σ_j represents an outward edge, and β_j represents an odd number of alternating edges starting with an inward edge.

Theorem 3 [19] A solution of a group equation exists if the pattern $P^{(v)}_{r(t)}$ is pre-stable of **Type 3** for some v satisfying the following condition:

• There exists a distinct index i such that $\beta_i \geq \beta_j$ for every $j \neq i$.

Definition 3 A relative (group) presentation is a triple $\Gamma = \langle A, \ t \mid \mathbf{w} \rangle$, where $t \notin A$, and $\mathbf{w}$ is a collection of all cyclically reduced words in the free product $A * \langle t \rangle$.

This idea was introduced by Pride and Bogley [25], where aspherical relative presentations were discussed. See [10, 12] for essential definitions related to relative presentations and further details.

Theorem 4 In [25], Bogley and Pride proved that whenever the relative presentation Γ is aspherical and orientable (that is no element of $\mathbf{w}$ is a cyclic permutation of its inverse), the canonical map $\tau : A \longrightarrow \dfrac{A * \langle t \rangle}{C}$ is injective, where C is the conjugate closure of $r(t)$ in $A * \langle t \rangle$.

In our case, $\mathbf{w}$ contains only one element $r(t)$, and hence, Γ is orientable. Therefore, proving the asphericity of Γ is sufficient to establish the solvability of the equation $r(t) = 1$.

2.2 *Tests for asphericity*

We use the weight test and the curvature distribution method to establish asphericity. To explain these methods, we begin by introducing some key definitions. Here, we present only the main ideas. For a detailed discussion of both methods, see [25] for the weight test and [22] for the curvature distribution method.

Definition 4 [25] The co-initial graph Λ of the group presentation $\Gamma = \langle A, \ t \mid \mathbf{w} \rangle$ is a directed graph with vertices t and t^{-1}, and edge set $\mathbf{w}'$, consisting of all cyclic permutations of elements in $\mathbf{w} \cup \mathbf{w}^{-1}$ that begin with t or t^{-1}. For $S \in \mathbf{w}'$, we write $S = Ta_j$, where $a_j \in A$, and T begins and ends with either t or t^{-1}. We define $\iota(S)$ and $\rho(S)$ as the inverses of the first and last symbols of T, respectively, and let $\lambda(S) = a_j$.

Definition 5 [25] A *weight function* ϑ on the co-initial graph Λ is a real-valued function defined on its edges. For any edge $c \in \Lambda$, we require that $\vartheta(c) = \vartheta(\overline{c})$, where $\overline{c}$ denotes the edge inverse to c.

2.2.1 *Weight test*

A weight function ϑ on the co-initial graph Λ is called *aspherical* if it satisfies the following conditions:

1. Let $S \in \mathbf{w}$, where $S = t_1^{\xi_1} a_1 t_2^{\xi_2} a_2 \cdots t_n^{\xi_n} a_n$. Then

$$\sum_{j=1}^{n} \left[1 - \vartheta \left(t_j^{\xi_j} a_j \cdots t_n^{\xi_n} a_n t_1^{\xi_1} a_1 \cdots t_{j-1}^{\xi_{j-1}} a_{j-1} \right) \right] \geq 2.$$

2. Every admissible closed path in the co-initial graph Λ has weight at least 2.
3. The weight of every edge in the co-initial graph Λ is non-negative.

Theorem 5 [25] If the co-initial graph Λ admits an aspherical weight function ϑ, then the presentation Γ is aspherical.

2.2.2 *Curvature distribution method*

Let K be a reduced spherical diagram with an angle function δ defined on its corners [10]. The curvature function ϕ at a vertex h, with corners κ in the diagram K, is given by

$$\phi(h) = 2\pi - \sum_{\kappa \in h} \delta(\kappa),$$

and the curvature of a region Ω in K is defined as

$$\phi(\Omega) = 2\pi - \sum_{\kappa \in \Lambda} (\pi - h(\kappa)).$$

For a vertex h of degree d, we assign an angle $\frac{2\pi}{d}$ to each corner κ of the vertex. Then the curvature at h is

$$2\pi - \frac{2\pi}{d} \cdot d = 0.$$

On the other hand, the curvature of a region Ω of degree m is given by

$$\phi(\Omega) = 2\pi - m\pi + \sum_{j=1}^{m} \frac{2\pi}{d_j},$$

where d_j is the degree of the vertex h_j associated with the j-th corner of Ω.

This method asserts that if a diagram K contains no dipole over Γ, then, by the Gauss-Bonnet (or Euler) formula, the total curvature of all the regions in K is equal to 4π. It follows that K must contain at least one region with positive curvature. For each region Ω of K with positive curvature $\phi(\Omega)$, there exists a neighboring region $\widehat{\Omega}$, uniquely associated with Ω, such that

$$\phi(\widehat{\Omega}) + \phi(\Omega) \leq 0.$$

Hence, the total curvature of all regions in K is not positive, implying that Γ is aspherical.

3. A proof of the theorem

We now prove the main theorem concerning the solvability of group equations of length 8 over torsion-free groups.

Theorem 6 Each of the following group equations is solvable over a torsion-free group:

$$r_1(t) = atbtctdtetftgtht = 1$$

$$r_2(t) = atbtctdtetftght^{-1} = 1$$

$$r_3(t) = atbtctdtetftgt^{-1}ht^{-1} = 1$$

$$r_4(t) = atbtctdtetft^{-1}gt^{-1}ht^{-1} = 1$$

$$r_5(t) = atbtctdtet^{-1}ft^{-1}gt^{-1}ht^{-1} = 1$$

$$r_6(t) = atbtctdtetft^{-1}ght^{-1} = 1$$

$$r_7(t) = atbtctdt^{-1}etft^{-1}ght^{-1} = 1$$

Proof. The equation

$$r_1(t) = atbtctdtetftgtht = 1$$

is a positive group equation, and therefore has a solution over a torsion-free group [5]. If the exponent of t changes sign exactly twice in a group equation, then the equation is solvable over a torsion-free group [11]. Since this condition is satisfied by each of the following equations, it follows that they are solvable:

$$r_2(t) = atbtctdtetftgtht^{-1} = 1,$$

$$r_3(t) = atbtctdtetftgt^{-1}ht^{-1} = 1,$$

$$r_4(t) = atbtctdtetft^{-1}gt^{-1}ht^{-1} = 1,$$

$$r_5(t) = atbtctdtet^{-1}ft^{-1}gt^{-1}ht^{-1} = 1.$$

The group equation

$$r_6(t) = atbtctdtetft^{-1}ght^{-1} = 1,$$

with $p = 2$ and $q-1 = 4$, has the form

$$r(t) = \left(\prod_{n=1}^{q-1} c_n t\right)\left(\prod_{m=1}^{p} a_m t b_m t^{-1}\right) = 1,$$

where $a_m,\ b_m,\ c_n \in A$. Therefore, the equation $r_6(t) = 1$ has a solution over a torsion-free group. Similarly, the group equation

$$r_7(t) = atbtctdt^{-1}etft^{-1}ght^{-1} = 1,$$

with $p = q = 3$, also fits the same form, so $r_7(t) = 1$ has a solution over a torsion-free group. □

