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Research Article

A Characterization of Backward Bounded Solutions

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Abstract: We prove that the collection $\mathscr{M}_{-\infty}$ of backward bounded solutions for a semilinear evolution equation is the graph of an upper hemicontinuous set-valued function from the low Fourier modes to the higher Fourier modes, which is invariant and contains the global attractor. We also show that there exists a limit $\mathscr{M}_{\infty}$ of finite dimensional Lipschitz manifolds $\mathscr{M}_t$ generated by the time t-maps ($t > 0$) from the flat manifold $\mathscr{M}_0$ with the Hausdorff distance and we find $\mathscr{M}_{\infty} \subset \mathscr{M}_{-\infty}$. No spectral gap conditions are assumed.

***Keywords*:** inertial manifold, invariant attracting set, asymptotic behaviour of solution, infinite dimensional dynamical system

MSC: 35B40, 35B42, 37L05, 37L25

1. Introduction

During the last few years, we have seen several illustrations of finite dimensional structures within the infinite dimensional dynamical systems. For example, many dissipative systems have global attractors which, oftentimes, has finite Hausdorff and fractal dimensions. In an attempt to describe the long time dynamics more adequately, an inertial manifold is introduced. The inertial manifold is a positively invariant finite dimensional manifold which attracts every solutions at the exponential rates in the ambient infinite dimensional phase space. Thus the dynamics of the evolution equation is reduced to finite-dimensional system of ordinary differential equations (e.g., see [1–9]).

The existence of inertial manifolds is proved in several directions. Among others, the most typical one is the Lyapunov Perron method, where we find a fixed point of a nonlinear integral operator. The other is the Hadamard method or graph transform method, where we find a limit of time-dependent t-maps of a flat manifold. We note that the former requires backward and the latter requires forward arguments. However, the theory, even after enormous efforts, does not apply to several important examples including the Navier-Stokes equations simply because a spectral gap condition is not satisfied. In fact, arbitrary large spectral gaps in the spectrum of the leading differential operator are required but it does not hold even in the simple geometry of domain unless the dynamics are rather trivial.

This paper is an attempt to justify that the objects constructed by two different previously-mentioned methods are real candidates of inertial manifolds. Notably we find out backward-in-time solutions is important in the construction of Lyapunov Perron integral and two objects constructed in complete different methods coincide without any limitation of gap conditions.

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More precisely, let us consider a semilinear evolution equation in a Hilbert space H of the form

$$u_t + Au = F(u), \tag{1}$$

where A is a linear closed unbounded positive self-adjoint operator in H, and the nonlinearity $F(\cdot)$ is bounded and Lipschitz, i.e., there are positive constants K_0 and K_1 satisfying

$$|F(u)| \leq K_0, \tag{2}$$

$$|F(u) - F(v)| \leq K_1 |u - v| \text{ for all } u,\ v \in H.$$

We also assume that $F(\cdot)$ has a compact support after truncating (as usual) outside of the absorbing ball (e.g., see [3, 10, 11]). Hence there exists $R > 0$ such that $F(u) = 0$ for $|u| \geq R$. We also assume that A^{-1} is compact in H and thus there exist eigenvalues of A satisfying

$$0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N < \lambda_{N+1} \leq \cdots \to \infty, \text{ and } K_1 < \lambda_{N+1}.$$

Let P denote the spectral projection onto the finite dimensional space spanned by the first N-eigenfunctions of A, and $Q = I - P$ the complementary orthogonal projection. Then each $u \in H$ has a unique orthogonal decomposition as the sum of low Fourier modes and higher Fourier modes

$$u = p + q \in PH \oplus QH.$$

One of the most important questions is whether the ultimate dynamics of (1) is governed by a finite number of Fourier modes and thus one can reduce to a finite dimensional system of ordinary differential aligns, which completely describe the long-time dynamics. The inertial manifold theory has been developed for this purpose. Hence this manifold contains the global attractor and when restricted to the inertial manifold, the long-time dynamics can be reduced to a finite system of ODE.

