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Research Article

Exploring the New Exponentiated Inverse Weibull Distribution: Properties, Estimation, and Analysis via Classical and Bayesian Approaches

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Abstract: The "New Exponentiated Inverse Weibull" (NEIW) distribution is a novel probability distribution with versatile applications in reliability engineering, survival analysis, and related fields. This study explores the properties of the NEIW distribution, including moments, hazard function behavior, and reliability measures, through rigorous theoretical analysis and simulation studies. Additionally, maximum likelihood estimators (MLEs) for NEIW parameters are investigated via simulation studies to evaluate their performance under various scenarios. Empirical analysis using real-world data sets demonstrates the applicability of the NEIW model in capturing the underlying data structure and extracting meaningful insights. Parameter estimation and analysis are conducted using classical and Bayesian approaches, showcasing the robustness and flexibility of the NEIW distribution. The latest Bayesian analysis software STAN is used to perform the Bayesian analysis using Hamiltonian Monte Carlo (HMC) under No-U-Turn Sampler (NUTS). Overall, this research contributes to the advancement of statistical modeling and analysis by providing a comprehensive framework for utilizing the NEIW distribution in practice, with implications for diverse fields and potential for further exploration and innovation.

***Keywords*:** Inverse Weibull, G-family, Bayesian analysis, Hamiltonian Monte Carlo, posterior distribution

MSC: 35A01, 65L10, 65L12, 65L20, 65L70

Abbreviation

CDF	Cumulative probability Distribution Function
HMC	Hamiltonian Monte Carlo
NEIW	New Exponentiated Inverse Weibul
PDF	Probability Density Function

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1. Introduction

The diverse and complex nature of data in applied disciplines often surpasses the capabilities of traditional statistical models, necessitating the development of more flexible distributions. To address these challenges, researchers have created new families of distributions designed to accommodate the unique characteristics of data across the natural, applied, and social sciences. The ongoing development of versatile distribution families is crucial for capturing the complexities inherent in applied fields. Despite significant progress, there remains a pressing need for new and innovative distribution families to address the evolving challenges of modern data analysis.

The process of creating new distribution families often involves augmenting existing distributions with additional parameters, enabling the definition of novel distributions. For example, the Weibull family arose through a power transformation of the exponential distribution, while the gamma distribution was derived via Laplace transformations applied to the exponential distribution [1]. Similarly, new families can be generated by adding parameters to the survival function of an existing base distribution [1]. This approach of incorporating additional parameters into the survival function of a base distribution to generate a new family of distributions is widely popular. Exponentiating a baseline distribution is another approach that can be used to add an additional parameter [2]. Lehmann type-I (L-I) and type-II (L-II) models [3] are simple and adaptable techniques for incorporating additional parameters into a baseline distribution. The L-I model is often discussed in the context of the Power Function (PF) distribution. The PF distribution is created by simply raising any baseline model to a power. The corresponding cumulative distribution function (CDF) is then given by

$$F(x;\, \alpha,\, \psi) = (G(x;\, \psi))^{\alpha};\;\; \alpha > 0,\;\; x \in R. \tag{1}$$

where ψ is a parameter vector of the baseline distribution $G(x;\, \psi)$. For $\alpha = 1$, the L-I model reduces back to the original baseline distribution. Utilizing equation (1), various lifetime distribution models have been introduced. These include the exponentiated Gumbel (EGu) [4], the exponentiated generalized class [5], the exponentiated Kumaraswamy Dagum (EKu-D) [6], the exponentiated Weibull-G (EW-G) [7], and the exponentiated power function distribution [8].

The simplicity and utility of the PF distribution have led researchers to investigate its applications, extensions, and generalizations in various scientific fields. Lehmann Type-II (L-II) models are also well-represented in the literature, with notable examples including the Lehmann Type II Inverse Weibull distribution [9] and the Lehmann Type II-Teissier distribution [10].

With some modification on Lehmann Type II type distribution, Cordeiro and Castro [11], and Corderio et al. [12] presented the concept of adding two parameters to the baseline distribution where the cumulative distribution function (CDF) is given by

$$F(x;\, \alpha,\, \beta,\, \psi) = [1-(1-G(x;\, \psi))^{\alpha}]^{\beta};\;\; \alpha > 0,\;\; \beta > 0,\;\; x \in R.$$

Building on the concept of exponentiation, we propose a novel method to expand existing distributions by introducing an additional parameter, forming what we call the ‘A New Exponentiated G-Family of Distributions’. This new family demonstrates greater robustness compared to traditional compound probability distributions and holds significant potential for modeling complex empirical datasets. A key special member of this family includes three parameters, allowing it to effectively capture a wider range of dataset characteristics, such as skewness, kurtosis, failure rates, and mathematical tractability. This enhanced flexibility enables a more accurate representation of intricate data patterns and distributional properties.

To further empower data analysis, both classical and Bayesian estimation methods were developed, with maximum likelihood estimation applied to estimate the parameters. The model’s validity was rigorously tested through Monte Carlo simulations. When applied to real-world datasets and compared against several existing distributions, the proposed model consistently outperformed its counterparts, offering a superior fit for complex data scenarios.

For Bayesian analysis, we have used the Bayesian analysis software STAN, which uses Hamiltonian Monte Carlo (HMC) under the No-U-Turn Sampler (NUTS) [13]. For more information on Bayesian analysis, the reader can go through [14–16].

The structure of the manuscript is organized as follows: Section 2 introduces the New Exponentiated G-Family (NEG) of distributions, and Section 3 focuses on the New Exponentiated Inverse Weibull (NEIW) distribution. Section 4 discusses various statistical properties, while Section 5 explores statistical inferences related to the NEIW distribution. Sections 6 and 7 present the simulation study and a practical application, respectively. Bayesian analysis of the proposed model, using STAN, is detailed in Section 8. Finally, Section 9 concludes the manuscript.