Theorem 7 Each of the following group equations is solvable over a torsion-free group:

$$r_8(t) = atbtctdtet^{-1}ftgt^{-1}ht^{-1} = 1$$

$$r_9(t) = atbtctdt^{-1}etftgt^{-1}ht^{-1} = 1$$

$$r_{10}(t) = tbtct^{-1}dtetft^{-1}gtht^{-1} = 1$$

Proof. The derivative $P'_{r_8(t)}$ of the pattern $P_{r_8(t)}$, associated with the equation $r_8(t) = atbtctdtet^{-1}ftgt^{-1}ht^{-1} = 1$, is shown in Figure 2. The group equation associated with $P'_{r_8(t)}$ has the form

$$a_1tb_1tc_1td_1t^{-1},$$

which is solvable when we substitute $p = 1$ and $q = 3$ into the form

$$r(t) = \left(\prod_{m=1}^{p} a_m t^{-1} b_m t\right)\left(\prod_{n=1}^{q-1} c_n t\right) = 1,$$

with $a_m,\ b_m,\ c_n \in A$. Thus, the equation $r_8(t) = 1$ is also solvable over a torsion-free group.

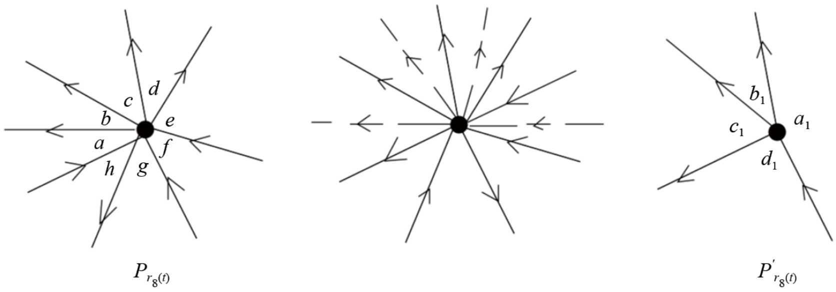

Figure 2. Pattern $P_{r_8(t)}$ and its derivative $P'_{r_8(t)}$

The derivative $P'_{r_9(t)}$ of the pattern $P_{r_9(t)}$, associated with the equation

$$r_9(t) = atbtctdt^{-1}etftgt^{-1}ht^{-1} = 1,$$

is shown in Figure 3. The group equation corresponding to $P'_{r_9(t)}$ has the form

$$a_1tb_1tc_1td_1t^{-1},$$

which is solvable when we substitute $p = 1$ and $q = 3$ into the general form

$$r(t) = \left(\prod_{m=1}^{p} a_m t^{-1} b_m t \right) \left(\prod_{n=1}^{q-1} c_n t \right) = 1,$$

where $a_m,\ b_m,\ c_n \in A$. Therefore, the equation $r_9(t) = 1$ is solvable over a torsion-free group.

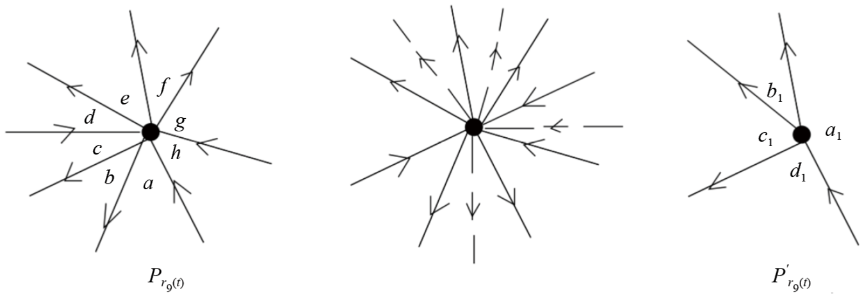

Figure 3. Pattern $P_{r_9(t)}$ and its derivative $P'_{r_9(t)}$

The derivative $P'_{r_{10}(t)}$ of the pattern $P_{r_{10}(t)}$, associated with the equation

$$r_{10}(t) = atbtct^{-1}dtetft^{-1}gtht^{-1} = 1,$$

is shown in Figure 4. Since the pattern $P_{r_{10}(t)} = (3,\ 1,\ 2,\ 3)$ is pre-stable of **Type 3**, it follows that the equation $r_{10}(t) = 1$ has a solution over a torsion-free group.

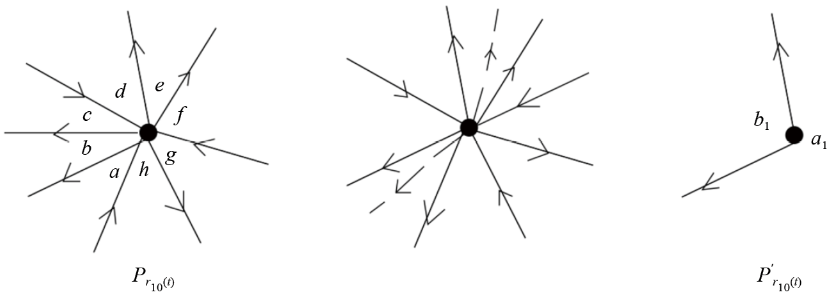

Figure 4. Pattern $P_{r_{10}(t)}$ and its derivative $P'_{r_{10}(t)}$

□

Before proceeding further, we introduce some notational conventions that will be used consistently throughout the remainder of the paper.

Remark 1 From now, we will use the following notations.

1. The inverse of an element $a \in A$ is denoted by either a^{-1} or $\bar{a}$.

2. A vertex of each region Ω is denoted by v_j, where $j \in \{a^{\pm 1},\ b^{\pm 1},\ c^{\pm 1},\ d^{\pm 1},\ e^{\pm 1},\ f^{\pm 1},\ g^{\pm 1},\ h^{\pm 1}\}$ is the corner label in Ω.

3. The label of a vertex v_j is denoted by $l_\Omega(v_j)$.

4. The degree of a vertex (the number of edges incident to it) is denoted by $d_\Omega(v_j)$.

5. Set notation may be used when a vertex has more than one possible labeling. For example, $l_\Omega(v_d) = fd\{bc^{-1}, bg\}$ indicates that $l_\Omega(v_d)$ may be either $fdbc^{-1}$ or $fdbg$.

Theorem 8 The following group equation is solvable over a torsion-free group:

$$r_{11}(t) = atbtctdt^{-1}etft^{-1}gt^{-1}ht^{-1} = 1$$

Proof. If we apply the change of variable $u = tb$, $u \notin A \cup \{t\}$, then the equation $r_{11}(t) = 1$ becomes

$$r_{11}(t) = au^2b^{-1}cub^{-1}dbu^{-1}eub^{-1}fbu^{-1}gbu^{-1}hbu^{-1}$$

$$= a_1u^2a_2ua_3u^{-1}a_4ua_5u^{-1}a_6u^{-1}a_7u^{-1}$$

where $a_1, ..., a_7 \in A$ and $a_1, a_3, a_4, a_5 \neq 1$. So without loss of generality, we can say $b = 1$ in A. We make this assumption from now on, but retain the symbol b for notational simplicity.

