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Research Article

Common Fixed Point Theorems for Generalized Contractions in Ordered M-Metric Spaces

Nawab Hussain*, Abdulaziz Alofi, Osama Alhindi

Department of Mathematics, King Abdulaziz University, P.O. Box, 80203, Jeddah, 21589, Saudi Arabia
E-mail: nhusain@kau.edu.sa

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Abstract: This paper seeks to investigate and derive fixed point and common fixed point results for almost generalized contractive mappings and Reich contractions within the framework of partially ordered M-metric spaces. Furthermore, the practical utility and applicability of these findings are demonstrated through illustrative examples. The results presented herein build upon and substantially generalize several foundational studies in the existing literature.

***Keywords*:** partial metric space, M-metric space, fixed point, ordered space, contraction mapping, Reich contraction

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1. Introduction

Within the domain of metric fixed point theory, the renowned Banach contraction principle [1] remains a cornerstone result, providing a robust framework for ensuring the existence and uniqueness of fixed points for contraction mappings in the setting of complete metric spaces. Over time, various researchers have introduced generalizations and extensions of this principle by refining the conditions for contractions, incorporating auxiliary functions, and broadening the scope of metric spaces where such fixed point results hold. These advancements have significantly expanded the applicability of the principle to more complex and diverse mathematical frameworks. In particular, we concentrate on the partial metric spaces and the M-metric spaces.

The concept of partial metric spaces, first introduced by Matthews in 1994 [2], represents a generalization of traditional metric spaces by allowing the self-distance of points to be nonzero. Partial metric spaces have found extensive applications, particularly in the development of topological frameworks relevant to information science and computer science. In 2014, Asadi et al. [3] further developed the concept by extending partial metric spaces to M-metric spaces and established the Banach contraction principle and the Kannan contraction for complete M-metric spaces. An M-metric space serves as a generalization of a partial metric space, where the nonzero small self-distance, and the triangle inequality are modified. This structure broadens the scope of fixed point theorems and has proven useful in various applications. This foundational work has spurred significant interest, leading to the derivation of numerous fixed point theorems in M-metric spaces by various researchers (see, for example, [4–10]). We aim to provide a deeper understanding of M-metric space, especially in the context of partially ordered structures, which opens new possibilities for fixed and common fixed point results.

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Alternatively, fixed point theory has seen significant advances within the framework of partially ordered metric spaces. The initial breakthrough in this area was provided by Ran and Reurings [11], who applied their findings to matrix equations. Later, Nieto and Rodríguez-López [12] expanded upon this result, using it to establish a unique solution for periodic boundary value problems. Numerous subsequent contributions have emerged from various researchers [13–18].

Reich contraction [19] have been studied extensively in various settings, the current paper introduces new fixed and common fixed point results in the context of partially ordered M-metric spaces. These results not only generalize the previous findings but also provide new insights into the interplay between Reich contractions and partial orderings, offering a more comprehensive understanding of fixed points in such spaces.

The initial section of this paper presents fundamental definitions and principal results concerning partial metric spaces and M-metric spaces. In addition, we explore notable contractions, such as almost generalized contractions and Reich contractions, within the realm of partially ordered metric spaces. In the subsequent section, we derive common fixed point results utilizing the concept of partially ordered M-metric spaces, proving and extending established common fixed point theorems, all backed by examples in partially ordered M-metric spaces. Lastly, we discuss fixed point results for such contractions in partially ordered M-metric spaces, illustrated through examples, and provides pertinent corollaries.

2. Preliminaries

We begin this section by introducing the key notations and definitions that will be essential to our discussion.

Definition 1 [20] Consider a self-mapping F on a metric space (Y, d). Then F is called a weak contraction if there are two constants $\delta \in (0, 1)$ and $L \geq 0$ such that

$$d(F(s), F(t)) \leq \delta d(s, t) + Ld(F(s), t) \text{ for any } s, t \in Y.$$

In [20], Berinde employed the concept of weak contraction mappings and proved the subsequent theorem concerning fixed points.

Theorem 1 Assume that F is a weak contraction self-mapping in a complete metric space (Y, d). Then F has a fixed point.

It is important to note that, as shown in Example 1 of [20], a weak contraction does not always guarantee a unique fixed point.

Definition 2 [21] Let F be a self-mapping on a metric space (Y, d), then F satisfy almost generalized contractive condition if there exist $\delta \in [0, 1)$ and $L \geqslant 0$, such that for all $s, t \in Y$:

$$d(F(s), F(t)) \leqslant \delta M(s, t) + LN(s, t),$$

where

$$M(s, t) = \max\left\{d(s, t), d(s, F(s)), d(t, F(t)), \frac{d(s, F(t)) + d(t, F(s))}{2}\right\}$$

and

$$N(s, t) = \min\{d(s, F(s)), d(t, F(t)), d(s, F(t)), d(t, F(s))\}.$$

Reich [19] introduced the subsequent definition in 1971.

Definition 3 A mapping F in a metric space (Y, d) is called Reich contraction if there exist constants $\alpha, \beta, \gamma \geq 0$ with $\alpha+\beta+\gamma<1$ such that for all $s, t \in Y$,

$$d(F(s), F(t)) \leq \alpha d(s, t)+\beta d(s, F(s))+\gamma d(t, F(t)).$$

It guarantees that a mapping F, that satisfy Reich contraction, in a complete metric space, has a unique fixed point. Jungck [22] introduced the notion of compatible mappings, after which generalized compatibility to the notion of weakly compatible mappings. The pair $\{F, G\}$ is called weakly compatible if they commute at their coincidence points; i.e., if $F(s)=G(s)=c$ for some $s \in Y$, then $FG(s)=GF(s)$, and s is called a coincidence point of F and G, and c is called a point of coincidence.

Definition 4 [23] Let $(Y, \preceq)$ be a partially ordered set, and let F be a mapping. Then F is called a weak annihilator of G if $FG(s) \preceq s$ for all $s \in Y$. And F is called a dominating mapping if $s \preceq F(s)$ for all $s \in Y$. In addition, if $F(s) \preceq s$ for all $s \in Y$, then F is called dominated mapping.