The structure of the remaining sections of this manuscript is delineated as follows: Section 2 introduces the New Exponentiated G-Family (NEG) of distribution, while Section 3 focuses on the New Exponentiated Inverse Weibull (NEIW) distribution. Section 4 discusses various statistical properties, and Section 5 provides a discussion on the statistical inferences pertinent to the NEIW distribution. Sections 6 and 7 present the simulation study and a practical application, respectively. Bayesian analysis of the proposed model, using STAN, is detailed in Section 8. Finally, Section 9 concludes the manuscript.

2. A new exponentiated G-Family of distribution

Let $G(x;\ \psi)$ be the baseline CDF, with ψ as a vector of associated parameters. The CDF $F(x)$ of the New Exponentiated G-family (NEG) of distributions is defined as:

$$F(x;\ \alpha,\ \psi) = \frac{(1+G(x;\ \psi))^{\alpha}-1}{2^{\alpha}-1},\ \ \alpha > 0,\ \ x \in \mathbb{R}, \tag{2}$$

where α is the shape parameter. By differentiating the CDF in equation (2), the PDF $f(x)$ of the family is obtained as:

$$f(x;\ \alpha,\ \psi) = \frac{\alpha(1+G(x;\ \psi))^{\alpha-1}}{2^{\alpha}-1} g(x;\ \psi);\ \ \alpha > 0,\ \ x \in R. \tag{3}$$

The corresponding survival/reliability $R(x)$ and hazard $H(x)$ functions are respectively given by

$$R(x;\ \alpha,\ \psi) = 1 - \left[\frac{(1+G(x;\ \psi))^{\alpha}-1}{2^{\alpha}-1}\right];\ \ \alpha > 0,\ \ x \in R. \tag{4}$$

and

$$H(x;\ \alpha,\ \psi) = \frac{\alpha(1+G(x;\ \psi))^{\alpha-1} g(x;\ \psi)}{2^{\alpha}-1}\left[1-\left\{\frac{(1+G(x;\ \psi))^{\alpha}-1}{2^{\alpha}-1}\right\}\right]^{-1};\ \ \alpha > 0,\ \ x \in R. \tag{5}$$

2.1 *Quantile function*

For any p in the range (0, 1), the p-th quantile function, denoted by $Q(p)$, for the NEG family is defined as the value $Q(p)$ that satisfies $F(Q(p)) = p$, with the condition that $Q(p) > 0$. This function can be expressed in a simplified form as:

$$Q_x(p) = G^{-1}\left[\left\{(p(2^\alpha - 1) + 1)^{1/\alpha} - 1\right\}\right]; \quad p \in (0, 1),$$

where G^{-1} refers to the baseline quantile function. By setting $p = 0.5$, we can find the median of the NEG family. The impact of the shape parameters on skewness and kurtosis can be investigated through $Q(p)$. The random deviates can be generated by:

$$x = G^{-1}\left[\left\{(w(2^\alpha - 1) + 1)^{1/\alpha} - 1\right\}\right]; \quad w \in (0, 1).$$

2.2 *Linear form of new exponentiated-G family of distribution*

The linear form of the PDF is crucial for deriving the statistical properties of the NEG family of distributions. By applying a Taylor series expansion, the CDF of the NEG, as presented in equation (2), transforms into

$$F(x; \alpha, \psi) = \frac{1}{2^\alpha - 1}\left\{\sum_{i=0}^{\infty}\binom{\alpha}{i} G^i(x; \psi) - 1\right\}; \quad x \in R. \tag{6}$$

Since the base CDF $G(x)$ is well defined and valid CDF $F(x)$ holds the property of continuity and differentiability hence by differentiating the CDF in equation (6), the PDF $f(x)$ of the family is obtained as

$$f(x; \alpha, \psi) = \frac{1}{2^\alpha - 1}\sum_{i=0}^{\infty}\binom{\alpha}{i} iG^{i-1}(x; \psi)g(x; \psi); \ x \in R. \tag{7}$$

3. Special member

3.1 *A New Exponentiated Inverse Weibull (NEIW) distribution*

The CDF and PDF of the baseline Inverse Weibull distribution are given by:

$$G(x; \beta, \delta) = e^{-\beta x^{-\delta}} \quad \text{where} \quad \beta > 0, \ \delta > 0, \ x > 0,$$

and

$$g(x; \beta, \delta) = \beta\delta x^{-(\delta+1)} e^{-\beta x^{-\delta}} \quad \text{where} \quad \beta > 0, \ \delta > 0, \ x > 0.$$

By substituting $G(x; \beta, \delta)$ and $g(x; \beta, \delta)$ into equations (2), (3), (4), and (5), we derive the CDF, PDF, reliability function, and hazard function of the NEIW distribution. The CDF and PDF of the NEIW distribution are expressed as follows:

$$F(x;\,\alpha,\,\beta,\,\delta)=\frac{\left(1+e^{-\beta x^{-\delta}}\right)^{\alpha}-1}{2^{\alpha}-1} \quad \text{where} \quad \alpha>0,\ \beta>0,\ \delta>0,\ x>0.$$

$$f(x;\,\alpha,\,\beta,\,\delta)=\frac{\alpha\beta\delta}{2^{\alpha}-1}\left(1+e^{-\beta x^{-\delta}}\right)^{\alpha-1}e^{-\beta x^{-\delta}}x^{-(\delta+1)} \quad \text{where} \quad \alpha>0,\ \beta>0,\ \delta>0,\ x>0.$$

The reliability and hazard functions can be presented as:

$$R(x;\,\alpha,\,\beta,\,\delta)=1-F(x)=1-\left\{\frac{\left(1+e^{-\beta x^{-\delta}}\right)^{\alpha}-1}{2^{\alpha}-1}\right\} \quad \text{for} \quad x>0,$$

and

$$H(x;\,\alpha,\,\beta,\,\delta)=\frac{\alpha\beta\delta}{2^{\alpha}-1}\left(1+e^{-\beta x^{-\delta}}\right)^{\alpha-1}e^{-\beta x^{-\delta}}x^{-(\delta+1)}\left\{\frac{2^{\alpha}-\left(1+e^{-\beta x^{-\delta}}\right)^{\alpha}}{2^{\alpha}-1}\right\}^{-1} \quad \text{for} \quad x>0.$$