Now, the presentation Γ is $< A, t \mid r_{11}(t) >$, where

$$r_{11}(t) = atbtctdt^{-1}etft^{-1}gt^{-1}ht^{-1} = 1 \text{ } (a, e, d, f \in A\backslash\{1\} \text{ and } b = 1, c, g, h \in A)$$

Moreover, A is not cyclic and it is generated by $\{a, b, c, d, e, f, g, h\}$. Figure 5a illustrates the co-initial graph Λ for Γ. The region of a reduced spherical diagram K is given by Ω in 5b.

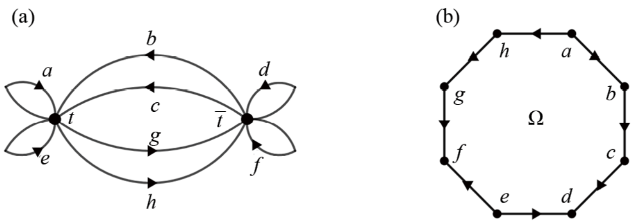

Figure 5. Co-initial graph Λ for Γ and region Ω of K

By using the co-initial graph Λ and the assumption that group A is torsion-free, the probable labels of vertices having degree 2 for a given region Ω of diagram are (up to inversion and cyclic permutation)

$$V = \{ae^{\pm 1}, cg, ch, bg, bh, bc^{-1}, gh^{-1}, f^{\pm 1}d\}.$$

We can continue over work with $t \leftrightarrow t^{-1}$, modulo inversion, cyclic permutation, and the equivalences

$$a \leftrightarrow a^{-1}, b \leftrightarrow h^{-1}, c \leftrightarrow g^{-1}, d \leftrightarrow f^{-1}, e \leftrightarrow e^{-1}.$$

We'll continue in accordance with the number of elements of V that are admissible [21] and categorise the cases properly [12]. The Remark 2 minimize the numbers of cases to be studied.

Remark 2 We will consider the following observations while consider our list of cases.

(a) Since $\phi(2,\ 2,\ 3,\ 3,\ 3,\ 3,\ 3,\ 3) = 0$, at least 3 vertices must be of degree 2 in order to have a region of positive curvature.

(b) If ae^{-1} and ae are both admissible then $e^2 = 1$, a contradiction.

(c) If df^{-1} and df are both admissible then $f^2 = 1$, a contradiction.

(d) If any two of either $\{bh,\ bc^{-1},\ ch\}$, $\{bg,\ bc^{-1},\ cg\}$, $\{cg,\ gh^{-1},\ ch\}$, $\{bg,\ gh^{-1},\ bh\}$ are admissible, then so is the third.

(e) Different from the triplets in (d), if any three of $\{ch,\ bh,\ bc^{-1},\ gh^{-1},\ cg,\ bg\}$ are admissible, then the remaining are also admissible.

Checking shows that there are 84 cases for $r_{11}(t) = 1$. From these 84 cases, 81 cases are solved by weight test and the remaining 3 cases are solved by using the curvature distribution method.

Remark 3 In the following, we will use the assumption that the group A is torsion-free, non-cyclic and $a,\ e,\ d,\ f$ are non-trivial elements in group A.

The Lemmas 1-8 are proved by using the weight test. We show that there exists an aspherical weight function ϑ for the co-initial graph Λ of Γ.

Lemma 1 If any of the following sets of relations holds in A, then the relative (group) presentation $\Gamma = < A,\ t \mid r_{11}(t) >$ is aspherical:

1. $a = e^{-1}, d = f^{-1}$
2. $a = e, d = f^{-1}$
3. $a = e, d = f$
4. $a = e^{-1}, d = f$.

Proof. Consider $a = e$, $d = f$. Then, the relator is

$$r_{11}(t) = atbtctdt^{-1}etft^{-1}gt^{-1}ht^{-1} = 1.$$

The presentation Γ has co-initial graph Λ, which is shown in Figure 6, where

$$\rho_1 \leftrightarrow a,\ \rho_2 \leftrightarrow b,\ \rho_3 \leftrightarrow c,\ \rho_4 \leftrightarrow d,\ \rho_5 \leftrightarrow e,\ \rho_6 \leftrightarrow f,\ \rho_7 \leftrightarrow g,\ \rho_8 \leftrightarrow h.$$

Assign the weights

$$\vartheta(\rho_1) = \vartheta(\rho_4) = \vartheta(\rho_5) = \vartheta(\rho_6) = 1, \text{ and } \vartheta(\rho_2) = \vartheta(\rho_3) = \vartheta(\rho_7) = \vartheta(\rho_8) = \frac{1}{2}$$

to the edges of the co-initial graph Λ. Then,

$$\sum_{j=1}^{8}(1 - \vartheta(\rho_j)) = 2,$$

which confirms that criterion 1 of the weight test is met. Furthermore, any admissible cycle in Λ with weight less than 2 leads to a contradiction. For example, the paths

$$cg,\ bg,\ bh,\ ch,\ bc^{-1}$$

are admissible closed paths of length 2, each with weight less than 2. This would imply

$$cg = bg = bh = ch = bc^{-1} = 1,$$

which contradicts the assumption that A is torsion-free. Hence, criterion 2 is also satisfied. Finally, since each edge has positive weight, criterion 3 is clearly met. □

Lemma 2 If any of the following sets of relations holds in A, then the relative (group) presentation $\Gamma = < A,\ t \mid r_{11}(t) >$ is aspherical:

1. $b^{-1} = h = c^{-1}$
2. $b^{-1} = g = c^{-1}$
3. $g = h = c^{-1}$
4. $g = h = b^{-1}$.

Proof. We will consider $g = h = b^{-1}$ and the rest of the cases will follows in a similar way. The relator is

$$r_{11}(t) = atbtctdt^{-1}etft^{-1}gt^{-1}ht^{-1} = 1.$$

The presentation Γ has co-initial graph Λ, shown in Figure 6, where

$$\rho_1 \leftrightarrow a,\ \rho_2 \leftrightarrow b,\ \rho_3 \leftrightarrow c,\ \rho_4 \leftrightarrow d,\ \rho_5 \leftrightarrow e,\ \rho_6 \leftrightarrow f,\ \rho_7 \leftrightarrow g,\ \rho_8 \leftrightarrow h.$$

Assign weights to the edges of the co-initial graph Λ as follows:

$$\vartheta(\rho_1) = \vartheta(\rho_4) = \vartheta(\rho_5) = \vartheta(\rho_6) = 1, \text{ and } \vartheta(\rho_2) = \vartheta(\rho_3) = \vartheta(\rho_7) = \vartheta(\rho_8) = \frac{1}{2}$$

Then,

$$\sum_{j=1}^{8}(1 - \vartheta(\rho_j)) = 2,$$

which confirms that criterion 1 of the weight test is met. Furthermore, any admissible closed path in Λ with weight less than 2 leads to a contradiction, so criterion 2 is also satisfied. Finally, since each edge has positive weight, criterion 3 is clearly met. □

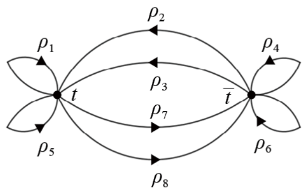

Figure 6. Co-initial graph

Lemma 3 If any of the following sets of relations holds in A, then the relative (group) presentation $\Gamma = < A,\ t \mid r_{11}(t) >$ is aspherical:

1. $b = c,\ f = d^{-1}$
2. $g = b^{-1},\ f = d^{-1}$
3. $h = b^{-1},\ f = d^{-1}$
4. $c = g^{-1},\ f = d^{-1}$
5. $e = a^{-1},\ b = c$ and $f = d^{-1}$
6. $e = a^{-1},\ g = b^{-1}$ and $f = d^{-1}$
7. $a = e^{-1},\ h = b^{-1}$ and $f = d^{-1}$
8. $a = e^{-1},\ c = g^{-1}$ and $f = d^{-1}$
9. $a = e,\ b = c$ and $f = d^{-1}$
10. $a = e,\ g = b^{-1}$ and $f = d^{-1}$
11. $a = e,\ h = b^{-1}$ and $f = d^{-1}$
12. $a = e,\ c = g^{-1}$ and $f = d^{-1}$
13. $f = d^{-1},\ b = c$ and $g = h$
14. $f = d^{-1},\ g = b^{-1}$ and $c = h^{-1}$
15. $f = d^{-1},\ h = b^{-1}$ and $c = g^{-1}$
16. $a = e^{-1},\ h = b^{-1},\ c = g^{-1}$ and $f = d^{-1}$
17. $a = e,\ h = b^{-1},\ c = g^{-1}$ and $f = d^{-1}$
18. $b^{-1} = h = c^{-1}$ and $f = d^{-1}$
19. $b^{-1} = g = c^{-1}$ and $f = d^{-1}$
20. $a = e^{-1},\ f = d^{-1}$ and $b^{-1} = h = c^{-1}$
21. $a = e,\ f = d^{-1}$ and $b^{-1} = h = c^{-1}$
22. $a = e^{-1},\ f = d^{-1}$ and $b^{-1} = g = c^{-1}$
23. $a = e,\ f = d^{-1}$ and $b^{-1} = g = c^{-1}$
24. $h = g = c^{-1} = b{-}1$ and $f = d^{-1}$
25. $a = e^{-1},\ f = d^{-1}$ and $h = g = c^{-1} = b^{-1}$
26. $a = e,\ f = d^{-1}$ and $h = g = c^{-1} = b^{-1}$.

Proof. We consider the case $a = e$, $f = d^{-1}$, and $b^{-1} = g = c^{-1}$. Then, the relator is

$$r_{11}(t) = atbtctdt^{-1}etft^{-1}gt^{-1}ht^{-1} = 1.$$

We define $u = t^{-1}dt$ to obtain

$$S_1 = at^2uau^{-1}t^{-1}ht^{-1}, \text{ and } \quad S_2 = u^{-1}tdt^{-1}.$$

The presentation Γ has co-initial graph Λ, which is shown in Figure 7, where

$$\rho_1 \leftrightarrow a,\ \rho_2 \leftrightarrow 1,\ \rho_3 \leftrightarrow 1,\ \rho_4 \leftrightarrow a,\ \rho_5 \leftrightarrow 1,\ \rho_6 \leftrightarrow h,$$

$$\varsigma_1 \leftrightarrow 1,\ \varsigma_2 \leftrightarrow 1,\ \varsigma_3 \leftrightarrow d,\ \varsigma_4 \leftrightarrow 1.$$

Assign the weights

$$\vartheta(\varsigma_1) = \vartheta(\varsigma_2) = \vartheta(\rho_1) = \vartheta(\rho_4) = 0,\ \vartheta(\rho_2) = \vartheta(\rho_3) = \vartheta(\rho_5) = \vartheta(\rho_6) = \vartheta(\varsigma_3) = 1$$

to the edges of the co-initial graph. Then, the equations

$$\sum_{j=1}^{6}(1-\vartheta(\rho_j)) = \sum_{j=1}^{3}(1-\vartheta(\varsigma_j)) = 2$$

confirm that criterion 1 of the weight test is satisfied. We consider the case $a = e,\ f = d^{-1}$, and $b^{-1} = g = c^{-1}$. Furthermore, every closed path in Λ having weight less than 2 has label a^j or d^j, where $j \in \mathbb{Z}\setminus\{0\}$ and $a,\ d \in A\setminus\{1\}$. This implies that a and d have finite order, contradicting the assumption that A is torsion-free. Hence, criterion 2 is also satisfied. Finally, since every edge has non-negative weight, criterion 3 is clearly satisfied. □

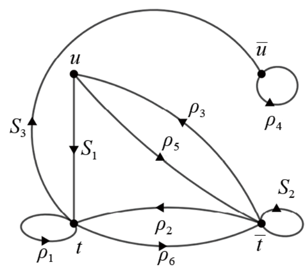

Figure 7. Co-initial graph for $\hat{\Gamma} =< A,\ t,\ u \mid S_1,\ S_2 >$ where $S_1 = at^2uau^{-1}t^{-1}ht^{-1}$ and $S_2 = u^{-1}tdt^{-1}$

Lemma 4 If any of the following sets of relations holds in A, then the relative (group) presentation $\Gamma =< A,\ t \mid r_{11}(t) >$ is aspherical:

1. $b = c,\ f = d$
2. $g = b^{-1},\ f = d$
3. $h = b^{-1},\ f = d$
4. $c = g^{-1},\ f = d$
5. $e = a^{-1},\ b = c$ and $f = d$
6. $e = a^{-1},\ g = b^{-1}$ and $f = d$
7. $a = e^{-1},\ h = b^{-1}$ and $f = d$
8. $a = e^{-1},\ c = g^{-1}$ and $f = d$
9. $a = e,\ b = c$ and $f = d$
10. $a = e,\ g = b^{-1}$ and $f = d$
11. $a = e,\ h = b^{-1}$ and $f = d$
12. $a = e,\ c = g^{-1}$ and $f = d$

13. $f = d,\ b = c$ and $g = h$
14. $f = d,\ g = b^{-1}$ and $c = h^{-1}$
15. $f = d,\ h = b^{-1}$ and $c = g^{-1}$
16. $a = e^{-1},\ h = b^{-1},\ c = g^{-1}$ and $f = d$
17. $a = e,\ h = b^{-1},\ c = g^{-1}$ and $f = d$
18. $b^{-1} = h = c^{-1}$ and $f = d$
19. $b^{-1} = g = c^{-1}$ and $f = d$
20. $a = e^{-1},\ f = d$ and $b^{-1} = h = c^{-1}$
21. $a = e,\ f = d$ and $b^{-1} = h = c^{-1}$
22. $a = e^{-1},\ f = d$ and $b^{-1} = g = c^{-1}$
23. $a = e,\ f = d$ and $b^{-1} = g = c^{-1}$
24. $h = g = c^{-1} = b{-}1$ and $f = d$
25. $a = e^{-1},\ f = d$ and $h = g = c^{-1} = b^{-1}$
26. $a = e,\ f = d$ and $h = g = c^{-1} = b^{-1}$.