Example 1 Let $Y=[0, 1]$ endowed with usual ordering and $F, G: Y \rightarrow Y$ given by

$$F(s)=\sqrt{s}, \ \ G(s)=s^{4}.$$

Then, $\forall s \in Y$ $s \preceq F(s)$, so F is a dominating map. In addition, F is a weak annihilator of G; i.e., $FG(s)=s^{2} \preceq s$ for all $s \in Y$.

Definition 5 [2] A partial metric on a non-empty set Y is defined as a mapping $\rho: Y \times Y \rightarrow [0, \infty)$ that adheres to the following properties for any elements $s, t, c \in Y$:

(ρ_1) $s=t$ if and only if $\rho(s, t)=\rho(s, s)=\rho(t, t)$ (T_0-separation);

(ρ_2) $\rho(s, s) \leq \rho(s, t)$ (small self-distance);

(ρ_3) $\rho(s, t)=\rho(t, s)$ (symmetry);

(ρ_4) $\rho(s, t) \leq \rho(s, c)+\rho(c, t)-\rho(c, c)$ (triangular inequality).

A partial metric space is a pair (Y, ρ).

Note that all metric spaces qualify as partial metric spaces. Nonetheless, the subsequent example demonstrates that the converse is not always true.

Example 2 [2] Consider $Y=\{[s, t]: s, t \in \mathbb{R}, s \leq t\}$ and define $\rho: Y \times Y \rightarrow [0, \infty)$ such that

$$\rho([s, t], [j, k])=\max\{t, k\}-\min\{s, j\}.$$

The pair (Y, ρ) constitutes a partial metric space, but it does not meet the criteria for a metric space because $\rho([s, t], [s, t]) \neq 0$.

Definition 6 [3] An M-metric on a non-empty set Y is a function $m: Y \times Y \longrightarrow [0, \infty)$ fulfilling the subsequent conditions for all $s, t, c \in Y$:

(m_1) $s=t$ if and only if $m(s, t)=m(s, s)=m(t, t)$ (T_0-separation);

(m_2) $m_{\{s, t\}} \leq m(s, t)$ (modified small self-distance);

(m_3) $m(s, t)=m(t, s)$ (symmetry);

(m_4) $m(s, t)-m_{\{s, t\}} \leq (m(s, c)-m_{\{s, c\}})+(m(c, t)-m_{\{c, t\}})$ (modified triangular inequality), where $m_{\{s, t\}}:=\min\{m(s, s), m(t, t)\}$. Then, an M-metric space is a pair (Y, m).

As stated in [3], the M-metric described in m within Y induces a T_0 topology τ_m on Y characterized by a basis consisting of open m-balls $\{B_m(s,\ \varepsilon),\ \ s\in Y,\ \ \varepsilon>0\}$, where $B_m(s,\ \varepsilon)=\{t\in Y: m(s,\ t)<m_{\{s,\ t\}}+\varepsilon\}$, holds for every $s\in Y$ and $\varepsilon>0$. Additionally, suppose $(Y,\ m)$ is an M-metric space, and let $m^u,\ m^v: Y\times Y\to[0,\ \infty)$ be defined as follows:

1. $m^u(s,\ t)=m(s,\ t)-m_{\{s,\ t\}}$ whenever $s\neq t$ and if $s=t$, then $m^u(s,\ t)=0$.
2. $m^v(s,\ t)=m(s,\ t)-2m_{\{s,\ t\}}+M_{\{s,\ t\}}$, where $M_{\{s,\ t\}}:=\max\{m(s,\ s),\ m(t,\ t)\}$.

Thus, the functions $m^u,\ m^v$ constitute usual metrics on Y.

Remark 1 [3] For all $s,\ t,\ c\in Y$,

1. $0\leq M_{\{s,\ t\}}+m_{\{s,\ t\}}=m(s,\ s)+m(t,\ t)$,
2. $0\leq M_{\{s,\ t\}}-m_{\{s,\ t\}}=|m(s,\ s)-m(t,\ t)|$,
3. $M_{\{s,\ t\}}-m_{\{s,\ t\}}\leq\left(M_{\{s,\ c\}}-m_{\{s,\ c\}}\right)+\left(M_{\{c,\ t\}}-m_{\{c,\ t\}}\right)$.

Example 3 [3] Assume $Y=[0,\ \infty)$. Then the function $m: Y\times Y\to[0,\ \infty)$ defined by $m(s,\ t)=\dfrac{s+t}{2}$ constitutes an M-metric on Y.

The subsequent definitions are vital for what follows.

Definition 7 [3] Consider an M-metric space $(Y,\ m)$ and a sequence $\{s_n\}_{n\in\mathbb{N}}$ in this space. Then:

1. The sequence converges to a point $s\in Y$ if and only if

$$\lim_{n\to\infty}\left(m\left(s_n,\ s\right)-m_{\{s_n,\ s\}}\right)=0.$$

2. $\{s_n\}$ is called an m-Cauchy sequence if

$$\lim_{n,\ m\to\infty}\left(m\left(s_n,\ s_m\right)-m_{\{s_n,\ s_m\}}\right),\ \ \lim_{n,\ m\to\infty}\left(M_{\{s_n,\ s_m\}}-m_{\{s_n,\ s_m\}}\right)$$

exist and are finite.

3. The space $(Y,\ m)$ is called complete if every m-Cauchy sequence $\{s_n\}$ in Y converges to a point $s\in Y$, such that

$$\lim_{n\to\infty}\left(m\left(s_n,\ s\right)-m_{\{s_n,\ s\}}\right)=0=\lim_{n\to\infty}\left(M_{\{s_n,\ s\}}-m_{\{s_n,\ s\}}\right).$$

Lemma 1 [3] If $\{s_n\}$ and $\{t_n\}$ are two sequences such that $s_n\to s$ and $t_n\to t$ as $n\to\infty$ in an M-metric space $(Y,\ m)$. Then

$$\lim_{n\to\infty}\left(m\left(s_n,\ t_n\right)-m_{\{s_n,\ t_n\}}\right)=m(s,\ t)-m_{\{s,\ t\}}.$$

Lemma 2 [3] Suppose that $\{s_n\}$ is a sequence in an M-metric space $(Y,\ m)$ such that s_n converges to $s\in Y$ as $n\to\infty$. Then,

$$\lim_{n\to\infty}\left(m\left(s_n,\ t\right)-m_{\{s_n,\ t\}}\right)=m(s,\ t)-m_{\{s,\ t\}}.$$

Lemma 3 [3] Let $\{s_n\}$ be a sequence in an M-metric space $(Y,\ m)$, satisfying

$$m(s_{n+1},\ s_n) \leq \bar{\alpha} m(s_n,\ s_{n-1}),\ \text{for some } \bar{\alpha} \in [0,\ 1), \forall n \in \mathbb{N}.$$

Then,

(i). $\lim_{n\to\infty} m(s_n,\ s_{n-1}) = 0$,

(ii). $\lim_{n\to\infty} m(s_n,\ s_n) = 0$,

(iii). $\lim_{n\to\infty} m_{s_m,\ s_n} = 0$,

(iv). $\{s_n\}$ is an m-Cauchy sequence.