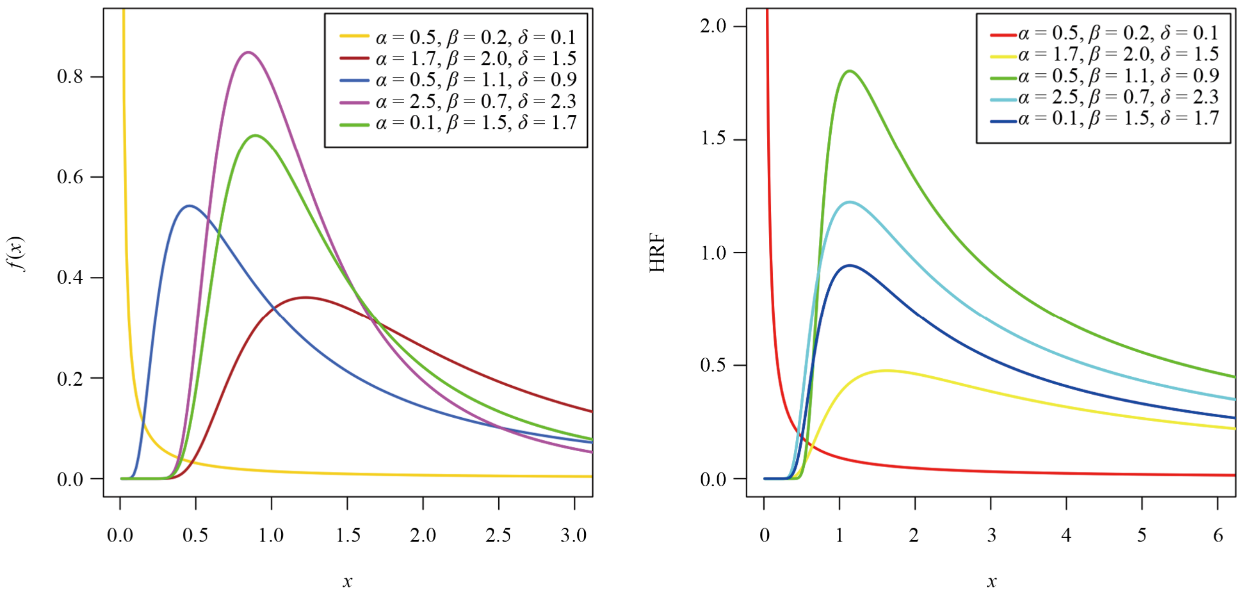


Figure 1. Possible shapes of PDF and HRF of NEIW

The NEIW distribution exhibits a variety of shapes for its density function, such as increasing, decreasing, unimodal, and right-skewed forms. Depending on the parameter values, it can display both decreasing and increasing failure rates, as shown in Figure 1. The quantile function $Q_X(p)$ and random deviate for the NEIW distribution are given by

$$Q_X(p)=\left\{-\beta\left\{\log\left\{\{p(2^\alpha-1)+1\}^{1/\alpha}-1\right\}\right\}^{-1}\right\}^{1/\delta};\quad 0<p<1, \tag{8}$$

and

$$x=\left\{-\beta\left\{\log\left\{\{u(2^\alpha-1)+1\}^{1/\alpha}-1\right\}\right\}^{-1}\right\}^{1/\delta};\quad 0<u<1. \tag{9}$$

3.2 *The Linear form of NEIW distribution*

The Linear form of NEIW distribution The linear form of CDF and PDF of NEIW distribution are given respectively

$$F(x;\alpha,\beta,\delta)=\frac{1}{2^\alpha-1}\left\{\sum_{i=0}^{\infty}\binom{\alpha}{i}e^{-i\beta x^{-\delta}}-1\right\};\quad x>0.$$

and

$$f(x;\alpha,\beta,\delta)=\sum_{i=0}^{\infty}\Delta_i e^{-i\beta x^{-\delta}}x^{-(\delta+1)};\quad x>0$$

where $\Delta_i=\dfrac{\beta\delta i}{2^\alpha-1}\binom{\alpha}{i}$.

4. Statistical properties of NEIW distribution

4.1 *Moments*

The rth non-central moment, mean, and variance for the NEIW distributions are given respectively as

$$E[X^r]=\sum_{i=0}^{\infty}\Delta_i\int_0^{\infty}x^{r-\delta-1}e^{-i\beta x^{-\delta}}dx=\sum_{i=0}^{\infty}\Delta_i\int_0^{\infty}t^{-\frac{r}{\delta}+1-1}e^{-i\beta t}dt;\quad \text{where}\quad x^{-\delta}=t$$

$$=\sum_{i=0}^{\infty}\Delta_i\frac{\delta^{-1}\Gamma\left(-\frac{r}{\delta}+1\right)}{\{i\beta\}^{-\frac{r}{\delta}+1}}=\sum_{i=0}^{\infty}\Delta_i\frac{\delta^{-1}\Gamma\left(\frac{\delta-r}{\delta}\right)}{\{i\beta\}^{\frac{\delta-r}{\delta}}};\quad \delta>r. \tag{10}$$

Now, the first four raw moments are given by:

$$E[X]=\sum_{i=0}^{\infty}\Delta_i\frac{\delta^{-1}\Gamma\left(\frac{\delta-1}{\delta}\right)}{\{i\beta\}^{\frac{\delta-1}{\delta}}};\quad \delta>1,$$

$$E[X^2]=\sum_{i=0}^{\infty}\Delta_i\frac{\delta^{-1}\Gamma\left(\frac{\delta-2}{\delta}\right)}{\{i\beta\}^{\frac{\delta-2}{\delta}}};\quad \delta>2,$$

$$E[X^3]=\sum_{i=0}^{\infty}\Delta_i\frac{\delta^{-1}\Gamma\left(\frac{\delta-3}{\delta}\right)}{\{i\beta\}^{\frac{\delta-3}{\delta}}};\quad \delta>3,$$

$$E[X^4]=\sum_{i=0}^{\infty}\Delta_i\frac{\delta^{-1}\Gamma\left(\frac{\delta-4}{\delta}\right)}{\{i\beta\}^{\frac{\delta-4}{\delta}}};\quad \delta>4.$$

The first four central moments are:

$$\mu_1=0,$$

$$\mu_2=E\left[X^2\right]-[E(X)]^2,$$

$$\mu_3=E\left[X^3\right]-3E[X]E\left[X^2\right]+2(E[X])^3,$$

$$\mu_4=E\left[X^4\right]-4E[X]E\left[X^3\right]+6(E[X])^2E\left[X^2\right]-3(E[X])^4.$$

Table 1 presents the mean, variance, the third and fourth moments for various combinations of parameter values.

