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Research Article

Study of a Mild Solution of a Stochastic Integrodifferential Equation with Non-lipschitzian Coefficients in a Complex Hilbert Space

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Abstract: In this work we consider a system of stochastic integrodifferential equations in a complex Hilbert space. We first establish the existence and uniqueness of mild solutions for our equation (1) under non-Lipschitz conditions. Then we show under certain assumptions that the mild solution found is asymptotically stable in mean order n. We obtain our existence and uniqueness results by using the Lipschitz global and growth conditions and applying the properties of the analytic semigroup with those of stochastic calculus. The application of the fixed point theorem together with the properties of the stochastic integral gives us the asymptotic stability result.

***Keywords*:** analytical semi-groups, stochastic integral, analytical resolving operator, equation integrodifferential, mild solution, asymptotic stability

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1. Introduction

Stochastic Integro-differential Equations (SIE) are a family of mathematical equations that combine both integral and differential terms, while taking into account the randomness of the process under study. They are used to model dynamic systems subject to random influences, such as Brownian motion or diffusion processes. SIE are widely used in many fields, such as statistical physics, mathematical biology and quantitative finance, to understand and predict the behaviour of complex systems.

Note that the same equation has been studied in [1] where the authors found the solution in a real Hilbert space; we give a generalisation of this result in a complex Hilbert space. For the success of the work we have considered that the delayed part of our equation admits in the sense of Grimmer [2] an analytic resolvent operator and is locally non-Lipschitzian. The literature on the properties of SIE is vast, and many authors have made these properties explicit under various conditions. The sufficient conditions for the exponential stability of mild solutions of a stochastic partial differential equation with delay were studied in [3] by using the Gronwall inequality. With the proper application of the fixed point theorem J. Luo proves that stability of solutions to stochastic partial differential equations with finite delays in [4]. A result of the exponential stability of weak solutions of semi-linear stochastic evolution equations is found by Khasminskii [5] using the properties of the stochastic convolution integral. The comparison theorem has shown the stability of the mild solution of stochastic partial differential equations in [6].

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In [7] without integrating the random term and by supposing that the retarded part of the equation as much as locally Lipschitzian admits also a resolvent operator, the existence and the regularity of the solutions of the integrodifferential equations have been proved. Thanks to the use of the resolvent operator theory and a successive approximation method, results of existence and uniqueness of mild solution for the stochastic partial integrodifferential functional equation in a real separable Hilbert space under non-Lipschitzian conditions on the coefficients have been obtained in [8].

The existence and uniqueness of energy solutions to stochastic evolution equations under the local non-Lipschitzian condition have been studied in [9].

By using the solver operator theory coupled with a stopping time technique we obtain the result of existence and uniqueness of mild solution for a stochastic evolution integrodifferential equation under non-Lipschitzian conditions on the coefficients on a real Hilbert space [1]. Dans Diop et al. [10] consider a class of random partial integro-differential equations with unbounded delay where they study the existence of weak solutions using a random fixed point theorem with a stochastic domain combined with Schauder's fixed point theorem and Grimmer's resolvent operator theory. Their authors in [11] deal with existence, uniqueness and controllability results for stochastic partial integro-differential equations with non-local conditions. They use Kuratowski's noncompactness measure, Darbo's generalized fixed point theorem, Mönch's fixed point theorem and Grimmer's resolvent operator theory. To extend the analysis of the problem studied to the real case, we turn to the complex Hilbert space. Introducing the complex structure allows us to benefit from more flexible or powerful mathematical tools, and a better understanding of the spectral and analytical properties of the operators concerned. In numerical analysis or in the solution of differential equations, complex Hilbert spaces can sometimes provide a better analysis of stability and convergence. In addition, complex Hilbert spaces are highly suitable for dealing with dynamical systems with stochastic perturbations. These spaces offer a mathematical structure that allows solutions to be defined within a rigorous framework, even in the presence of random noise. This leads us to the following question: can we generalize the result of [8] in a complex Hilbert space? What about the asymptotic stability of this weak solution? The answer to these questions leads us to the formulation of our problem "Study of a mild solution of a stochastic integrodifferential equation with non-lipschitzian coefficients in a complex Hilbert space".

We want to study the existence, uniqueness and stability of a mild solution to the following equation:

$$\begin{cases} dZ(\zeta) = \Big[\mathcal{A}Z(\zeta) + \int_0^\zeta \mathcal{B}(\zeta-\gamma)Z(\gamma)d\gamma \\ \qquad\qquad + k(\zeta,\, Z(\zeta))\Big]d\zeta + l(\zeta,\, Z(\zeta))dB(\zeta) \\ Z(0) = x_0 \in \mathbb{X}_1 \end{cases} ; \tag{1}$$

where $Z(\zeta)$ is a stochastic process, $\mathcal{A} : D(\mathcal{A}) \subset \mathbb{X}_1 \to \mathbb{X}_1$ a closed analytic operator, of domain $D(\mathcal{A})$. For any $\zeta \geq 0$, $\mathcal{B}(\zeta)$ is a bounded analytic operator of domain $D(\mathcal{B}(\zeta))$ with $D(\mathcal{B}(\zeta)) \supset D(\mathcal{A})$; the functions $k : [0,\, +\infty) \times \mathbb{X}_1 \to \mathbb{X}_1$; $l : [0,\, +\infty) \times \mathbb{X}_1 \to L_2^0(\mathbb{X}_2,\, \mathbb{X}_1)$ are continuous; $\mathbb{X}_1$ and $\mathbb{X}_2$ are separable Hilbert spaces.

Our analysis uses the principle of analytic resolving operators in the sense of Grimmer [2] with a technical skill of stopping time. Our work is organised as follows: we shall present the essential notions and definitions of the Wiener process in section 2. In section 3 we will gather some useful results on the analytic solver operator in order to establish the existence and uniqueness of mild solution for our equation (1). Section 4 will be devoted to studying the asymptotic behaviour of this mild solution.

2. Preliminaries

Let $(\Omega,\, \mathcal{F},\, (\mathcal{F}_s)_{s\in[0,\,\infty]},\, \mathbb{P})$ be a filtered probability space with $\mathcal{F}_0$ containing all negligible sets of $\mathcal{F}$.

Let $\mathbb{X}_1$ be a separable complex Hilbert space admitting the norm $\|.\|_{\mathbb{X}_1}$ and the scalar product $\langle .,\, .\rangle_{\mathbb{X}_1}$ respectively.

Let $\mathbb{X}_2$ be a separable complex Hilbert space admitting respectively the norm $|.|_{\mathbb{X}_2}$ and the scalar product $\langle .,\, .\rangle_{\mathbb{X}_2}$. Let $\delta_{n(n=1,\,2,\,3,\,\ldots)}$ be a complete orthonormal basis on $\mathbb{X}_2$ and ψ_n be a family of elements of $\mathbb{R}^+$.

Consider $L(\mathbb{X}_2,\ \mathbb{X}_1)$ the set of all bounded linear operators and $\psi \in L(\mathbb{X}_2,\ \mathbb{X}_2)$ a covariance operator defined by $\psi = \psi_n \delta_n$ with finite trace $\left(Tr(\psi) = \sum_{n=1}^{\infty} \psi_n < \infty \right)$.

The ψ-Wiener process is defined by:

$$B(s) = \sum_{n=1}^{\infty} \psi_n^{1/2} \beta_n(s) \delta_n,\ s \geq 0,$$

where $\beta_n(s)_{(n=1,\ 2,\ 3,\ .....)}$ is an independent real family of standard Brownian motions defined on the triplet $(\Omega,\ \mathcal{F},\ \mathbb{P})$. In the following, we will denote $L_2^0(\mathbb{X}_2,\ \mathbb{X}_1)$ the space of all $(\mathbb{X}_2,\ \mathbb{X}_1)$ such that $\xi \psi^{1/2}$ is a finite Hilbert- operator. Schmidt operator $(Tr(\xi \psi_n \delta_n \xi^*) < \infty)$ whose norm is given by $||\xi||^2_{L_2^0} = ||(\psi_n)^{1/2} \xi \delta_n||^2_{HS} = Tr(\xi \psi_n \xi^*)$.