However, the theory requires a spectral gap condition $\lambda_{N+1} - \lambda_N > 2K_1$, which holds only for special domains and $F(\cdot)$. Even much worse in the case of Navier-Stokes aligns. There were enormous efforts for relaxing these conditions with a few gains. For example, Fang and Kwak [12] showed that there exists an inertial manifold for a parabolic system which is related to the transformed Navier-Stokes aligns. Moreover, Kostianko and Zelik [13] gave sharp spectral gap conditions for existence of inertial manifolds for abstract semilinear parabolic aligns with non-self-adjoint leading part, and Zelik [9] investigated that when the spectral gap is violated, the inertial manifold may fail to exist.

As another attempt, we first consider the collection $\mathscr{M}_{-\infty}$ of backward bounded solutions of (1). The existence of such a solution is proved by using the degree theory and Arzela-Ascoli theorem in Section 2.

In Section 3, we consider the collection of finite dimensional Lipschitz manifolds $\mathscr{M}_t$ generated by the time t-map ($t > 0$) from the flat manifold $\mathscr{M}_0$, and construct a limit $\mathscr{M}_\infty$ of $\mathscr{M}_t$ with the Hausdorff distance.

In Section 4, we prove that $\mathscr{M}_\infty \subset \mathscr{M}_{-\infty}$, and $\mathscr{M}_{-\infty}$ is the graph of an upper hemicontinuous set-valued function from PH to QH, which is invariant and contains the global attractor.

2. Backward bounded solutions

We will show that a backward bounded solution of (1) is approximated by a sequence of solutions for boundary value problems

$$\begin{cases} \dfrac{dp}{dt} + Ap = PF(p+q), \ p(0) = p_0, \\ \\ \dfrac{dq}{dt} + Aq = QF(p+q), \ q(-n) = 0, \end{cases} \tag{3}$$

where $n \in \mathbb{N}$, $p(t) \in PH$, $q(t) \in QH$ for $t \in [-n, 0]$. Equivalently, (3) can be expressed by

$$\begin{cases} \dfrac{dp}{dt} + Ap = PF(p+q), \ p(n) = p_0, \\ \\ \dfrac{dq}{dt} + Aq = QF(p+q), \ q(0) = 0. \end{cases} \tag{4}$$

Inspired from Lemma 2.4 in [7], the existence of a solution of (4) can be proved by using the degree theory.

Lemma 1 Given $p_0 \in PH$, there exists a solution $(p(t), q(t))$of (4) for $t \in [0, n]$ with $p(n) = p_0$ and $q(0) = 0$.

Proof. For any $n \in \mathbb{N}$ and $p_{-n} \in PH$, let $(p_n(t), q_n(t))$ be the solution of

$$\begin{cases} \dfrac{dp}{dt} + Ap = PF(p+q), \ p(0) = p_{-n}, \\ \\ \dfrac{dq}{dt} + Aq = QF(p+q), \ q(0) = 0, \end{cases}$$

where $t \in [0, n]$. For convenience, we use the notations $p_n(t) = p_n(t, p_{-n}, 0)$ and $q_n(t) = q_n(t, p_{-n}, 0)$ for $n \in \mathbb{N}$. Then we complete the proof by showing that for any $p_0 \in PH$ and $n \in \mathbb{N}$, there exists a point $p_{-n} \in PH$ satisying $p_n(n, p_{-n}, 0) = p_0$. Let $\mathscr{B}_n$ denote the ball

$$\mathscr{B}_n = \{p \in PH : |p| \leq 2e^{n\lambda_N}(R + |p_0|)\}.$$

By the truncation, if $p_{-n} \in \partial\mathscr{B}_n$ we have

$$p_n(t, p_{-n}, 0) = e^{-At}p_n(0) = e^{-At}p_{-n}, \text{ and } q_n(t, p_{-n}, 0) = 0$$

for all $t \in [0, n]$, where $\partial\mathscr{B}_n$ denotes the boundary of $\mathscr{B}_n$. Consequently we get $p_0 \notin p_n(t, \partial\mathscr{B}_n, 0)$ for $t \in [0, n]$. For each $t \in [0, n]$, consider a function $f_t : \mathscr{B}_n \to PH$ defined by

$$f_t(p) = p_n(t, p, 0) \text{ for } p \in \mathscr{B}_n.$$

Then the degree of f_t with respect to the point p_0 is well-defined, and $\deg f_t = \deg f_0$ for all $t \in [0,\ n]$. Since $p_0 \in \mathscr{B}_n$ and f_0 is the identity map on $\mathscr{B}_n$, we see that

$$\deg f_t = 1 \text{ for all } t \in [0,\ n].$$

This implies that $p_0 \in f_n(\mathscr{B}_n) = p_n(n,\ \mathscr{B}_n,\ 0)$, and so there exists a point $p_{-n} \in \mathscr{B}_n$ satisfying $p_0 = p_n(n,\ p_{-n},\ 0)$. $\square$

