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Research Article

Yukawa Cut-Offs to Model the Muon Self-Interaction

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Abstract: The present study introduce a regularization scheme yielding divergent free exact solutions to the classical field equations underlying the muon self-interaction. Exact results for the self-energy and anomalous g-factor of the muon and upper bound on the sum of the muon and electron neutrinos rest masses are reported. Yukawa cut-offs with unique screening constants regularizing the electromagnetic field inherent to the self-interacting muon are presented. Equations of motion and a transcendental equation satisfied by the muon anomalous g-factor are derived, with solution $a_\mu = 0.001165920530996(3)$ matching the most recent experimental value found in the literature to 60 ppt showing an excellent agreement.

***Keywords*:** regularization method, classical field, yukawa potential, self-interaction, muon

MSC: 81V10, 81T15, 81Q05

1. Introduction

The study of the electron's anomalous magnetic moment have played an essential role in the development of quantum theory [1–4]. Its successful theoretical description led to the expectation that the followed calculation of the anomalous component in the muon's intrinsic magnetic moment will strengthen and crystallize the established knowledge [5–10]. Over the decades, however, with the improvement of the experimental setup and the consequent collection of more data from highly precise measurements the gap between the measured and calculated values not only remained but thickened [11–17]. The difference between the most recent experimental average [18, 19] and the most recent consensus on the average theoretical value [20] is about 2.49 ppb, which is significantly larger than the relevant uncertainty. It is expected that the analysis of more experimental data will only result in a negligible variation in the measured average value reaffirming the obtained gap. As a result, a variety of theoretical approaches to the calculation of the corresponding anomalous component were proposed, see [21] and the references therein.

In the light of the apparent discrepancy many efforts to revise and improve the hadronic vacuum polarization corrections [22–26] and the hadronic light-by-light scattering one [25–29] have been considered, see also the revision of Dyson-Schwinger approach [30]. Despite the increasing number of fitting parameters and relevant ambiguity in the choice of observables needed to calculate these contributions, it is believed that the used quantum field theory approaches have the prospect to reduce the obtained tension.

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Besides the standard regularization and renormalization procedures used in the above mentioned quantum field theory calculations [31, 32], any progress in regularizing the muon's self-interaction at a classical limit is expected to favor a progress to the resolution of named tension. Yukawa cut-offs [33–36] are the most prominent tools for removing radial singularities at a microscopic scale in classical theory approaches. The application of Yukawa regulators [37] implies minimal number of effective parameters and leads to high precision results. Over the years number of attempts to establish an inherent to the non-composite particles regularization scheme yielding quantitative description of the vacuum in classical electrodynamics and field theory in general ware discussed [38–49]. Recently an exact regularization scheme with Yukawa potentials describing a massive off shell photons and no free mass parameters was successfully applied to quantify the electron $g-2$ value and the associated self-interaction [50]. To the best of our knowledge calculations of the muon electromagnetic self-energy and $g-2$ value via the application of Yukawa regulators in the classical theory have not been undertaken yet.

The present study implements the regularization technique proposed in [50] with the aim to quantify exactly the self-interaction energy (self-energy) and anomalous component in the muon's intrinsic magnetic moment. The applied technique regularizes the electrodynamics of non-composite particles yielding a classical stationary description of the vacuum consistent with the quantum theory beyond the corresponding principle. Accordingly, exact results for the muon's self-energy, anomalous g-factor and all intrinsic characteristics underlying the dynamics of its self-interaction are reported. Improved accuracy in the calculation of the muon anomalous g-factor is obtained (60 ppt), overcoming the existing gap between the average value predicted by the quantum theory [20] and the measured one [18, 19]. An essential outcome of the obtained accuracy is an exact bound on the sum of the muon and electron neutrinos rest masses.

2. Theoretical framework

In this section we set-out the mathematical notation of all physical quantities and present all ab-initio relations underlying the obtained results. We find it convenient to restrict all representations within the mathematical framework of three-dimensional vector formalism.