Definition 8 Let Y be a non-empty set. Then $(Y,\ m,\ \preceq)$ is called a partially ordered M-metric space if $(Y,\ m)$ is an M-metric space, and $(Y,\ \preceq)$ is a partially ordered set. Also, $s,\ t \in Y$ are called comparable if $s \preceq t$ or $t \preceq s$ holds.

3. Main results

In this part, we establish common fixed point results for four mappings in the setting of partially ordered complete M-metric space.

Theorem 2 Consider $(Y,\ \preceq,\ m)$ as a partially ordered complete M-metric space with self-mappings $f,\ g,\ T$, and S on the set Y, where $f(Y) \subseteq T(Y)$ and $g(Y) \subseteq S(Y)$. Let f and g be dominating mappings that function as weak annihilators of T and S, respectively. Suppose there exist $\delta \in [0,\ 1)$ and $L \geq 0$ such that

$$m(f(s),\ g(t)) \leqslant \delta M(s,\ t) + LN(s,\ t), \tag{1}$$

where

$$M(s,\ t) = \max\Bigg\{ m(S(s),\ T(t)),\ m(f(s),\ S(s)),\ m(g(t),\ T(t)),$$

$$\frac{[m(S(s),\ f(s)) + 1] m(g(t),\ T(t))}{m(f(s),\ g(t)) + 1} \Bigg\}$$

and

$$N(s,\ t) = \min\{m^u(f(s),\ S(s)),\ m^u(g(t),\ T(t)),\ m^u(S(s),\ g(t)),\ m^u(f(s),\ T(t))\},$$

for all comparable elements $s,\ t \in Y$. For a non-decreasing sequence $\{s_n\}$ with $s_n \preceq t_n$ for all n and $t_n \to z$ as $n \to \infty$, it follows that $s_n \preceq z$. Additionally, if $\{f,\ S\}$ and $\{g,\ T\}$ are weakly compatible, and at least one of $f(Y),\ g(Y),\ S(Y),$ or $T(Y)$ is a complete subspace of Y, then $f,\ g,\ S,$ and T share a common fixed point.

Proof. We select an arbitrary point $s_0 \in Y$. Given that $f(Y) \subseteq T(Y)$, it follows that $f(s_0) = T(s_1)$ and $s_1 \in Y$. Additionally, for $g(s_1) \in S(Y)$, we find there exists $s_2 \in Y$ such that $g(s_1) = S(s_2)$. Generally, let $s_{2n+1} \in Y$ such that $f(s_{2n}) = T(s_{2n+1})$ and $s_{2n+2} \in Y$ such that $g(s_{2n+1}) = S(s_{2n+2})$. We then establish a sequence $\{t_n\}$ in Y where $t_{2n} = f(s_{2n}) = T(s_{2n+1})$ and $t_{2n+1} = g(s_{2n+1}) = S(s_{2n+2})$ for $n \geq 0$. Consequently, based on the above assumption, we derive:

$$s_{2n} \preceq f(s_{2n}) = T(s_{2n+1}) \preceq f(T(s_{2n+1})) \preceq s_{2n+1},$$

and

$$s_{2n+1} \preceq g(s_{2n+1}) = S(s_{2n+2}) \preceq g(S(s_{2n+2})) \preceq s_{2n+2}.$$

Therefore, for all $n \geq 0$, we observe $s_n \preceq s_{n+1}$. If $m(t_{2k},\ t_{2k+1}) = 0$ for some $k \geq 0$, then $t_{2k} = t_{2k+1}$, meaning g and T share a coincidence point. From equation (1), we infer that

$$m(t_{2k+1},\ t_{2k+2}) = m(f(s_{2k+2}),\ g(s_{2k+1})) \leqslant \delta M(s_{2k+2},\ s_{2k+1}) + LN(s_{2k+2},\ s_{2k+1}),$$

where

$$\begin{aligned} M(s_{2k+2},\ s_{2k+1}) = \ & \max\Bigg\{ m(S(s_{2k+2}),\ T(s_{2k+1})),\ m(f(s_{2k+2}),\ S(s_{2k+2})), \\ & m(g(s_{2k+1}),\ T(s_{2k+1})),\ \frac{[m(S(s_{2k+2}),\ f(s_{2k+2})) + 1] m(g(s_{2k+1}),\ T(s_{2k+1}))}{m(f(s_{2k+2}),\ g(s_{2k+1})) + 1} \Bigg\} \\ = \ & \max\Bigg\{ m(t_{2k+1},\ t_{2k}),\ m(t_{2k+2},\ t_{2k+1}),\ m(t_{2k+1},\ t_{2k}), \\ & \frac{[m(t_{2k+1},\ t_{2k+2}) + 1] m(t_{2k+1},\ t_{2k})}{m(t_{2k+2},\ t_{2k+1}) + 1} \Bigg\} \\ = \ & \max\{0,\ m(t_{2k+2},\ t_{2k+1})\} = m(t_{2k+2},\ t_{2k+1}), \end{aligned}$$

and

$$\begin{aligned} N(s_{2k+2},\ s_{2k+1}) = \ & \min\{m^u(t_{2k+2},\ t_{2k+1}),\ m^u(t_{2k+1},\ t_{2k}),\ m^u(t_{2k+1},\ t_{2k+1}), \\ & m^u(t_{2k+2},\ t_{2k})\} = 0. \end{aligned}$$