The above lemma implies that for any $p_0 \in PH$ and $n \in \mathbb{N}$, there exists a sequence $\{(p_n,\ q_n) : [-n,\ 0] \to H\}_{n\in\mathbb{N}}$ of solutions of (4) such that

$$p_n(0) = p_0 \text{ and } q_n(-n) = 0.$$

We now apply the Arzela-Ascoli theorem to get a backward bounded solution as we see in the following lemma.

Lemma 2 Let $\mathscr{F} = \{(p_n,\ q_n) : [-n,\ 0] \to H\}_{n\in\mathbb{N}}$ be a sequence of solutions of (3) such that $p_n(0) = p_0$, $q_n(0) = q_n$ and $q_n(-n) = 0$. Then there exists a subsequence $\{(p_{n_k},\ q_{n_k})\}_{k\in\mathbb{N}}$ of $\mathscr{F}$ which converges uniformly to a function, say $(p,\ q)$, on every compact subset in $(-\infty,\ 0]$. In particular, $u(t) = p(t) + q(t)$ is a backward bounded solution of (1) with $p(0) = \lim_{k\to\infty} p_{n_k}(0)$ and $q(0) = \lim_{k\to\infty} q_{n_k}(0)$.

Proof. From the variation of constants formula, we see that

$$p_n(t) = e^{-PAt}p_0 - \int_t^0 e^{-PA(t-s)}PF(p_n(s)+q_n(s))ds, \tag{5}$$

$$q_n(t) = \int_{-n}^t e^{-QA(t-s)}QF(p_n(s)+q_n(s))ds.$$

For each $n_0 \in \mathbb{N}$, let $I_{n_0} = [-n_0,\ 0]$. Then for any $n \geq n_0$ and $t \in I_{n_0}$, we have

$$|p_n(t)| \leq \left(|p_0| + \frac{K_0}{\lambda_N}\right)e^{-\lambda_N t}, \tag{6}$$

$$|q_n(t)| \leq \int_{-n}^t \|e^{-QA(t-s)}\|_{op}|QF|ds \tag{7}$$

$$\leq K_0 \int_{-\infty}^t \|e^{-QA(t-s)}\|_{op}ds$$

$$= \frac{K_0}{\lambda_{N+1}}$$

and

$$|A^{1/2}q_n(t)| \leq \int_{-n}^{t} \|A^{1/2}e^{-QA(t-s)}\|_{op}|QF|ds \tag{8}$$

$$\leq K_0 \int_{-\infty}^{t} \|A^{1/2}e^{-QA(t-s)}\|_{op} ds$$

$$= \frac{2K_0}{\sqrt{2}} \frac{1}{\sqrt{\lambda_{N+1}}}.$$

We note that $\{p_n(t)\}$ is uniformly bounded by (6). Since $D(A^{1/2})$ is compactly imbedded in H, $\{q_n(t)\}$ is precompact in H (for more details, see [10]).

For any $\varepsilon > 0$ and $t \in I_{n_0}$, choose $\delta = \delta(\varepsilon) > 0$ so that

$$|p_n(t) - p_n(t-\delta)| = \left| \int_{t-\delta}^{t} \frac{dp_n(s)}{ds} ds \right| \leq \int_{t-\delta}^{t} \left| \frac{dp_n(s)}{ds} \right| ds$$

$$\leq \lambda_N e^{\lambda_N \delta} \delta \left(|p_0| + 1 + \frac{K_0}{\lambda_N} + K_0 \delta \right) < \varepsilon$$

and

$$|q_n(t) - q_n(t-\delta)| \leq |(e^{-QA\delta} - 1)q_n(t-\delta)| + \left| \int_{t-\delta}^{t} e^{-QA(t-s)} QF ds \right|$$

$$\leq \frac{2\sqrt{2}K_0\sqrt{\delta}}{e\sqrt{\lambda_{N+1}}} + K_0\delta < \varepsilon.$$

