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Research Article

Local Derivations on the Planar Galilean Conformal Algebra

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Abstract: Local derivation is a significant concept for various algebras, which measures some kind of local property of the algebras. This paper aims to study the local derivations on the planar Galilean conformal algebra. We determine all local derivations on the planar Galilean conformal algebra. Unlike the case of the Virasoro algebra and $W(2, 2)$, there indeed exists a nontrivial local derivation on the planar Galilean conformal algebra. The key construction and some methods will help to do such researches for some other Lie (super)algebras.

***Keywords*:** Virasoro algebra, local derivation, $W(2, 2)$, planar Galilean conformal algebra

MSC: 15A06, 17A36, 17B40

1. Introduction

Let L be an algebra, a linear map $\Delta : L \to L$ is called a local derivation if for every $v \in L$, there exists a derivation $D_v : L \to L$ (depending on v) such that $\Delta(v) = D_v(v)$. Local derivation for Banach (or associative) algebras was introduced by Kadison [1], Larson and Sourour [2] in 1990 aroused from studying the reflexivity of the space of linear maps from an algebra to itself. To study the cohomology program for operator algebras [3], it requires the decompositions (norm-continuous) of linear mappings of one operator algebra into another. The key point is to determine whether any local derivation is a derivation [1]. Due to the different characteristics of different algebras, it requires some special skills to determine all local derivations on various algebras.

Local derivations on algebras are some kinds of local properties for the algebras, which turn out to be very interesting (see [4–6]). Recently, local derivations for Lie (super)algebras have drawn some mathematicians' attention. For example, it was proved that any local derivation on the semisimple Lie algebra, the Virasoro algebra, the Lie algebra $W(2, 2)$, the super Virasoro algebra and the super $W(2, 2)$ algebra is a derivation in [6–10].

The Galilean conformal algebras (GCAs) have recently been studied in the context of the nonrelativistic limit of the Anti-de Sitter/Conformal Field Theory (AdS/CFT) conjecture correspondence [11]. It was found that the infinite-dimensional Galilean conformal algebra in 2D turned out to be related to the symmetries of non-relativistic hydrodynamic equations [12], the Bondi-Metzner-Sachs / Galilean Conformal Algebra (BMS/GCA) correspondence [13]. The planar Galilean conformal algebra is the infinite-dimensional Galilean conformal algebra in $(2+1)$ dimensional space-time which was initially introduced by Bagchi and Gopakumar [11] and named by Aizawa in [12]. Recently, many researches study its structure and representation theory ([12, 14–20]).

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Based on [7, 8], we study local derivations on the planar Galilean conformal algebra. Using some linear algebra methods in [7, 8] , we determine all local derivations on the planar Galilean conformal algebra $\mathfrak{g}$. However, unlike the case of the Virasoro algebra and the Lie algebra $W(2, 2)$, there indeed exists a nontrivial local derivation on this algebra.

The present paper is arranged as follows. In Section 2, we recall some known results and establish some related properties of the planar Galilean conformal algebra. In Section 3 and Section 4, we prove that every local derivation on the planar Galilean conformal algebra is a derivation. Throughout this paper, we denote by $\mathbb{Z}$, $\mathbb{Z}^*$, $\mathbb{C}$, $\mathbb{C}^*$ the sets of all integers, nonzero integers, complex numbers, nonzero complex numbers respectively. All algebras are defined over $\mathbb{C}$.

2. Preliminaries

In this section we recall definitions, symbols and establish some auxiliary results for later use in this paper.

A derivation on a Lie algebra L is a linear map $D : L \to L$ which satisfies the Leibniz law, that is,

$$D([v, w]) = [D(v), w] + [v, D(w)]$$

for all $v, w \in L$. The set of all derivations of L, denoted by Der (L), is a Lie algebra with respect to the commutation operation. For $u \in L$, the map

$$\text{ad } u : L \to L, \ \ \text{ad } u(v) = [u, v], \ \ \forall v \in L$$

is a derivation and a derivation of this form is called *inner derivation*. The set of all inner derivations of L, denoted by Inn (L), is an ideal of Der (L).

Recall that a linear map $\Delta : L \to L$ is called a *local derivation* if for every $v \in L$, there exists a derivation $D_v : L \to L$ (depending on v) such that

$$\Delta(v) = D_v(v).$$

Denoted by LDer (L) is the linear space of all local derivations of L.