Proof. We consider the case $a = e^{-1}$, $f = d$, and $h = g = c^{-1} = b^{-1}$. Then, the relator is

$$r_{11}(t) = atbtctdt^{-1}etft^{-1}gt^{-1}ht^{-1} = 1.$$

Define $u = t^{-1}dt$, so we obtain the relators

$$S_1 = at^2ua^{-1}u^{-1}t^{-2}, \text{ and } \quad S_2 = u^{-1}tdt^{-1}.$$

The presentation Γ has co-initial graph Λ, which is shown in Figure 8, where

$$\rho_1 \leftrightarrow a,\ \rho_2 \leftrightarrow 1,\ \rho_3 \leftrightarrow 1,\ \rho_4 \leftrightarrow a^{-1},\ \rho_5 \leftrightarrow 1,\ \rho_6 \leftrightarrow 1,$$

$$\varsigma_1 \leftrightarrow 1,\ \varsigma_2 \leftrightarrow d,\ \varsigma_3 \leftrightarrow 1.$$

Assign the weights

$$\vartheta(\varsigma_2) = \vartheta(\varsigma_3) = \vartheta(\rho_1) = \vartheta(\rho_3) = 0,\ \vartheta(\varsigma_1) = \vartheta(\rho_2) = \vartheta(\rho_4) = \vartheta(\rho_5) = \vartheta(\rho_6) = 1$$

to the edges of the co-initial graph Λ. Then,

$$\sum_{j=1}^{6}(1 - \vartheta(\rho_j)) = \sum_{j=1}^{3}(1 - \vartheta(\varsigma_j)) = 2,$$

which confirms that criterion 1 of the weight test is satisfied. Furthermore, every closed path in Λ having weight less than 2 has label a^j or d^j, where $j \in \mathbb{Z} \setminus \{0\}$ and $a,\ d \in A \setminus \{1\}$. This implies that the orders of a and d are finite, which contradicts the assumption that A is torsion-free. Therefore, criterion 2 of the weight test is also satisfied. Finally, since each edge has non-negative weight, criterion 3 is clearly satisfied. The rest of the cases follows in a similar way. □

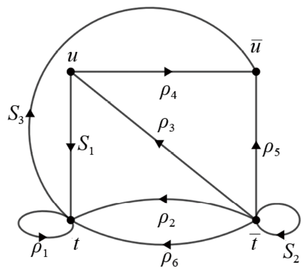

Figure 8. Co-initial graph for $\hat{\Gamma} = < A,\ t,\ u \mid S_1,\ S_2 >$ where $S_1 = at^2ua^{-1}u^{-1}t^{-1}t^{-1}$ and $S_2 = u^{-1}tdt^{-1}$

Lemma 5 If any of the following sets of relations holds in A, then the relative (group) presentation $\Gamma = < A,\ t \mid r_{11}(t) >$ is aspherical:
1. $a = e^{-1}$, $g = b^{-1}$
2. $a = e^{-1}$, $c = g^{-1}$
3. $a = e^{-1}$, $b = c$
4. $a = e^{-1}$, $h = b^{-1}$
5. $a = e^{-1}$, $g = b^{-1}$ and $c = h^{-1}$
6. $a = e^{-1}$, $b = c$ and $g = h$
7. $b^{-1} = g = c^{-1}$ and $a = e^{-1}$
8. $b^{-1} = h = c^{-1}$ and $a = e^{-1}$
9. $h = g = c^{-1} = b^{-1}$ and $a = e^{-1}$
10. $a = e^{-1}$, $h = b^{-1}$ and $c = g^{-1}$.

Proof. We consider the case $h = g = c^{-1} = b^{-1}$ and $a = e^{-1}$. The relator is

$$r_{11}(t) = atbtctdt^{-1}etft^{-1}gt^{-1}ht^{-1} = 1.$$

Define $u = t^{-1}at$, so we obtain the relators

$$S_1 = ut^2du^{-1}ft^{-2}, \text{ and } \quad S_2 = u^{-1}t^{-1}at.$$

The presentation Γ has co-initial graph Λ, which is shown in Figure 9, where

$$\rho_1 \leftrightarrow 1,\ \rho_2 \leftrightarrow 1,\ \rho_3 \leftrightarrow d,\ \rho_4 \leftrightarrow f,\ \rho_5 \leftrightarrow 1,\ \rho_6 \leftrightarrow 1,$$

$$\varsigma_1 \leftrightarrow 1,\ \varsigma_2 \leftrightarrow a,\ \varsigma_3 \leftrightarrow 1.$$

Assign the weights

$$\vartheta(\rho_1) = \vartheta(\rho_3) = \vartheta(\rho_4) = \vartheta(\rho_6) = \vartheta(\varsigma_1) = \vartheta(\varsigma_3) = \frac{1}{2},\ \vartheta(\rho_2) = \vartheta(\rho_5) = 1,\ \vartheta(\varsigma_2) = 0$$

to the edges of the co-initial graph Λ. Then,

$$\sum_{j=1}^{6}(1-\vartheta(\rho_j)) = \sum_{j=1}^{3}(1-\vartheta(\varsigma_j)) = 2,$$

which confirms that criterion 1 of the weight test is satisfied. Furthermore, every closed path in Λ with weight less than 2 has label a^j, d^j, or f^j, where $j \in \mathbb{Z} \setminus \{0\}$ and $a,\ d,\ f \in A \setminus \{1\}$. This implies that the orders of $a,\ d$, and f are finite, which contradicts the assumption that A is torsion-free. Hence, criterion 2 of the weight test is also satisfied. Finally, since each edge has non-negative weight, criterion 3 is clearly satisfied. The rest of the cases will follows in a similar way. □

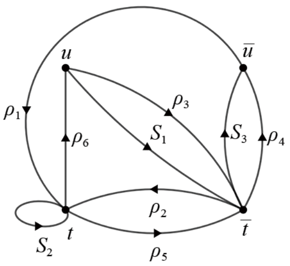

Figure 9. Co-initial graph for $\hat{\Gamma} = < A,\ t,\ u \mid S_1,\ S_2 >$ where $S_1 = ut^2du^{-1}ft^{-1}t^{-1}$ and $S_2 = u^{-1}t^{-1}at$

Lemma 6 If any of the following sets of relations holds in A, then the relative (group) presentation $\Gamma = < A,\ t \mid r_{11}(t) >$ is aspherical:

1. $a = e,\ g = b^{-1}$
2. $a = e,\ b = c$
3. $a = e,\ h = b^{-1}$
4. $a = e,\ c = g^{-1}$
5. $a = e,\ g = b^{-1}$ and $c = h^{-1}$
6. $a = e,\ b = c$ and $g = h^{-1}$
7. $b^{-1} = g = c^{-1}$ and $a = e$
8. $b^{-1} = h = c^{-1}$ and $a = e$
9. $h = g = c^{-1} = b^{-1}$ and $a = e$ 10. $a = e,\ h = b^{-1}$ and $c = g^{-1}$.