Hence $\{(p_n(t),\ q_n(t))\}$ is equicontinuous on I_{n_0}. By the Arzela-Ascoli theorem, there exist a subsequence $\{(p_{n_k}(t), q_{n_k}(t))\}_{k \in \mathbb{N}}$ which converges uniformly to a function $(p(t),\ q(t))$, i.e.,

$$p_{n_k}(t) \to p(t) \text{ and } q_{n_k}(t) \to q(t) \text{ for } t \in I_{n_0}.$$

By increasing $n_0 \to \infty$ and applying the diagonal argument, we can deduce that

$$p_{n_k}(t) \to p(t) \text{ and } q_{n_k}(t) \to q(t) \text{ for } t \in (-\infty, 0].$$

Note that $p(0) = \lim_{k\to\infty} p_{n_k}(0)$ and $q(0) = \lim_{k\to\infty} q_{n_k}(0)$. □

Let us denote the collection of backward bounded solutions by

$$\mathscr{M}_{-\infty} = \{(p_0,\, q_0) \in H : u(t) = p(t) + q(t) \text{ is a solution of (1) such that}$$

$$p(0) = p_0,\, q(0) = q_0, \text{ and } q(t) \text{ is bounded on } (-\infty,\, 0]\}.$$

Clearly it is invariant and contains the global attractor.

3. The graph transform method

The classical graph transform method due to Hadamard is a standard technique used in finite dimensional dynamical systems to construct locally invariant manifolds, such as the center manifold and the center unstable manifold. Starting from the flat manifold $\mathscr{M}_0 = PH$, one then lets the dynamics of the given evolution align act on M_0, thereby obtaining a Lipschitz manifold

$$\mathscr{M}_t = \{u(t) \in H : u(t) = p(t) + q(t)$$

is a solution of (1) with

$$p(0) = p_0 \in PH,\, q(0) = 0\},$$

at each time $t > 0$. Note that $\mathscr{M}_t$ is Lipschitz in p_0 from the Lipschitz continuity of solution with respect to the initial data. We are interested in a limit of $\mathscr{M}_t$ as $t \to \infty$. Not much is known in an infinite dimensional setting except some special case (e.g., see [7]).

In this section, we reveal that $\{Q\mathscr{M}_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence with the Hausdorff distance d_H. To show this, we first prove the following lemma which is crucial throughout the paper.

Lemma 3 Let $u(t)$, $v(t)$ be two solutions of (1), and let $\rho(t) = Pu(t) - Pv(t)$ and $\sigma(t) = Qu(t) - Qv(t)$. Then for any $0 \le t_0 \le t$, we have

$$|\sigma(t)| \le K_2|\rho(t_0)|e^{(K_1-\lambda_1)(t-t_0)} + K_3|\sigma(t_0)|e^{-(\lambda_{N+1}-K_1-\alpha)(t-t_0)} \text{ and} \tag{9}$$

$$|\rho(t)| \le |\rho(t_0)|(1 + K_4(t-t_0))e^{(K_1-\lambda_1)(t-t_0)} + K_5|\sigma(t_0)|e^{(K_1-\lambda_1)(t-t_0)}, \tag{10}$$

where K_2, K_3, K_4, K_5 are positive constants depending on K_1, λ_1 and λ_{N+1}.

Proof. We first note that

$$\frac{d|\rho|}{dt} + \lambda_1|\rho| \le K_1|\rho| + K_1|\sigma|,$$

$$\frac{d|\sigma|}{dt} + \lambda_{N+1}|\sigma| \le K_1|\rho| + K_1|\sigma|,$$

and

$$\frac{d}{dt}\left(e^{(\lambda_1-K_1)t}|\rho|\right)\le K_1e^{(\lambda_1-K_1)t}|\sigma|,$$

$$\frac{d}{dt}\left(e^{(\lambda_{N+1}-K_1)t}|\sigma|\right)\le K_1e^{(\lambda_{N+1}-K_1)t}|\rho|.$$