By the definition, the planar Galilean conformal algebra $\hat{\mathfrak{g}}$ [15] is an infinite-dimensional Lie algebra with a basis

$$\{L_m, H_m, I_m, J_m, C_1, C_2, C_3 \mid m \in \mathbb{Z}\}$$

and the nontrivial Lie brackets defined by

$$[L_m,\ L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,\ 0}C_1,$$

$$[L_m,\ H_n] = -nH_{m+n} + \delta_{m+n,\ 0}(m^2+m)C_2,$$

$$[H_m,\ H_n] = \delta_{m+n,\ 0}mC_3,$$

$$[L_m,\ I_n] = (m-n)I_{m+n},$$

$$[L_m,\ J_n] = (m-n)J_{m+n},$$

$$[H_m,\ I_n] = I_{m+n},$$

$$[H_m,\ J_n] = -J_{m+n}$$

Clearly, the subalgebras $\widehat{\mathcal{S}} = \mathrm{span}\{L_i,\ I_i,\ C_1 \mid i \in \mathbb{Z}\}$ and $\widehat{\mathcal{S}}' = \mathrm{span}\{L_i,\ J_i,\ C_1 \mid i \in \mathbb{Z}\}$ are isomorphic to the Lie algebra $W(2,\ 2)$ [21]. Set $\mathfrak{g} := \hat{\mathfrak{g}}/\langle C_1,\ C_2,\ C_3\rangle$, the quotient of $\hat{\mathfrak{g}}$ by its center,

$$\mathcal{H} = \mathrm{span}\{H_i,\ J_i \mid i \in \mathbb{Z}\}$$

and

$$\mathcal{S} = \mathrm{span}\{L_i,\ I_i \mid i \in \mathbb{Z}\},$$

the subalgebras of $\mathfrak{g}$.

Lemma 1 [14] The derivation algebra of $\mathfrak{g}$ is

$$\mathrm{Der}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{g}) \bigoplus \mathbb{C}\delta,$$

where δ is an outer derivation defined by $\delta(L_m) = \delta(H_m) = 0,\ \delta(I_m) = I_m$ and $\delta(J_m) = J_m$ for any $m \in \mathbb{Z}$.

The following result plays a key role in our research.

Theorem 1 [8] Every local derivation on the Lie algebra $W(2,\ 2)$ is a derivation.

3. Local derivations on the subalgebra $\mathcal{S}$

In this section, we shall determine all local derivations on the subalgebra $\mathcal{S}$ with the same methods in [7, 8].

For a local derivation $\Delta : \mathfrak{g} \to \mathfrak{g}$ and $x \in \mathfrak{g}$, we always use the symbol D_x for the derivation of $\mathfrak{g}$ satisfying $\Delta(x) = D_x(x)$ and D_x given by Lemma 1 in the following sections.

For a given $m \in \mathbb{Z}^*$, recall that $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$ is the modulo m residual ring of $\mathbb{Z}$. Then for any $i \in \mathbb{Z}$ we have $\bar{i} \in \mathbb{Z}_m$, where $\bar{i} = \{i + km \mid k \in \mathbb{Z}\}$.

Let Δ be a local derivation on $\mathfrak{g}$ with $\Delta(L_0) = 0$. Clearly, by Theorem 1, there exists $D \in \mathrm{Der}(\mathfrak{g})$ such that $(\Delta - D)|_{\mathcal{S}} : \mathcal{S} \to \mathcal{H}$. Replaced Δ by $\Delta - D$ we can get $\Delta|_{\mathcal{S}} : \mathcal{S} \to \mathcal{H}$ and $\Delta(L_0) = 0$.

For L_m with $m \neq 0$, suppose that

$$\Delta(L_m) = \sum_{n \in \mathbb{Z}} (a_n H_n + b_n J_n), \tag{1}$$

where $a_n,\ b_n \in \mathbb{C}$ for any $n \in \mathbb{Z}$, the sum is finite.

Note that

$$\mathbb{Z} = \bar{0} \cup \bar{1} \cup \cdots \cup \overline{m-1}.$$

Therefore, (1) can be written as follows:

$$\Delta(L_m) = \sum_{\bar{i} \in A} \sum_{k=p_i}^{q_i} a_{i+km} H_{i+km} + \sum_{\bar{i} \in B} \sum_{k=g_i}^{h_i} b_{i+km} J_{i+km}, \tag{2}$$

where $p_i \leq q_i \in \mathbb{Z},\ g_i \leq h_i \in \mathbb{Z}$, and $A,\ B \subset \mathbb{Z}_m$.

