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Research Article

A Modification of the Efficient Decomposition Shooting Method for Two-Point Boundary-Value Problems

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Abstract: This research aims to present two modified numerical approaches based on the efficient decomposition shooting method (EDSM) to find approximate and reliable solutions to the class of ordinary differential equations (ODEs), featuring the two-point boundary-value problems (BVPs). In fact, the two famous modifications of the Adomian decomposition method (ADM) are sought and further infused in the classical shooting method to unravel the two schemes of interest. Further, the proposed schemes are applied to several second and third-order BVPs of both the linear and nonlinear cases. Moreover, the proposed schemes perfumed superbly in comparison with some contending approaches in the literature. What is more, the convergence rate of the devised schemes is found to be very high.

Keywords: ODEs, BVPs, wazwaz modifications, shooting method, efficient decomposition shooting method

MSC: 65L05, 34K06, 34K28

1. Introduction

Second and third-order differential equations featuring two-point boundary data occur in a variety of real-life applications in engineering, thermodynamics, optimization theory, and mathematical physics to state a few. These equations-two-point boundary-value problems (BVPs) are usually solved analytically unless when the analytical techniques fail; in such a situation, appropriate numerical schemes are sought. Many attempts have been made in the past literature to study methods for solving BVPs. Among these methods are the shooting method [1], and the Adomian decomposition method (ADM) [2, 3], which are the most widely utilized and recognized methods.

Even though the shooting method requires a massive amount of numerical task, especially while tackling nonlinear problems in getting a reasonably truthful approximate solution; it nevertheless, has so many benefits including a reduced size of the system and a fast solver to get a hold of a perfect approximate solution. In fact, there exist some related works in the literature, where the shooting method and the Adomian decomposition method are combined to solve BVPs in both the nonlinear and linear cases of the second, third and higher-order ODEs with the help of a standard inverse operator; read [4–6] for more information. Additionally, it is notable to mention the work of Attili and Syam [7], where the mixture of the aforementioned methods was proposed and utilized to solve dissimilar BVPs featuring second-order nonlinear differential equations with a specific inverse operator.

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However, the main objective of this paper is to proposes two modifications for the efficient decomposition shooting method (EDSM) [4–6] to find approximate and reliable solutions to the class of ODEs modeled through the two-point BVPs. In fact, we termed these procedures as the first modification of the efficient decomposition shooting method (MEDSM1) and the second modification of the efficient decomposition shooting method (MEDSM2). More so, various applications of the modified schemes will be demonstrated on several models of linear and nonlinear cases of the second and third-order differential equations. Lastly, the effectiveness of the two methods will be evaluated by comparing the exact and the acquired approximate solutions, in addition to the deployment of other well-known numerical methods from the open literature. Besides, an all-inclusive conclusion with regard to the effectiveness of the proposed methods will be deduced in the end. The present paper is arranged as follows: Section 2 outlines the modified decomposition method by Wazwaz, while Section 3 gives a modification of the efficient decomposition shooting method. Section 4 demonstrates the presented method given in Section 3 on certain test problems. Lastly, Section 5 presents some concluding points.

2. Modified decomposition method by Wazwaz

Consider the following generalized differential equation [2]

$$Ly+Ry+Ny=r(x), \tag{1}$$

together with the following initial conditions

$$y^{(i)}(a)=\alpha_i, \quad y^{(n-1)}(a)=t, \quad i=0,\,1,\,\ldots,\,n-2, \tag{2}$$

where L represents the highest order derivative which is assumed to be easily invertible, R is also a linear differential operator, but of order less than that of L; N denotes the nonlinear differential operator, and $r(x)$ is a given source function. Lastly, α_i for $i=0,\,1,\,\ldots,\,n-2$, and t are given constants (real).

Furthermore, ADM proceeds by suggesting that the solution of the differential equation (1) be expressed as a sum of infinite series of components expressed as $y(x)=\sum_{m=0}^{\infty}y_m(x)$, while the nonlinear term N is obtained with the aid of the following recurrent formula

$$A_m=\frac{1}{m!}\frac{d^m}{d\lambda^m}\left[N\left(\sum_{i=0}^{m}\lambda^i y_i\right)\right]_{\lambda=0} \quad \text{where } m=0,\,1,\,2,\,\ldots.$$

By using the ADM, the recursive relation can be obtained as follows

$$\begin{aligned} y_0 &= \phi(x)+L^{-1}\left(r(x)\right), \\ y_{m+1} &= -L^{-1}\left(Ry_m\right)-L^{-1}\left(A_m\right), \quad m\geq 0, \end{aligned} \tag{3}$$

where the function $\phi(x)$ represents the terms arising from integrate Ly in (1) and from using the given conditions (2), so $L\phi(x)=0$. After that, we use the following relationship, $y_{M+1}=\sum_{m=0}^{M}y_m(x)$, to get the approximate solution.