4.2 *Moment generating function (MGF)*

The moment-generating function $M_X(t)$ for the NEIW distributions is given by

$$M_X(t)=\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\Delta_i\frac{t^k}{k!}\int_0^{\infty}x^{r-(\delta+1)}e^{-i\beta x^{-\delta}}\,dx$$

$$=\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\Delta_i\frac{t^k}{k!}\int_0^{\infty}t^{-\frac{r}{\delta}+1-1}e^{-i\beta t}\,dt$$

$$= \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\Delta_i \frac{t^k}{k!}\frac{\delta^{-1}\Gamma\left(-\frac{r}{\delta}+1\right)}{(i\beta)^{-\frac{r}{\delta}+1}}$$

$$= \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\Delta_i \frac{t^k}{k!}\frac{\delta^{-1}\Gamma\left(\frac{\delta-r}{\delta}\right)}{(i\beta)^{\frac{\delta-r}{\delta}}}, \quad \delta > r.$$

Table 1. Computed mean, variance, third and fourth moments for various values of α, β, and δ

No.	α	β	δ	Mean	Variance	μ_3	μ_4
1	0.25	0.5	4.2	0.9759	0.1350	0.2295	4.4154
2	0.25	0.5	2.75	1.0120	0.5820	-5.5585	18.8247
3	0.25	0.5	2.5	1.0310	0.9606	-4.8369	15.2444
4	0.25	1.2	4.2	1.2021	0.2048	0.4289	10.1643
5	0.25	1.2	2.75	1.3913	1.1001	-14.4455	67.2611
6	0.25	1.2	2.5	1.4633	1.9350	-13.8298	61.8656
7	0.25	1.75	4.2	1.3151	0.2451	0.5616	14.5590
8	0.25	1.75	2.75	1.5960	1.4474	-21.8014	116.4396
9	0.25	1.75	2.5	1.7017	2.6168	-21.7493	113.1410
10	0.5	0.5	4.2	0.9948	0.1490	0.2846	5.8319
11	0.5	0.5	2.75	1.0445	0.6875	-7.5971	25.4120
12	0.5	0.5	2.5	1.0686	1.1638	-6.8025	19.5649
13	0.5	1.2	4.2	1.2254	0.2261	0.5318	13.4250
14	0.5	1.2	2.75	1.4360	1.2995	-19.7436	90.7973
15	0.5	1.2	2.5	1.5167	2.3446	-19.4501	79.3993
16	0.5	1.75	4.2	1.3405	0.2706	0.6963	19.2295
17	0.5	1.75	2.75	1.6472	1.7098	-29.7974	157.1847
18	0.5	1.75	2.5	1.7638	3.1706	-30.5878	145.2069
19	1.2	0.5	4.2	1.0380	0.1712	0.3467	7.4933
20	1.2	0.5	2.75	1.1175	0.8327	-10.0552	35.7012
21	1.2	0.5	2.5	1.1522	1.4336	-9.2226	27.9517
22	1.2	1.2	4.2	1.2786	0.2597	0.6479	17.2497
23	1.2	1.2	2.75	1.5364	1.5741	-26.1317	127.5608
24	1.2	1.2	2.5	1.6354	2.8880	-26.3698	113.4347
25	1.2	1.75	4.2	1.3987	0.3109	0.8483	24.7079
26	1.2	1.75	2.75	1.7624	2.0711	-39.4385	220.8281
27	1.2	1.75	2.5	1.9018	3.9056	-41.4700	207.4516

5. Statistical inference

5.1 *Method of estimation*

Let $X_i \sim NEIW(x; \alpha, \beta, \delta)$, $i = 1, \ldots, n$, then their log density and log-likelihood function are respectively given by

$$\log L = \log\left(\frac{\alpha\beta\delta}{2^{\alpha}-1}\right) + (\alpha-1)\log(1+e^{-\beta x^{-\delta}}) - (\delta+1)\log(x) - \beta x^{-\delta},$$

and

$$\ell(\underline{x};\, \alpha,\, \beta,\, \delta) = n\log\left(\frac{\alpha\beta\delta}{2^{\alpha}-1}\right) + (\alpha-1)\sum_{i=1}^{n}\log(1+e^{-\beta x_i^{-\delta}}) - (\delta+1)\sum_{i=1}^{n}\log(x_i) - \beta\sum_{i=1}^{n}x_i^{-\delta}. \tag{11}$$

Differentiating equation (11) with respect to α, β, and δ, taken equal to zero, we have obtained $\dfrac{\partial \ell}{\partial \alpha} = 0$, $\dfrac{\partial \ell}{\partial \beta} = 0$ and $\dfrac{\partial \ell}{\partial \delta} = 0$. To solve these non-linear equations for α, β, and δ we used the `maxLik` package [17] in R-software [18].