2.1 *Generalities*

Consider a free muon with rest mass and electric charge denoted by m_μ and $\bar{e} = -e$, respectively, where e is the elementary charge. Let $\mathbf{R}$ be the muon's rest frame of reference and $r_{c\mu} = \alpha\bar{\lambda}_{c\mu}$ be its electromagnetic radius in $\mathbf{R}$, where α and $\bar{\lambda}_{c\mu}$ are the fine structure constant and associated reduced Compton wavelength, respectively. Let $\mathbf{r}_\mu$ be the intrinsic field vector associated to the muon, with magnitude r_μ, and $\tilde{u}_\mu$ be the magnitude of the tangential velocity $\tilde{\mathbf{u}}_\mu$ related to its rotation about the origin of $\mathbf{R}$ in the plane perpendicular to the muon's relative velocity $\mathbf{u}_\mu = u_\mu \boldsymbol{\kappa}$ defined with respect to an observer with frame of reference $\mathbf{O}$, where $\boldsymbol{\kappa}$ is the respective unit vector (see Figure 1). Since the system is closed, we have the constraint $u_\mu = \tilde{u}_\mu$. Furthermore, the oscillation of $\mathbf{r}_\mu$ is characterized by an angular velocity $\boldsymbol{\omega}_\mu$, with magnitude $\omega_\mu = \tilde{u}_\mu r_\mu^{-1}$ representing the angular frequency of the corresponding circularly polarized field $\boldsymbol{\Phi}_\mu \in \mathbb{R}^3$, with $||\cdot||_2 = r_\mu$. The classical representation of the field reads

$$\boldsymbol{\Phi}_\mu(\mathrm{x}) = \frac{A_\mu}{\sqrt{2}}(\Phi_\mu(\mathrm{x})\mathbf{n}^* + \Phi_\mu^*(\mathrm{x})\mathbf{n}),$$

where the amplitude $A_\mu \equiv r_\mu$, $\mathbf{n} \in \mathbb{C}^3$, is the field's unit vector and $\Phi_\mu(\mathrm{x}) \in \mathbb{C}$ is a phase factor, with $\mathrm{x} \in \mathbb{R}^{1,\,3}$ denoting the field's four-position vector with respect to $\mathbf{O}$. The star symbol indicates complex conjugate. The unit vector components satisfy equations $\mathbf{n}\cdot\mathbf{n}^* = 1$ and $\mathbf{n}\cdot\mathbf{n} = 0$ yielding a trivial solution for the component with real part collinear to $\boldsymbol{\kappa}$. Accordingly, for the respective field vector, tangential and angular velocity, we have

$$\mathbf{r}_\mu = \mathbf{\Phi}_\mu(t),\ \tilde{\mathbf{u}}_\mu = \dot{\mathbf{\Phi}}_\mu(t),\ \boldsymbol{\omega}_\mu = \frac{1}{r_\mu^2}(\mathbf{\Phi}_\mu(t) \times \dot{\mathbf{\Phi}}_\mu(t)),$$

where the time variable t is defined in $\mathbf{R}$, the dot symbol stands for time derivative and $\Phi_\mu(\mathrm{x} = ct) \to \Phi_\mu(t) = e^{i\omega_\mu t}$. In the case shown in Figure 1, we have $\mathbf{n} = \left(\frac{1}{\sqrt{2}},\ \frac{i}{\sqrt{2}},\ 0\right)$ and $\mathbf{\Phi}_\mu(t) = (r_\mu \cos\omega_\mu t,\ r_\mu \sin\omega_\mu t,\ 0)$.

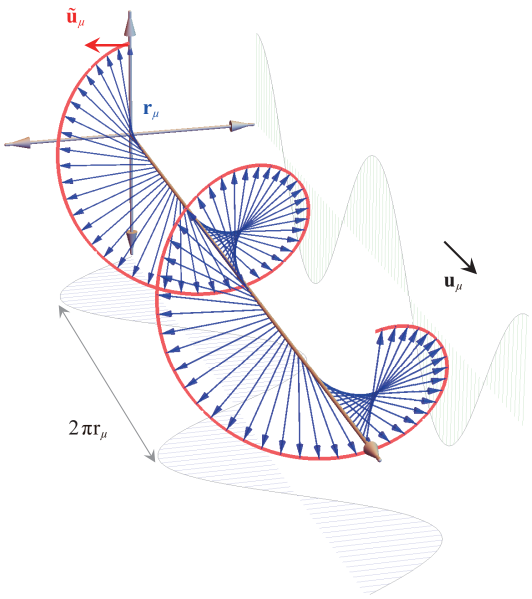

Figure 1. Sketch of a right circularly polarized spatial wave representing a free self-interacting muon. The corresponding field vector $\mathbf{r}_\mu$ (blue arrows) depicts a helix as the particle moves relative to an observer with velocity $\mathbf{u}_\mu$