Similarly, all the above steps for the nonlinear case remain valid for the linear case with the exception of the decomposition of the nonlinear term. Indeed, the referred Adomian polynomials will be absent here; hence, the components $y_m(x)$ are now found recurrently with regard to the liner case as

$$y_0 = \phi(x) + L^{-1}(r(x)),$$

$$y_{m+1} = -L^{-1}(Ry_m), \quad m \geq 0. \tag{4}$$

In an attempt to fatherly advance the accuracy or convergence of ADM, various ways have been put forward by different mathematicians to propose dissimilar modifications of the method [8–11]. Besides, we also recall the modification by Wazwaz [12], where in 1999 he modified the assumptions made by Adomian. In addition, Wazwaz and El-Sayed [13] proposed yet another new modification in 2002. More precisely, the two modifications in [12, 13] will be utilized in the present study after being infused in EDSM to tackle the governing second and third-order models.

2.1 *Reliable modification*

Recall that ADM begins by defining the zeroth component y_0 as follows

$$f = \phi(x) + L^{-1}(r(x)),$$

that is, after taking the net sum of the terms emanating from the prescribed initial data, and that of the source function. Therefore, Wazwaz [12] suggests that this new function f should be systematically split into two parts as follows

$$f = f_0 + f_1.$$

Hence, with the current development, the modified recursive relation for the nonlinear case is as follows

$$y_0 = f_0,$$

$$y_1 = f_1 - L^{-1}(Ry_0) - L^{-1}(A_0),$$

$$y_{m+1} = -L^{-1}(Ry_m) - L^{-1}(A_m), \quad m \geq 1, \tag{5}$$

for the linear case, the nonlinear part A_m, $m \geq 0$ will be removed while maintaining the same recursive relation (5).

Moreover, it is noted from the above schemes that this modification only affects y_0 and y_1 components, however, its significance is hugely observed to majorly minimize the computational size, and on the other hand, accelerate the convergence rate. Additionally, among the advantages of this modification is that the need to compute the Adomian polynomials A_m in the case of nonlinear equations is many at times disregarded. In fact, the accomplishment of the current scheme relies barely on the correct selection of f_0 and f_1.

2.2 *New modification*

In this approach, Wazwaz and El-Sayed [13] make use of Taylor's series expansion to express f as a sum of infinite components-against just the two components f_0 and f_1 in [12] as follows

$$f = \sum_{m=0}^{\infty} f_m.$$

In addition, the resulting recursive relation for nonlinear models is thus suggested as follows [13]

$$\begin{aligned} &y_0 = f_0, \\ &y_{m+1} = f_{m+1} - L^{-1}\left(Ry_m\right) - L^{-1}\left(A_m\right), \quad m \geq 0, \end{aligned} \tag{6}$$

similarly, we will eliminate the nonlinear part A_m, $m \geq 0$ to obtain the recursive relation in the linear case.

Remarkably, it is computationally noted that the algorithm expressed in (6) significantly lessen the size of the calculations, as against the standard ADM, by reducing the number of terms used in each component; indeed, the construction of the polynomials (Adomian polynomials) belonging to the nonlinear operators becomes easier.

3. A modification of the efficient decomposition shooting method

The shooting method is an iterative method that is used for solving BVP by recasting it to a system of initial-value problems (IVPs) with specified initial conditions. Even though, none of these IVPs is tackled exactly; so, the solution will be approximated using the one-step methods or multistep methods or even directly using ADM. However, in this paper, we will make use of the two aforementioned modifications directly to solve linear and nonlinear IVPs of the second and third orders. In fact, we will be coupling the two modifications for ADM [12, 13] with the shooting method [1, 4–6, 14, 15] and termed these procedures as the first and second modification of the efficient decomposition shooting method (MEDSM1) and (MEDSM2), respectively, to examine the said two-point BVPs.

3.1 *Linear case*

Consider the following nth-order linear two-point BVP

$$y^{(n)}(x) = \sum_{j=1}^{n-1} P_{n-j+1}(x) y^{(n-j)}(x) + P_1(x) y(x) + r(x), \quad a \leq x \leq b, \tag{7}$$

subject to the two-point boundary data as follows

$$y^{(i)}(a) = \alpha_i, \quad y(b) = \beta, \quad i = 0, 1, \ldots, n-2, \tag{8}$$

with $P_{n-j+1}(x)$, $j = 1, 2, \ldots, n$ as known functions.