For $L_m + xL_0$, where $x \in \mathbb{C}^*$, since Δ is a local derivation, there exists $\sum_{n \in \mathbb{Z}} (a'_n H_n + b'_n J_n) \in \mathfrak{g}$, where $a'_n,\ b'_n \in \mathbb{C}$ for any $n \in \mathbb{Z}$, such that

$$\begin{aligned} \Delta(L_m) &= \Delta(L_m + xL_0) \\ &= \left[\sum_{n \in \mathbb{Z}} a'_n H_n + b'_n J_n,\ L_m + xL_0 \right] \\ &= \sum_{\bar{i} \in A} \sum_{k=p'_i+1}^{q'_i+1} (i + (k-1)m) a'_{i+(k-1)m} H_{i+km} + \sum_{\bar{i} \in A} \sum_{k=p'_i}^{q'_i} x(i+km) a'_{i+km} H_{i+km} \\ &\quad + \sum_{\bar{i} \in B} \sum_{k=g'_i+1}^{h'_i+1} (i + (k-2)m) b'_{i+(k-1)m} J_{i+km} + \sum_{\bar{i} \in B} \sum_{k=g'_i}^{h'_i} x(i+km) b'_{i+km} J_{i+km}. \end{aligned} \tag{3}$$

Note that the subset $A,\ B$ in (2) is same as that of (3).

Lemma 2 Let Δ be a local derivation on $\mathfrak{g}$ such that $\Delta(L_0) = 0$. Then $A = B = \{\bar{0}\}$ in (2) and (3).

Proof. It is essentially same as that of Lemma 3 in [8].

Assumed that $m \neq 0,\ a_{i+p_i m} \neq 0,\ a_{i+q_i m} \neq 0$ and $a'_{i+p'_i m} \neq 0,\ a'_{i+q'_i m} \neq 0$ for some $\bar{i} \neq \bar{0}$. Comparing the right hand sides of (2) and (3) we see that $p_i = p'_i$ and $q_i \leq q'_i + 1$. If $q_i < q'_i + 1$, from (2) and (3), we deduce that

$$(i+q_i'm)a'_{i+q_i'm}=0.$$

Then $i+q_i'm=0$, i.e. $\bar{i}=\bar{0}$, a contradiction. Thus $q_i=q_i'+1$, and $p_i<q_i$.

Comparing (2) and (3), we deduce that

$$\begin{aligned}
a_{i+p_im}= &\ x(i+p_im)a'_{i+p_im};\\
a_{i+(p_i+1)m}= &\ (i+p_im)a'_{i+p_im}+x(i+(p_i+1)m)a'_{i+(p_i+1)m};\\
\vdots &\\
a_{i+(q_i-1)m}= &\ (i+(q_i-2)m)a'_{i+(q_i-2)m}+x(i+(q_i-1)m)a'_{i+(q_i-1)m};\\
a_{i+q_im}= &\ (i+(q_i-1)m)a'_{i+(q_i-1)m}.
\end{aligned}$$

Since $i+km\neq 0$ for $k\in\mathbb{Z}$, eliminating $a'_{i+p_im},\ \cdots,\ a'_{i+(q_i-1)m}$ in this order by substitution we see that

$$a_{i+q_im}+*x^{-1}+\cdots+*x^{-q_i+p_i}=0, \tag{4}$$

where $*$ are independent of x. We always find some $x\in\mathbb{C}^*$ not satisfying (4), and then get a contradiction. Therefore, $A=\{\bar{0}\}$. Similarly, $B=\{\bar{0}\}$. The lemma follows. □

Motivated by Lemma 3 in [8], we have the following lemma.

Lemma 3 Let Δ be a local derivation on $\mathfrak{g}$ such that $\Delta(L_0)=0$. Then for any $m\in\mathbb{Z}^*$, we have

$$\Delta(L_m)=b_mJ_m$$

for some $b_m\in\mathbb{C}$.

Proof. The proof is essentially same as that of Lemmas 4, 5 in [8].

By Lemma 2, we can suppose that

$$\Delta(L_m)=\sum_{k\in\mathbb{Z}}(a_{km}H_{km}+b_{km}J_{km}), \tag{5}$$

where $a_k,\ b_k\in\mathbb{C}$ for any $k\in\mathbb{Z}$, the sum is finite.

Without loss of generality, we can suppose that $m\geq 1$. Using the same way in [8], we can easily get $\sum\limits_{k\in\mathbb{Z}}b_{km}J_{km}=b_mJ_m$ in (5). By Lemma 2, (2) and (3), we have

$$\sum_{k=p}^{q} a_{km}H_{km} = \sum_{k=p'+1}^{q'+1} (k-1)ma'_{(k-1)m}H_{km} + \sum_{k=p\prime}^{q\prime} xkma'_{km}H_{km}, \tag{6}$$

We may assume that $a_{pm},\ a_{qm},\ a'_{p'm},\ a'_{q'm} \neq 0$. Clearly, $p' \leq p \leq q \leq q'+1$, and $p' = p$ if $p' \neq 0$. Our goal is to prove that $b_{km} = 0$ for any $k \in \mathbb{Z}$.