The magnitudes of muon field vector and tangential velocity are a conjugate quantities satisfying

$$r_\mu \tilde{u}_\mu = \bar{\lambda}_{c\mu} c, \tag{1}$$

where c is the light speed in vacuum. Moreover, in the considered system the following identity holds $\mathbf{u}_\mu \cdot \boldsymbol{\omega}_\mu (u_\mu \omega_\mu)^{-1} = 1$, with u_μ, $\tilde{u}_\mu$ and ω_μ being intrinsic quantities. The normal mode of the muon field is unique and even in the case of an arbitrary motion, u_μ remains invariant, with $\mathbf{u}_\mu$ being a component of the total relative velocity. Therefore, for an arbitrary motion of $\mathbf{R}$ with respect to $\mathbf{O}$, the mode will appear as a superposition of different modes, with field dynamics resembling the one of a wave packet.

The muon's charge ρ_e and rest mass ρ_{m_μ} densities satisfying the equality $\rho_e \rho_{m_\mu}^{-1} = e m_\mu^{-1}$ are defined within the spherically symmetric spatial domain $\Omega_{c\mu} \subset \mathbb{R}^3$, with radius $r_{c\mu}$, boundary $\partial\Omega_{c\mu}$ and volume $V_{c\mu}$. The muon effective rest mass density $\rho_{M_\mu} = \rho_{M_\mu}(r)$ is a smooth function of the radial distance r from the origin of $\mathbf{R}$, with $r \in (0,\ +\infty)$ and $\rho_{M_\mu} > \rho_{m_\mu}$ for all r. The effective mass density results from the particle's electromagnetic self-interaction [51–57] and therefore in the considered case, if $\nexists\, e$, then $\rho_{M_\mu} = \rho_{m_\mu}$. The corresponding effective rest mass reads

$$M_\mu = \int_{\Omega_{c\mu}} \rho_{M_\mu} \mathrm{d}v = \frac{1}{2} g_\mu m_\mu,$$

where $g_\mu = 2(1+a_\mu)$ is the g-factor of the muon and a_μ is the anomalous component. In particular, we have

$$g_\mu = \frac{2}{V_{c\mu}} \int_{\Omega_{c\mu}} G_\mu \mathrm{d}v, \quad G_\mu = \frac{e\rho_{M_\mu}}{m_\mu \rho_e}. \tag{2}$$

The inherent dynamics of $\mathbf{r}_\mu$ underpin the occurrence of intrinsic magnetic moment $\boldsymbol{\mu}_\mu = -\frac{1}{2} g_\mu \mu_\mu \boldsymbol{\kappa}$, where $\mu_\mu = e\hbar(2m_\mu)^{-1}$ is the corresponding magneton. The spin magnetic moment $\boldsymbol{\mu}_\mu$ occur due to the fact that the corresponding field is polarized. Neither the electric charge nor mass of the muon are spinning. In other words, the muon is characterized by an intrinsic charge density pseudocurrent $\mathbf{j}_\mu = \rho_e r_\mu \boldsymbol{\omega}_\mu$ defined in $\mathbf{R}$ and satisfying

$$\boldsymbol{\mu}_\mu = -\frac{1}{2} r_\mu \int_{\Omega_{c\mu}} G_\mu \mathbf{j}_\mu \mathrm{d}v$$

$$= \frac{g_\mu \bar{e}}{2m_\mu} \mathbf{s}_\mu,$$

where

$$\mathbf{s}_\mu = \frac{1}{2} m_\mu (\boldsymbol{\Phi}_\mu(t) \times \dot{\boldsymbol{\Phi}}_\mu(t))$$

is the corresponding spin angular momentum. In the case depicted in Figure 1 the helicity is positive.