Consequently, we have $m(t_{2k+2},\ t_{2k+1}) \leq \delta m(t_{2k+2},\ t_{2k+1})$. Given $\delta \in [0,\ 1)$, it follows that $m(t_{2k+2},\ t_{2k+1}) = 0$, implying $t_{2k+1} = t_{2k+2}$. Similarly, it can be shown that $t_{2k+2} = t_{2k+3}$. Therefore, t_{2n} is a common fixed point of f and S. Now, assume that $m(t_n,\ t_{n+1}) > 0$ for each n. Our objective is to demonstrate that for each $n \geq 1$,

$$m(t_n, t_{n+1}) \leq \delta m(t_{n-1}, t_n) \tag{2}$$

Since s_{2n} and s_{2n+1} are comparable, from equation (1), we obtain

$$m(t_{2n}, t_{2n+1}) = m(f(s_{2n}), g(s_{2n+1})) \leqslant \delta M(s_{2n}, s_{2n+1}) + LN(s_{2n}, s_{2n+1}),$$

where

$$\begin{aligned} M(s_{2n}, s_{2n+1}) = & \max\{m(S(s_{2n}), T(s_{2n+1})), m(f(s_{2n}), S(s_{2n})), m(g(s_{2n+1}), T(s_{2n+1})), \\ & \left.\frac{[m(S(s_{2n}), f(s_{2n})) + 1]m(g(s_{2n+1}), T(s_{2n+1}))}{m(f(s_{2n}), g(s_{2n+1})) + 1}\right\} \\ = & \max\{m(t_{2n-1}, t_{2n}), m(t_{2n}, t_{2n-1}), m(t_{2n+1}, t_{2n}), \\ & \left.\frac{[m(t_{2n-1}, t_{2n}) + 1]m(t_{2n+1}, t_{2n})}{m(t_{2n-1}, t_{2n}) + 1}\right\} \\ \leq & \max\{m(t_{2n-1}, t_{2n}), m(t_{2n+1}, t_{2n})\} \end{aligned}$$

and

$$\begin{aligned} N(s_{2n}, s_{2n+1}) & = \min\{m^u(t_{2n}, t_{2n-1}), m^u(t_{2n+1}, t_{2n}), m^u(t_{2n-1}, t_{2n+1}), m^u(t_{2n}, t_{2n})\} \\ & = 0. \end{aligned}$$

If $\max\{m(t_{2n-1}, t_{2n}), m(t_{2n+1}, t_{2n})\} = m(t_{2n}, t_{2n+1}) \geq m(t_{2n-1}, t_{2n}) > 0$, then we have $m(t_{2n}, t_{2n+1}) \leq \delta m(t_{2n}, t_{2n+1})$. This leads to a contradiction because $\delta \in [0, 1)$. Therefore, $m(t_{2n}, t_{2n+1}) < m(t_{2n-1}, t_{2n})$, indicating that $m(t_{2n}, t_{2n+1}) \leq \delta m(t_{2n-1}, t_{2n})$. Similarly, we can prove that $m(t_{2n}, t_{2n-1}) \leq \delta m(t_{2n-1}, t_{2n-2})$. Thus, (2) holds true for each n. Consequently, for each $n \in \mathbb{N}$, it follows that $m(t_n, t_{n+1}) \leq \delta m(t_{n-1}, t_n) \leq \cdots \leq \delta^n m(t_0, t_1)$. Now, we prove that the sequence $\{t_n\}$ is an m-Cauchy sequence in Y. By (m_4), we have

$$\begin{aligned} m(t_{2n+1}, t_{2n+3}) - m_{\{t_{2n+1}, t_{2n+3}\}} \leqslant & \left(m(t_{2n+1}, t_{2n+2}) - m_{\{t_{2n+1}, t_{2n+2}\}}\right) \\ & + \left(m(t_{2n+2}, t_{2n+3}) - m_{\{t_{2n+2}, t_{2n+3}\}}\right) \\ \leqslant & \, m(t_{2n+1}, t_{2n+2}) + m(t_{2n+2}, t_{2n+3}). \end{aligned}$$

Similarly,

$$\begin{aligned} m(t_{2n+1},\ t_{2n+4}) - m_{\{t_{2n+1},\ t_{2n+4}\}} &\leqslant (m(t_{2n+1},\ t_{2n+2}) - m_{\{t_{2n+1},\ t_{2n+2}\}}) \\ &\quad + (m(t_{2n+2},\ t_{2n+3}) - m_{\{t_{2n+2},\ t_{2n+3}\}}) \\ &\quad + (m(t_{2n+3},\ t_{2n+4}) - m_{\{t_{2n+3},\ t_{2n+4}\}}) \\ &\leqslant m(t_{2n+1},\ t_{2n+2}) + m(t_{2n+2},\ t_{2n+3}) + m(t_{2n+3},\ t_{2n+4}) \end{aligned}$$

In general, for $n > m$ with $m = 2k+1$, we obtain

$$m(t_m,\ t_n) - m_{\{t_m,\ t_n\}} \leq \sum_{i=m}^{n-1} m(t_i,\ t_{i+1}) \leq \sum_{i=m}^{n-1} \delta^i m(t_0,\ t_1),$$

since $\delta \in [0,\ 1)$. The convergence of the series $\sum_{i=m}^{n-1} \delta^i m(t_0,\ t_1)$ implies that

$$\lim_{n,\ m\to\infty} m(t_m,\ t_n) - m_{\{t_m,\ t_n\}}.$$

exists and is finite. In a similar manner, it can be inferred that

$$\begin{aligned} M_{\{t_m,\ t_n\}} - m_{\{t_m,\ t_n\}} &\leq \sum_{i=m}^{n-1} M_{\{t_i,\ t_{i+1}\}} - m_{\{t_i,\ t_{i+1}\}} \leq \sum_{i=m}^{n-1} M_{\{t_i,\ t_{i+1}\}} \\ &\leq \sum_{i=m}^{n-1} m(t_i,\ t_{i+1}) \leq \sum_{i=m}^{n-1} \delta^i m(t_0,\ t_1), \end{aligned}$$

which implies that

$$\lim_{n,\ m\to\infty} M_{\{t_m,\ t_n\}} - m_{\{t_m,\ t_n\}}$$

exists and is finite. Hence, $\{t_n\}$ is an m-Cauchy sequence, and since Y is complete, there exists a point t in Y such that

$$\lim_{n\to\infty} \left(m(t_n,\ t) - m_{\{t_n,\ t\}}\right) = 0 = \lim_{n\to\infty} \left(M_{\{t_n,\ t\}} - m_{\{t_n,\ t\}}\right).$$