Now, based on the shooting method, the nth-order BVP given above will turn into two IVPs where the boundary conditions given in (8) will be replaced with specific initial conditions for each IVP as

$$u^{(n)}(x) = \sum_{j=1}^{n-1} P_{n-j+1}(x) u^{(n-j)}(x) + P_1(x) u(x) + r(x),$$

$$u^{(i)}(a) = \alpha_i, \quad u^{(n-1)}(a) = 0, \tag{9}$$

and

$$v^{(n)}(x) = \sum_{j=1}^{n-1} P_{n-j+1}(x) v^{(n-j)}(x) + P_1(x) v(x),$$

$$v^{(i)}(a) = 0, \quad v^{(n-1)}(a) = 1. \tag{10}$$

Next, using ADM we get

$$\sum_{m=0}^{\infty} u_m(x) = \phi_1(x) + L^{-1} \left(\sum_{j=1}^{n-1} P_{n-j+1}(x) \sum_{m=0}^{\infty} u_m^{(n-j)}(x) + P_1(x) \sum_{m=0}^{\infty} u_m(x) + r(x) \right),$$

and

$$\sum_{m=0}^{\infty} v_m(x) = \phi_2(x) + L^{-1} \left(\sum_{j=1}^{n-1} P_{n-j+1}(x) \sum_{m=0}^{\infty} v_m^{(n-j)}(x) + P_1(x) \sum_{m=0}^{\infty} v_m(x) \right).$$

Now, we are going to use the Wazwaz [12] and Wazwaz and El-Sayed [13] modifications to recurrently define the components $u_m(x)$ and $v_m(x)$; while letting

$$f = \phi_1(x) + L^{-1}(r(x)), \quad \text{and} \quad g = \phi_2(x)$$

MEDSM1: Here, upon using the **reliable modification** by Wazwaz [12], the functions f and g can be divided into two parts. Therefore, the resulting recursive relations will be as follows

$$\begin{cases} u_0 = f_0 \\ u_1 = f_1 + L^{-1} \left(\sum_{j=1}^{n-1} P_{n-j+1}(x) u_0^{(n-j)}(x) + P_1(x) u_0(x) \right), \\ u_{m+1} = L^{-1} \left(\sum_{j=1}^{n-1} P_{n-j+1}(x) u_m^{(n-j)}(x) + P_1(x)\, u_m(x) \right), \quad m \geq 1, \end{cases} \tag{11}$$

and

$$
\begin{cases}
v_0 = g_0 \\
v_1 = \; g_1 + \; L^{-1}\left(\sum_{j=1}^{n-1} P_{n-j+1}(x)v_0^{(n-j)}(x) + P_1(x)v_0(x)\right), \\
v_{m+1} = \; L^{-1}\left(\sum_{j=1}^{n-1} P_{n-j+1}(x)v_m^{(n-j)}(x) + P_1(x)\; v_m(x)\right), \quad m \geq 1.
\end{cases}
\tag{12}
$$

MEDSM2: In a similar approach, we make use of the **new modification** by Wazwaz and El-Sayed [13] to decompose the functions f and g into several parts with the help of Taylor's expansion. Therefore, the resulting recursive relations take the following forms

$$
\begin{cases}
u_0 = f_0 \\
u_{m+1} = f_{m+1} + L^{-1}\left(\sum_{j=1}^{n-1} P_{n-j+1}(x)u_m^{(n-j)}(x) + P_1(x)\; u_m(x)\right), \quad m \geq 0,
\end{cases}
\tag{13}
$$

and

$$
\begin{cases}
v_0 = g_0 \\
v_{m+1} = g_{m+1} + L^{-1}\left(\sum_{j=1}^{n-1} P_{n-j+1}(x)v_m^{(n-j)}(x) + P_1(x)\; v_m(x)\right), \quad m \geq 0.
\end{cases}
\tag{14}
$$

Further, by letting $u(x)$ and $v(x)$ to denote the solutions to the nth-order linear IVPs given in (9) and (10) respectively, then we define

$$
z(x) = \; u(x) + \frac{\beta - u(b)}{v(b)} v(x), \quad v(b) \neq 0,
\tag{15}
$$

where $z(x)$ is the solution to the nth-order linear BVP given (7) and (8).

3.2 *Nonlinear case*

Let us consider the nth-order nonlinear two-point BVP as follows

$$
y^{(n)} = \; f\left(x, y, y', \ldots, \; y^{(n-1)}\right), \quad a \leq x \leq b,
\tag{16}
$$

together with the specific boundary data as

$$y^{(i)}(a) = \alpha_i, \quad y(b) = \beta, \quad i = 0, 1, \ldots, n-2. \tag{17}$$

Then, with the help of the shooting technique, it is worth mentioning here that the procedure for solving nonlinear BVP via shooting technique is similar to that of the linear BVP, only that the solution to a nonlinear problem is not expressed as a linear combination of the solutions of the two resulting IVPs. Certainly, the solution to the BVP is approximated through the solutions to a sequence of IVPs of the parameter t. So, we will convert the nth-order BVP into IVPs, where we replace the boundary condition with specific initial conditions. Indeed, the problem has the following format

$$y^{(n)} = f\left(x, y, y', \ldots, y^{(n-1)}\right), \quad a \le x \le b, \tag{18}$$

with the initial conditions given by

$$y^{(i)}(a) = \alpha_i, \quad y^{(n-1)}(a) = t. \tag{19}$$

We will then use the **MEDSM1** or **MEDSM2** directly to solve the IVP expressed in (18) and (19); in fact, we do this by choosing the parameters $t = t_s$ in a manner to ensure that