Assume that $p' < 0$, then $p' = p$. If further $q' \geq -1$, from (6) we get a set of aligns

$$\begin{aligned} a_{pm} =&\ xpma'_{pm};\\ a_{(p+1)m} =&\ pma'_{pm} + x(p+1)ma'_{(p+1)m};\\ \vdots&\\ a_{-m} =&\ -2ma'_{-2m} - xma'_{-m};\\ a_0 =&\ -ma'_{-m}. \end{aligned} \tag{7}$$

If $a_0 \neq 0$, using the same arguments as for (4), the aligns in (7) make a contradiction. So we consider the case that $a_0 = 0$. From (7), we see that $a'_{-m} = 0$. We continue upwards in (7) in this manner to some steps. We can get

$$a_0 = a_{-m} = \cdots = a_{pm} = 0.$$

Hence $p' \geq 0$.

If $p' > 0$, then $p = p' \geq 1$. So $q = q'+1$ and $p < q$. We get a set of aligns from (6)

$$\begin{aligned} a_{pm} =&\ xpma'_{pm};\\ a_{(p+1)m} =&\ pma'_{pm} + x(p+1)mb'_{(q+1)m};\\ \vdots&\\ a_{(q-1)m} =&\ (q-2)ma'_{(q-2)m} + x(q-1)ma'_{(q-1)m};\\ a_{qm} =&\ (q-1)ma'_{(q-1)m}. \end{aligned} \tag{8}$$

Using the same arguments again, aligns in (8) make a contradiction.

Now we have $p' = 0$ and $p \geq 1$ and $q = q'+1$. If $q > 1$, We get a set of aligns from (6)

$$a_{pm} = xpma'_{pm};$$

$$a_{(p+1)m} = pma'_{pm} + x(p+1)ma'_{(p+1)m};$$

$$\vdots$$

$$a_{(q-1)m} = (q-2)ma'_{(q-2)m} + x(q-1)ma'_{(q-1)m};$$

$$a_{qm} = (q-1)ma'_{(q-1)m}. \tag{9}$$

Using the same arguments again, aligns in (9) make a contradiction. Now we have $q = 1$, and then $q' = 0$. So $a_m = 0$. The lemma follows. □

Lemma 4 Let Δ be a local derivation on $\mathfrak{g}$ such that $\Delta(L_0) = \Delta(L_1) = 0$. Then $\Delta(L_m) = 0$ for any $m \in \mathbb{Z}$.

Proof. If $m \geq 2$, by Lemma 3, there exist $b_m \in \mathbb{C}$ and $\sum\limits_{k \in K} b'_k J_k \in \mathfrak{g}$, where $b'_k \in \mathbb{C}$, K is finite subset of $\mathbb{Z}$, such that

$$b_m J_m = \Delta(L_m) = \Delta(L_m + L_1)$$

$$= \left[\sum_{k \in K} b'_k J_k,\ L_m + L_1\right].$$

Clearly $K \subset \{0,\ 1,\ m\}$. By easy calculations we have $b_m = 0$. Similarly, if $m < 0$, we can also get $b_m = 0$. The proof is completed. □

Lemma 5 Let Δ be a local derivation on $\mathfrak{g}$ such that $\Delta(L_m) = 0$ for any $m \in \mathbb{Z}$. Then $\Delta(I_m) = 0$ for any $m \in \mathbb{Z}$.

Proof. By the definition of local derivation and Lemma 1, we have

$$\Delta(I_m) \in \oplus_{k \in \mathbb{Z}} \mathbb{C} I_k, \ \ \forall m \in \mathbb{Z}.$$

By the definition $\Delta|_{\mathcal{S}} : \mathcal{S} \to \mathcal{H}$, it is clear that $\Delta(I_m) = 0$. □

Theorem 2 Let Δ be a local derivation on $\mathfrak{g}$. Then there exists $D \in \mathrm{Der}(\mathfrak{g})$ such that $\Delta(L_m) = D(L_m)$, $\Delta(I_m) = D(I_m)$ for any $m \in \mathbb{Z}$.