2.2 *Field equation*

The electromagnetic field in the considered system is inherent to the particle and do not represent a collection of massive [58–62] and massless on shell photons. Since $\dot{\boldsymbol{\omega}}_\mu = 0$ and $\dot{\mathbf{u}}_\mu = 0$, in both frame of references, $\mathbf{R}$ and $\mathbf{O}$, the classical representation of the electromagnetic field is time independent and is characterized by a finite energy depending on the distance from the origin of $\mathbf{R}$. In other words, it correspond to a collection of off shell photons [38, 63–68] represented in the classical theory via static angular independent scalar and vector potentials including Yukawa cut-offs [37]. Thus, the electromagnetic field potentials are not retarded, the radial singularity is removed and the Lorenz gauge is trivially satisfied.

The classical field equation describing the radial dependence of the electromagnetic field in the considered system reads

$$\Delta_r \psi_\mu(r) - \chi_\mu^2 \phi_\mu(r) - \tilde{\chi}_\mu^2 \tilde{\phi}_\mu(r) = 0, \tag{3}$$

where Δ_r represents the radial spherically symmetric Laplace operator, the real functions $\phi_\mu(r)$ and $\tilde{\phi}_\mu(r)$ are the cut-off terms with screening constants $\chi_\mu \in \mathbb{R}_{>0}$ and $\tilde{\chi}_\mu \in \mathbb{R}_{>0}$, respectively, implying the following boundary conditions

$$\psi_\mu(r) = \begin{cases} 0, & r \to \infty, \quad u_\mu < c, \\ (\chi_\mu + \tilde{\chi}_\mu)\dfrac{\bar{e}}{4\pi\varepsilon_o}, & r \to 0, \quad u_\mu < c. \end{cases}$$

where ε_o is the electric constant. The solution of Equation (3) is a real function regularized to the origin of **R** with respect to **O** and reads

$$\psi_\mu(r) = \frac{\bar{e}}{4\pi\varepsilon_o r}\left(2 - e^{-\chi_\mu r} - e^{-\tilde{\chi}_\mu r}\right). \tag{4}$$

For all $r \in (0, +\infty)$, both χ_μ and $\tilde{\chi}_\mu$ represent a fundamental bound to the wave numbers of the effectively massive off energy shell photons coupled to the muon. Both Yukawa potentials are a regulators to the Coulomb terms that are implicitly accounted for in Equation (3), since the corresponding screening constants equal zero. Therefore, the function given in Equation (4) is an amplitude of the oscillating vector field $\mathbf{\Lambda}_\mu(r, t) = c^{-2}\psi_\mu(r)\dot{\mathbf{\Phi}}_\mu(t)$ and has a Fourier transform

$$\psi_\mu(r) = \frac{\bar{e}}{\varepsilon_o}\int \frac{\chi_\mu^2\, e^{i\mathbf{k}\cdot\mathbf{r}}}{k^2(k^2+\chi_\mu^2)}\frac{\mathrm{d}^3k}{(2\pi)^3} + \frac{\bar{e}}{\varepsilon_o}\int \frac{\tilde{\chi}_\mu^2\, e^{i\tilde{\mathbf{k}}\cdot\mathbf{r}}}{\tilde{k}^2(\tilde{k}^2+\tilde{\chi}_\mu^2)}\frac{\mathrm{d}^3\tilde{k}}{(2\pi)^3}, \tag{5}$$

with off shell amplitudes bounded by the regulators $\chi_\mu^2(k^2+\chi_\mu^2)^{-1}$ and $\tilde{\chi}_\mu^2(\tilde{k}^2+\tilde{\chi}_\mu^2)^{-1}$ adding the singularities at $k = i\chi_\mu$ and $\tilde{k} = i\tilde{\chi}_\mu$ for the removal of the radial one $r = 0$ in Equation (4). Here, $\mathbf{r}$ is the radial vector taken with respect to **R** and the unit vectors of $\mathbf{k} \in \mathbb{C}^3$ and $\tilde{\mathbf{k}} \in \mathbb{C}^3$ are defined in $\mathbb{R}^3$.