Let $\mathbb{H}$ be the space of all processes adapted to the filtration $\mathfrak{F}_\zeta \in \mathbb{X}_1$, the adapted processes $Z_\zeta(u) : \mathbb{R}^+ \times \Omega \to \mathbb{X}_1$ which are almost certainly continuous on ζ and u fixed satisfying $\|Z\| = \sup_{\zeta \geq 0} \mathrm{E} \|Z_\zeta(u)\|^n_{\mathbb{X}_1}$.

3. Existence and uniqueness of a mild solution

In this section it is necessary to give the sine qua non conditions for the existence of the operator analytically solving the equation (1) .

To do this we consider the following equation:

$$\begin{cases} \dfrac{\partial}{\partial \zeta} \varphi(\zeta) = \mathcal{A}\varphi(\zeta) + \displaystyle\int_0^\zeta B(\zeta - \gamma)\varphi(\gamma)\mathrm{d}\gamma + h(\zeta),\ \text{for}\ \zeta \geq 0 \\ \varphi(0) = \varphi_0 \in \mathrm{D}(\mathcal{A}) \subset \mathbb{X}_1, \end{cases} \tag{2}$$

where $\mathbb{X}_1$ is a complex Hilbert space, $\mathcal{A}$ is a generator of the analytic semigroup, for all $\beta > 0,\ (-\mathcal{A})^\beta$ is a dense domain operator $\mathbb{X}_2^\beta \in \mathbb{X}_1$. $\mathbb{X}_2^\beta$ has the graph norm $\|.\|_\beta$ so that $\left(\mathbb{X}_2^\beta,\ \|.\|_\beta\right)$ is a complex Hilbert space.

3.1 *Resolvent operator*

Definition 1 We define an operator solving the equation (2) as a bounded linear operator of function $\mathcal{S}(\zeta) \in \mathcal{L}(\mathbb{X}_2^\beta)$ satisfying the following properties:

(i). For any $\zeta \geq 0, \mathcal{S}(\zeta)$ is strongly continuous, with $\mathcal{S}(0) = I$ and there exist constants $N,\ \eta$ such that $\| \mathcal{S}(\zeta) \| \leq N e^{\eta \zeta}$.

(ii). For all $\zeta \geq 0$ et $z \in \mathbb{X}_2^\beta,\ \mathcal{S}(\zeta)z$ is strongly continuous on $\mathbb{X}_2^\beta$.

(iii). For $z \in \mathbb{X}_2^\beta$, and $\zeta \geq 0,\ \mathcal{S}(\zeta)z$ is continuously differentiable,

$$\frac{\partial}{\partial \zeta}\mathcal{S}(\zeta)z = \mathcal{A}_0\mathcal{S}(\zeta)z + \int_0^\zeta \mathcal{B}(\zeta - \gamma)\mathcal{S}(\gamma)z\mathrm{d}\gamma,$$

and

$$\frac{\partial}{\partial \zeta}\mathcal{S}(\zeta)z = \mathcal{S}(\zeta)\mathcal{A}_0 z + \int_0^\zeta \mathcal{S}(\zeta - \gamma)\mathcal{B}(\gamma)z\mathrm{d}\gamma.$$

In a similar way, researchers such as Grimmer and Pitchard in [2] have generalized the existence of an analytic resolving operator in the complex case. The following theorem gives us the certainty of the existence of the analytic resolving operator.

Theorem 1 See [2] Equation (2) admits a solving operator $(\mathcal{S}(\zeta))_{\zeta\geq 0}$. However, we have the existence of a constant $N \geq 0$ which satisfies the following inequality $\|\mathcal{S}(\zeta)\| \leq N$ with $\mathcal{S}(\zeta)$ the analytical extension in the region $\{\zeta \in \mathbb{C};\ |arg(\zeta)| < \phi\}$.

Theorem 2 See [2] Let $b(\zeta)$ be bounded on any interval of the form $0 < \tau_1 \leq \zeta \leq \tau_2 < \infty$. Then for all $\beta \in [0,\ 1]$ and $\zeta > 0$, $\mathcal{S}(\zeta) \in \mathcal{L}(\mathbb{X}_2^{\beta},\ \mathbb{X}_2)$ with $\|\mathcal{S}(\zeta)\|_{\beta,\ 1} \leq \dfrac{M}{\zeta^{1-\beta}}$, M a positive constant. In addition, for all $z \in \mathbb{X}_2^{\beta}$; $\beta \in [0,\ 1]$ we have:

$$\mathcal{S}'(\zeta)z = \mathcal{A}(\zeta)z + \int_0^{\zeta} \mathcal{B}(\zeta-\gamma)\mathcal{S}(\gamma)z\mathrm{d}\gamma,\ \zeta > 0.$$

3.2 *Existence and uniqueness of a mild solution*

This subsection is dedicated to the study of the existence and uniqueness of mild solutions of the equation (1). Authors have already treated the same problem in a real Hilbert space. We aim to prove the same result in a complex Hilbert space. To carry out the study, we will first present some useful definitions and lemmas that we will use to prove our results. Then we will give some assumptions (note that these same assumptions have been used in [1, 9]) on the Lipschitz conditions of the coefficients of f and g. Finally, we state the main theorem of our study.

Definition 2 A process $\{Z(\zeta),\ \zeta \in [0,\ \tau],\ 0 \leq \tau < +\infty\}$ is called a mild solution of the equation (1) if:

(i). For any $\zeta \geq 0$, $Z(\zeta)$ is $\mathcal{F}$-adapted with $\int_0^{\tau} |Z(u)|^2_{\mathbb{X}_1}\mathrm{d}u < \infty$ p.s;

(ii). For all $\zeta \in [0,\ \tau]$, $Z(\zeta) \in \mathbb{X}_1$ is almost surely continuous and satisfies the following integral equation:

$$\begin{cases} Z(\zeta) = \mathcal{S}(\zeta)z_0 + \displaystyle\int_0^{\zeta} \mathcal{S}(\zeta-\gamma)k(\gamma,\ Z(\gamma))\mathrm{d}\gamma \\ \qquad\qquad + \displaystyle\int_0^{\zeta} \mathcal{S}(\zeta-\gamma)l(\gamma,\ Z(\gamma))\mathrm{d}B(\gamma),\ \forall\zeta \geq 0 \\ Z(0) = z_0 \in \mathbb{X}_1. \end{cases} \tag{3}$$

Lemma 1 See [2] Assume $\mathcal{S}(\zeta)$ is an operator solving the equation (2), $z_0 \in \mathbb{X}_2$ and $k \in \mathcal{C}([0,\ \infty],\ \mathbb{X}_1)$. If φ is a solution of the equation (2), then the method for varying the parameters is given by:

$$\varphi(\zeta) = \mathcal{S}(\zeta)\varphi_0 + \int_0^{\zeta} \mathcal{S}(\zeta-\gamma)h(\gamma)\mathrm{d}\gamma \text{ pour } \zeta \geq 0. \tag{4}$$

Lemma 2 See [2] We assume that $\mathcal{S}(\zeta)$ is an operator solving the equation (2). Let $z_0 \in \mathbb{X}_2$ and $k \in \mathcal{C}([0,\ \infty],\ \mathbb{X}_1)$ then

$$\varphi(\zeta) = \mathcal{S}(\zeta)\varphi_0 + \int_0^{\zeta} \mathcal{S}(\zeta-\gamma)h(\gamma)\mathrm{d}\gamma \text{ for } \zeta \geq 0. \tag{5}$$

is a solution of the equation (2).

Definition 3 The function $\varphi : [0,\ +\infty] \to \mathbb{X}_1$ is a mild solution of (2) if

1. $\varphi \in \mathcal{C}([0,\ \tau];\ \mathbb{X}_1) \cap \mathcal{C}^1([0,\ \tau];\ \mathbb{X}_1)$,

2. φ satisfies the conditions of the equation (2) and the method of variation of the parameters (5) for $\zeta \geq 0$,
3. $h \in \mathcal{C}\left([0,\ \tau],\ \mathbb{X}_2^{\beta}\right)$.

Lemma 3 See [2] Suppose that the conditions of theorem 1 are satisfied and $z_0 \in \mathbb{X}_2^{\beta}$, $k \in \mathcal{C}\left([0,\ T],\ \mathbb{X}_2^{\beta}\right)$, then φ is a solution of (2) in the sense that $\varphi \in \mathcal{C}([0,\ \tau]; \mathbb{X}_1) \cap \mathcal{C}^1([0,\ \tau];\ \mathbb{X}_1)$ and

$$\frac{\partial}{\partial \zeta}\varphi(\zeta) = \mathcal{A}\varphi(\zeta) + \int_0^{\zeta} \mathcal{B}(\zeta - \gamma)\varphi(\gamma)\mathrm{d}\gamma$$

$$+ h(\zeta,) \quad \text{for } \zeta \geq 0 \text{ and } \varphi(0) = \varphi_0.$$

H1 the equation (2) has an analytic resolving operator.