5.2 *Confidence interval for large sample*

Under certain conditions, the first derivative of the logarithm of the likelihood function with respect to parameter θ viz., $\dfrac{\partial}{\partial \theta}\log L$, is asymptotically normally distributed with mean zero and variance is given by:

$$Var\left(\frac{\partial}{\partial \theta}\log L\right) = E\left(\frac{\partial}{\partial \theta}\log L\right)^2 = -E\left(\frac{\partial^2}{\partial \theta^2}\log L\right).$$

Hence for large n,

$$Z = \frac{\dfrac{\partial}{\partial \theta}\log L}{\sqrt{Var\left(\dfrac{\partial}{\partial \theta}\log L\right)}} \sim N(0,\, 1).$$

The result allows us to determine a confidence interval for the parameter θ in a large sample. Consequently, for a large sample, the confidence interval for θ with a confidence level of $(1-c)\%$ is derived by transforming the inequalities into

$$P(|Z| \leq \tau_c) = 1 - c$$

where τ_c is given by

$$\frac{1}{\sqrt{2\pi}}\int_{-\tau_c}^{\tau_c} \exp\left(-t^2/2\right)dt = 1 - c.$$

Thus Confidence interval for α, β, and δ are given by $\widehat{\alpha} \pm SE(\widehat{\alpha})$, $\widehat{\beta} \pm SE\left(\widehat{\beta}\right)$ and $\widehat{\delta} \pm SE\left(\widehat{\delta}\right)$ at the confidence coefficient $(1-c)\%$.

6. Simulation

Using the `maxLik` R package introduced by [17], we generated samples from the quantile function defined in equation (8) for various parameter combinations of the NEIW distribution. We then calculated the Maximum Likelihood Estimates (MLEs) for each sample using the `maxLik()` function with the BFGS algorithm. This approach allows us to assess parameter estimation issues such as the sharpness or flatness of the likelihood function, as well as to estimate the bias in the MLEs, including their size and direction (underestimate or overestimate).

The simulation used sample sizes of 300, 350, 400, 450, 500, 550, 600, 650, and 700, repeated 10,000 times. We computed the average estimate value, bias, and mean square error (MSE). The initial values of the model parameters are listed in Table 2. The results are summarized in Tables 3, 4, 5, and 6, which display the bias (B_α, B_β, B_δ), MSEs (M_α, M_β, M_δ), and confidence intervals (CI) for each parameter. Our findings indicate that the MLE method effectively estimates the parameters α, β, and δ of the proposed model.

Table 2. Initial values taken for simulation study

Table	Initial value
Table 3	$(\alpha = 1.5,\ \beta = 0.5,\ \delta = 0.75)$
Table 4	$(\alpha = 2.5,\ \beta = 1.5,\ \delta = 0.5)$
Table 5	$(\alpha = 5.5,\ \beta = 1.75,\ \delta = 1.5)$
Table 6	$(\alpha = 3.5,\ \beta = 1.25,\ \delta = 0.25)$

Table 3. Simulation study of the parameters estimated using MLE

Sample	B_α	B_β	B_δ	M_α	M_β	M_δ	CI_α	CI_β	CI_δ
300	0.2785	0.0052	-0.002	3.3893	0.0248	0.0031	(0.00 7.1352)	(0.1783 0.7664)	(0.6466 0.8559)
350	0.1987	0.0087	-0.0023	2.6709	0.0216	0.0028	(0.00 5.0333)	(0.2250 0.7625)	(0.6500 0.8475)
400	0.2145	0.0034	-5.0E-04	2.6695	0.0210	0.0026	(0.00 5.0585)	(0.2265 0.7490)	(0.6580 0.8422)
450	0.1722	0.0076	-0.0023	2.4724	0.0196	0.0026	(0.00 4.8287)	(0.2384 0.7516)	(0.6564 0.8449)
500	0.1512	0.0066	-0.0013	2.0242	0.0184	0.0023	(0.00 4.596)	(0.2479 0.7542)	(0.6591 0.8388)
550	0.1863	0.0038	-4.0E-04	2.4539	0.0186	0.0022	(0.00 4.7345)	(0.2478 0.7418)	(0.6642 0.8355)
600	0.1682	0.0022	2.0E-04	1.8645	0.0159	0.002	(0.00 4.1002)	(0.2710 0.7366)	(0.6651 0.8334)
650	0.1186	0.0069	-0.0018	1.705	0.0165	0.0021	(0.00 4.2325)	(0.2682 0.7391)	(0.6655 0.8348)
700	0.1188	0.0031	-9.0E-04	1.4507	0.0139	0.0018	(0.00 3.6426)	(0.3051 0.7321)	(0.6677 0.8253)

Table 4. Simulation study of the parameters estimated using MLE

Sample	B_α	B_β	B_δ	M_α	M_β	M_δ	CI_α	CI_β	CI_δ
300	0.1942	0.0323	-0.0051	4.1715	0.1973	0.0012	(0.0000 8.787)	(0.5390 2.3583)	(0.4235 0.5565)
350	0.2305	0.0192	-0.005	4.1753	0.1923	0.0011	(0.0000 8.8032)	(0.5508 2.3466)	(0.4240 0.5541)
400	0.2093	0.0078	-0.0042	3.2446	0.1699	9.0E-04	(0.0000 7.6299)	(0.6235 2.3093)	(0.4335 0.5538)
450	0.1667	0.0193	-0.0052	3.4043	0.162	9.0E-04	(0.0000 7.4655)	(0.6032 2.3050)	(0.4309 0.5486)
500	0.1178	0.0213	-0.0041	2.5324	0.1386	8.0E-04	(9.0E-04 6.4729)	(0.7613 2.2847)	(0.4369 0.5470)
550	0.1211	0.0164	-0.0041	2.4403	0.1294	7.0E-04	(0.0064 5.8389)	(0.8090 2.2413)	(0.4368 0.5414)
600	0.1265	0.0168	-0.0036	2.4301	0.1212	7.0E-04	(0.1465 5.9763)	(0.8108 2.2479)	(0.4389 0.5397)
650	0.1774	7.0E-04	-0.0029	2.4802	0.1127	6.0E-04	(0.1779 5.8845)	(0.8227 2.1821)	(0.4381 0.5385)
700	0.1739	0.0062	-0.0034	2.6574	0.1151	6.0E-04	(0.4146 6.4350)	(0.7504 2.1480)	(0.4399 0.5368)