Integrating these inequailities from t_0 to t, we find that

$$e^{(\lambda_1-K_1)t}|\rho(t)|\le e^{(\lambda_1-K_1)t_0}|\rho(t_0)|+K_1\int_{t_0}^{t}e^{(\lambda_1-K_1)s}|\sigma(s)|ds,\text{ and} \tag{11}$$

$$e^{(\lambda_{N+1}-K_1)t}|\sigma(t)|\le e^{(\lambda_{N+1}-K_1)t_0}|\sigma(t_0)|+K_1\int_{t_0}^{t}e^{(\lambda_{N+1}-K_1)s}|\rho(s)|ds. \tag{12}$$

The last integral of (12) is bounded by

$$K_1\int_{t_0}^{t}e^{(\lambda_{N+1}-\lambda_1)\tau}\left\{e^{(\lambda_1-K_1)t_0}|\rho(t_0)|+K_1\int_{t_0}^{\tau}e^{(\lambda_1-K_1)s}|\sigma(s)|ds\right\}d\tau$$

$$=K_1e^{(\lambda_1-K_1)t_0}|\rho(t_0)|\int_{t_0}^{t}e^{(\lambda_{N+1}-\lambda_1)\tau}d\tau+K_1^2\int_{t_0}^{t}e^{(\lambda_{N+1}-\lambda_1)\tau}\int_{t_0}^{\tau}e^{(\lambda_1-K_1)s}|\sigma(s)|dsd\tau$$

$$:=I+II$$

The term I is bounded by

$$\frac{K_1}{\lambda_{N+1}-\lambda_1}e^{(\lambda_1-K_1)t_0}|\rho(t_0)|e^{(\lambda_{N+1}-\lambda_1)t}.$$

If we apply the Fubini theorem, the term II becomes

$$\frac{K_1^2}{\lambda_{N+1}-K_1}\int_{t_0}^{t}\left(e^{(\lambda_{N+1}-\lambda_1)t}-e^{(\lambda_{N+1}-\lambda_1)s}\right)e^{(\lambda_1-K_1)s}|\sigma(s)|ds.$$

Thus we have

$$e^{(\lambda_{N+1}-K_1)t}|\sigma(t)| \le e^{(\lambda_{N+1}-K_1)t_0}|\sigma(t_0)| + \frac{K_1}{\lambda_{N+1}-\lambda_1}e^{(\lambda_1-K_1)t_0}|\rho(t_0)|e^{(\lambda_{N+1}-\lambda_1)t}$$

$$+\frac{K_1^2}{\lambda_{N+1}-\lambda_1}e^{(\lambda_{N+1}-\lambda_1)t}\int_{t_0}^{t}e^{(\lambda_1-K_1)s}|\sigma(s)|ds$$

$$-\frac{K_1^2}{\lambda_{N+1}-\lambda_1}\int_{t_0}^{t}e^{(\lambda_{N+1}-K_1)s}|\sigma(s)|ds.$$

Put

$$\phi(t) = \int_{t_0}^{t} e^{(\lambda_1-K_1)s}|\sigma(s)|ds.$$

Then

$$\phi'(t) = e^{(\lambda_1-K_1)t}|\sigma(t)|,$$

and

$$e^{(\lambda_{N+1}-\lambda_1)t}\phi'(t) \le e^{(\lambda_{N+1}-K_1)t_0}|\sigma(t_0)| + \frac{K_1}{\lambda_{N+1}-\lambda_1}e^{(\lambda_1-K_1)t_0}|\rho(t_0)|e^{(\lambda_{N+1}-\lambda_1)t} \tag{13}$$

$$+\frac{K_1^2}{\lambda_{N+1}-\lambda_1}e^{(\lambda_{N+1}-\lambda_1)t}\phi(t) - \frac{K_1^2}{\lambda_{N+1}-\lambda_1}\int_{t_0}^{t}e^{(\lambda_{N+1}-\lambda_1)s}\phi'(s)ds.$$

The last two terms of (13) are reduced to

$$K_1^2\int_{t_0}^{t}e^{(\lambda_{N+1}-\lambda_1)s}\phi(s)ds.$$

We now introduce another auxiliary function

$$\psi(t) = \int_{t_0}^{t}e^{(\lambda_{N+1}-\lambda_1)s}\phi(s)ds.$$

Then we have

$$\psi'(t) = e^{(\lambda_{N+1}-\lambda_1)t}\phi(t),$$

$$\psi''(t) = (\lambda_{N+1}-\lambda_1)\psi'(t) + e^{(\lambda_{N+1}-\lambda_1)t}\phi'(t)$$