Proof. We consider the case $b^{-1} = g = c^{-1}$ and $a = e$. The relator is

$$r_{11}(t) = atbtctdt^{-1}etft^{-1}gt^{-1}ht^{-1} = 1.$$

Define $u = t^{-1}at$, so we obtain the relators

$$S_1 = ut^2duft^{-2}h, \text{ and } \quad S_2 = u^{-1}t^{-1}at.$$

The new presentation $\hat{\Gamma}$ has co-initial graph Λ, shown in Figure 10, where

$$\rho_1 \leftrightarrow 1,\ \rho_2 \leftrightarrow 1,\ \rho_3 \leftrightarrow f,\ \rho_4 \leftrightarrow d,\ \rho_5 \leftrightarrow 1,\ \rho_6 \leftrightarrow h,$$

$$\varsigma_1 \leftrightarrow 1,\ \varsigma_2 \leftrightarrow a,\ \varsigma_3 \leftrightarrow 1.$$

Assign the weights

$$\vartheta(\rho_1) = \vartheta(\rho_3) = \vartheta(\rho_4) = \vartheta(\rho_6) = \vartheta(\varsigma_1) = \vartheta(\varsigma_3) = \frac{1}{2},\ \vartheta(\rho_2) = \vartheta(\rho_5) = 1,\ \vartheta(\varsigma_2) = 0$$

to the edges of the co-initial graph Λ. Then,

$$\sum_{j=1}^{6}(1-\vartheta(\rho_j)) = \sum_{j=1}^{3}(1-\vartheta(\varsigma_j)) = 2,$$

which confirms that criterion 1 of the weight test is satisfied. Furthermore, every closed path in Λ with weight less than 2 has label a^j, d^j, or f^j, where $j \in \mathbb{Z} \setminus \{0\}$ and $a,\ d,\ f \in A \setminus \{1\}$. This implies that $a,\ d$, and f have finite order, which contradicts the assumption that A is torsion-free. Therefore, criterion 2 of the weight test is also satisfied. Finally, since each edge has non-negative weight, criterion 3 is clearly satisfied. The remaining cases follow in a similar manner. $\square$

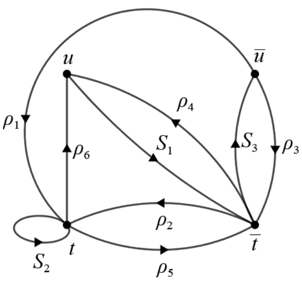

Figure 10. Co-initial graph for $\hat{\Gamma} = < A,\ t,\ u \mid S_1,\ S_2 >$ where $S_1 = ut^2duft^{-1}t^{-1}h$ and $S_2 = u^{-1}t^{-1}at$

Lemma 7 If the set of relation $h = b^{-1}, c = g^{-1}$ is hold in A, then the relative (group) presentation $\Gamma = < A,\ t \mid r_{11}(t) >$ is aspherical.

Proof. The relator is

$$r_{11}(t) = atbtctdt^{-1}etft^{-1}gt^{-1}ht^{-1} = 1.$$

Define $u = ttct$, so we obtain the relators

$$S_1 = audt^{-1}etfu^{-1}, \text{ and } S_2 = u^{-1}ttct.$$

The presentation Γ has co-initial graph Λ, which is shown in Figure 11, where

$$\rho_1 \leftrightarrow a,\ \rho_2 \leftrightarrow d,\ \rho_3 \leftrightarrow e,\ \rho_4 \leftrightarrow f,$$

$$\varsigma_1 \leftrightarrow 1,\ \varsigma_2 \leftrightarrow 1,\ \varsigma_3 \leftrightarrow c,\ \varsigma_4 \leftrightarrow 1.$$

Assign the weights

$$\vartheta(\rho_1) = 0,\ \vartheta(\rho_2) = \vartheta(\rho_4) = \vartheta(\varsigma_1) = \vartheta(\varsigma_2) = \vartheta(\varsigma_3) = \vartheta(\varsigma_4) = \frac{1}{2},\ \vartheta(\rho_3) = 1$$

to the edges of the co-initial graph Λ. Then,

$$\sum_{j=1}^{4}(1-\vartheta(\rho_j)) = \sum_{j=1}^{4}(1-\vartheta(\varsigma_j)) = 2,$$

which confirms that criterion 1 of the weight test is satisfied. Furthermore, every closed path in Λ with weight less than 2 has label a^j or d^j, where $j \in \mathbb{Z} \setminus \{0\}$ and $a,\ d \in A \setminus \{1\}$. This implies that a and d have finite order, contradicting the assumption that A is torsion-free. Therefore, criterion 2 of the weight test is also satisfied. Finally, since each edge has non-negative weight, criterion 3 is clearly satisfied. □

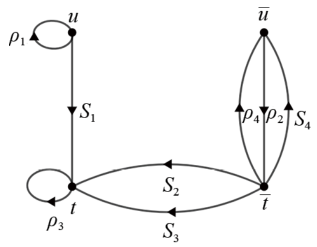

Figure 11. Co-initial graph for $\hat{\Gamma} =< A,\ t,\ u \mid S_1,\ S_2 >$ where $S_1 = audt^{-1}etfu^{-1}$ and $S_2 = u^{-1}ttct$.

The following Lemma is proved by using curvature distribution method.

Lemma 8 If any one of the following set of relations holds in A, then the relative (group) presentation $\Gamma =< A,\ t \mid r_{11}(t) >$ is aspherical:

1. $b = c,\ g = h$
2. $g = b^{-1},\ c = h^{-1}$
3. $h = g = c^{-1} = b^{-1}$.

Proof. In each of these case we show that $\phi(\Omega) \leq 0$ for all regions Ω in K, which contradicts $\phi(K) = 4\pi$.

1. In this case region Ω is given by Figure 12. Since $d_\Omega(v_b)$ and $d_\Omega(v_c)$ cannot both equal 2, also $d_\Omega(v_g)$ and $d_\Omega(v_h)$ cannot both equal 2, therefore in this case it follows that $\phi(\Omega) \leq 0$.

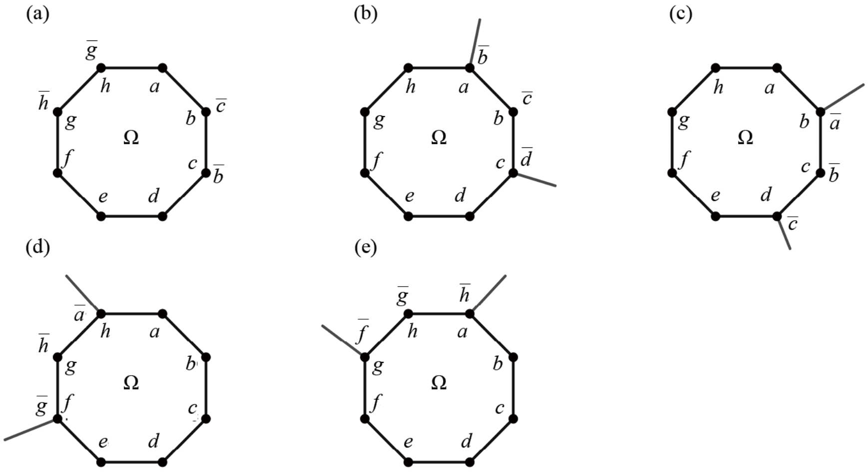

Figure 12. Region Ω when $b = c$, $g = h$

2. In this case region Ω is given by Figure 13. Since $d_\Omega(v_b)$ and $d_\Omega(v_c)$ cannot both equal 2, also $d_\Omega(v_g)$ and $d_\Omega(v_h)$ cannot both equal 2, therefore in this case it follows that $\phi(\Omega) \leq 0$.