Table 5. Simulation study of the parameters estimated using MLE

Sample	B_α	B_β	B_δ	M_α	M_β	M_δ	CI_α	CI_β	CI_δ
300	-0.2355	0.3142	-0.0198	9.8327	0.7454	0.0064	(1.0516 12.4549)	(0.7034 3.7979)	(1.3167 1.6216)
350	-0.2178	0.3155	-0.0170	9.4709	0.7478	0.0053	(1.0307 12.4795)	(0.7018 3.9130)	(1.3341 1.6146)
400	-0.3193	0.3066	-0.0150	6.9963	0.6561	0.0047	(1.3797 11.9078)	(0.7190 3.6473)	(1.3564 1.6174)
450	-0.0685	0.2298	-0.0161	6.794	0.5892	0.0041	(1.5539 11.6766)	(0.7637 3.6052)	(1.3554 1.5985)
500	-0.0365	0.2053	-0.0150	6.6153	0.5549	0.0039	(1.6539 11.4443)	(0.7820 3.5652)	(1.3644 1.5986)
550	-0.1129	0.2253	-0.0164	6.2770	0.5531	0.0035	(1.6103 11.8952)	(0.7226 3.4749)	(1.3695 1.5915)
600	-0.0473	0.1805	-0.0126	5.5848	0.4597	0.0032	(2.0034 11.0979)	(0.8255 3.2740)	(1.3739 1.5890)
650	-0.1453	0.2028	-0.0131	5.0647	0.4665	0.0030	(1.8341 11.287)	(0.7890 3.3238)	(1.3809 1.5855)
700	-0.0442	0.1687	-0.0122	4.9512	0.4184	0.0026	(2.1161 10.8385)	(0.8074 3.2140)	(1.3898 1.5832)

Table 6. Simulation study of the parameters estimated using MLE

Sample	B_α	B_β	B_δ	M_α	M_β	M_δ	CI_α	CI_β	CI_δ
300	-0.0078	0.1025	-0.005	4.8588	0.2139	3.0E-04	(4.0E-04 9.4469)	(0.4863 2.2654)	(0.2111 0.2741)
350	0.0385	0.0717	-0.0031	3.7313	0.1746	2.0E-04	(0.1793 8.6547)	(0.5434 2.2371)	(0.2125 0.2714)
400	0.1577	0.0507	-0.0030	4.0553	0.1691	2.0E-04	(0.1009 9.4105)	(0.5098 2.2215)	(0.2147 0.2706)
450	0.1394	0.0461	-0.0028	3.7846	0.1496	2.0E-04	(0.7877 9.1854)	(0.5206 2.0792)	(0.2199 0.2692)
500	0.0852	0.0455	-0.0022	3.1249	0.1343	1.0E-04	(0.6498 8.3738)	(0.5613 2.0867)	(0.2203 0.2681)
550	0.1613	0.0267	-0.0020	2.9974	0.1196	1.0E-04	(1.0656 8.2249)	(0.5752 1.9647)	(0.2245 0.2686)
600	0.1489	0.0344	-0.0024	3.3097	0.1289	1.0E-04	(0.9011 8.3526)	(0.5559 2.0331)	(0.2221 0.2670)
650	0.1200	0.0383	-0.0017	3.0897	0.1246	1.0E-04	(1.0449 8.7353)	(0.5362 2.0008)	(0.2257 0.2675)
700	0.0597	0.0499	-0.0020	2.9550	0.1213	1.0E-04	(0.9875 8.4402)	(0.5509 1.9903)	(0.2248 0.2663)

7. Application

7.1 *Data set*

The real data set, sourced from Gross and Clark [19], provides the relief times of 20 patients receiving an analgesic. The data are as follows: "1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3, 1.7, 2.3, 1.6, 2.0".

7.2 *Model analysis*

To analyze the data set, we calculated several well-known goodness-of-fit statistics and evaluated the fitted models using the Akaike information criterion (AIC), log-likelihood value (-2logL), Hannan-Quinn information criterion (HQIC), Kolmogorov-Smirnov (KS), Anderson-Darling (AD), with p-values, and Cramér-von Mises (CVM). All computations were performed using R software. For comparison, we selected several models: Inverse Weibull (IW), KM-transformation of IW (KMIW) [20], Exponentiated Exponential IW (EEIW) [21], Lindley Weibull (LW) [22], and Exponentiated Weibull (EW) [2].

Table 7 presents the estimated parameter values and their associated standard errors (SE) for the models under study, obtained using the MLE method for the relief time data. Table 8 displays the model selection and goodness-of-fit statistics, including log-likelihood, HQIC, AIC, KS, AD, and CVM, for the data set. It was observed that the suggested model had the lowest statistics compared to IW, KMIW, EEIW, LW, and EW, indicating that NEIW is more flexible and provides a better fit. Graphical illustrations of the fitted models are shown in Figure 2, along with probability-probability plots for all models under study (See Figure 3). These figures confirm that the NEIW model performs well compared to the other candidate models.

Table 7. Estimated parameters using MLE method along with SE

Model	Parameter	SE	Parameter	SE	Parameter	SE	Parameter	SE
NEIW	2.3871	5.4224	5.1682	4.2398	4.3152	1.1347	-	-
IW	4.0175	0.7060	6.0224	2.0083	-	-	-	-
KMIW	3.5562	0.6375	1.6907	0.1093	-	-	-	-
EEIW	1.6530	5.3178	3.4486	3.9779	1.2227	1.9090	3.9199	14.5563
LW	9.2825	21.6417	2.0201	0.3020	0.0053	0.0241	-	-
EW	138.1805	14.7326	0.7415	0.1190	3.4746	0.2403	-	-