Consequently the inequality (13) is reduced to

$$\psi''(t) - (\lambda_{N+1}-\lambda_1)\psi'(t) - K_1^2\psi(t) \tag{14}$$

$$\leq e^{(\lambda_{N+1}-K_1)t_0}|\sigma(t_0)| + \frac{K_1^2}{\lambda_{N+1}-\lambda_1}e^{(\lambda_1-K_1)t_0}|\rho(t_0)|e^{(\lambda_{N+1}-\lambda_1)t}.$$

With the choice of

$$\alpha = \frac{-(\lambda_{N+1}-\lambda_1) + \sqrt{(\lambda_{N+1}-\lambda_1)^2 + 4K_1^2}}{2}$$

and

$$\beta = \frac{(\lambda_{N+1}-\lambda_1) + \sqrt{(\lambda_{N+1}-\lambda_1)^2 + 4K_1^2}}{2},$$

the left hand side of (14) becomes

$$\psi''(t) - \alpha\psi'(t) + \beta(\psi'(t) - \alpha\psi(t)).$$

Multiply an integrating factor $e^{\beta t}$ and integrate to obtain

$$\psi' - \alpha\psi \leq \frac{1}{\beta}e^{(\lambda_{N+1}-K_1)t_0}|\sigma(t_0)|$$

$$+ \frac{K_1^2}{(\lambda_{N+1}-\lambda_1)(\lambda_{N+1}-\lambda_1+\beta)}e^{(\lambda_1-K_1)t_0}|\rho(t_0)|e^{(\lambda_{N+1}-\lambda_1)t}.$$

Integrating again, we obtain

$$\psi \le \frac{1}{\alpha\beta} e^{(\lambda_{N+1}-K_1)t_0} |\sigma(t_0)| e^{\alpha(t-t_0)}$$

$$+ \frac{K_1^2}{(\lambda_{N+1}-\lambda_1)(\lambda_{N+1}-\lambda_1+\beta)(\lambda_{N+1}-\lambda_1-\alpha)} e^{(\lambda_1-K_1)t_0} |\rho(t_0)| e^{(\lambda_{N+1}-\lambda_1)t}.$$

From the inequality (12), we get

$$|\sigma(t)| \le e^{-(\lambda_{N+1}-K_1)(t-t_0)} |\sigma(t_0)| + \frac{K_1}{\lambda_{N+1}-\lambda_1} e^{(\lambda_1-K_1)t_0} |\rho(t_0)| e^{(K_1-\lambda_1)t}$$

$$+ \frac{K_1^2}{\alpha\beta} e^{(\lambda_{N+1}-K_1-\alpha)t_0} |\sigma(t_0)| e^{-(\lambda_{N+1}-K_1-\alpha)t}$$

$$+ \frac{K_1^2}{(\lambda_{N+1}-\lambda_1)(\lambda_{N+1}-\lambda_1+\beta)(\lambda_{N+1}-\lambda_1-\alpha)} e^{(\lambda_1-K_1)t_0} |\rho(t_0)| e^{(K_1-\lambda_1)t}.$$

In short notation, we have

$$|\sigma(t)| \le K_2 |\sigma(t_0)| e^{-(\lambda_{N+1}-K_1-\alpha)(t-t_0)} + K_3 |\rho(t_0)| e^{(K_1-\lambda_1)(t-t_0)} \tag{15}$$

for some positive constants K_2, K_3 depending on K_1, λ_1 and λ_{N+1}. Plugging (15) into the inequality (11), we obtain

$$|\rho(t)| \le |\rho(t_0)|(1+K_4(t-t_0)) e^{(K_1-\lambda_1)(t-t_0)} + K_5 |\sigma(t_0)| e^{(K_1-\lambda_1)(t-t_0)}.$$

□

As an immediate consequence of Lemma 3, we observe that if $\rho(t_0) = 0$, then $\sigma(t)$ decays to zero exponentially.

Lemma 4 $\{Q\mathscr{M}_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence with the Hausdorff distance d_H.