Alternatively,

$$\lim_{n\to\infty} m(t_n,\ t) = \lim_{n\to\infty} m_{\{t_n,\ t\}} = \lim_{n\to\infty} M_{\{t_n,\ t\}} = 0,$$

Therefore,

$$\lim_{n\to\infty} m(f(s_{2n}),\ t) = \lim_{n\to\infty} m(T(s_{2n+1}),\ t) = \lim_{n\to\infty} m(g(s_{2n+1}),\ t) = \lim_{n\to\infty} m(S(s_{2n+2}),\ t) = 0.$$

We now demonstrate that t is the fixed point of the mappings g and T. Let us assume $T(Y)$ is complete. Consequently, there exists an element $s \in Y$ such that $t = T(s)$. We need to show $g(s) = t$. For contradiction, assume $m(g(s),\ t) > 0$. Furthermore, $s_{2n+1} \preceq f(s_{2n+1})$, and $f(s_{2n+1}) \to t$ as $n \to \infty$, therefore $s_{2n+1} \preceq t$. Given that the dominating map f acts as a weak annihilator of T, as a consequence, $s_{2n+1} \preceq t = T(s) \preceq f(T(s)) \preceq s$. Thus, from (1), we get

$$m(f(s_{2n+1}),\ g(s)) \leqslant \delta M(s_{2n+1},\ s) + LN(s_{2n+1},\ s)$$

where

$$\begin{aligned} M(s_{2n+1},\ s) = {} & \max\{m(S(s_{2n+1}),\ T(s)),\ m(f(s_{2n+1}),\ S(s_{2n+1})),\ m(g(s),\ T(s)), \\ & \left.\frac{[m(S(s_{2n+1}),\ f(s_{2n+1})) + 1] \cdot m(g(s),\ T(s))}{m(g(s),\ f(s_{2n+1})) + 1}\right\} \\ = {} & \max\{m(t_{2n+1},\ t),\ m(t_{2n+1},\ t_{2n}),\ m(g(s),\ t), \\ & \left.\frac{[m(t_{2n},\ t_{2n+1}) + 1] \cdot m(g(s),\ t)}{m(g(s),\ t_{2n+1}) + 1}\right\}. \end{aligned}$$

and

$$\begin{aligned} N(s_{2n+1},\ s) = {} & \min\{m^u(f(s_{2n+1}),\ S(s_{2n+1})),\ m^u(g(s),\ T(s)), \\ & m^u(S(s_{2n+1}),\ g(s)),\ m^u(f(s_{2n+1}),\ T(s))\} \\ = {} & \min\{m^u(t_{2n+1},\ t_{2n}),\ m^u(g(s),\ t),\ m^u(t_{2n},\ g(s)),\ m^u(t_{2n+1},\ t)\}. \end{aligned}$$

By taking the limit as $n \to \infty$, we get $M(s_{2n+1},\ s) = m(g(s),\ t)$ and $\lim_{n\to\infty} N(s_{2n+1},\ s) = 0$. Thus, we have $m(g(s),\ t) \leqslant \delta m(g(s),\ t)$, which is sensible only if $m(g(s),\ t) = 0$. This outcome contradicts the assumption that $m(g(s),\ t) > 0$. Consequently, $g(s) = t$, leading to the conclusion that $g(s) = T(s) = t$. Since the mappings g and T are weakly compatible, it follows that $g(t) = g(T(s)) = T(g(s)) = T(t)$. In a similar manner, we will demonstrate that $g(t) = t$. Assume, in contradiction, that $m(g(t),\ t) > 0$. From (1) we have

$$m\left(t_{2n+1},\ g(t)\right) = m\left(f\left(s_{2n+1}\right),\ g(t)\right) \leqslant \delta M\left(s_{2n+1},\ t\right) + LN\left(s_{2n+1},\ t\right)$$

where

$$\begin{aligned} M\left(s_{2n+1},\ t\right) = &\ \max\left\{m\left(S(s_{2n+1}),\ T(t)\right),\ m\left(f(s_{2n+1}),\ S(s_{2n+1})\right),\ m\left(g(t),\ T(t)\right),\right. \\ &\left.\frac{[m\left(S(s_{2n+1}),\ f(s_{2n+1})\right)+1]m\left(g(t),\ T(t)\right)}{m\left(g(t),\ f(s_{2n+1})\right)+1}\right\} \\ = &\ \max\left\{m\left(t_{2n},\ g(t)\right),\ m\left(t_{2n+1},\ t_{2n}\right),\ m\left(g(t),\ T(t)\right),\right. \\ &\left.\frac{[m\left(t_{2n},\ t_{2n+1}\right)+1]m\left(g(t),\ t\right)}{m\left(g(t),\ t_{2n+1}\right)+1}\right\}. \end{aligned}$$

and

$$\begin{aligned} N\left(s_{2n+1},\ t\right) = &\ \min\{m^u\left(f(s_{2n+1}),\ S(s_{2n+1})\right),\ m^u\left(g(t),\ T(t)\right), \\ &\ m^u\left(S(s_{2n+1}),\ g(t)\right),\ m^u\left(f(s_{2n+1}),\ T(t)\right)\} \\ = &\ \min\left\{m^u\left(t_{2n+1},\ t_{2n}\right),\ m^u\left(g(t),\ T(t)\right),\ m^u\left(t_{2n},\ g(t)\right),\ m^u\left(t_{2n+1},\ T(t)\right)\right\} \\ = &\ 0. \end{aligned}$$

Taking $n \to \infty$, we obtain

$$\lim_{n\to\infty} M(s_{2n+1},\ t) = m(t,\ g(t)),$$

leading to $m(t,\ g(t)) \leqslant \delta m(t,\ g(t))$. This inequality holds only if $m(t,\ g(t)) = 0$, which contradicts $m(t,\ g(t)) > 0$. Consequently, $g(t) = t$. Similarly, it can be demonstrated that t is also a fixed point of f and S. Therefore, we have $f(t) = g(t) = S(t) = T(t) = t$. The arguments for the cases where $S(Y)$, $f(Y)$, or $g(Y)$ are complete follow the same procedure. □

The subsequent example demonstrates the validity of Theorem 2.