2.3 *Field's potentials*

In the muon's rest frame of reference the system's electromagnetic field is described by the centrally symmetric scalar and pseudovector potentials $\psi_\mu(r)$ and $\mathbf{A}'_\mu(r) = c^{-2}r_\mu\boldsymbol{\omega}_\mu\psi_\mu(r)$, respectively. Here, we take into account that $\mathbf{A}'_\mu(r) = (\boldsymbol{\rho}(t) \times \mathbf{\Lambda}_\mu(r,t))$, where the unit vector $\boldsymbol{\rho} = \mathbf{r}_\mu r_\mu^{-1}$. According to the Lorentz transformations, in the observer's frame of reference the system's electromagnetic field relative to the origin of **R** is characterized by the scalar and vector potentials

$$\varphi_\mu(r) = \gamma_\mu\eta_\mu\psi_\mu(r), \quad \mathbf{A}_\mu(r) = 2\gamma_\mu\frac{\mathbf{u}_\mu}{c^2}\psi_\mu(r), \tag{6}$$

where γ_μ is the corresponding Lorentz factor and

$$\eta_\mu = 1 + \frac{u_\mu^2}{c^2}, \quad \nabla\cdot\mathbf{A}_\mu = 0.$$

As discussed above, the electromagnetic field do not propagate independently from the particle and in **O** it is further characterized by the Umov-Poynting vector $\mu_o^{-1}(\mathbf{E}_\mu \times \mathbf{B}_\mu) = 2\eta_\mu^{-1}\varepsilon_o E_\mu^2\mathbf{u}_\mu$, where μ_o is the vacuum magnetic permeability, $\mathbf{E}_\mu = -\nabla\varphi_\mu(r)$ and $\mathbf{B}_\mu = 2c^{-2}\eta_\mu^{-1}(\mathbf{u}_\mu \times \mathbf{E}_\mu)$ are the corresponding electric and magnetic field components, respectively.

2.4 *Field's energy*

The energy of the electromagnetic field in the considered system, $W_\mu(r)$, is also time independent and regularized. In general, integrating the corresponding energy density $2\eta_\mu^{-1}\varepsilon_o|\nabla\varphi_\mu(r)|^2$ over $\mathbb{R}^3$, we obtain

$$W_\mu(r) = C_\mu \frac{r_{c\mu}}{2r}\Bigg(8\big(e^{2(\chi_\mu+\tilde{\chi}_\mu)r} - e^{(2\chi_\mu+\tilde{\chi}_\mu)r} - e^{(\chi_\mu+2\tilde{\chi}_\mu)r}\big)$$

$$+(2+\chi_\mu r)e^{2\tilde{\chi}_\mu r} + (2+\tilde{\chi}_\mu r)e^{2\chi_\mu r}$$

$$+4\frac{(\chi_\mu+\tilde{\chi}_\mu+\chi_\mu\tilde{\chi}_\mu r)}{\chi_\mu+\tilde{\chi}_\mu}\Bigg)e^{-2(\chi_\mu+\tilde{\chi}_\mu)r},$$

where $C_\mu = 2\gamma_\mu^2\eta_\mu m_\mu c^2$. At the origin of $\mathbf{R}$ and for $u_\mu < c$ the electromagnetic field energy in the considered system is finite. Thus, we have

$$\lim_{r\to 0} W_\mu(r) = C_\mu r_{c\mu}\frac{\chi_\mu^2 + 6\chi_\mu\tilde{\chi}_\mu + \tilde{\chi}_\mu^2}{2(\chi_\mu+\tilde{\chi}_\mu)}. \tag{7}$$

In contrast to the non-regularized electrodynamics, here for $r \to 0$ the discussed electromagnetic field energy vanish when the particle's rest mass is negligible. In other words, the charge screening (Yukawa cloud) is nearly complete making the bare particle to appear as a dressed one.

3. Results

3.1 *The hamiltonian*

The energy of a free self-interacting muon do not depend explicitly on time. The considered system is not characterized by a potential energy and hence there is no net self-force acting on the muon [69–71]. Moreover, since the particle is at rest with respect to the origin of $\mathbf{R}$, the vector field in Equation (6) does not contribute to the self-interaction. In both frame of references the self-interaction depends only on the electromagnetic field scalar potential. Therefore, in the observational frame of reference the system's dynamics is entirely intrinsic and described only by an effective Hamiltonian. The latter reads

$$H_\mu = \gamma_\mu m_\mu c^2 + \Sigma_\mu, \tag{8}$$

where Σ_μ is the energy of electromagnetic self-interaction. This energy is not a potential energy of a gradient field and equals the spatial average over the domain $\Omega_{c\mu}$ of the interaction density $-\rho_e\varphi_\mu \equiv -\gamma_\mu(\rho_e\psi_\mu + \mathbf{j}_\mu\cdot\mathbf{A}'_\mu)$. In particular, with respect to Equation (6), we have the representation