Proof. For any $m,\ n \in \mathbb{N}$ with $m \le n$, let $q_m \in Q\mathscr{M}_m$ and $q_n \in Q\mathscr{M}_n$. Then by definition, there exist solutions $u_m(t)$ and $u_n(t)$ of (1) such that

$$Qu_m(0) = Qu_n(0) = 0,\ Qu_m(m) = q_m,\ \text{and}\ Qu_n(n) = q_n.$$

Let $v(t)$ be the solution of (1) for $t \in [n-m,\ n]$ with $v(n-m) = Pu_n(m)$. Then from Lemma 3, we find

$$|\sigma(n)| = |Qu_n(n) - Qv(n)| = |q_n - Qv(n)| \le K_2 |\sigma(m)| e^{-(\lambda_{N+1}-K_1-\alpha)m},$$

and thus

$$\mathrm{dist}(q_n,\ Q\mathscr{M}_m) \leq \frac{K_0 K_2}{\lambda_{N+1}} e^{-(\lambda_{N+1}-K_1-\alpha)m}.$$

On the other hand, from Lemma 1, there exists a solution $u(t)$ of (1) for $t \in [m-n,\ m]$ such that

$$Qu(m-n) = 0 \text{ and } Pu(0) = Pu_m(0).$$

By Lemma 3, we get

$$|\sigma(m)| = |Qu(m) - Qu_m(m)| = |Qu(m) - q_m| \leq K_2 |\sigma(0)| e^{-(\lambda_{N+1}-K_1-\alpha)m}.$$

Hence

$$\mathrm{dist}(Q\mathscr{M}_n,\ q_m) \leq \frac{K_0 K_2}{\lambda_{N+1}} e^{-(\lambda_{N+1}-K_1-\alpha)m}.$$

Consequently we obtain

$$d_H(Q\mathscr{M}_n,\ Q\mathscr{M}_m) \leq \frac{K_0 K_2}{\lambda_{N+1}} e^{-(\lambda_{N+1}-K_1-\alpha)m},$$

which completes the proof. □

We recall that the space of closed and bounded subsets of the Banach space QH with the Hausdorff distance is complete (e.g., see [14]). Hence the sequence $\{QM_n\}$ converges to a closed bounded set, say D_∞. For any $q_\infty \in D_\infty$, choose a sequence $\{q_n \in QM_n\}$ satisfying $q_n \to q_\infty$. Then for each $n \in \mathbb{N}$, there exists $p_n \in PH$ such that $(p_n,\ q_n) \in \mathscr{M}_n$. Moreover we observe that the dynamics of (1) are trivial outside of the absorbing ball by the truncation, and thus we may assume

$$p_n \in B_R := \{p \in PH : |p| \leq R\}.$$

In fact, for any $(p, q) \in M_n$,

$$q = 0 \text{ if } |p| > R. \tag{16}$$

Since B_R is compact in PH, $\{p_n\}_{n=1}^{\infty}$ has a convergent subsequence, say $p_{n_k} \to p_\infty$. We finally define

$$M_\infty = \{(p_\infty,\ q_\infty) : \text{there exists a sequence } \{(p_{n_k},\ q_{n_k})\} \text{ in } \mathscr{M}_{n_k}$$

$$\text{such that } (p_{n_k},\ q_{n_k}) \to (p_\infty,\ q_\infty) \text{ as } k \to \infty\}.$$

4. Main results

We recall that $\mathscr{M}_{-\infty}$ is constructed as the collection of backward bounded solutions whereas M_∞ is a forward accumulation of solutions. Surprisingly, we first have;

Theorem 1 $\mathscr{M}_\infty \subset \mathscr{M}_{-\infty}$.

Proof. For any $(p_\infty, q_\infty) \in \mathscr{M}_\infty$, choose $(p_{n_k},\ q_{n_k}) \in \mathscr{M}_{n_k}$ such that $p_{n_k} \to p_\infty$ and $q_{n_k} \to q_\infty$. For each $k \in \mathbb{N}$, there exists a solution $u_{n_k}(t) = p_{n_k}(t) + q_{n_k}(t)$ of (3) satisfying

$$p_{n_k}(0) = p_{n_k},\ q_{n_k}(0) = q_{n_k} \text{ and } q_{n_k}(-n_k) = 0.$$

By Lemma 2, $\{u_{n_k}(t)\}$ has a subsequence which converges to a backward bounded solution $u(t)$ with $Pu(0) = p_\infty$ and $Qu(0) = q_\infty$, which completes the proof. □

Now we are in a position to state a characterization of backward bounded solutions.