H2 (I). **Growth condition:** for any $\zeta \in [0,\ \tau]$ and $s \geq 0$ fixed; there exists a function $J(\zeta,\ s) : \mathbb{R}_+^2 \to \mathbb{R}_+$ locally integrable with respect to ζ and s. It is also continuous, strictly increasing and concave in s and $\zeta \in [0,\ \tau]$ fixed. Furthermore, for all $\mu \in \mathbb{X}_1$ the following inequality is satisfied:

$$|k(\zeta,\ \mu)|^2 + |l(\zeta,\ \mu)|_{L_2^0}^2 \leq J(\zeta,\ |\mu|^2),\ \zeta \in [0,\ \tau], \tag{6}$$

(II). for any constant $\beta > 0$, the differential equation

$$\frac{\mathrm{d}\phi}{\mathrm{d}\zeta} = \beta J(\zeta,\ \phi),\ \zeta \in [0,\ \tau],$$

has a solution $\phi(\zeta) = \phi(\zeta;\ 0,\ \phi_0)$ on $[0,\ \tau]$ for any initial value ϕ_0.

H3 (I). **Global condition:** for all $\zeta \in [0,\ \tau]$, $s \geq 0$ fixed; there exists a function $Q(\zeta,\ s) : \mathbb{R}_+^2 \to \mathbb{R}_+$ locally integrable with respect to ζ and s fixed. $Q(\zeta,\ s)$ is also continuous, monotonically strictly increasing and concave. Furthermore, the following inequality is satisfied: for all $\mu,\ \nu \in \mathbb{X}_1$,

$$|k(\zeta,\ \mu) - k(\zeta,\ \nu)|^2 + |l(\zeta,\ \mu) - l(\zeta,\ \nu)|_{L_2^0}^2$$

$$\leq Q(\zeta,\ |\mu - \nu|^2);$$

(II). for any constant $\eta > 0$, if a positive function $y(\zeta)$ satisfies:

$$y(\zeta) \leq \eta \int_0^{\zeta} Q(\gamma,\ y(\gamma))\mathrm{d}\gamma,$$

for all $y(\zeta) \in [0,\ \tau]$, then $y(\zeta) \equiv 0$.

Theorem 3 Assume that assumptions **H1, H2** and **H3** are satisfied. Then there is a unique mild solution $Z(\zeta)$ for the equation (1).

Proof. We will prove this theorem in two stages. The first will consist of proving the existence in the solution and the uniqueness of the solution in the second stage. In this first stage two elements are essential, namely the analytic resolving operator and the formula for varying the parameters.

(I.) Let $Z(\zeta)$, $\zeta \geq 0$ be a stochastic process solution of the equation (1). From Theorem 2 we have the certainty of the existence of an analytic solver operator $\mathcal{S}(\zeta)$ for our equation (1). With this solver operator lemma 1 gives us point (2) of definition 2. Points (1) and (3) of the definition are satisfied by Lemma 3. Since all the conditions of Definition 2 are satisfied, we can conclude at this stage that our equation has a mild solution.

Following the existence of a mild solution for our equation, we will use the properties of the resolving operator of the analytic semi-group simultaneously with the technical procedures of stopping time to prove the uniqueness of the mild solution in our second stage of proof.

(II.) Let two processes $Z_\zeta,\ \tilde{Z}_\zeta,\ \zeta \in [0,\ \tau],\ 0 \leq \tau < +\infty$, all mild solutions of the equation (1); such that $Z_0 = \tilde{Z}_0$. With $M > 0$ fixed, let $\zeta_1 = \inf\{|Z_\zeta| \geq M\ or;\ |\tilde{Z}_\zeta| \geq M\}_{\zeta \geq 0}$.

Let $\mathcal{S}(\zeta) = (2\pi i)^{-1} \int_\Delta e^{\psi\zeta} \rho(\psi) \mathrm{d}\psi$ be an analytic resolving operator in the region $\Delta = \psi \in \mathbb{C},\ |\arg(\psi)| < \frac{\pi}{2} + \theta\}$, and $0 < \theta < \frac{\pi}{2}$;

$$Z_\zeta = \mathcal{S}(\zeta)Z_0 + \int_0^\zeta \mathcal{S}(\zeta-\gamma)k\left(\gamma,\ Z_\gamma\right)\mathrm{d}\gamma + \int_0^\zeta \mathcal{S}(\zeta-\gamma)l\left(\gamma,\ Z_\gamma\right)\mathrm{d}B_\gamma,$$

$$\tilde{Z}_\zeta = \mathcal{S}(\zeta)\tilde{Z}_0 + \int_0^\zeta \mathcal{S}(\zeta-\gamma)k\left(\gamma,\ \tilde{Z}_\gamma\right)\mathrm{d}\gamma + \int_0^\zeta \mathcal{S}(\zeta-\gamma)l\left(\gamma,\ \tilde{Z}_\gamma\right)\mathrm{d}B_\gamma.$$

We have noted:

$$\begin{aligned}
\tilde{Z}_\zeta - Z_\zeta &= \mathcal{S}(\zeta)Z_0 - \mathcal{S}(\zeta)\tilde{Z}_0 + \int_0^\zeta \mathcal{S}(\zeta-\gamma)k(\gamma,\ \tilde{Z}_\gamma)\mathrm{d}\gamma - \int_0^\zeta \mathcal{S}(\zeta-\gamma)k(\gamma,\ Z_\gamma)\mathrm{d}\gamma \\
&\quad + \int_0^\zeta \mathcal{S}(\zeta-\gamma)l(\gamma,\ \tilde{Z}_\gamma)\mathrm{d}B_\gamma - \int_0^\zeta \mathcal{S}(\zeta-\gamma)l(\gamma,\ Z_\gamma)\mathrm{d}B_\gamma \\
&= \int_0^\zeta \mathcal{S}(\zeta-\gamma)\Big(k(\gamma,\ \tilde{Z}_\gamma) - k(\gamma,\ Z_\gamma)\Big)\mathrm{d}\gamma + \int_0^\zeta \mathcal{S}(\zeta-\gamma)\left(l(\gamma,\ \tilde{Z}_\gamma) - l(\gamma,\ Z_\gamma)\right)\mathrm{d}B_\gamma \\
&= \int_0^\zeta (2\pi i)^{-1} \int_\Gamma e^{\psi(\zeta-\gamma)}\rho(\psi)\left(k(\gamma,\ \tilde{Z}_\gamma) - k(\gamma,\ Z_\gamma)\right)\mathrm{d}\psi\mathrm{d}\gamma \\
&\quad + \int_0^\zeta (2\pi i)^{-1} \int_\Gamma e^{\psi(\zeta-\gamma)}\rho(\psi)\left(l(\gamma,\ \tilde{Z}_\gamma) - l(\gamma,\ Z_\gamma)\right)\mathrm{d}\psi\mathrm{d}B_\gamma \\
&= (2\pi i)^{-1} \int_\Gamma \int_0^\zeta e^{\psi(\zeta-\gamma)}\rho(\psi)\left(k(\gamma,\ \tilde{Z}_\gamma) - k(\gamma,\ Z_\gamma)\right)\mathrm{d}\psi\mathrm{d}\gamma \\
&\quad + (2\pi i)^{-1} \int_\Gamma \int_0^\zeta e^{\psi(\zeta-\gamma)}\rho(\psi)\left(l(\gamma,\ \tilde{Z}_\gamma) - l(\gamma,\ Z_\gamma)\right)\mathrm{d}\psi\mathrm{d}B_\gamma.
\end{aligned}$$

We continue the proof with the following lemma.

Lemma 4 $\mathbb{X}_1$ being a Hilbert space, for all $\vartheta_k \in \mathbb{X}_1$, we have:

1. $\left\| \sum_{k=1}^{m} \vartheta_k \right\|^2 \leq m \left(\sum_{k=1}^{m} \|\vartheta_k\|^2 \right)$,

2. if the family $(\vartheta_k)_{k\geq 0}$ is orthogonal then, $\left\| \sum_{k=1}^{m} \vartheta_k \right\|^2 = \sum_{k=1}^{m} \|\vartheta_k\|^2$ (Pythagorean relation).