$$\lim_{s \to \infty} y(b, t_s) = y(b) = \beta,$$

where $y(x, t_s)$ represents the solution to the IVP (18) and (19), while $t = t_s$, and $y(x)$ represents the solution to the BVP (16) and (17). More so, the solution of the first IVP in the required sequence will be determined by imposing initial guess $t_0 = \dfrac{\beta - \alpha}{b - a}$. Then, we make use of the Newton's method to determine the value t_1 as follows

$$t_1 = t_0 - \frac{y(b, t_0) - \beta}{\dfrac{dy}{dt}(b, t_0)}.$$

Moreover, to determine the solutions to remaining sequence, the following guess points t_s, $s = 2, 3, \ldots$, with nonlinear function $y(b, t) - \beta = 0$ are found with the help of the Secant iterative procedure, which takes the following expression

$$t_s = t_{s-1} - \frac{(y(b, t_{s-1}) - \beta)(t_{s-1} - t_{s-2})}{y(b, t_{s-1}) - y(b, t_{s-2})}, \quad s = 2, 3, \ldots .$$

More so, the computational process will be stopped when

$$|y(b, t_s) - \beta| \le \text{tolerance}.$$

4. Numerical example

This section examines the proposed methodology for the linear and nonlinear cases of the second and third-order BVPs by demonstrating its application on some selected test problems. The modified methods are also compared with the mixture of the shooting method with the Runge-Kutta method of order-fourth (SRKM4) to further assess the performance of the proposed MEDSM1 and MEDSM2. Similarly, we will compare the proposed results with others as reported in [16–26].

Further, we present certain supportive Tables 1-6 and Figures 1-3 reporting the absolute error difference between the exact analytical solution and, on the other hand, the obtained numerical results using the MEDSM1 and MEDSM2, further validated with SRKM4.

Example 1 Consider the second-order linear BVP [16–20]

$$y^{''}(x)-\pi^2 y(x)=-2\pi^2\sin(\pi x),\quad y(0)=0,\quad y(1)=0.$$

The actual analytical solution is given by $y(x)=\sin(\pi x)$.

Now we consider the following two IVPs

$$u^{''}=\pi^2 u-2\pi^2\sin(\pi x),\quad u(0)=0,\quad u^{'}(0)=0, \tag{20}$$

and

$$v^{''}=\pi^2 v,\quad v(0)=\ 0,\quad v^{'}(0)=1. \tag{21}$$

Applying ADM in (20) and (21), we obtain

$$u(x)=-2\pi x+2\sin(\pi x)+\pi^2 L^{-1}(u). \tag{22}$$

and

$$v(x)=x+\pi^2 L^{-1}\left(v(x)\right). \tag{23}$$

MEDSM1: Using the modified recursive relation (11) in (22) and by choosing $f_0=2\sin(\pi x)$, and $f_1=-2\pi x$, then the recursive relation takes the following pattern

$$\begin{cases} u_0=2\sin(\pi x), \\ u_1=-2\pi x+\pi^2\ L^{-1}\left(u_0\right), \\ u_{m+1}=\pi^2\ L^{-1}\left(u_m\right),\quad m\geq 1. \end{cases}$$

and using the modified recursive relation (12) in Equation (23), since $g = x$ consists of one terms, then relation (12) reduces to relation (4), so the recursive relation of Equation (23) is

$$\begin{cases} v_0 = x, \\ v_{m+1} = \pi^2 L^{-1}(v_m), \quad m \geq 0. \end{cases} \tag{24}$$

MEDSM2: Using Taylor's expansion for $f = -2\pi x + 2\sin(\pi x)$, we obtain the following recursive relation

$$\begin{cases} u_0 = -\dfrac{2}{3!}\pi^3 x^3, \\ u_{m+1} = f_{m+1} + \pi^2 L^{-1}(u_m), \quad m \geq 0, \end{cases}$$

similarly when using the modified recursive relation (14) in Equation (23), so the recursive relation of Equation (23) is the same as the recursive relation of (24).

Finally, the approximate solution $z(x)$ is obtained while setting $M = 10$, and $h = \dfrac{1}{10}$ from $z(x_k) = u(x_k) + \dfrac{-u(1)}{v(1)} v(x_k)$, with $x_k = kh$ for $k = 0, 1, \dots, M$. Indeed, the corresponding numerical results are reported in Tables 1 and 2.