$$\Sigma_\mu = -\int_{\Omega_{c\mu}} \rho_e \varphi_\mu \mathrm{d}v$$

$$= \gamma_\mu c^2 \int_{\Omega_{c\mu}} (\rho_{M_\mu} - \rho_{m_\mu}) \mathrm{d}v, \tag{9}$$

where the effective mass density reads

$$\rho_{M_\mu} = \rho_{m_\mu} \left(1 + \frac{\eta_\mu \bar{e}}{m_\mu c^2} \psi_\mu\right)$$

$$= \rho_{m_\mu} \left(1 + \eta_\mu \frac{r_{c\mu}}{r} \left(2 - e^{-\chi_\mu r} - e^{-\tilde{\chi}_\mu r}\right)\right). \tag{10}$$

Here, we take into account the equality $e^2 = 4\pi\varepsilon_o r_{c\mu} m_\mu c^2$.

3.2 *The Hamiltonian density*

The information about the muon filed vector dynamics, with helix trajectory shown in Figure 1, is embedded in the Hamiltonian density $\mathscr{H}_\mu = \gamma_\mu \rho_{M_\mu} c^2$ associated to Equation (8) as follows $H_\mu = \int_{\Omega_{c\mu}} \mathscr{H}_\mu \mathrm{d}v$. Taking into account Equations (1) and (10), we get

$$\mathscr{H}_\mu = c^2 \rho_{m_\mu} \left(\gamma_\mu + \frac{\alpha}{m_\mu c} \mathscr{P}_\mu\right),$$

where

$$\mathscr{P}_\mu = \frac{\bar{e}}{\alpha c} \varphi_\mu(r)$$

$$= \gamma_\mu \eta_\mu m_\mu \tilde{u}_\mu \frac{r_\mu}{r} \left(2 - e^{-\chi_\mu r} - e^{-\tilde{\chi}_\mu r}\right)$$

is the corresponding generalized momentum. Accordingly, we have the equations of motion

$$\tilde{u}_\mu = \int_{\Omega_{c\mu}} \frac{\partial \mathscr{H}_\mu}{\partial \mathscr{P}_\mu} \mathrm{d}v, \quad \dot{\mathscr{P}}_\mu = 0$$

and subsequently the exact values

$$\tilde{u}_\mu = \alpha c, \quad \eta_\mu \to \eta_\alpha = 1 + \alpha^2, \quad \gamma_\mu \to \gamma_\alpha = \frac{1}{\sqrt{1 - \alpha^2}}. \tag{11}$$

3.3 *Effective mass-energy equivalence*

Taking into account Equations (2) and (9), we obtain the self-interaction energy as a function of the muon's rest mass. Thus, we have

$$\Sigma_\mu = \gamma_\alpha a_\mu m_\mu c^2.$$

As a result, from Equation (8) we get the muon's effective relativistic energy

$$\begin{aligned} \mathscr{E}_\mu &= \gamma_\alpha (1 + a_\mu) m_\mu c^2 \\ &= \sqrt{P_\mu^2 c^2 + M_\mu^2 c^4}, \end{aligned} \tag{12}$$

where $P_\mu = \alpha \gamma_\alpha M_\mu c$ is the muon's momentum. Therefore, as a result of the self-interaction the total energy of a free muon is $\gamma_\alpha (1 + a_\mu)$ times higher than its rest energy and the system's total rest mass will be $(1 + a_\mu)$ times higher than the muon's rest mass. As the frame of reference **R** moves relatively to an observer in **O** and the muon field vector oscillates, the muon will appear as circularly polarized traveling spatial wave with unique mode characterized by the phase factor $\Phi_\mu(\mathrm{x}) \equiv \Phi_\mu(r,\ \tau) = e^{i\frac{\alpha^2}{(1+a_\mu)\hbar}(\mathscr{E}_\mu \tau - P_\mu r)}$, where we take into account the relation $\hbar\omega_\mu = \alpha^2 m_\mu c^2$, see Section 2.1. Given the obtained phase factor the classical and quantum representations of the free self-interacting muon are interchangeable. Thus, the corresponding spinor field $\Psi_\mu(\mathrm{x})$ that accounts for the obtained muon's self-energy and anomalous magnetic moment satisfies the Dirac equation

$$(ic\partial\!\!\!/ - \omega_\mu)\Psi_\mu(\mathrm{x}) = 0,$$

with solution given in Equation (12).