Proof. Let Δ be the local derivation on $\mathfrak{g}$. Then $\Delta(L_0) = 0$. By Lemma 3, there exists $b_1 \in \mathbb{C}$ such that

$$\Delta(L_1) = b_1 J_1.$$

Set $\Delta_1 = \Delta + b_1 \mathrm{ad}(J_0)$. Then Δ_1 is a local derivation such that

$$\Delta_1(L_0) = 0,\ \Delta_1(L_1) = 0.$$

By Lemma 4, we have $\Delta_1(L_m) = 0$ for any $m \in \mathbb{Z}$. The theorem follows from Lemma 5. □

Lemma 6 Let Δ be a local derivation on $\mathfrak{g}$ such that $\Delta(L_m) = 0$ for any $m \in \mathbb{Z}$. Then $\Delta(H_0) = 0$.

Proof. For the local derivation Δ, suppose that

$$\Delta(H_0) = \sum_{n=s}^{t} a_n I_n + \sum_{k=e}^{f} b_k J_k. \tag{10}$$

For $H_0 + xL_1,\ x \in \mathbb{C}$, there exists $a_x = \sum_{n\in N} a'_n I_n + \sum_{k\in K} b'_k J_k \in \mathfrak{g}$, where $a'_n,\ b'_k \in \mathbb{C}$ for any $n \in N,\ k \in K$, such that

$$\begin{aligned}
\sum_{n=s}^{t} a_n I_n + \sum_{k=e}^{f} b_k J_k &= \Delta(H_0 + xL_1) \\
&= \left[\sum_{n=s'}^{t'} a'_n I_n + \sum_{k=e'}^{f'} b'_k J_k,\ H_0 + xL_1 \right] \\
&= -\sum_{n=s'}^{t'} a'_n I_n + x\sum_{n=s'}^{t'} a'_n (n-1) I_{n+1} + \sum_{k=e'}^{f'} b'_k J_k + x\sum_{k=e'}^{f'} b'_k (k-1) J_{k+1}.
\end{aligned} \tag{11}$$

Now we shall prove that $t' \le 1$. Assume that $t' > 1$, if further $s' \le 2$, then we get a set of aligns

$$\begin{aligned}
a_t &= xa'_{t-1}(t-2); \\
a_{t-1} &= -a'_{t-1} + xa'_{t-2}(t-3); \\
&\vdots \\
a_3 &= -a'_3 + xa'_2; \\
a_2 &= -a'_2.
\end{aligned} \tag{12}$$

Using the same arguments in Lemma 2, aligns in (12) make a contradiction.

If $s' > 2$, we can also get a set of aligns

$$a_t = \; xa'_{t-1}(t-2);$$

$$a_{t-1} = \; -a'_{t-1} + xa'_{t-2}(t-3);$$

$$\vdots$$

$$a_{s+1} = \; -a'_{s+1} + xa'_s(s-1);$$

$$a_s = \; -a'_s. \tag{13}$$

Using the same arguments in Lemma 2, aligns in (13) make a contradiction. Thus $t' \leq 1$, and then $t \leq 1$.

For $H_0 + xL_{-1},\ x \in \mathbb{C}$, there exists $a_x = \sum_{n \in N'} a''_n I_n + \sum_{k \in K'} b''_k J_k \in \mathfrak{g}$, where $a''_n,\ b''_k \in \mathbb{C}$ for any $n \in N',\ k' \in K$, such that

$$\sum_{n=s}^{t} a_n I_n + \sum_{k=e}^{f} b_k J_k = \Delta(H_0 + xL_{-1})$$

$$= \left[\sum_{n=s''}^{t''} a''_n I_n + \sum_{k=e''}^{f''} b''_k J_k,\ H_0 + xL_{-1} \right]$$

$$= -\sum_{n=s''}^{t''} a''_n I_n + x \sum_{n=s''}^{t''} a''_n (n+1) I_{n-1} + \sum_{k=e''}^{f''} b''_k J_k + x \sum_{k=e''}^{f''} b'_k (k+1) J_{k-1}. \tag{14}$$

Using the same way above we can get $s \geq -1$. Similarly, $e \geq -1,\ f \leq 1$. So

$$\Delta(H_0) = a_{-1} I_{-1} + a_0 I_0 + a_1 I_1 + b_{-1} J_{-1} + b_0 J_0 + b_1 J_1.$$

For $H_0 + L_0$, there exists $\sum\limits_{n=-1}^{1} c'_n I_n + \sum\limits_{l=-1}^{1} d'_k J_k$ such that

$$\Delta(H_0) = a_{-1} I_{-1} + a_0 I_0 + a_1 I_1 + b_{-1} J_{-1} + b_0 J_0 + b_1 J_1$$

$$= \Delta(H_0 + L_0) = \left[\sum_{n=-1}^{1} c'_n I_n + \sum_{k=-1}^{1} d'_k J_k,\ H_0 + L_0 \right].$$