Table 1. The absolute error for SRKM4, MEDSM1, and MEDSM2 when $h = 0.1$

x	SRKM4	MEDSM1	MEDSM2
0	0	0	0
0.1	4.8×10^{-6}	3.0×10^{-11}	3.0×10^{-13}
0.2	1.1×10^{-5}	6.2×10^{-11}	6.2×10^{-13}
0.3	1.8×10^{-5}	1.0×10^{-10}	1.0×10^{-12}
0.4	2.4×10^{-5}	1.5×10^{-10}	1.5×10^{-12}
0.5	2.8×10^{-5}	2.1×10^{-10}	2.1×10^{-12}
0.6	3.0×10^{-5}	3.0×10^{-10}	3.0×10^{-12}
0.7	2.8×10^{-5}	4.1×10^{-10}	4.1×10^{-12}
0.8	2.2×10^{-5}	5.6×10^{-10}	5.6×10^{-12}
0.9	1.3×10^{-5}	6.6×10^{-10}	6.9×10^{-12}
1.0	0	0	0

Table 2. Comparison between different methods when $h = 0.1$

Numerical methods	NPSM [16]	QBIM [17]	HCBSM [18]	HPM [19]	ECBIM [20]	SRKM4	MEDSM1	MEDSM2
Maximum error	3.7×10^{-3}	2.0×10^{-5}	1.7×10^{-8}	8.6×10^{-9}	4.0×10^{-9}	3.0×10^{-5}	6.6×10^{-10}	6.9×10^{-12}

In Table 1, we report the absolute error difference between the exact analytical solution and the proposed solutions **MEDSM1** and **MEDSM2**, and further validated with **SRKM4**. From Table 2, we can see that **MEDSM2** is the most competent technique for solving the governing model in comparison with the **SRKM4**, **MEDSM1**, and the methods used in [16–20]. Again, we portray the exact analytical and the contending approximate solutions graphically in Figure 1, where one would notice an ideal agreement between the solutions.

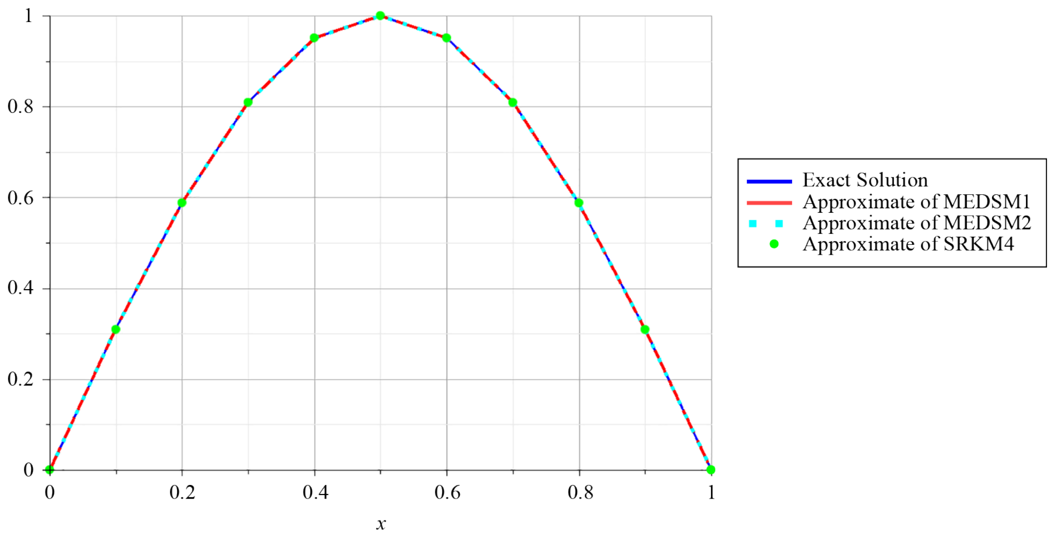

Figure 1. Graphical comparison, depicting the exact and contending approximate solutions with $h = 0.1$

Example 2 Consider the third-order linear BVP as follows [21]

$$y'''(x) + 2y''(x) - 4y'(x) + y(x) = \frac{1}{4}\left(8x - x^2\right) + \frac{9e^{-2x} + 2x - 9}{4\left(1 - e^{-2}\right)},$$

$$y(0) = 1, \quad y(1) = 1, \quad y'(0) = 1.$$

The actual analytical solution is given by

$$y(x) = K_1 e^x + K_2 e^{\frac{-x}{2}\left(3 - \sqrt{13}\right)} + K_3 e^{\frac{-x}{2}\left(3 + \sqrt{13}\right)} + \frac{1}{4}\left(4 - x^2\right) + \frac{e^{-2x} + 2x - 1}{4\left(1 - e^{-2}\right)}$$

where K_1, k_2, and K_3 are constants determined such that the boundary data are satisfied.

Now we consider the two IVPs as follows

$$u'''(x) = -2u''(x) + 4u'(x) - u(x) + \frac{1}{4}\left(8x - x^2\right) + \frac{9e^{-2x} + 2x - 9}{4\left(1 - e^{-2}\right)},$$

$$u(0) = u'(0) = 1, \quad u''(0) = 0, \tag{25}$$

and

$$v'''(x) = -2v''(x) + 4v'(x) - v(x), \quad v(0) = v'(0) = 0, \quad v''(0) = 1. \tag{26}$$

MEDSM1: We use the modified recursive relation (11) in (25) and suitably construct f_0 and f_1 as follows

$$f_0 = \frac{1}{480}\left(\frac{10\left(4e^{-2}-5\right)x^4 - 270x^2 + 135e^{-2x}}{-1+e^{-2}}\right),$$

$$f_1 = \frac{1}{480}\left(\frac{2\left(1-e^{-2}\right)x^5 + 180x^3 + 30x\left(16e^{-2}-7\right) + 15\left(32e^{-2}-41\right)}{-1+e^{-2}}\right).$$

MEDSM2: We will use the modified recursive relation (13) in (25) and we choose $f_0 = 1$, $f_1 = x$, $f_2 = \frac{1}{12}\left(\frac{1+e^{-2}}{-1+e^{-2}}\right)x^4, \cdots$.