Theorem 2 The collection of backward bounded solutions of (1) is the graph of an upper hemicontinuous set-valued function from PH to QH, which is invariant and contains the global attractor.

Proof. Consider a set-valued function $\Phi : PH \to QH$ defined by

$$\Phi(p_0) = \{q_0 \in QH : (p(t),\ q(t)) \text{ is a backward bounded solution of (1)}$$

$$\text{such that } p(0) = p_0,\ q(0) = q_0, \text{ and } q(t) \text{ is bounded on } (-\infty,\ 0]\}.$$

Then we see that the collection $\mathscr{M}_{-\infty}$ of backward bounded solutions of (1) is the graph of Φ, i.e.,

$$\mathscr{M}_{-\infty} = \{(p_0,\ q_0) : p_0 \in PH,\ q_0 \in \Phi(p_0)\}.$$

It is clear that $\mathscr{M}_{-\infty}$ is invariant and contains the global attractor.

Furthermore, from (16), Φ has a compact support in B_R. The estimate (8) holds for any backward bounded solution and since the space $D(A^{1/2})$ is compactly imbedded in H, the image of Φ is contained in a compact subset of QH.

We now show that the graph of Φ is closed in H. For any $(p_\infty,\ q_\infty) \in \overline{\mathscr{M}}_\infty$, we choose a sequence $\{(p_{\infty,\ n},\ q_{\infty,\ n})\}$ in $\mathscr{M}_\infty$ such that

$$(p_{\infty,\ n},\ q_{\infty,\ n}) \to (p_\infty,\ q_\infty) \text{ as } n \to \infty.$$

By the definition of $\mathscr{M}_\infty$, there is a sequence $\{(p_{m_{k,\ n}},\ q_{m_{k,\ n}})\}$ in $\mathscr{M}_{m_{k,\ n}}$ such that

$$(p_{m_{k,\ n}},\ q_{m_{k,\ n}}) \to (p_{\infty,\ n},\ q_{\infty,\ n}) \text{ as } k \to \infty.$$

We note here that $m_{k,\ l} \le m_{k,\ l'}$ if $l \le l'$, and for each l, $m_{k,\ l} \to \infty$ as $k \to \infty$. By the diagonal argument, we can derive that

$$(p_{m_{n,\ n}},\ q_{m_{n,\ n}}) \to (p_\infty,\ q_\infty) \text{ as } n \to \infty,$$

and so $(p_\infty,\ q_\infty) \in \mathscr{M}_\infty$.

Similarly we can show that for any $p_0 \in PH$, $\Phi(p_0)$ is closed in QH. If we apply the closed graph theorem (e.g., see Prop. 1.4.8 in [15]), we may conclude that Φ is upper hemicontinuous. This completes the proof. □

We finally observe that

Theorem 3 If Φ is a single valued function, then it holds

$$\mathscr{M}_\infty = \mathscr{M}_{-\infty}.$$

Proof. It suffices to prove that $\mathscr{M}_\infty$ is also a graph of a set-valued function from lower Fourier modes to higher. Then Theorem 1 together with assumption implies its result. In fact, Let $p_\infty \in PH$. As argued in Lemma 2, we may take a sequence $\{u_{n_k}(t) = p_{n_k}(t) + q_{n_k}(t)\}$ of solutions of (3) for $t \in [-n_k,\ 0]$ such that

$$p_{n_k}(0) = p_\infty \text{ and } q_{n_k}(-n_k) = 0,$$

and $u_{n_k}(t)$ converges uniformly to a backward bounded solutuon as $k \to \infty$. Now we have

$$(p_{n_k}(0),\ q_{n_k}(0)) \in \mathscr{M}_{n_k}, \text{ and } (p_{n_k}(0),\ q_{n_k}(0)) \to (p_\infty,\ q_\infty) \text{ as } k \to \infty,$$

for some $q_\infty \in QH$. □

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Conflict of interest

The authors declare no conflict of interest.

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