Comparing with the coefficients of $I_n,\ J_n$, we get $a_1 = 0,\ b_{-1} = 0$. So

$$\Delta(H_0) = a_{-1}I_{-1} + a_0I_0 + b_0J_0 + b_1J_1.$$

For $H_0 + L_2 + L_4$, there exists $c''_{-1}I_{-1} + c''_0I_0 + d''_{-1}J_{-1} + d''_0J_0$ such that

$$\Delta(H_0) = a_{-1}I_{-1} + a_0I_0 + b_0J_0 + b_1J_1$$

$$= \Delta(H_0 + L_2 + L_4) = [c''_{-1}I_{-1} + c''_0I_0 + d''_{-1}J_{-1} + d''_0J_0,\ H_0 + L_2 + L_4].$$

Comparing with the coefficients of $I_n,\ J_n$, we can get $\Delta(H_0) = 0$. □

Lemma 7 Let Δ be a local derivation on $\mathfrak{g}$ such that $\Delta(L_m) = 0$ for any $m \in \mathbb{Z}$. Then $\Delta(H_m) = \sum_{k\in K} c_kH_k$, where $c_k \in \mathbb{C}$, K is a finite subset of $\mathbb{Z}$, for any $m \in \mathbb{Z}^*$.

Proof. By Lemma 6, we have $\Delta(H_0) = 0$.

For $m \neq 0$, by the definition of local derivation and Lemma 1, we have

$$\Delta(H_m) = \sum_{n=s}^{t} a_nI_n + \sum_{l=e}^{f} b_lJ_l + \sum_{k\in K} c_kH_k \tag{15}$$

for some $a_n,\ b_l,\ c_k \in \mathbb{C},\ s,\ t,\ e,\ f \in \mathbb{Z},$ and finite subset K of $\mathbb{Z}$.

For $H_m + xH_0,\ x \in \mathbb{C}$, there exists $\sum_{n=s'}^{t'} a'_nI_n + \sum_{l=e'}^{f'} b'_lJ_l + \sum_{k\in K'} c'_kL_k$ for some $a'_n,\ b'_l,\ c'_k \in \mathbb{C},\ s',\ t',\ e',\ f' \in \mathbb{Z},$ and finite subset K' of $\mathbb{Z},$ such that

$$\Delta(H_m) = \sum_{n=s}^{t} a_nI_n + \sum_{l=e}^{f} b_lJ_l + \sum_{k\in K} c_kH_k$$

$$= \Delta(H_m + xH_0) = \left[\sum_{n=s'}^{t'} a'_nI_n + \sum_{l=e'}^{f'} b'_lJ_l + \sum_{k\in K'} c'_kL_k,\ H_m + xH_0\right].$$

As Lemma 6, we can get $a_n = b_l = 0$ for any $n,\ l \in \mathbb{Z}$. So

$$\Delta(H_m) = \sum_{k\in K} c_kH_k. \tag{16}$$

□

Lemma 8 Let Δ be a local derivation on $\mathfrak{g}$ such that $\Delta(L_m) = 0$ for any $m \in \mathbb{Z}$. Then $\Delta(H_m) = 0$ for any $m \in \mathbb{Z}$.

Proof. $\Delta(H_0) = 0$ follows from Lemma 6.

For $m \neq 0$, by Lemma 7, we can suppose that

$$\Delta(H_m) = \sum_{k=s}^{t} c_k H_k. \tag{17}$$

for some $s,\ t \in \mathbb{Z}$.

Choose $q >> t$ and $q > 0$, $p << s$ and $p < 0$, such that $q+m > t,\ p+m < s$.

For $H_m + L_q + L_p$, there exists $a'_x = \sum_{i\in I'} a'_i L_i + \sum_{n\in J'} b'_n H_n \in \mathfrak{g}$, where $a'_i,\ b'_n \in \mathbb{C}^*$ for any $i \in I',\ n \in J'$, such that

$$\sum_{k=s}^{t} c_k H_k = \Delta(H_m + L_p + L_q) = \left[\sum_{i\in I'} a'_i L_i + \sum_{n\in J'} b'_n H_n,\ H_m + L_p + L_q\right]. \tag{18}$$

So $\sum_{i\in I'} a'_i L_i = a'(L_q + L_p)$ for some $a' \in \mathbb{C}$. Moreover, we have $J' \subset \{m,\ 0\}$.