Example 4 Consider self-mappings $f,\ g,\ T$ and S defined on the set $Y = [0,\ \infty)$ endowed with partial order ($s,\ t \in [0,\ 1]$ with $s \leq t$) or $s \preceq t \iff s = t$. We define f and T as follows:

$$f(s)=g(s)=\begin{cases} e^s-1, & s\in[0,\ 1], \\ e-1, & s>1, \end{cases} \quad T(s)=S(s)=\begin{cases} \ln(1+s), & s\in[0,\ 1], \\ \ln 2, & s>1. \end{cases}$$

Let $m: Y\times Y\to[0,\ \infty)$ be defined by $m(s,\ t)=\dfrac{s+t}{2}$. Then, $(Y,\ \preceq,\ m)$ is a complete partially ordered M-metric space. Now, let us verify whether these mappings satisfy the conditions given in Theorem 2. Initially, we show that f is a dominating map and a weak annihilator of T. Indeed, for $s\in[0,\ 1)$, $s\preceq e^s-1=f(s)$, while for $s>1$, $s\preceq f(s)=s$. Also, $fT(s)=s\preceq s$, $\forall s\in Y$.

Next, we verify that f and T satisfy condition 1. We choose $\delta=9/10$, and $L=2$. Suppose that $t\preceq s$, then we have two cases. Case I: if $s\in[0,\ 1]$ (and so $t\in[0,\ 1]$). We derive

$$\begin{aligned} m(f(s),\ g(t)) &= \frac{f(s)+g(t)}{2}=\frac{e^s+e^t-2}{2} \\ &\le \delta\cdot\frac{f(s)+S(s)}{2}+L\cdot\left(\frac{f(s)+S(s)}{2}-\min\{f(s),\ S(s)\}\right) \\ &= \delta\cdot\frac{e^s-1+\ln(1+s)}{2}+L\cdot\left(\frac{e^s-1+\ln(1+s)}{2}-\ln(1+s)\right). \end{aligned}$$

Consequently, $m(f(s),\ g(t))\le\delta M(s,\ t)+LN(s,\ t)$, for any $s,\ t\in[0,\ 1]$. Case II: if $s>1$ (and so $s=t$), then

$$m(f(s),\ g(s))=e-1\le\delta\cdot\frac{e-1+\ln 2}{2}+L\cdot\left(\frac{e-1+\ln 2}{2}-\ln 2\right).$$

Hence, the mappings f and T meet condition 1 in Theorem 2 and possess a common fixed point, that is $s=0$.

Corollary 1 Consider a partially ordered complete M-metric space $(Y,\ \preceq,\ m)$, with self-mappings f and T in Y such that $f(Y)\subseteq T(Y)$. Let f be a dominating mapping and act as a weak annihilator of T. Suppose there exist $\delta\in[0,\ 1)$ and $L\ge 0$ so that

$$m(f(s),\ f(t))\le\delta M(s,\ t)+LN(s,\ t),$$

where

$$M(s,\ t)=\max\left\{m(T(s),\ T(t)),\ m(f(s),\ T(s)),\ m(f(t),\ T(t)),\ \frac{[m(T(s),\ f(s))+1]m(f(t),\ T(t))}{m(T(t),\ f(s))+1}\right\}$$

and

$$N(s,\ t)=\min\left\{m(f(s),\ T(s)),\ m(f(t),\ T(t)),\ m(T(s),\ f(t)),\ m(f(s),\ T(t))\right\},$$

for any comparable elements $s,\ t\in Y$. Given a nondecreasing sequence $\{s_n\}$ with $s_n\preceq t_n$ for all n and $t_n\to c$ as $n\to\infty$, if $s_n\preceq c$ holds, then f and T share a common fixed point.

Next, we give fixed point results for almost m-generalized contractions and Reich contractions in a complete partially ordered M-metric space. In addition, we substantiate our findings with appropriate examples to verify their validity. First, we define the concept of an almost m-generalized contraction for a self-mapping F on Y in a partially ordered M-metric space as follows:

$$m(F(s),\ F(t))\leqslant\delta M(s,\ t)+LN(s,\ t),\tag{3}$$

for some $\delta\in[0,\ 1)$ and $L\geqslant 0,$ and for every comparable $s,\ t\in Y,$ where

$$M(s,\ t)=\max\left\{m(s,\ t),\ m(s,\ F(s)),\ m(t,\ F(t)),\ \frac{m(t,\ F(t))[1+m(s,\ F(s))]}{1+m(s,\ t)},\ \frac{m(s,\ F(s))m(t,\ F(t))}{1+m(s,\ t)}\right\}$$

and

$$N(s,\ t)=\min\{m^u(s,\ F(s)),\ m^u(t,\ F(t)),\ m^u(s,\ F(t)),\ m^u(t,\ F(s))\}.$$

Theorem 3 Let $(Y,\ \preceq,\ m)$ be a complete partially ordered M-metric space, and a mapping $F:Y\to Y$ satisfying almost m-generalized contraction, and let F be a continuous, dominating mapping. Then F admits at least one fixed point in Y.

Proof. We choose any point $s_0\in Y$. Then, we have $F(s_0)=s_1,$ if $F(s_0)\neq s_0,$ then $s_0\prec F(s_0)$. We construct a sequence $\{s_n\}\subseteq Y$ given by $s_{n+1}=F(s_n),\ n\in\mathbb{N}$. As F is a dominating mapping, we demonstrate

$$s_0\prec F(s_0)=s_1\preceq F(s_1)\preceq\cdots\preceq s_n\preceq s_{n+1},\ \cdots.$$