Proof. For $m = 1$ it is obvious.
For $m = 2$,

$$\begin{aligned}
\|\vartheta_1 + \vartheta_2\|^2 &= \|\vartheta_1\|^2 + \|\vartheta_2\|^2 + 2\Re(\langle \vartheta_1, \vartheta_2 \rangle) \\
&\leq \|\vartheta_1\|^2 + \|\vartheta_2\|^2 + 2|\langle \vartheta_1, \vartheta_2 \rangle| \\
&\leq \|\vartheta_1\|^2 + \|\vartheta_2\|^2 + 2\|\vartheta_1\|.\|\vartheta_2\|
\end{aligned}$$

according to Cauchy Schawrz's inequality

$$\begin{aligned}
&\leq \|\vartheta_1\|^2 + \|\vartheta_2\|^2 + \|\vartheta_1\|^2 + \|\vartheta_2\|^2 \\
&\leq 2\left(\|\vartheta_1\|^2 + \|\vartheta_2\|^2\right).
\end{aligned}$$

Hence $|\vartheta_1 + \vartheta_2|^2 \leq 2\left(|\vartheta_1|^2 + |\vartheta_2|^2\right)$ and the property is true for $m = 2$.
Assuming the property is true at rank m, let's prove the property at rank $m+1$.

$$\begin{aligned}
\left\| \sum_{k=1}^{m+1} \vartheta_k \right\|^2 &= \left\| \sum_{k=1}^{m} \vartheta_k + \vartheta_{m+1} \right\|^2 \\
&\leq \left\| \sum_{k=1}^{m} \vartheta_i \right\|^2 + \|\vartheta_{m+1}\|^2 + 2\left| \left\langle \sum_{k=1}^{m} \vartheta_k, \vartheta_{m+1} \right\rangle \right| \\
&\leq \left\| \sum_{k=1}^{m} \vartheta_k \right\|^2 + \|\vartheta_{m+1}\|^2 + 2|\langle \vartheta_1 + \vartheta_2 + \vartheta_3 + \cdots + \vartheta_m, \vartheta_{m+1} \rangle| \\
&\leq \left\| \sum_{k=1}^{m} \vartheta_k \right\|^2 + \|\vartheta_{m+1}\|^2 + 2|\langle \vartheta_1 + \vartheta_2 + \vartheta_3 + \cdots + \vartheta_m, \vartheta_{m+1} \rangle| \\
&\leq \left\| \sum_{k=1}^{m} \vartheta_k \right\|^2 + \|\vartheta_{m+1}\|^2 + 2|\langle \vartheta_1, \vartheta_{m+1} \rangle| + 2|\langle \vartheta_2, \vartheta_{m+1} \rangle| + \\
&\quad \cdots + 2|\langle \vartheta_m, \vartheta_{m+1} \rangle|.
\end{aligned}$$

From the recurrence relation we have: $\left\|\sum_{k=1}^{m}\vartheta_k\right\|^2 \leq m\left(\sum_{k=1}^{m}\|\vartheta_k\|^2\right)$.
For all $k \in \overline{1,\ m}$,

$$2|\langle\vartheta_k, \vartheta_{m+1}\rangle| \leq \|\vartheta_k\|^2 + \|\vartheta_{m+1}\|^2.$$

As a result,

$$\begin{aligned}
\left\|\sum_{k=1}^{m+1}\vartheta_k\right\|^2 \leq m\left(\sum_{k=1}^{m}\|\vartheta_k\|^2\right) + \|\vartheta_{m+1}\|^2 + \|\vartheta_1\|^2 + \|\vartheta_{m+1}\|^2 + \|\vartheta_2\|^2 \\
+\|\vartheta_{m+1}\|^2 + \cdots + \|\vartheta_m\|^2 + \|\vartheta_{m+1}\|^2 \\
\leq m\left(\sum_{k=1}^{m}\|\vartheta_k\|^2\right) + \|\vartheta_{m+1}\|^2 + \sum_{k=1}^{m}\|\vartheta_k\|^2 + m\|\vartheta_{m+1}\|^2 \\
\leq m\left(\sum_{k=1}^{m}\|\vartheta_k\|^2 + \|\vartheta_{m+1}\|^2\right) + \left(\sum_{k=1}^{m}\|\vartheta_k\|^2 + \|\vartheta_{m+1}\|^2\right) \\
\leq (m+1)\left(\sum_{k=1}^{m}\|\vartheta_k\|^2 + \|\vartheta_{m+1}\|^2\right) \\
\leq (m+1)\left(\sum_{k=1}^{m+1}\|\vartheta_i\|^2\right).
\end{aligned}$$

Hence the relation is true at rank $m+1$; consequently it is true for all $m \in \mathbb{N}$.
The second point of the lemma is obvious because $\forall k \in \mathbb{N}^*$; $\langle\vartheta_k, \vartheta_{m+1}\rangle = 0$ if $\vartheta_k \perp \vartheta_{m+1}$.
Using lemma 4 we get:

$$\mathtt{E}\left(\sup_{0\le\gamma\le\zeta\wedge\zeta_1}\|Z_\gamma-\tilde{Z}_\gamma\|^2\right)$$

$$=\mathtt{E}\left(\sup_{0\le\gamma\le\zeta\wedge\zeta_1}\left\|(2\pi i)^{-1}\int_0^\gamma\int_\Gamma e^{\psi(\gamma-r)}\rho(\psi)\left[k(r,\,\tilde{Z}_r)-k(r,\,Z_r)\right]\mathrm{d}r\mathrm{d}\psi\right.\right.$$

$$\left.\left.+(2\pi i)^{-1}\int_0^\gamma\int_\Gamma e^{\psi(\gamma-r)}\rho(\psi)\left[l(r,\,\tilde{Z}_r)-l(r,\,Z_r)\right]\mathrm{d}B_r\mathrm{d}\psi\right\|^2\right)$$

$$\le|(\sqrt{2}\pi i)^{-2}|\mathtt{E}\left(\sup_{0\le\gamma\le\zeta\wedge\zeta_1}\left\|\int_0^\gamma\int_\Gamma e^{\psi(\gamma-r)}\rho(\psi)\left[k(r,\,\tilde{Z}_r)-k(r,\,Z_r)\right]\mathrm{d}r\mathrm{d}\psi\right\|_{\mathbb{X}_1}^2\right)$$

$$+|(\sqrt{2}\pi i)^{-2}|\mathtt{E}\left(\sup_{0\le\gamma\le\zeta\wedge\zeta_1}\left\|\int_0^\gamma\int_\Gamma e^{\psi(\gamma-r)}\rho(\psi)\left[l(r,\,\tilde{Z}_r)-l(r,\,Z_r)\right]\mathrm{d}B_r\mathrm{d}\psi\right\|^2\right).$$

$$\mathtt{E}\left(\sup_{0\le\gamma\le\zeta\wedge\zeta_1}\|Z_\gamma-\tilde{Z}_\gamma\|^2\right)$$

$$\le|(\sqrt{2}\pi)^{-2}|\mathtt{E}\left(\sup_{0\le\gamma\le\zeta\wedge\zeta_1}\left\|\int_0^\gamma\int_\Gamma e^{\psi(\gamma-r)}\rho(\psi)\left[k(r,\,\tilde{Z}_r)-k(r,\,Z_r)\right]\mathrm{d}r\mathrm{d}\psi\right\|_{\mathbb{X}_1}^2\right)\tag{7}$$

$$+|(\sqrt{2}\pi)^{-2}|\mathtt{E}\left(\sup_{0\le\gamma\le\zeta\wedge\zeta_1}\left\|\int_0^\gamma\int_\Gamma e^{\psi(\gamma-r)}\rho(\psi)\left[l(r,\,\tilde{Z}_r)-l(r,\,Z_r)\right]\mathrm{d}B_r\mathrm{d}\psi\right\|^2\right).\tag{8}$$