3.4 *The anomalous g-factor*

The implemented regularization scheme yields a single-parametric transcendental equation for the calculation of the muon's anomalous g-factor. In particular, accounting for Equations (2) and (10), we obtain

$$a_\mu = \frac{\eta_\alpha}{V_{c\mu}} \int_{\Omega_{c\mu}} \frac{r_{c\mu}}{r} \left(2 - e^{-\chi_\mu r} - e^{-\tilde{\chi}_\mu r}\right) \mathrm{d}v. \tag{13}$$

The screening constants are given by

$$\chi_\mu = \frac{\gamma_\alpha m_\mu c}{(1 + a_\mu) h}, \quad \tilde{\chi}_\mu = \frac{\gamma_\alpha \tilde{m} c}{(1 + a_\mu) h}, \tag{14}$$

where h is Plancks constant and the residual rest mass $\tilde{m} > 0$ for all $r \in (0,\ +\infty)$.

Essentially, taking into account Equations (11) and (14), from Equation (13) we get

$$a_\mu = 3\eta_\alpha \left[1 - \left(\frac{1 - e^{-\frac{\alpha\gamma_\alpha}{2\pi(1+a_\mu)}} \left(1 + \frac{\alpha\gamma_\alpha}{2\pi(1+a_\mu)}\right)}{\left(\frac{\alpha\gamma_\alpha}{2\pi(1+a_\mu)}\right)^2} \right) - \left(\frac{1 - e^{-\frac{\alpha\gamma_\alpha\xi_\mu}{2\pi(1+a_\mu)}} \left(1 + \frac{\alpha\gamma_\alpha\xi_\mu}{2\pi(1+a_\mu)}\right)}{\left(\frac{\alpha\gamma_\alpha\xi_\mu}{2\pi(1+a_\mu)}\right)^2} \right) \right], \tag{15}$$

where

$$\xi_\mu = \frac{\tilde{m}}{m_\mu}. \tag{16}$$

Table 1. Theoretical and experimental (EXP) data for the muon's anomalous g-factor (second column).

Approach	a_μ	ξ_μ	Refs.
RCED	0.001165920530996(3)	0.0048363318963797	Eq. (15)
QFT	0.00116591810(43)	-	[20]
EXP	0.00116592059(22)	-	[18, 19]

The regularized classical electrodynamics (RCED) and quantum field theory (QFT) results in Table 1 are given in the second and third rows, respectively. Fourth row represents the most recent experimental average value. The value of mass ratio ξ_μ follows from Equation (16). For additional details see Figure 2.

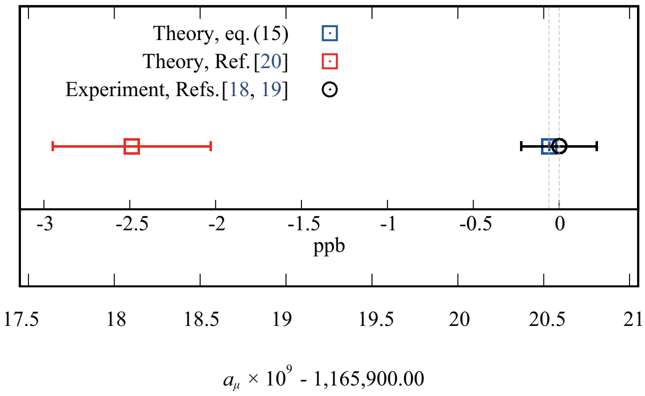

Figure 2. Comparison between the most recent experimental result (black circle) for the muon's anomalous g-factor and its value obtained from Equation (15) (blue square). In addition the latest result predicted by the quantum theory (red square) is also shown. The depicted data is further provided in Table 1