Table 8. Model fit and selection statistics

Model	-2logL	AIC	HQIC	KS	p (KS)	CVM	p (CVM)	AD	p (AD)
NEIW	30.7559	36.7559	37.3391	0.0924	0.9956	0.0249	0.9914	0.1493	0.9988
IW	30.8174	34.8174	35.2062	0.1020	0.9854	0.0266	0.9880	0.1545	0.9984
KMIW	30.8867	34.8867	35.2754	1.0000	0.0000	0.1272	0.4705	2.6801	0.0405
EEIW	30.8034	38.8034	39.5809	0.0974	0.9914	0.0257	0.9899	0.1512	0.9986
LW	38.6683	44.6683	45.2514	0.1811	0.5282	0.1456	0.4060	0.8604	0.4380
EW	31.7197	37.7197	38.3028	0.1243	0.9168	0.0396	0.9395	0.2396	0.9756

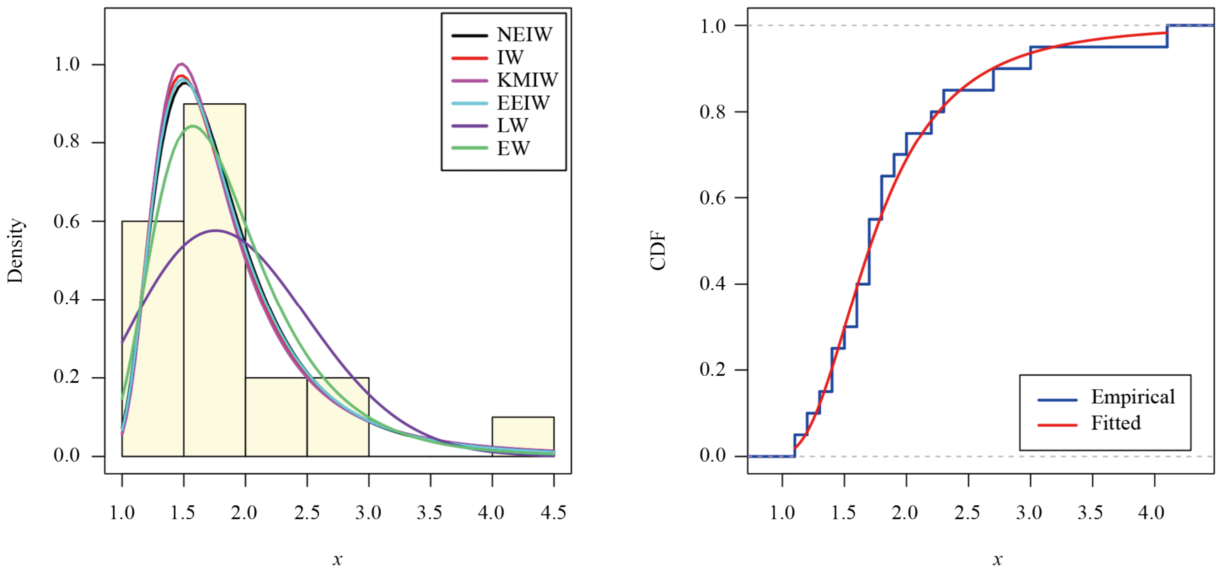


Figure 2. PDF and CDF fit of the models under study

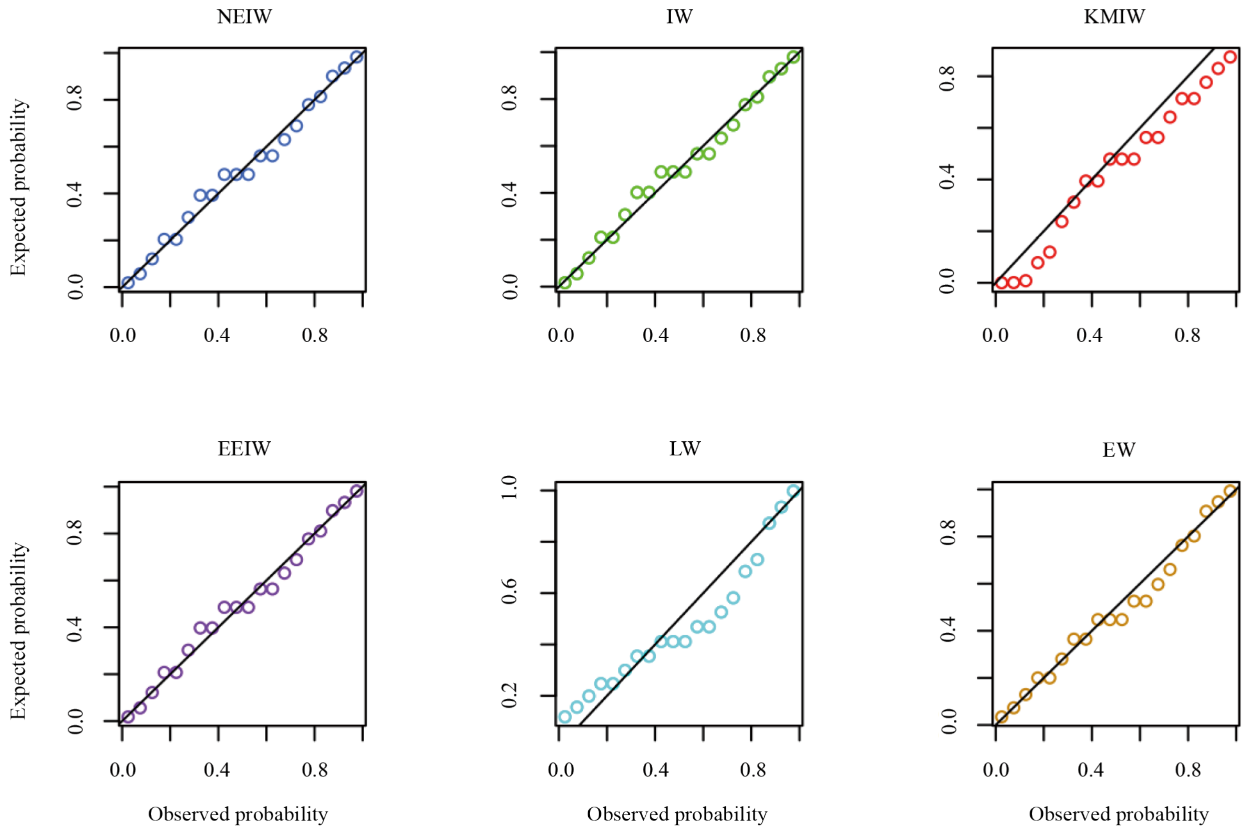


Figure 3. PP plot of the models under study

8. Bayesian analysis

Bayesian inference encompasses the procedure of adjusting a probability model to a provided dataset and summarizing the result using a probability distribution on the model parameters, referred to as the posterior distribution. In this section, we have used the dataset presented in the application section. From a Bayesian viewpoint, both the observed variables (data) and the parameters are treated as stochastic variables [23]. In this study, we have taken the gamma prior along with the hyper-parameter as $\alpha \sim gamma(10,\ 100)$, $\beta \sim gamma(790,\ 800)$ and $\delta \sim gamma(880,\ 900)$. Hyper-parameters were selected based on prior and posterior density plots (see Figure 4). We have used the STAN software a probabilistic programming language for Bayesian analysis. The application of STAN and the Hamiltonian Monte Carlo (HMC) algorithm has provided valuable insights into the posterior distribution of model parameters in our study [13]. 10,000 samples were generated for each chain (A total of 40,000 for all four chains) using the HMC algorithm and No-U-Turn sampling (NUTS) [24]. By default, STAN uses 20,000 samples as warm-up samples and 20,000 samples were used in this study.

Assuming observed data $\underline{x} = (x_1,\ \dots,\ x_n)$ and a parameter ψ, the connection between x and the prior distribution $h(\psi)$ is expressed by means of the likelihood function $L(\underline{x}|\psi)$, given as:

$$L(\underline{x}|\alpha,\ \beta,\ \delta) = \left(\frac{\alpha\beta\delta}{2^\alpha - 1}\right)^n \prod_{i=1}^{n} (1 + e^{-\beta x_i^{-\delta}})^{\alpha-1} e^{-\beta x_i^{-\delta}} x_i^{-(\delta+1)}.$$

The joint distribution of $\underline{x} = (x_1, ..., x_n)$ and $\psi = (\alpha, \beta, \delta)$ can therefore be represented as the product of the likelihood and the prior distribution.

$$g(\underline{x};\, \alpha,\, \beta,\, \delta) = \left\{ \left(\frac{\alpha\beta\delta}{2^{\alpha}-1} \right)^{n} \prod_{i=1}^{n} (1+e^{-\beta x_i^{-\delta}})^{\alpha-1} e^{-\beta x_i^{-\delta}} x_i^{-(\delta+1)} \right\}$$