From [2] there exists a constant N such that $\|\rho(\psi)\|\le N|\psi|^{-1}$ in (7) and in (8), which subsequently gives,

$$\mathtt{E}\left(\sup_{0\le\gamma\le\zeta\wedge\zeta_1}\|Z_\gamma-\tilde{Z}_\gamma\|^2\right)$$

$$\le|(\sqrt{2}\psi\pi)^{-2}N^2|\mathtt{E}\left(\sup_{0\le\gamma\le\zeta\wedge\zeta_1}\int_0^\gamma\int_\Gamma\left\|e^{\psi(\gamma-r)}\left[k(r,\,\tilde{Z}_r)-k(r,\,Z_r)\right]\mathrm{d}r\mathrm{d}\psi\right\|_{\mathbb{X}_1}^2\right)$$

$$+|(\sqrt{2}\psi\pi)^{-2}N^2|\mathtt{E}\left(\sup_{0\le\gamma\le\zeta\wedge\zeta_1}\left\|\int_0^\gamma\int_\Gamma e^{\psi(\gamma-r)}\left[l(r,\,\tilde{Z}_r)-l(r,\,Z_r)\right]\mathrm{d}B_r\mathrm{d}\psi\right\|^2\right)$$

$$\le|(\sqrt{2}\psi\pi)^{-2}N^2|\mathtt{E}\left(\sup_{0\le\gamma\le\zeta\wedge\zeta_1}\int_0^\gamma\int_\Gamma\left\|e^{\psi(\gamma-r)}\left[k(r,\,\tilde{Z}_r)-k(r,\,Z_r)\right]\mathrm{d}r\mathrm{d}\psi\right\|_{\mathbb{X}_1}^2\right)$$

$$+|(\sqrt{2}\psi\pi)^{-2}N^2|\mathtt{E}\left(\sup_{0\le\gamma\le\zeta\wedge\zeta_1}\left\|\int_\Gamma e^{\psi(\gamma-r)}\left((\int_0^\gamma\left[l(r,\,\tilde{Z}_r)-l(r,\,Z_r)\right]\mathrm{d}B_r)^2\right)^{1/2}\mathrm{d}\psi\right\|^2\right).$$

By using the It's Lemma

$$\mathtt{E}\left(\sup_{0\le\gamma\le\zeta\wedge\zeta_1}\|Z_\gamma-\tilde{Z}_\gamma\|^2\right)$$

$$\le|(\sqrt{2}\psi\pi)^{-2}N^2|\mathtt{E}\left(\sup_{0\le\gamma\le\zeta\wedge\zeta_1}\int_0^\gamma\int_\Gamma\left\|e^{\psi(\gamma-r)}\left[k(r,\,\tilde{Z}_r)-k(r,\,Z_r)\right]\mathrm{d}r\mathrm{d}\psi\right\|^2_{\mathbb{X}_1}\right)$$

$$+|(\sqrt{2}\psi\pi)^{-2}N^2|\mathtt{E}\left(\sup_{0\le\gamma\le\zeta\wedge\zeta_1}\left\|\int_\Gamma e^{\psi(\gamma-r)}\left(\int_0^\gamma\left[l(r,\,\tilde{Z}_r)-l(r,\,Z_r)\right]^2\mathrm{d}_r\right)^{1/2}\mathrm{d}\psi\right\|^2\right)$$

$$\le|(\sqrt{2}\psi\pi)^{-2}N^2|\mathtt{E}\left(\sup_{0\le\gamma\le\zeta\wedge\zeta_1}\int_0^\gamma\int_\Gamma\|e^{\psi(\gamma-r)}\|^2\left\|\left[k(r,\,\tilde{Z}_r)-k(r,\,Z_r)\right]\right\|^2_{\mathbb{X}_1}\mathrm{d}r\mathrm{d}\psi\right)$$

$$+|(\sqrt{2}\psi\pi)^{-2}N^2|\mathtt{E}\left(\sup_{0\le\gamma\le\zeta\wedge\zeta_1}\int_\Gamma\|e^{\psi(\gamma-r)}\|^2\left\|\left(\int_0^\gamma\left[l(r,\,\tilde{Z}_r)-l(r,\,Z_r)\right]^2\mathrm{d}_r\right)^{1/2}\right\|^2\mathrm{d}\psi\right)$$

$$\le|(\sqrt{2}\psi\pi)^{-2}N^2|\mathtt{E}\left(\sup_{0\le\gamma\le\zeta\wedge\zeta_1}\int_0^\gamma\int_\Gamma\|e^{\psi(\gamma-r)}\|^2\left\|k(r,\,\tilde{Z}_r)-k(r,\,Z_r)\right\|^2_{\mathbb{X}_1}\mathrm{d}r\mathrm{d}\psi\right)$$

$$+|(\sqrt{2}\psi\pi)^{-2}N^2|\mathtt{E}\left(\sup_{0\le\gamma\le\zeta\wedge\zeta_1}\int_\Gamma\int_0^\gamma\|e^{\psi(\gamma-r)}\|^2\left\|l(r,\,\tilde{Z}_r)-l(r,\,Z_r)\right\|^2\mathrm{d}_r\mathrm{d}\psi\right)$$

$$\le|(\sqrt{2}\psi\pi)^{-2}N^2|\mathtt{E}\left(\sup_{0\le\gamma\le\zeta\wedge\zeta_1}\int_0^\gamma\int_\Gamma e^{2|\psi|\|\gamma-r\|}\left\|k(r,\,\tilde{Z}_r)-k(r,\,Z_r)\right\|^2_{\mathbb{X}_1}\mathrm{d}r\mathrm{d}\psi\right)$$

$$+|(\sqrt{2}\psi\pi)^{-2}N^2|\mathtt{E}\left(\sup_{0\le\gamma\le\zeta\wedge\zeta_1}\int_\Gamma\int_0^\gamma e^{2|\psi|\|\gamma-r\|}\left\|l(r,\,\tilde{Z}_r)-l(r,\,Z_r)\right\|^2\mathrm{d}_r\mathrm{d}\psi\right).$$

By the hypothesis **H3** we have:

$$\mathbb{E}\left(\sup_{0\leq\gamma\leq\zeta\wedge\zeta_1}\|Z_\gamma-\tilde{Z}_\gamma\|^2\right)$$

$$\leq|(\sqrt{2}\psi\pi)^{-2}N^2|\mathbb{E}\left(\sup_{0\leq u\leq r}\int_\Gamma\int_0^{\zeta\wedge\zeta_1}e^{2|\psi|\|r-u\|}Q\left(r,\ \|\tilde{Z}_u-Z_u\|^2_{\mathbb{X}_1}\right)\mathrm{d}r\mathrm{d}\psi\right)$$

$$\leq|(\sqrt{2}\psi\pi)^{-2}N^2|\int_\Gamma\int_0^{\zeta\wedge\zeta_1}\sup_{0\leq u\leq r}e^{2|\psi|\|r-u\|}\mathbb{E}\left(\sup_{0\leq u\leq r}Q\left(r,\ \|\tilde{Z}_u-Z_u\|^2_{\mathbb{X}_1}\right)\mathrm{d}r\mathrm{d}\psi.\right.$$

We now apply Jensen's inequality and obtain:

$$\mathbb{E}\left(\sup_{0\leq\gamma\leq\zeta\wedge\zeta_1}\|Z_\gamma-\tilde{Z}_\gamma\|^2\right)$$

$$\leq|(\sqrt{2}\psi\pi)^{-2}|N^2\int_\Gamma\int_0^{\zeta\wedge\zeta_1}\sup_{0\leq u\leq r}e^{2|\psi|\|r-u\|}Q\left(r,\ \mathbb{E}\left(\sup_{0\leq u\leq r}\|\tilde{Z}_u-Z_u\|^2_{\mathbb{X}_1}\right)\right)\mathrm{d}r\mathrm{d}\psi$$

$$\leq|(\sqrt{2}\psi\pi)^{-2}|N^2\int_\Gamma\int_0^{\zeta\wedge\zeta_1}e^{2|\psi|\|r\|}Q\left(r,\ \mathbb{E}\left(\sup_{0\leq u\leq r}\|\tilde{Z}_u-Z_u\|^2_{\mathbb{X}_1}\right)\right)\mathrm{d}r\mathrm{d}\psi$$

$$\leq|(\sqrt{2}\psi\pi)^{-2}|N^2\int_0^{\zeta\wedge\zeta_1}Q\left(r,\ \mathbb{E}\left(\sup_{0\leq u\leq r}\|\tilde{Z}_u-Z_u\|^2_{\mathbb{X}_1}\right)\right)\int_\Gamma e^{2|\psi|\|r\|}\mathrm{d}\psi\mathrm{d}r.$$