Now we can suppose that

$$a'_x = a'(L_p + L_q) + b'_0 H_0 + b'_m H_m.$$

Comparing with the coefficients of $H_k,\ s \le k \le t$, we get $c_k = 0$ for any $k \in \mathbb{Z}$. The proof is completed. □

Corollary 1 Let Δ be a local derivation on $\mathfrak{g}$. Then there exists $D \in \mathrm{Der}\ \mathfrak{g}$ such that $\Delta(L_m) = D(L_m)$, $\Delta(I_m) = D(I_m)$ and $\Delta(H_m) = D(H_m)$ for any $m \in \mathbb{Z}$.

Proof. It follows form Theorem 2, Lemmas 5, 8. □

4. Local derivations on $\mathfrak{g}$

In this section we determine all local derivations on $\mathfrak{g}$ and $\hat{\mathfrak{g}}$.

Lemma 9 Let Δ be a local derivation on $\mathfrak{g}$ such that $\Delta(L_m) = 0$ for any $m \in \mathbb{Z}$. Then $\Delta(J_m) \in \mathbb{C}J_m$.

Proof. By the definition of local derivation and Lemma 1, we have

$$\Delta(J_m) \in \oplus_{k\in\mathbb{Z}} \mathbb{C}J_k,\ \ \forall m \in \mathbb{Z}. \tag{19}$$

Set

$$\Delta(J_m) = \sum_{i=s}^{t} a_i J_i,$$

where $s \le t,\ a_i \in \mathbb{C}$ for any i and $a_s,\ a_t \ne 0$.

Choose $p < s,\ p < 0,\ q >> t,\ q > 0$ such that $p+q > t$. For $J_m + L_p + L_q$, there exist $a_x = \sum_{i\in I} a'_i L_i + \sum_{k\in K} c'_k J_k + \sum_{j\in J} b'_j H_j \in \mathfrak{g},\ \ d' \in \mathbb{C}$, where $a'_i,\ b'_j,\ c'_k \in \mathbb{C}^*$ for any $i \in I,\ j \in J,\ k \in K$, such that

$$\sum_{i=s}^{t} a_i J_i = \Delta(J_m) = \Delta(J_m + L_p + L_q)$$

$$= \left[\sum_{i\in I} a_i' L_i + \sum_{j\in J} b_j' H_j + \sum_{k\in K} c_k' J_k,\ J_m + L_p + L_q\right] + d'\delta(J_m). \tag{20}$$

So $\sum_{i\in I} a_i' L_i = a'(L_p + L_q)$ for some $a' \in \mathbb{C}$, $\sum_{j\in J} b_j' H_j = b_0' H_0$ for some $b_0' \in \mathbb{C}$ and $\max\{k \mid c_k' \neq 0\} \leq q$ and $\min\{k \mid c_k' \neq 0\} \geq p$.

In this case we claim that $K \subset \{p, q\}$. In fact, if $l = \max\{k \in K \mid c_k' \neq 0,\ k \neq q\} \geq 0$, then there exists a nonzero term $c_l'(q-l)J_{q+l}$, where $q + l > t$ in the right hand side of (20). It is a contradiction. If $l = \min\{k \in K \mid c_k \neq 0,\ k \neq p\} < 0$, then there exists a nonzero term $c_l'(p-l)I_{p+l}$, where $p + l < s$ in the right hand side of (20). It is also a contradiction. So the claim holds.

Now we can suppose that

$$a_x = a'(L_p + L_q) + b_0' H_0 + c_p' J_p + c_q' J_q.$$

Comparing with the coefficients of $J_i,\ s \leq i \leq t$, we get $a_i = 0$ for any $i \neq m$. The proof is completed. □

Lemma 10 Let Δ_J be a linear map on $\mathfrak{g}$ such that $\Delta_J(L_m) = \Delta_J(I_m) = \Delta_J(H_m) = 0$ and $\Delta_J(J_m) = J_m$ for any $m \in \mathbb{Z}$. Then Δ_J is local derivation on $\mathfrak{g}$.

Proof. Since $\Delta_J(L_m) = \Delta_J(I_m) = \Delta_J(H_m) = 0,\ \Delta_J(J_m) = \delta(J_m) = J_m$.

For any $x,\ y,\ z,\ a_i,\ b_j,\ c_k \in \mathbb{C}$, $\Delta_J(J_m + \sum x a_i L_i + \sum y b_j H_j + \sum z c_k I_k) = [J_m + \sum_x a_i L_i + \sum_y b_j H_j + \sum_z c_k I_k,\ J_m + \sum_x a_i L_i + \sum_y b_j H_j + \sum_z c_k I_k] + \delta(J_m)$. So the lemma holds. □

Set $L_m' := L_m - mJ_m$, $H_m' := H_m - J_m$, we can easily see that $[L_m',\ L_n'] = (m-n)L_{m+n}'$, $[L_m',\ I_n] = (m-n)I_{m+n}$, $[L_m',\ H_n'] = -nH_{m+n}'$, $[L_m',\ J_n] = (m-n)J_{m+n}$, $[H_m',\ I_n] = I_{m+n}$, and $[H_m',\ J_n] = -J_{m+n}$. So we can get a new construction of $\mathfrak{g}$.