If for some $n\in\mathbb{N},\ s_{n+1}=s_n$, then F possess a fixed point. Suppose that $m(s_n,\ s_{n+1})>0$ for each n. Then, from (3) we obtain

$$m(s_n,\ s_{n+1})=m(F(s_{n-1}),\ F(s_n))\leqslant\delta M(s_{n-1},\ s_n)+LN(s_{n-1},\ s_n),\tag{4}$$

where

$$
\begin{aligned}
M(s_{n-1},\ s_n) = &\ \max\{m(s_{n-1},\ s_n),\ m(s_{n-1},\ F(s_{n-1})),\ m(s_n,\ F(s_n)),\\
&\left.\frac{m(s_n,\ F(s_n))[1+m(s_{n-1},\ F(s_{n-1}))]}{1+m(s_{n-1},\ s_n)},\ \frac{m(s_{n-1},\ F(s_{n-1}))m(s_n,\ F(s_n))}{1+m(s_{n-1},\ s_n)}\right\}\\
= &\ \max\{m(s_{n-1},\ s_n),\ m(s_{n-1},\ s_n),\ m(s_n,\ s_{n+1}),\\
&\left.\frac{m(s_n,\ s_{n+1})[1+m(s_{n-1},\ s_n)]}{1+m(s_{n-1},\ s_n)},\ \frac{m(s_{n-1},\ s_n)m(s_n,\ s_{n+1})}{1+m(s_{n-1},\ s_n)}\right\}\\
\leq &\ \max\{m(s_{n-1},\ s_n),\ m(s_n,\ s_{n+1})\}
\end{aligned}
$$

and

$$
\begin{aligned}
N(s_{n-1},\ s_n) &= \min\{m^u(s_{n-1},\ Fs_{n-1}),\ m^u(s_n,\ Fs_n),\ m^u(s_{n-1},\ Fs_n),\ m^u(s_n,\ Fs_{n-1})\}\\
&= \min\{m^u(s_{n-1},\ s_n),\ m^u(s_n,\ s_{n+1}),\ m^u(s_{n-1},\ s_{n+1}),\ m^u(s_n,\ s_n)\} = 0.
\end{aligned}
$$

Assume that $\max\{m(s_{n-1},\ s_n),\ m(s_n,\ s_{n+1})\} = m(s_n,\ s_{n+1})$ for some n. From statement (4), it follows that $m(s_n,\ s_{n+1}) \leqslant \delta m(s_n,\ s_{n+1})$, which is a contradiction because $\delta \in [0,\ 1)$. Hence, the relation $\max\{m(s_{n-1},\ s_n),\ m(s_n,\ s_{n+1})\} = m(s_{n-1},\ s_n)$ must hold. Therefore,

$$
m(s_n,\ s_{n+1}) \leqslant \delta m(s_{n-1},\ s_n).
$$

According to Lemma 3 (i), we have

$$
\lim_{n\to\infty} m(s_n,\ s_{n+1}) = 0. \tag{5}
$$

Thus, the sequence $\{s_n\}$ is an m-Cauchy sequence. Since $(Y,\ m)$ is a complete M-metric space, there exists some $z \in Y$ such that $\{s_n\}$ converges to z as $n \to \infty$ with respect to τ_m. This implies

$$
\lim_{n\to\infty}\left(m(s_n,\ z) - m_{\{s_n,\ z\}}\right) = 0 = \lim_{n\to\infty}\left(M_{\{s_n,\ z\}} - m_{\{s_n,\ z\}}\right).
$$

Equivalently,

$$
\lim_{n\to\infty} m(s_n,\ z) = \lim_{n\to\infty} m_{\{s_n,\ z\}} = \lim_{n\to\infty}\min\{m(s_n,\ s_n),\ m(z,\ z)\} = 0, \tag{6}
$$

We now demonstrate that z is a fixed point of F. From (3), we derive

$$m(s_{n+1},\ F(z)) = m(F(s_n),\ F(z)) \leq \delta M(s_n,\ z) + LN(s_n,\ z), \tag{7}$$

where

$$M(s_n,\ z) = \max\left\{m(s_n,\ z),\ m(s_n,\ s_{n+1}),\ m(z,\ F(z)),\right.$$

$$\left.\frac{m(z,\ F(z))[1+m(s_n,\ s_{n+1})]}{1+m(s_n,\ z)},\ \frac{m(s_n,\ s_{n+1})m(z,\ F(z))}{1+m(s_n,\ z)}\right\}$$

and

$$N(s_n,\ z) = \min\{m^u(s_n,\ s_{n+1}),\ m^u(z,\ F(z)),\ m^u(s_n,\ F(z)),\ m^u(z,\ s_{n+1})\}.$$

By letting $n \to \infty$ in equation (7) and applying equations (5) and (6), we find that $M(s_n,\ z) = m(z,\ F(z))$ and $N(s_n,\ z) = 0$. Thus, we conclude that $m(z,\ F(z)) \leq \delta m(z,\ F(z)) < m(z,\ F(z))$. Therefore, $m(z,\ F(z)) = m_{F(z),\ z}$. Furthermore, from (3) we have

$$m(F(z),\ F(z)) \leq \delta m(z,\ F(z)) = m_{\{F(z),\ z\}}.$$

Consequently,

$$m(F(z),\ F(z)) = m(z,\ F(z)) = m(z,\ z) = m_{\{F(z),\ z\}}.$$

Thus, $F(z) = z$ is indeed a fixed point. □

We now provide the following example to substantiate our previously mentioned result.

Example 5 Consider $Y = \{s = (s_1,\ s_2,\ \cdots,\ s_n) \in \mathbb{R}^n \mid 0 < s_1 \leq 1 \text{ and } s_i = 0 \text{ for all } i > 1\}$ endowed with the usual order $\preceq$. Let $m : Y \times Y \to [0,\ \infty)$ be an M-metric space defined as follows:

$$m(s,\ t) = \frac{s+t}{2},\ \forall s,\ t \in Y.$$

Let $F : Y \to Y$ defined by

$$F(s) = \left(1 - \frac{s_1}{2},\ 0,\ 0,\ \ldots,\ 0\right).$$

It can be shown that F is an almost m-generalized contraction. Specifically, choose $\delta = 1/2$ and $L = 2$, and we analyze the subsequent three cases:

Case 1: If $s_1 = t_1 = 1$, then

$$m(F(s),\ F(t)) = 1/2 \leq \delta \cdot 1 + L \cdot 1/4,$$

Case 2: If $s_1 = t_1 \in (0,\ 1)$, then

$$m(F(s),\ F(t)) = 1 - \frac{s_1}{2} \leq \delta \cdot 1/2 + L \cdot 1/2,$$

Case 3: If $s_1 \neq t_1 \in (0,\ 1)$, then

$$m(F(s),\ F(t)) = 1 - \frac{s_1}{4} - \frac{t_1}{4} < \delta \left(\frac{1 + s_1/2}{2} \right) + L \left(\frac{1 + s_1 - t_1/2}{2} \right).$$

Therefore, according to Theorem 3, F possess a fixed point, which is $s = (2/3,\ 0,\ 0,\ \cdots,\ 0)$.