Similar to this, we will employ the recursive relation (4) when solving (26) using the **MEDSM1** and **MEDSM2** procedures since $g = x$ only contains one term.

The approximate solution $z(x)$ with $M = 15$, $h = \frac{1}{15}$, is given by

$$z(x_k) = u(x_k) + \frac{1-u(1)}{v(1)}v(x_k),$$

see Tables 3 and 4 for the numerical results.

Table 3. The absolute error for SRKM4, MEDSM1, and MEDSM2 when $h = 1/15$

x	SRKM4	MEDSM1	MEDSM2
$1/15$	1.2×10^{-6}	2.8×10^{-14}	3.8×10^{-13}
$3/15$	2.3×10^{-6}	2.3×10^{-13}	3.1×10^{-12}
$5/15$	2.4×10^{-6}	6.1×10^{-13}	8.2×10^{-12}
$7/15$	2.0×10^{-6}	1.1×10^{-12}	1.5×10^{-11}
$9/15$	1.5×10^{-6}	1.8×10^{-12}	2.5×10^{-11}
$11/15$	9.4×10^{-7}	2.6×10^{-12}	3.6×10^{-11}
$13/15$	4.4×10^{-7}	3.2×10^{-12}	4.5×10^{-11}
1	1.0×10^{-39}	1.0×10^{-39}	1.0×10^{-39}

Table 4. Comparison between different methods when $h = 1/15$

Numerical methods	MFDMs [21]	SRKM4	MEDSM1	MEDSM2
Maximum error	2.0×10^{-6}	2.4×10^{-6}	3.2×10^{-12}	4.5×10^{-11}

In Table 3, we report the absolute error difference between the exact analytical solution and the proposed solutions via **MEDSM1** and **MEDSM**, and further validated with **SRKM4**. From Table 4, we can see that **MEDSM1** is the most competent technique for solving the governing model in comparison with the **SRKM4**, **MEDSM2**, and the method used in [21]. Again, we portray the exact analytical and the contending approximate solutions in Figure 2, where one would notice an ideal agreement between the solutions.

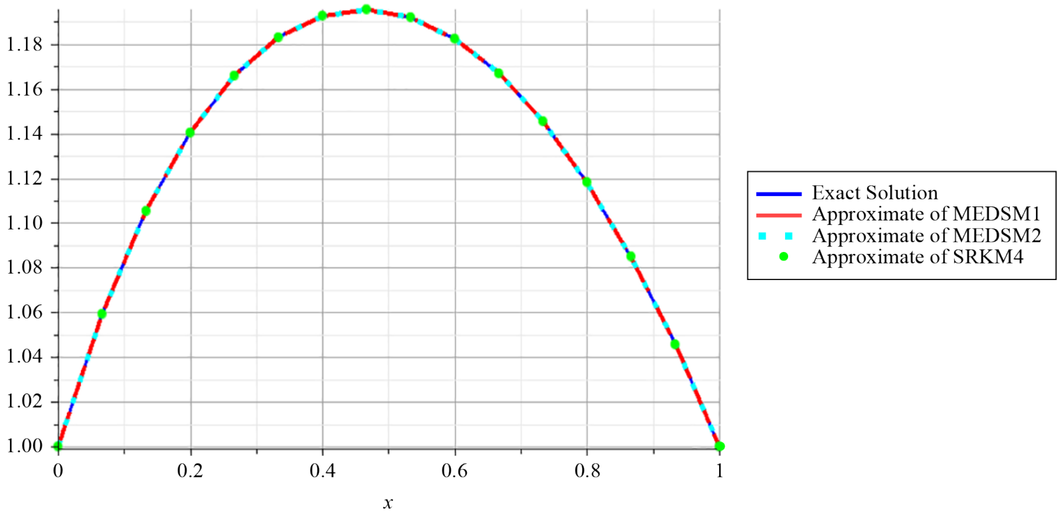

Figure 2. Graphical comparison, depicting the exact and contending approximate solutions with $h = 1/15$

Example 3 Consider the following second-order nonlinear BVP [22–25]

$$y''(x) = y^2(x) + 2\pi^2 \cos(2\pi x) - \sin^4(\pi x), \quad y(0) = y(1) = 0. \tag{27}$$