Remark 1 Obviously, Δ_J on $\mathfrak{g}$ can induce a local derivation on $\hat{\mathfrak{g}}$, which is still denoted by Δ_J, with $\Delta_J(C_i) = 0,\ i = 1,\ 2,\ 3$.

Lemma 11 The subalgebra generated by $\mathrm{span}_{\mathbb{C}}\{L_m',\ I_m,\ H_m',\ J_m \mid m \in \mathbb{Z}\}$ of $\mathfrak{g}$ is isomorphic to $\mathfrak{g}$.

Now we are in a proposition to get the main reult of this paper.

Theorem 3 For the centerless planar Galilean conformal algebra $\mathfrak{g}$, $\mathrm{LDer}\ (\mathfrak{g}) = \mathrm{Der}(\mathfrak{g}) \oplus \mathbb{C}\Delta_J$.

Proof. Let Δ be the local derivation on $\mathfrak{g}$. By Corollary 1, there exists $D \in \mathrm{Der}\ (\mathfrak{g})$ such that $(\Delta - D)(L_m) = (\Delta - D)(I_m) = (\Delta - D)(H_m) = 0$ for any $m \in \mathbb{Z}$. Replaced Δ by $\Delta - D$, we have

$$\Delta(L_m) = \Delta(I_m) = \Delta(H_m) = 0$$

for any $m \in \mathbb{Z}$.

By Lemma 9, we have

$$\Delta(J_m) = c_m J_m,\ \ c_m \in \mathbb{C},$$

for any $m \in \mathbb{Z}$.

By Lemma 11, we have $\text{span}_{\mathbb{C}}\{L'_m,\ I_m,\ H'_m,\ J_m \mid m\in\mathbb{Z}\}\cong\mathfrak{g}$. Using Corollary 1 again, there exists $D_1\in\text{Der}(\mathfrak{g})$ such that $\Delta(L'_m)=D_1(L'_m),\ \Delta(I_m)=D_1(I_m),\ \Delta(H'_m)=D_1(H'_m)$ for any $m\in\mathbb{Z}$. It is $-\Delta(J_m)=D_1(H_m-J_m)$ and then

$$D_1(H_m-J_m)=-c_mJ_m.$$

So there exist $a,\ b,\ c\in\mathbb{C}$ such that $D_1=a\text{ad}H_0+b\text{ad}J_0+c\delta$. It is $c_m=-a-b+c$ and then

$$\Delta(J_m)=(a+b-c)J_m,\ \ \forall m\in\mathbb{Z}.$$

Set $\Delta_1=\Delta-(a+b-c)\Delta_J$, then $\Delta_1(L_m)=\Delta_1(I_m)=\Delta_1(H_m)=\Delta_1(J_m)=0$ for any $m\in\mathbb{Z}$. The proof is completed. □

Corollary 2 For the planar Galilean conformal algebra $\hat{\mathfrak{g}}$, $\text{LDer}\ (\hat{\mathfrak{g}})=\text{Der}\ (\hat{\mathfrak{g}})\oplus\mathbb{C}\Delta_J$.

Proof. From [8], we just need to consider $C_2,\ C_3$. Let Δ be a local derivation of $\hat{\mathfrak{g}}$. By definition we have $\Delta(C_2)=\Delta(C_3)=0$. Now from Theorem 3 we can suppose that $\Delta(L_m)=\Delta(I_m)=\Delta(J_m)=0$ and $\Delta(H_m)=a_mC_2+b_mC_3$ for some $a_m,\ b_m\in\mathbb{C},\ m\in\mathbb{Z}$. By $a_mC_2+b_mC_3=\Delta(H_m)=[u,\ H_m]$ for some $u\in\hat{\mathfrak{g}}$, we can get $a_m=b_m=0$. □

In conclusion, a nontrivial local derivation of the planar Galilean conformal algebra is constructed (Lemma 10). It shows that

$$\dim\ \text{LDer}\ (\hat{\mathfrak{g}})/\text{Der}\ (\hat{\mathfrak{g}})=1.$$

Moreover, our research not only requires the use of the basic methods in [7, 8], but also some special methods in Lemma 5, Lemma 9 and Lemma 11. It will help to do such researches for some other Lie (super)algebras.

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Conflict of interest

The authors declare no competing financial interest.

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