If we again define the contour Γ' as being the portion of Γ located at a certain distance from the origin and the arc of a circle $\arg(\psi)<\dfrac{\pi}{2}+\delta$ such that we have for the following:

$$\mathbb{E}\left(\sup_{0\leq\gamma\leq\zeta\wedge\zeta_1}\|Z_\gamma-\tilde{Z}_\gamma\|^2\right)$$

$$\leq|(\sqrt{2}\psi\pi)^{-2}|N^2M^2\int_0^\zeta Q\left(r,\ \mathbb{E}\left(\sup_{0\leq u\leq r}\|\tilde{Z}_{u\wedge\zeta_1}-Z_{u\wedge\zeta}\|^2_{\mathbb{X}_1}\right)\right)\mathrm{d}r$$

$$\leq\left(\frac{NM}{\sqrt{2}\pi|\psi|}\right)^2\int_0^\zeta Q\left(r,\ \mathbb{E}\left(\sup_{0\leq u\leq r}\|\tilde{Z}_{u\wedge\zeta_1}-Z_{u\wedge\zeta_1}\|^2_{\mathbb{X}_1}\right)\right)\mathrm{d}r.$$

Point 2 of the hypothesis **H3** gives the following relationship:

$$\mathrm{E}\left(\sup_{0\le\gamma\le\zeta\wedge\zeta_1}\|Z_\gamma-\tilde{Z}_\gamma\|^2\right)=0.$$

We therefore obtain $Z_\zeta=\tilde{Z}_\zeta;\ \forall\zeta\in[0,\ \zeta\wedge\zeta_1]$ and $Z_\zeta=\mathcal{S}(\zeta)Z_0+\int_0^\zeta\mathcal{S}(\zeta-\gamma)k(\gamma,\ Z_\gamma)\mathrm{d}\gamma+\int_0^\zeta\mathcal{S}(\zeta-\gamma)l(\gamma,\ Z_\gamma)\mathrm{d}B_\gamma$. Hence the existence and uniqueness of the mild solution for the equation (1).

4. Stability of mild solutions

In the previous section we showed that (3) is a mild solution of our equation (1), which leads us to check the asymptotic stability of this solution. To this end, we formulate the following hypotheses:

H4: $\mathcal{S}(\zeta)$ being an analytic resolving operator on $\mathbb{X}_1$, from the definition of the resolving operator; there exist constants $N\ge 0$ and $\eta>0$ such that $\|\mathcal{S}(\zeta)\|\le Ne^{-\eta\zeta}$.

H5: $\|k(t,\ Z_t)-k(t-Y_t)\|\le L_1\|Z_t-Y_t\|;\ \forall t\ge 0,\ Z_t,\ Y_t\in\mathbb{X}_1$, where $L_1>0$.

H6: $\|l(t,\ Z_t)-l(t-Y_t)\|\le L_2\|Z_t-Y_t\|;\ \forall t\ge 0,\ Z_t,\ Y_t\in\mathbb{X}_1$, where $L_2>0$.

H7: $2^{m-1}N^m\left(L_1^m(1/\eta)^m+L_2^mC_m\left(\dfrac{2(m-1)}{m-2}\eta\right)^{\frac{2-m}{2}}\right)<1$.

Definition 4 Let $m\in\mathbb{N}$ with $m\ge 2$, the mild solution $Z(\zeta)$ of the equation (1) is said to be stable in moment of order m if for $\varepsilon>0$ given arbitrarily there exists α such that $\|Z_0\|<\alpha$ guarantees that:

$$\mathrm{E}\left(\sup_{s\ge\tau}\|Z(\zeta)\|^m\right)<\varepsilon$$

Definition 5 Let $m\in\mathbb{N}$ with $m\ge 2$, the mild solution $Z(\zeta)$ of the equation (1) is said to be asymptotically stable in moment of order m if it is stable in moment of order n and for all $X_0\in\mathbb{X}_1$,

$$\lim_{\tau\to\infty}\mathrm{E}\left(\sup_{s\ge\tau}\|Z(\zeta)\|^m\right)=0.$$

Lemma 5 See [12] For any $\beta\ge 1$ and for any arbitrary predictable process $\Phi(\cdot)$ with values in $\mathbf{L}_2^0(\mathbb{X}_2,\ \mathbb{X}_1)$,

$$\mathrm{E}\left(\sup_{0<s_1<s}\left\|\int_0^{s_1}\Phi(u)\mathrm{d}B(u)\right\|^{2\beta}\right)$$

$$\le c_\beta\sup_{0<s_1<s}\mathrm{E}\left\|\int_0^{s_1}\Phi(u)\mathrm{d}B(u)\right\|^{2\beta}$$

$$\le C_\beta\left(\int_0^s\left(\mathrm{E}\|\Phi(s)\|_{\mathbf{L}_2^0}^{2\beta}\right)^{1/\beta}\mathrm{d}s_1\right)^\beta,\ s\ge 0,$$

where, $c_\beta = \left(\dfrac{2\beta}{2\beta-1}\right)^{2\beta}$ et $C_\beta = (\beta(2\beta-1))^{\beta}\left(\dfrac{2\beta}{2\beta-1}\right)^{2\beta^2}$.

Theorem 4 Assuming assumptions **H4-H6** are true then Z_ζ of (3) is asymptotically stable in moment of order m with $m \geq 2$.

Proof. Let Ψ be an application defined from $\mathbb{F}$ into $\mathbb{F}$ by the following relation:

$$\Psi(Z_\zeta) = \mathcal{S}(\zeta)Z_0 + \int_0^\zeta \mathcal{S}(\zeta-\gamma)k(\gamma,\, Z_\gamma)\mathrm{d}\gamma + \int_0^\zeta \mathcal{S}(\zeta-\gamma)l(\gamma,\, Z_\gamma)\mathrm{d}B_\gamma. \tag{9}$$

First, let's check the m-order continuity of Ψ.

$$\mathtt{E}\Big(\big\|\Psi(Z_{\zeta+\ell}) - \Psi(Z_\zeta)\big\|_{\mathbb{X}_1}\Big)^m \leq I + J + K,$$

$$I = 3^{m-1}\mathtt{E}\Big(\big\|\mathcal{S}(\zeta+\ell)Z_0 - \mathcal{S}(\zeta)Z_0\big\|_{\mathbb{X}_1}\Big)^m \tag{10}$$

$$J = 3^{m-1}\mathtt{E}\Big(\Big\|\int_0^\zeta \mathcal{S}(\zeta+\ell-\gamma)k(\gamma,\, Z_\gamma)\mathrm{d}\gamma - \int_0^\zeta \mathcal{S}(\zeta-\gamma)k(\gamma,\, Z_\gamma)\mathrm{d}\gamma\Big\|_{\mathbb{X}_1}\Big)^m \tag{11}$$

$$K = 3^{m-1}\mathtt{E}\Big(\Big\|\int_0^\zeta \mathcal{S}(\zeta-\gamma)l(\gamma,\, Z_\gamma)\mathrm{d}B_\gamma - \int_0^\zeta \mathcal{S}(\zeta+\ell-\gamma)l(\gamma,\, Z_\gamma)\mathrm{d}B_\gamma\Big\|_{\mathbb{X}_1}\Big)^m. \tag{12}$$

$$\mathtt{E}\Big(\big\|\mathcal{S}(\zeta+\ell)Z_0 - \mathcal{S}(\zeta)Z_0\big\|_{\mathbb{X}_1}^m\Big) \longrightarrow 0, \text{ when } \ell \to 0.$$

Following

$$\mathtt{E}\Big(\Big\|\int_0^\zeta \mathcal{S}(\zeta+\ell-\gamma)k(\gamma,\, Z_\gamma)\mathrm{d}\gamma - \int_0^\zeta \mathcal{S}(\zeta-\gamma)k(\gamma,\, Z_\gamma)\mathrm{d}\gamma\Big\|_{\mathbb{X}_1}\Big)^m \longrightarrow 0, \text{ when } \ell \to 0.$$