Then, the actual analytical solution of the BVP is expressed as $y(x) = \sin^2(\pi x)$.
Now we consider the two IVPs as follows

$$y'' = y^2 + 2\pi^2 \cos(2\pi x) - \sin^4(\pi x), \quad y(0) = 0, \quad y'(0) = t_s, \tag{28}$$

and

$$z'' = (2y)z, \quad z(0) = 0, \quad z'(0) = 1. \tag{29}$$

MEDSM1: Using the modified recursive relation (5) in (28) and by selecting $f_0 = t_s x + \sin^2(\pi x)$, $f_1 = \frac{1}{16}\left(\frac{\sin^4(\pi x) - 3\pi^2 x^2 + 3\sin^2(\pi x)}{\pi^2}\right)$, then the recursive relation is obtained as follows

$$
\begin{cases}
y_0 = t_s x + \sin^2(\pi x), \\
y_1 = \dfrac{1}{16}\left(\dfrac{\sin^4(\pi x) - 3\pi^2 x^2 + 3\sin^2(\pi x)}{\pi^2}\right) + L^{-1}(A_0), \\
y_{m+1} = L^{-1}(A_m), \quad m \geq 1.
\end{cases}
$$

where A_m in the recursive scheme denote the Adomian polynomials in favor of the nonlinear term y^2. At the first iteration the value of $t_0 = \dfrac{\beta - \alpha}{b - a} = 0$, then

$$
\begin{cases}
y_0 = \sin^2(\pi x), \\
y_1 = \dfrac{1}{16}\left(\dfrac{\sin^4(\pi x) - 3\pi^2 x^2 + 3\sin^2(\pi x)}{\pi^2}\right) + L^{-1}(A_0) = 0, \\
y_{m+1} = 0, \quad m \geq 1.
\end{cases}
$$

Then, the exact solution is $y(x) = \sin^2(\pi x)$, is immediately obtained. Hence, there is no need to solve (29).

MEDSM2: We will use the modified recursive relation (6) in (28) to construct the following functions $f_0 = t_s x$, $f_1 = \pi^2 x^2$, $f_2 = -\dfrac{1}{3}\pi^4 x^4$, $f_3 = \left(\dfrac{2}{45}\pi^6 - \dfrac{1}{30}\pi^4\right) x^6$, $f_4 = \left(\dfrac{1}{84}\pi^6 - \dfrac{1}{315}\pi^8\right) x^8$, $\cdots$. Therefore, the recursive relation is obtained as follows is

$$
\begin{cases}
y_0 = t_s x, \\
y_{m+1} = f_{m+1} + L^{-1}(A_m), \quad m \geq 0.
\end{cases}
$$

Similarly, when using the modified recursive relation (6) in (29) reduces to relation (3), so the recursive relation of (29) takes the following form

$$
\begin{cases}
z_0 = x, \\
z_{m+1} = 2L^{-1}(y_m z_m), \quad m \geq 0,
\end{cases}
$$

where the solution (28) with $M = 20$ can be computed from the series $y(x) = \sum_{m=0}^{M} y_m = y_0 + y_1 + \ldots + y_M$.

Lastly, when using 7 iterations, then $y(x, t_6)$ represents the solution of the second-order BVP (27) with $t = t_6$; see Table 5 and 6 for the numerical results.

In Table 5, we report the absolute error difference between the exact analytical solution and the proposed solutions **MEDSM2**, and further validated with **SRKM4**. From Table 6, we can see that **MEDSM2** is the most competent technique for solving the governing model in comparison with the **SRKM4**, and the methods used in [22–25], but **MEDSM1** outperforms **MEDSM2** because **MEDSM1** gives the exact analytical solution without the need to solve the second IVP. Further, we portray the exact analytical and the contending approximate solutions in Figure 3, where one would notice an ideal agreement between the solutions.

Table 5. The absolute error for SRKM4, and MEDSM2 when $h = 0.05$

x	SRKM4	MEDSM2
0	0	0
0.1	7.1×10^{-7}	9.4×10^{-20}
0.2	2.7×10^{-6}	1.9×10^{-19}
0.3	5.2×10^{-6}	2.8×10^{-19}
0.4	7.3×10^{-6}	3.8×10^{-19}
0.5	8.1×10^{-6}	4.9×10^{-19}
0.6	7.3×10^{-6}	6.0×10^{-19}
0.7	5.2×10^{-6}	7.3×10^{-19}
0.8	2.6×10^{-6}	8.7×10^{-19}
0.9	5.7×10^{-6}	9.95×10^{-19}
1.0	0	0

Table 6. Comparison between different methods when $h = 0.05$

Numerical methods	EADM [22]	MDM [23]	HPM [24]	HPM [25]	SRKM4	MEDSM2
Maximum error	2.5×10^{-6}	1.8×10^{-8}	9.5×10^{-8}	1.5×10^{-8}	8.1×10^{-6}	9.95×10^{-19}