The value of a_μ calculated from Equation (15) is given in the second row of Table 1. The computations were carried out for $\alpha = 137.0359990849004^{-1}$, with value taken from [50]. The obtained accuracy with respect to the most recent experimental measurements [18] is about 0.06 ppb, see Figure 2. The value of mass ratio in Equation (16) is given in the third column in Table 1. We have $\tilde{m} = m_e + m_\nu$, where m_e denotes the electron's rest mass and $m_\nu = m_{\nu_\mu} + m_{\bar{\nu}_e}$. Here, m_{ν_μ} is the muon neutrino rest mass and $m_{\bar{\nu}_e}$ is the electron anti-neutrino rest mass. Numerically, we have $m_\nu = 3.8400015(1) \times 10^{-38}$ kg, where the values of electron's and muon's rest masses $m_e = 9.1093837015 \times 10^{-31}$ kg and $m_\mu = 1.883531627 \times 10^{-28}$ kg, respectively, are taken from NIST [72]. This result suggest that the sum of the muon neutrino and electron anti-neutrino rest energies is approximately 0.02154 eV, which is consistent with the reported upper bound on the sum of the three flavor neutrino rest energies of about 0.120 eV (see [73]). Moreover, it suggests that the electron anti-neutrino mass satisfies the inequality $m_{\bar{\nu}_e} < 3.84 \times 10^{-38}$ kg. This bound is approximately 40 times lower than the one set by KATRIN collaboration [74, 75].

3.5 *Electromagnetic field energy*

The total energy of the muon given in Equation (12) is part of the considered system's total energy. The latter accounts for the electromagnetic field energy associated to the muon. In the non-regularized electrodynamics, the electromagnetic field energy associated to the three flavors of charged leptons does not depend on their rest mass and is not defined at the origin of **R**. Consequently the corresponding energy density is quantitatively indistinguishable with respect to the flavor state of these particles. Within the current approach, as a result of the applied regularization the electromagnetic field energy and its density are flavor dependent. The larger the lepton's rest mass the higher corresponding electromagnetic field energy. In the considered case, substituting the obtained from Equation (15) values of m_ν and a_μ in Equation (7), we obtain $W_\mu(r \to 0) = 125.540 \times 10^3$ eV, which is 211.749 times the value of electromagnetic field energy associated to the electron at the same limit (see [50]) and about 1.18817×10^{-3} times the muon's rest energy.

4. Summary and conclusions

The present paper reports on the most recent progress in the application of the regularization technique in electrodynamics of electrically charged non-composite particles. In general, the regularization is based on the Yukawa theory with cut-off terms screening the Coulomb potential and describing massive off shell photons. The cut-offs are characterized by a unique to the particle screening constants that remove the radial singularity in the classical representation of the corresponding electromagnetic field yielding exact solutions to the system's equations of motion.

The effectiveness of proposed regularization scheme is demonstrated by quantifying the muon's intrinsic dynamics and calculating exactly the corresponding self-energy and anomalous g-factor. The comparison with the muon $g-2$ experimental value shows no sign of discrepancy between theory and experiment.

In particular, an exact divergence free classical representation of the electromagnetic field inherent to the muon is proposed. The electromagnetic field is described as a collection of massless and massive (see the wave vectors given in Equation (14)) off shell photons, with amplitudes given in Equation (5). At each point in space the energy of these photons is finite, see Equation (7). As a result, the muon's self-energy is precisely calculated showing the exact contribution of electromagnetic self-interaction into the muon's total mass and energy, see Equation (12). Moreover, the muon's anomalous g-factor is calculated with high precision and accuracy (see Equation (15)), improving the latter obtained from the latest quantum theory calculations, see Figure 2 and Table 1. The obtained accuracy implies that the contribution of electromagnetic field into the occurrence of anomalous magnetic moment is significantly larger than previously evaluated. It shows that the main contribution results from the massive off shell photons. The effective mass of these photons further implies that the muon and electron neutrinos have a rest mass, with upper bound on their sum equal to $3.8400015 \times 10^{-38}$ kg, see Equations (15) and (16). In addition, it points out that the contribution of both neutrinos' rest masses into the electron-muon mass ratio should be taken by the quantum theory approach in order to improve the relevant result.

The used regularization technique can be applied to quantify the intrinsic dynamics of the tau lepton and to fix the range of values of the corresponding anomalous g-factor determined by the multiplicity of branching fractions. It is, furthermore, applicable to composite particles with constant electric charge. Equation (12) shows that the total rest energy of a collection of free electrically charged particles is always greater than the anticipated value obtained by accounting for only the particles rest masses.

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Conflict of interest

The author declares no competing interests.

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