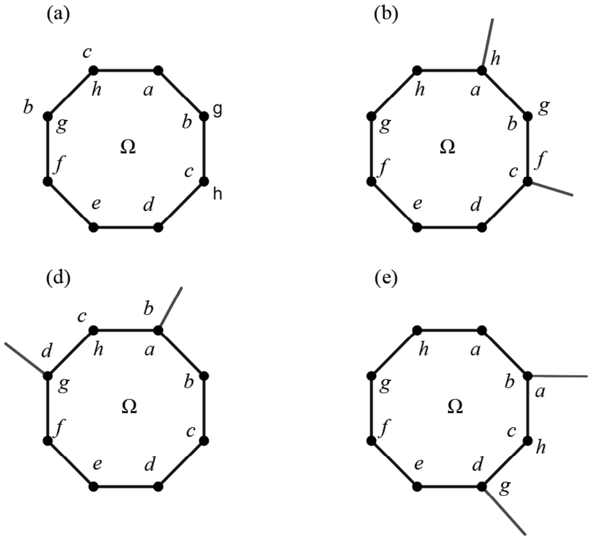

Figure 13. Region Ω when $g = b^{-1}$, $c = h^{-1}$

3. There are following four subcases to consider:

a. $d_\Omega(v_c) = d_\Omega(v_h) = d_\Omega(v_g) = 2$

b. $d_\Omega(v_b) = d_\Omega(v_c) = d_\Omega(v_g) = 2$

c. $d_\Omega(v_b) = d_\Omega(v_h) = d_\Omega(v_g) = 2$

d. $d_\Omega(v_b) = d_\Omega(v_c) = d_\Omega(v_h) = 2$.

(1) Here $d_\Omega(v_c) = d_\Omega(v_h) = d_\Omega(v_g) = 2$ implies $l_\Omega(v_g) = gc$ and $l_\Omega(v_h) = hb$. If $l_\Omega(v_c) = bc^{-1}$ in Ω as shown in Figure 14a, then this forces $d_\Omega(v_b)$ and $d_\Omega(v_d)$ to be greater than 3, otherwise $l_\Omega(v_b) = ba^{-1}\{c^{-1},\ g,\ h\}$ which implies that $a = 1$, and $l_\Omega(v_d) = c^{-1}d\{b,\ g^{-1},\ h^{-1}\}$ that implies $d = 1$. Therefore, in this case we have $\phi(\Omega) \leq (2,\ 2,\ 2,\ 3,\ 3,\ 3,\ 4,\ 4) = 0$.

If $l_\Omega(v_c) = ch$ in Ω as shown in Figure 14b, then $d_\Omega(v_b)$ and $d_\Omega(v_d)$ are greater than 3, otherwise $l_\Omega(v_b) = ba\{c^{-1},\ g,\ h\}$ which clearly implies $a = 1$, and $l_\Omega(v_d) = c^{-1}d\{b,\ g^{-1},\ h^{-1}\}$ which clearly implies $d = 1$. Therefore, in this case we have $\phi(\Omega) \leq (2,\ 2,\ 2,\ 3,\ 3,\ 3,\ 4,\ 4) = 0$.

Suppose that $l_\Omega(v_c) = cg$ as shown in Figure 14c. If $l_\Omega(v_c) = cg$, then $d_\Omega(v_b) > 3$ otherwise $l_\Omega(v_b) = bh\{d^{\pm 1},\ f^{\pm 1}\}$ which implies $d = 1$ or $f = 1$. In order to have a positive curvature $d_\Omega(v_a)$, $d_\Omega(v_d)$, $d_\Omega(v_e)$ and $d_\Omega(v_f)$ must be equal to 3. Now $l_\Omega(v_d) = \{fd^2,\ df^2\}$ and $l_\Omega(v_f) = \{fd^2,\ f^2d\}$ which implies $d_\Omega(v_e) \geq 4$. So $\phi(\Omega) \leq (2,\ 2,\ 2,\ 3,\ 3,\ 3,\ 4,\ 4) = 0$.

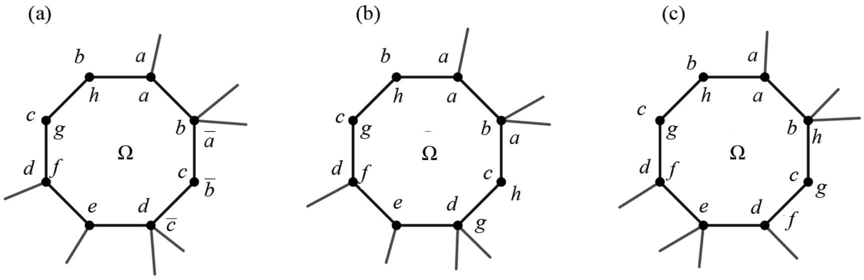

Figure 14. Region Ω when $d_\Omega(v_c) = d_\Omega(v_h) = d_\Omega(v_g) = 2$

(2) Here $d_\Omega(v_b) = d_\Omega(v_c) = d_\Omega(v_g) = 2$ implies $l_\Omega(v_b) = bh$ and $l_\Omega(v_c) = cg$. If $l_\Omega(v_g) = gh^{-1}$ in Ω as shown in Figure 15a, then this forces $d_\Omega(v_f)$ and $d_\Omega(v_h)$ to be greater than 3, otherwise $l_\Omega(v_f) = fg^{-1}\{h,\ b^{-1},\ c^{-1}\}$ that implies $f = 1$, and $l_\Omega(v_h) = a^{-1}h\{b,\ c,\ g^{-1}\}$ which implies $a = 1$. Therefore, in this case we have $\phi(\Omega) \leq (2,\ 2,\ 2,\ 3,\ 3,\ 3,\ 4,\ 4) = 0$.

If $l_\Omega(v_g) = bg$ in Ω as shown in Figure 15b, then $d_\Omega(v_f)$, and $d_\Omega(v_h)$ are greater than 3, otherwise $l_\Omega(v_f) = fc\{b^{-1},\ g,\ h\}$ which implies $f = 1$, and $l_\Omega(v_h) = ah\{b,\ c,\ g^{-1}\}$ which implies $a = 1$. Therefore, in this case we have $\phi(\Omega) \leq (2,\ 2,\ 2,\ 3,\ 3,\ 3,\ 4,\ 4) = 0$.

Suppose that $l_\Omega(v_g) = gc$. In this case region Ω is shown in Figure 15c. If $l_\Omega(v_g) = gc$ then $d_\Omega(v_h) > 3$ otherwise $l_\Omega(v_h) = hb\{a^{\pm 1},\ e^{\pm 1}\}$ which implies $a = 1$ or $e = 1$. In order to have a positive curvature, $d_\Omega(v_a)$, $d_\Omega(v_d)$, $d_\Omega(v_e)$ and $d_\Omega(v_f)$ must equals 3. Now $l_\Omega(v_d) = \{fd^2,\ df^2\}$ and $l_\Omega(v_f) = \{fd^2,\ f^2d\}$ which implies $d_\Omega(v_e) \geq 4$. So $\phi(\Omega) \leq (2,\ 2,\ 2,\ 3,\ 3,\ 3,\ 4,\ 4) = 0$.