Application of Hölder inequality and Lemma 5 we have:

$$\mathtt{E}\Big(\Big\|\int_0^\zeta \mathcal{S}(\zeta+\ell-\gamma)l(\gamma,\, Z_\gamma)\mathrm{d}B_\gamma - \int_0^\zeta \mathcal{S}(\zeta-\gamma)l(\gamma,\, Z_\gamma)\mathrm{d}B_\gamma\Big\|_{\mathbb{X}_1}\Big)^m$$

$$\leq 2^{n-1}\mathtt{E}\Big(\Big\|\int_0^\zeta \mathcal{S}(\zeta+\ell-\gamma)l(\gamma,\, Z_\gamma)\mathrm{d}B_\gamma - \int_0^\zeta \mathcal{S}(\zeta-\gamma)l(\gamma,\, Z_\gamma)\mathrm{d}B_\gamma\Big\|_{\mathbb{X}_1}\Big)^m$$

$$+ 2^{m-1}\mathtt{E}\Big(\Big\|\int_0^{\zeta+\ell} \mathcal{S}(\zeta+\ell-\gamma)l(\gamma,\, Z_\gamma)\mathrm{d}B_\gamma\Big\|_{\mathbb{X}_1}\Big)^m$$

$$\leq 2^{m-1}C_\beta\Big(\int_0^\zeta \big(\mathtt{E}\big\|(\mathcal{S}(\zeta+\ell-\gamma)\mathcal{S}(\zeta-\gamma))l(\gamma,\, Z_\gamma)\big\|_{\mathbb{X}_1}^m\big)^{2/m}\mathrm{d}\gamma\Big)^{m/2}$$

$$+2^{m-1}C_\beta\Big(\int_0^{\zeta+\ell}\big(\mathrm{E}\big\|\mathcal{S}(\zeta+\ell-\gamma)l(\gamma,\,Z_\gamma)\big\|_{\mathbb{X}_1}^m\big)^{2/m}\mathrm{d}\gamma\Big)^{m/2}\longrightarrow 0,\ \text{when } \ell\to 0.$$

This is what we have, $\mathrm{E}\Big(\big\|\Psi(Z_{\zeta+\ell})-\Psi(Z_\zeta)\big\|_{\mathbb{X}_1}\Big)^m\longrightarrow 0$ when ℓ is infinitesimally small.
This confirms the continuity of the application Ψ on average of order m.
From the relationship (9), we obtain that,

$$e^{\delta\zeta}\mathrm{E}\|\Psi(Z_\zeta)\|_{\mathbb{X}_1}^m\le 3^{m-1}e^{\delta\zeta}\mathrm{E}\big(\|\mathcal{S}(\zeta)Z_0\|_{\mathbb{X}_1}^m\big)+3^{m-1}e^{\delta\zeta}\mathrm{E}\Big\|\int_0^\zeta\mathcal{S}(\zeta-\gamma)k(\gamma,\,Z_\gamma)\mathrm{d}\gamma\Big\|_{\mathbb{X}_1}^m$$

$$+3^{m-1}e^{\delta\zeta}\mathrm{E}\Big\|\int_0^\zeta\mathcal{S}(\zeta-\gamma)l(\gamma,Z_\gamma)\mathrm{d}B_\gamma\Big\|_{\mathbb{X}_1}^m.$$

$$3^{m-1}e^{\delta\zeta}\mathrm{E}\big(\|\mathcal{S}(\zeta)Z_0\|_{\mathbb{X}_1}^n\big)=3^{m-1}e^{\delta\zeta}\mathrm{E}\big(\|\mathcal{S}(\zeta)\|_{\mathbb{X}_1}^m\|Z_0\|_{\mathbb{X}_1}^m\big)\le 3^{m-1}e^{\delta\zeta}N^me^{-n\eta\zeta}\|Z_0\|_{\mathbb{X}_1}^m,$$

$$3^{m-1}e^{\delta\zeta}\mathrm{E}\big(\|\mathcal{S}(\zeta)Z_0\|_{\mathbb{X}_1}^m\big)\longrightarrow 0,\ \text{when } \zeta\to\infty. \tag{13}$$

Furthermore, the

$$3^{m-1}e^{\delta\zeta}\mathrm{E}\Big\|\int_0^\zeta\mathcal{S}(\zeta-\gamma)k(\gamma,\,Z_\gamma)\mathrm{d}\gamma\Big\|_{\mathbb{X}_1}^m$$

$$\le 3^{m-1}e^{\delta\zeta}\mathrm{E}\Big(\int_0^\zeta\big\|\mathcal{S}(\zeta-\gamma)k(\gamma,\,Z_\gamma)\big\|_{\mathbb{X}_1}\mathrm{d}\gamma\Big)^m$$

$$\le 3^{m-1}e^{\delta\zeta}\mathrm{E}\Big(\int_0^\zeta Ne^{-\eta(\zeta-\gamma)}\big\|k(\gamma,\,Z_\gamma)\big\|_{\mathbb{X}_1}\mathrm{d}\gamma\Big)^m$$

$$\le 3^{m-1}N^mL_1^me^{\delta\zeta}\mathrm{E}\Big(\int_0^\zeta e^{-\eta((m-1+1)/m)(\zeta-\gamma)}\big\|Z_\gamma\big\|_{\mathbb{X}_1}\mathrm{d}\gamma\Big)^m$$

$$=3^{m-1}N^mL_1^me^{\delta\zeta}\mathrm{E}\Big(\int_0^\zeta e^{(-\eta(m-1)/m)(\zeta-\gamma)}e^{(-\eta/m)(\zeta-\gamma)}\big\|Z_\gamma\big\|_{\mathbb{X}_1}\mathrm{d}\gamma\Big)^m$$

$$\le 3^{m-1}N^mL_1^me^{\delta\zeta}\Big(\int_0^\zeta e^{-\eta(\zeta-\gamma)}\mathrm{d}\gamma\Big)^{m-1}\int_0^\zeta e^{\eta(\zeta-\gamma)}\mathrm{E}\big\|Z_\gamma\big\|_{\mathbb{X}_1}^m\mathrm{d}\gamma$$

$$\le 3^{m-1}N^mL_1^m(1/\eta)^{m-1}e^{\delta\zeta}\int_0^\zeta e^{-\eta(\zeta-\gamma)}\mathrm{E}\big\|Z_\gamma\big\|_{\mathbb{X}_1}^m\mathrm{d}\gamma$$

$$\le 3^{m-1}N^mL_1^m(1/\eta)^{m-1}e^{\delta\zeta}e^{-\eta\zeta}\int_0^\zeta e^{\eta\gamma}\mathrm{E}\big\|Z_\gamma\big\|_{\mathbb{X}_1}^m\mathrm{d}\gamma.$$

All $\varepsilon > 0$ for everyone $Z_\zeta \in \mathbb{X}_1$ there exists $\zeta_1 > 0$ such that $e^{\delta\gamma}\mathrm{E}\|Z_\gamma\|_{\mathbb{X}_1}^m < \varepsilon$ for $\zeta \geq \zeta_1$ then we obtain that

$$3^{m-1}e^{\delta\zeta}\mathrm{E}\left\|\int_0^\zeta \mathcal{S}(\zeta-\gamma)k(\gamma,\, Z_\gamma)\mathrm{d}\gamma\right\|^m$$

$$\leq 3^{m-1}N^mL_1^m(1/\eta)^{m-1}e^{\delta\zeta}\int_0^\zeta e^{-\eta(\zeta-\gamma)}\mathrm{E}\|Z_\gamma\|^m\mathrm{d}\gamma$$

$$\leq 3^{m-1}N^mL_1^m(1/\eta)^{m-1}e^{(\delta-\eta)\zeta}\int_0^{\zeta_1} e^{\eta\gamma}\mathrm{E}\|Z_\gamma\|^m\mathrm{d}\gamma$$

$$+3^{m-1}N^mL_1^m(1/\eta)^{m-1}e^{(\delta-\eta)\zeta}\int_{\zeta_1}^{\zeta} e^{\eta\gamma}\mathrm{E}\|Z_\gamma\|^m\mathrm{d}\gamma$$

$$3^{m-1}e^{\delta\zeta}\mathrm{E}\left\|\int_0^\zeta \mathcal{S}(\zeta-\gamma)k(\gamma,\, Z_\gamma)\mathrm{d}\gamma\right\|^m$$

$$\leq 3^{m-1}N^mL_1^m(1/\eta)^{m-1}e^{(\delta-\eta)\zeta}\int_0^{\zeta_1} e^{\eta\gamma}\mathrm{E}\|Z_\gamma\|^m\mathrm{d}\gamma$$

$$+3^{m-1}N^mL_1^m(1/\eta)^{m-1}e^{(\delta-\eta)\zeta}(1/\eta)\varepsilon. \tag{14}$$