Corollary 2 Consider $(Y,\ \preceq,\ m)$ as a complete partially ordered M-metric space. Let F be a continuous and dominating self-mapping on Y, if there exists $\delta \in [0,\ 1)$ and for any comparable elements $s,\ t \in Y$:

$$m(F(s),\ F(t)) \leqslant \delta \max \Bigg\{ m(s,\ t),\ m(s,\ F(s)),\ m(t,\ F(t)),$$

$$\frac{m(t,\ F(t))[1 + m(s,\ F(s))]}{1 + m(s,\ t)},\ \frac{m(s,\ F(s))m(t,\ F(t))}{1 + m(s,\ t)} \Bigg\}.$$

Then F admits at least one fixed point in Y.

Theorem 4 Let $(Y,\ \preceq,\ m)$ be a complete partially ordered M-metric space, and let F be a continuous, dominating self-mapping on Y. If for any two comparable elements $s,\ t \in Y$, we have

$$m(F(s),\ F(t)) \leqslant \alpha m(s,\ t) + \beta m(s,\ F(s)) + \zeta m(t,\ F(t)), \tag{8}$$

where $\alpha,\ \beta,\ \zeta \geq 0$ and $\alpha + \beta + \zeta < 1$. Then the mapping F possess at least one fixed point.

Proof. Consider a sequence $\{s_n\} \subseteq Y$ such that $s_{n+1} = F(s_n)$ for all $n \in \mathbb{N}$. Since F is a dominating map, we have

$$s_0 \prec F(s_0) = s_1 \preceq F(s_1) \preceq \cdots \preceq s_n \preceq s_{n+1},\ \cdots.$$

If there exists an integer $n \in \mathbb{N}$ such that $m(s_{n+1},\ s_n) = 0$, it follows that $F(s_n) = s_{n+1} = s_n$, thereby indicating that s_n is a fixed point. Suppose $m(s_n,\ s_{n+1}) > 0$ for each n. From (8), we get

$$m(s_n,\ s_{n+1}) = m(F(s_{n-1}),\ F(s_n)) \leqslant \alpha m(s_{n-1},\ s_n) + \beta m(s_{n-1},\ F(s_{n-1})) + \zeta m(s_n,\ F(s_n))$$

$$= \alpha m(s_{n-1},\ s_n) + \beta m(s_{n-1},\ s_n) + \zeta m(s_n,\ s_{n+1}),$$

$$m(s_n,\ s_{n+1})(1-\zeta) \leqslant (\alpha+\beta)m(s_{n-1},\ s_n)$$

$$m(s_n,\ s_{n+1}) \leqslant \frac{\alpha+\beta}{1-\zeta}m(s_{n-1},\ s_n).$$

Subsequently,

$$m(s_n,\ s_{n+1}) \leqslant \lambda m(s_{n-1},\ s_n),$$

where $\lambda = (\alpha+\beta)/(1-\zeta)$, with λ in the interval $[0,\ 1)$. Therefore, applying Lemma 3 (i), we infer that

$$\lim_{n\to\infty} m(s_n,\ s_{n+1}) = 0. \tag{9}$$

Thus, we conclude that the sequence $\{s_n\}$ serves as an m-Cauchy sequence. Given that $(Y,\ m)$ is a complete space, per Definition 7, it follows that

$$\lim_{n\to\infty} m(s_n,\ z) = \lim_{n\to\infty} m_{\{s_n,\ z\}} = \lim_{n\to\infty} M_{\{s_n,\ z\}} = 0, \tag{10}$$

the sequence $\{s_n\}$ converges to z as $n \to \infty$.

We now show that z is a fixed point of F. Assuming $m(z,\ F(z)) > 0$, we can apply the axiom (m_4) to arrive at a contradiction.

$$\begin{aligned} m(z,\ F(z)) - m_{\{z,\ F(z)\}} &\leq m(z,\ s_{n+1}) - m_{\{z,\ s_{n+1}\}} + m(s_{n+1},\ F(z)) - m_{\{s_{n+1},\ F(z)\}} \\ &\leq m(F(s_n),\ F(z)) \leqslant \alpha m(s_n,\ z) + \beta m(s_n,\ s_{n+1}) + \zeta m(z,\ F(z)). \end{aligned}$$

Using equations (9) and (10), we obtain $m(z,\ F(z)) \leqslant \zeta m(z,\ F(z)) < m(z,\ F(z))$. Hence, $m(z,\ F(z)) = 0$. Also, from (8) we have

$$m(F(z),\ F(z)) \leqslant \alpha m(z,\ z) + (\beta+\zeta)m(z,\ F(z)) = 0.$$

As a result,

$$m(F(z),\ F(z)) = m(z,\ F(z)) = m(z,\ z).$$

Thus, $F(z) = z$ is a fixed point. □

Corollary 3 Let $(Y, \preceq, m)$ denote a complete partially ordered M-metric space, and let $F : Y \to Y$ be a continuous, dominating self-mapping. Assume that for all comparable elements $s, t \in Y$, the following inequality holds:

$$m(F(s), F(t)) \leq \theta m(s, t),$$

where $\theta \in [0, 1)$. Under these conditions, the mapping F possess at least one fixed point.

Corollary 4 Let $(Y, \preceq, m)$ be a complete partially ordered M-metric space, and let $F : Y \to Y$ be a continuous, dominating self-mapping. Suppose that for all $s, t \in Y$ with s and t comparable under $\preceq$, the inequality

$$m(F(s), F(t)) \leq \varepsilon\big(m(s, F(s)) + m(t, F(t))\big)$$

is satisfied, where $\varepsilon \in [0, 1)$. Under these conditions, the mapping F admits at least one fixed point in Y.

4. Conclusions and future work

This paper provides a comprehensive investigation into almost m-generalized contractions and Reich contractions, alongside their associated fixed point theorems, within the framework of partially ordered M-metric spaces. The analysis culminates in the establishment of common fixed point results for generalized contractions in such spaces. As a direction for future research, we suggest exploring the extension of these results to partially ordered M_b-metric spaces, partially ordered rectangular M_b-metric spaces, and partially ordered fuzzy M-metric spaces.

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Conflict of interest

The authors declare no competing financial interest.

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