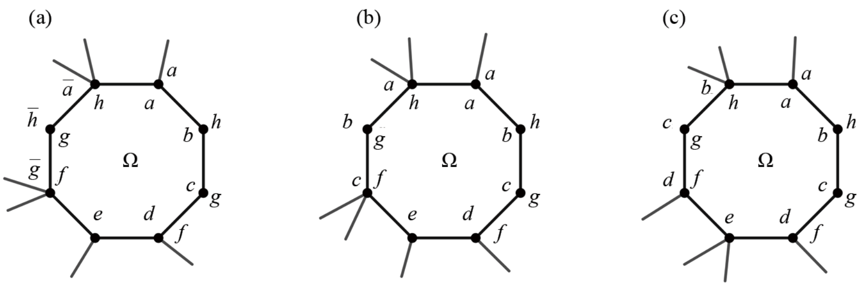

Figure 15. Region Ω when $d_\Omega(v_b) = d_\Omega(v_c) = d_\Omega(v_g) = 2$

(3) Here $d_\Omega(v_b) = d_\Omega(v_h) = d_\Omega(v_g) = 2$ implies $l_\Omega(v_g) = gc$ and $l_\Omega(v_h) = hb$. If $l_\Omega(v_b) = bc^{-1}$ in Ω as shown in Figure 16a, then $d_\Omega(v_a)$ and $d_\Omega(v_c)$ to be greater than 3, otherwise $l_\Omega(v_a) = ab^{-1}\{c,\ g^{-1},\ h^{-1}\}$ which clearly implies that $a = 1$, and $l_\Omega(v_c) = d^{-1}c\{b^{-1},\ g,\ h\}$ which clearly implies that $d = 1$. Therefore, in this case we have $\phi(\Omega) \leq (2,\ 2,\ 2,\ 3,\ 3,\ 3,\ 4,\ 4) = 0$.

Suppose that $l_\Omega(v_b) = bg$ in Ω as shown in Figure 16b. Observe that $d_\Omega(v_a)$ and $d_\Omega(v_c)$ are greater than 3, otherwise $l_\Omega(v_a) = ah\{b,\ c,\ g^{-1}\}$ which implies $a = 1$ and $l_\Omega(v_c) = fc\{b^{-1},\ g,\ h\}$ which implies $f = 1$. Therefore, in this case we have $\phi(\Omega) \leq (2,\ 2,\ 2,\ 3,\ 3,\ 3,\ 4,\ 4) = 0$.

Suppose that $l_\Omega(v_b) = bh$ as shown in Figure 16c, then obviously $d_\Omega(v_a) > 3$. If $d_\Omega(v_c) = 3$ then $l_\Omega(v_c) = cg\{d^{\pm 1},\ f^{\pm 1}\}$ which implies $d = 1$ or $f = 1$. Therefore $d_\Omega(v_c)$ must be greater than 3 and $\phi(\Omega) \leq (2,\ 2,\ 2,\ 3,\ 3,\ 3,\ 4,\ 4) = 0$.

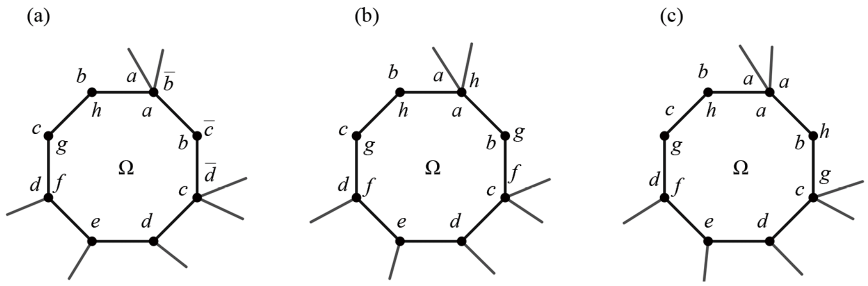

Figure 16. Region Ω when $d_\Omega(v_b) = d_\Omega(v_h) = d_\Omega(v_g) = 2$

(4) Here $d_\Omega(v_b) = d_\Omega(v_c) = d_\Omega(v_h) = 2$ implies $l_\Omega(v_b) = bh$ and $l_\Omega(v_c) = cg$. If $l_\Omega(v_h) = gh^{-1}$ in Ω as shown in Figure 17a, which forces $d_\Omega(v_g)$ and $d_\Omega(v_a)$ to be greater than 3, otherwise $l_\Omega(v_g) = gf^{-1}\{b,\ c,\ h^{-1}\}$ which implies $f = 1$, and $l_\Omega(v_a) = h^{-1}a\{b^{-1},\ c^{-1},\ g\}$ which implies $a = 1$. Therefore, in this case we have $\phi(\Omega) \leq (2,\ 2,\ 2,\ 3,\ 3,\ 3,\ 4,\ 4) = 0$.

Suppose that $l_\Omega(v_h) = ch$ in Ω as shown in Figure 17b. Notice that $d_\Omega(v_g)$ and $d_\Omega(v_a)3$ must be greater than 3, otherwise $l_\Omega(v_g) = gd\{b,\ c,\ h^{-1}\}$ which clearly implies $d = 1$ and $l_\Omega(v_a) = ba\{c^{-1},\ g,\ h\}$ which implies $a = 1$. Therefore, in this case we have $\phi(\Omega) \leq (2,\ 2,\ 2,\ 3,\ 3,\ 3,\ 4,\ 4) = 0$.

Suppose that $l_\Omega(v_h) = bh$ in Ω as shown in Figure 17c, then obviously $d_\Omega(v_a) > 3$. If $d_\Omega(v_g) = 3$ then $l_\Omega(v_g) = gc\{a^{\pm 1},\ e^{\pm 1}\}$ which implies $a = 1$ or $e = 1$. Therefore $d_\Omega(v_g)$ must be greater than 3 and $\phi(\Omega) \leq (2,\ 2,\ 2,\ 3,\ 3,\ 3,\ 4,\ 4) = 0$.

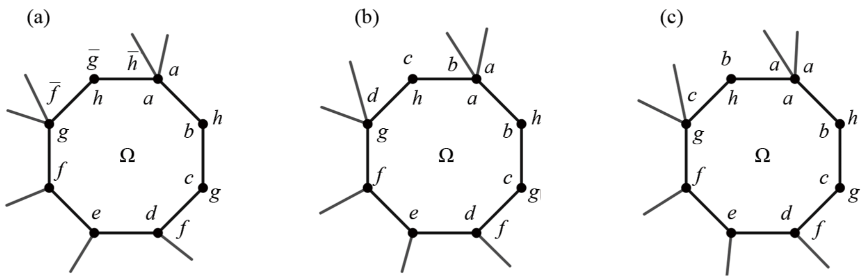

Figure 17. Region Ω when $d_\Omega(v_b) = d_\Omega(v_c) = d_\Omega(v_h) = 2$

□

4. Conclusion

In this research, we investigated the solvability of group equations over torsion-free groups. This work is motivated by Levin's conjecture, which states that every group equation over a torsion-free group is solvable. The conjecture has been proven for equations of length up to seven, as well as for some specific cases of length eight. We examined eleven group equations of length eight and analyzed their solvability over torsion-free groups. The results obtained contribute to the broader understanding of such equations and provide further support toward establishing Levin's conjecture for equations of length eight.

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Conflict of interest

The authors declare no competing financial interest.

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