Since $e^{-\eta\gamma} \to 0$ when $\zeta \to \infty$ by the assumptions of Theorem 0.4.1, there exists $\zeta_2 \geq \zeta_1$ such that for all $\zeta \geq \zeta_2$ we obtain:

$$3^{m-1}N^mL_1^m(1/\eta)^{m-1}e^{(\delta-\eta)\zeta}\int_0^{\zeta_1} e^{\eta\gamma}\mathrm{E}\|Z_\gamma\|^m\mathrm{d}\gamma$$

$$\leq \varepsilon - 3^{m-1}N^mL_1^m(1/\eta)^m e^{(\delta-\eta)\zeta}\varepsilon \tag{15}$$

From (14) and (15) we obtain for all $\zeta \geq \zeta_2$

$$3^{m-1}e^{\delta\zeta}\mathrm{E}\left\|\int_0^\zeta \mathcal{S}(\zeta-\gamma)k(\gamma,\, Z_\gamma)\mathrm{d}\gamma\right\|^m < \varepsilon.$$

When $s \to \infty$ then

$$3^{m-1}e^{\delta\zeta}\mathrm{E}\left\|\int_0^\zeta \mathcal{S}(\zeta-\gamma)k(\gamma,\, Z_\gamma)\mathrm{d}\gamma\right\|^m \longrightarrow 0. \tag{16}$$

Let's look at the remaining element of our sum.
For all $Z_\zeta \in \mathbb{X}_1$ with $\zeta \in \mathbb{R}^+$, we get:

$$
\begin{aligned}
& 3^{m-1} e^{\delta\zeta} \mathrm{E} \Big\| \int_0^\zeta \mathcal{S}(\zeta-\gamma) l(\gamma,\, Z_\gamma) \mathrm{d}B_\gamma \Big\|_{\mathbb{X}_1}^m \\
& \leq 3^{m-1} e^{\delta\zeta} C_\beta N^m \Big(\int_0^\zeta \big(e^{-\eta m(\zeta-\gamma)} \mathrm{E} \| l(\gamma,\, Z_\gamma) \|_{\mathbb{X}_1}^m \big)^{2/m} \mathrm{d}\gamma \Big)^{m/2} \\
& \leq 3^{m-1} e^{\delta\zeta} C_\beta N^m L_2^m \Big(\int_0^\zeta \big(e^{-\eta m(\zeta-\gamma)} \mathrm{E} \| Z_\gamma \|_{\mathbb{X}_1}^m \big)^{2/m} \mathrm{d}\gamma \Big)^{m/2} \\
& = 3^{m-1} e^{\delta\zeta} C_\beta N^m L_2^m \Big(\int_0^\zeta \big(e^{-\eta(m-1+1)(\zeta-\gamma)} \mathrm{E} \| Z_\gamma \|_{\mathbb{X}_1}^m \big)^{2/n} \mathrm{d}\gamma \Big)^{m/2} \\
& \leq 3^{m-1} e^{\delta\zeta} C_\beta N^m L_2^m \Big(\int_0^\zeta e^{-\frac{2(m-1)}{m-2}\eta(\zeta-\gamma)} \Big)^{\frac{m-2}{2}} \int_0^\zeta e^{-\eta(\zeta-\gamma)} \mathrm{E} \| X_\gamma \|_{\mathbb{X}_1}^m \mathrm{d}\gamma \\
& \leq 3^{m-1} e^{\delta\zeta} C_\beta N^m L_2^m \Big(\frac{2(m-1)}{m-2} \eta \Big)^{\frac{2-m}{2}} \int_0^\zeta e^{-\eta(\zeta-\gamma)} \mathrm{E} \| Z_\gamma \|_{\mathbb{X}_1}^m \mathrm{d}\gamma.
\end{aligned}
$$

That's why, when $\zeta \to \infty$ then

$$
3^{m-1} e^{\delta\zeta} \mathrm{E} \Big\| \int_0^\zeta \mathcal{S}(\zeta-\gamma) l(\gamma,\, Z_\gamma) \mathrm{d}B_\gamma \Big\|_{\mathbb{X}_1}^m \longrightarrow 0. \tag{17}
$$

The relationships (13), (16) and (17), allow us to deduce that when $\zeta \to \infty$ we get $e^{\delta\zeta} \mathrm{E} \| \Psi(Z_\zeta) \|_{\mathbb{X}_1}^m \longrightarrow 0$ and hence $\Psi(\mathbb{H}) \subset \mathbb{H}$.

We'll now show that Ψ is a contracting application, taking $Z_\zeta,\ Y_\zeta \in \mathbb{H}$. Proceeding in the same way as before, we obtain:

$$
\begin{aligned}
& \sup_{0 \leq s_1 \leq \tau} \mathrm{E} \Big(\big\| \Psi(Y_\zeta) - \Psi(Z_\zeta) \big\|_{\mathbb{X}_1} \Big)^m \\
& \leq 2^{m-1} \sup_{0 \leq s_1 \leq \tau} \mathrm{E} \Big(\Big\| \int_0^\zeta \mathcal{S}(\zeta-\gamma) \big(k(\gamma,\, Y_\gamma) - k(\gamma,\, Z_\gamma) \big) \mathrm{d}\gamma \Big\|_{\mathbb{X}_1} \Big)^m \\
& \quad + 2^{m-1} \sup_{0 \leq s_1 \leq \tau} \mathrm{E} \Big(\Big\| \int_0^\zeta \mathcal{S}(\zeta-\gamma) \big(l(\gamma,\, Y_\gamma) - l(\gamma,\, Z_\gamma) \big) \mathrm{d}B_\gamma \Big\|_{\mathbb{X}_1} \Big)^m \\
& \leq 2^{m-1} N^m L_1^m (1/\eta)^m \sup_{0 \leq s_1 \leq \tau} \mathrm{E} \| Y_\gamma - Z_\gamma \|_{\mathbb{X}_1}^m
\end{aligned}
$$

$$+2^{m-1}N^m L_2^m C_\beta \left(\frac{2(m-1)}{m-2}\eta\right)^{\frac{2-m}{2}} \sup_{0\leq s_1\leq \tau} \mathrm{E}\|Y_\gamma - Z_\gamma\|_{\mathbb{X}_1}^m$$

$$\leq 2^{m-1}N^m \left(L_1^m (1/\eta)^m + L_2^m C_\beta \left(\frac{2(m-1)}{m-2}\eta\right)^{\frac{2-m}{2}}\right)\mathrm{E}\|Y_\gamma - Z_\gamma\|_{\mathbb{X}_1}^m.$$

Consequently, the assumption **H7** allows us to assert by conviction that our Ψ application is contracting. To complete our proof, we now assume that Ψ is a contracting application, so it admits in $\mathbb{F}$ a unique fixed point Z_ζ, which is a solution of (3) with $Z_0 \in \mathbb{X}_1$ and $\mathrm{E}\|Z_\zeta\|_{\mathbb{X}_1}^m \longrightarrow 0$ when $\zeta \to \infty$.

From all the above we can say that the mild solution of our equation (1) is asymptotically stable.

5. Conclusion

In this work we have studied a stochastic integrodifferential equation in a complex Hilbert space. We have shown in this work the existence and uniqueness of a mild solution for the equation (1); as well as the asymptotic stability of this mild solution. For the existence and uniqueness of the mild solution, we obtain our results by using the assumptions on the Lipschitz growth conditions, the global Lipschitz conditions and the use of the analytic solver operator theory with a technical skill of stopping times in a finite measure contour C_n. Using the fixed-point theorem in conjunction with the properties of the stochastic integral, we can prove that this weak solution is asymptotically stable, ensuring that the solution obtained is robust to random perturbations and parameter variations. Our results complement and generalize the work of [8], and represent an important advance in our understanding of the problem, opening the way to several practical applications. Examples include the modeling of financial assets in the presence of stochastic noise, and the modeling of the spread of an epidemic in the presence of ecological phenomena... etc. The scope of EDIS applications covers a huge range of fields, from finance to biology, engineering and the physical sciences.

At the end of this work we expect to focus on a certain number of open problems, namely: The study of the almost certain asymptotic behaviour of this mild solution. The study of the exponential stability of this mild solution. The study of the exponential stability and the almost exponentially safe stability of this mild solution.

Modelling integrodifferential equations related to option prices in finance.

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