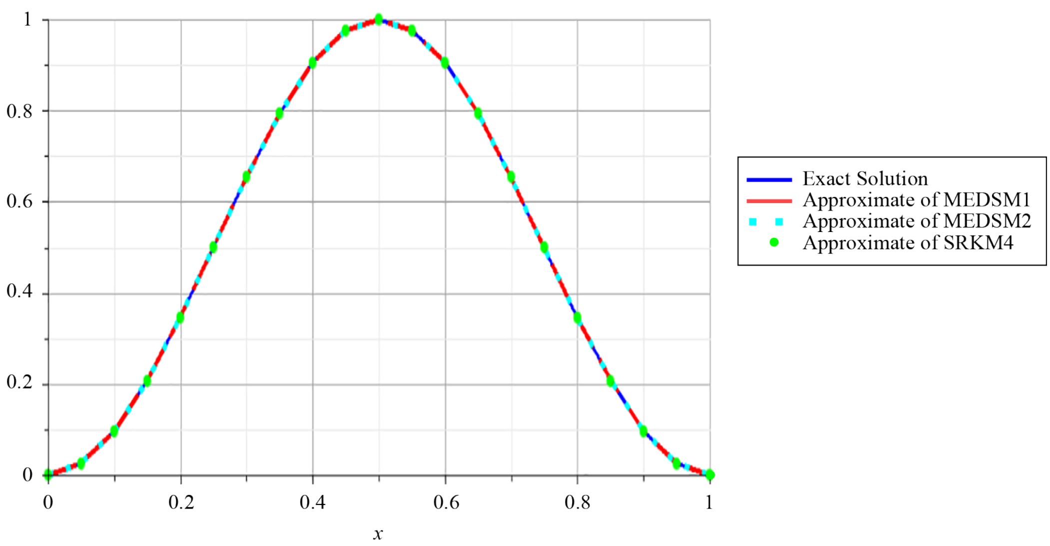

Figure 3. Graphical comparison, depicting the exact and contending approximate solutions with $h = 0.05$

Example 4 Consider the third-order nonlinear BVP as follows [26]

$$y^{'''}(x) - \ln(y(x)) = -ln\left(1 + \frac{x}{10{,}000}\right),$$

$$y(0) = 1, \quad y^{'}(0) = \frac{1}{10{,}000}, \quad y(1) = \frac{10{,}001}{10{,}000}. \tag{30}$$

The actual analytical solution is given by $y(x) = 1 + \dfrac{x}{10{,}000}$.

Now we consider the two IVPs as follows

$$y^{'''} = \ln(y(x)) - ln\left(1 + \frac{x}{10{,}000}\right), \quad y(0) = 1, \quad y^{'}(0) = \frac{1}{10{,}000}, \quad y^{''}(0) = t_k, \tag{31}$$

and

$$z^{'''} = \frac{1}{y}z, \quad z(0) = 0, \quad z^{'}(0) = 0, \quad z^{''}(0) = 1. \tag{32}$$

MEDSM1: Using the modified recursive relation (5) in (31) and by selecting

$$f_0 = 1 + \frac{x}{10{,}000} + \frac{2{,}000{,}000{,}000{,}000}{3}\ln(10),$$

$$f_1 = \frac{50{,}000{,}000}{3}x + \left(\frac{t_k}{2} + \frac{12{,}500}{3}\right)x^2 + \frac{11}{36}x^3 + \left(200{,}000{,}000x + 20{,}000x^2 + \frac{2}{3}x^3\right)\ln(10)$$

$$-\left(\frac{500{,}000{,}000{,}000}{3} + 50{,}000{,}000x + 5{,}000x^2 + \frac{1}{6}x^3\right)\ln(10{,}000 + x),$$

then the recursive relationship is

$$\begin{cases} y_0 = f_0, \\ y_1 = f_1 + L^{-1}(A_0), \\ y_{m+1} = L^{-1}(A_m), \quad m \geq 1, \end{cases}$$

where A_m in the recursive scheme denote the Adomian polynomials in favor of the nonlinear term $\ln(y(x))$. At the first iteration the value of $t_0 = \dfrac{\beta - \alpha}{b - a} = \dfrac{1}{10{,}000}$, then we will obtain the noise term $\left(\pm\dfrac{2{,}000{,}000{,}000{,}000}{3}\ln(10)\right)$; read [11, 27] for more information about noise term. Accordingly, by canceling the noise terms from y_0 and further justifying that the remaining non-canceled terms of y_0, which take the form $y_0 = 1 + \dfrac{x}{10{,}000}$ satisfy the equation, then, by using only two components, the exact solution taking the form $y(x) = 1 + \frac{x}{10{,}000}$ is obtained. Thus, there is no need to solve the second IVP (32).

5. Conclusions

In conclusion, two modified numerical methods have been introduced in the present study to efficiently tackle both the linear and nonlinear cases of the second and third-order two-point BVPs of ordinary differential equations. The proposed methods, which are numerically robust and economical, are further applied to several test problems and turned out to outperform SRKM4, and other available methods in the literature. Lastly, we have supported the findings of the present study with some comparison plots and tables-demonstrating the effectiveness of the devised approaches. In addition, the proposed methods can be applied to diverse models of real-life applications in mathematical physics.

Conflict of interest

The authors declare no competing financial interest.

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