$$\times \left\{ \frac{a^b}{\Gamma(b)} e^{-a\alpha} \alpha^{b-1} \right\} \left\{ \frac{c^d}{\Gamma(d)} e^{-c\beta} \beta^{d-1} \right\} \left\{ \frac{l^m}{\Gamma(m)} e^{-l\delta} \delta^{m-1} \right\}.$$

By applying Bayes' Theorem, one can update the distribution of $\psi = (\alpha, \beta, \delta)$ based on the information provided by the sample $\underline{x} = (x_1, ..., x_n)$. This yields the posterior distribution of $\psi = (\alpha, \beta, \delta)$, given by:

Posterior $\propto$ likelihood $\times$ prior.

$$f(\alpha,\, \beta,\, \delta|\underline{x}) \propto \left\{ \begin{array}{c} \left\{ \left(\frac{\alpha\beta\delta}{2^{\alpha}-1} \right)^{n} \prod_{i=1}^{n} (1+e^{-\beta x_i^{-\delta}})^{\alpha-1} e^{-\beta x_i^{-\delta}} x_i^{-(\delta+1)} \right\} \\ \times \left\{ \frac{a^b}{\Gamma(b)} e^{-a\alpha} \alpha^{b-1} \right\} \left\{ \frac{c^d}{\Gamma(d)} e^{-c\beta} \beta^{d-1} \right\} \left\{ \frac{l^m}{\Gamma(m)} e^{-l\delta} \delta^{m-1} \right\} \end{array} \right\}.$$

which can be interpreted as the proportional relationship between the posterior distribution and the product of the likelihood and the prior.

The full conditional density of parameter α is the term containing α in posterior distribution $f(\alpha, \beta, \delta|\underline{x})$ is given by:

$$f_1(\alpha|\underline{x},\, \beta,\, \delta) \propto \left(\frac{1}{2^{\alpha}-1} \right)^{n} e^{-a\alpha} \alpha^{b+n-1} \left\{ \prod_{i=1}^{n} (1+e^{-\beta x_i^{-\delta}})^{\alpha-1} e^{-\beta x_i^{-\delta}} x_i^{-(\delta+1)} \right\}.$$

The full conditional density of parameter β is the term containing β in posterior distribution $f(\alpha, \beta, \delta|\underline{x})$ is given by:

$$f_2(\beta|\underline{x},\, \alpha,\, \delta) \propto e^{-c\beta} \beta^{d+n-1} \left\{ \prod_{i=1}^{n} (1+e^{-\beta x_i^{-\delta}})^{\alpha-1} e^{-\beta x_i^{-\delta}} x_i^{-(\delta+1)} \right\}.$$

The full conditional density of parameter δ is the term containing β in posterior distribution $f(\alpha, \beta, \delta|\underline{x})$ is given by:

$$f_3(\delta|\underline{x},\, \alpha,\, \beta) \propto e^{-l\delta} \delta^{m+n-1} \left\{ \prod_{i=1}^{n} (1+e^{-\beta x_i^{-\delta}})^{\alpha-1} e^{-\beta x_i^{-\delta}} x_i^{-(\delta+1)} \right\}.$$

Given the posterior's complexity, it doesn't seem possible to draw any near-form conclusions. To facilitate sample-based on inferences, we propose using MCMC methods to generate samples from the posterior distribution. An effectively constructed Markov chain in Monte Carlo is employed to produce samples; over time, this chain converges to the target distribution, known as the stationary or equilibrium distribution, which corresponds to our posterior distribution.

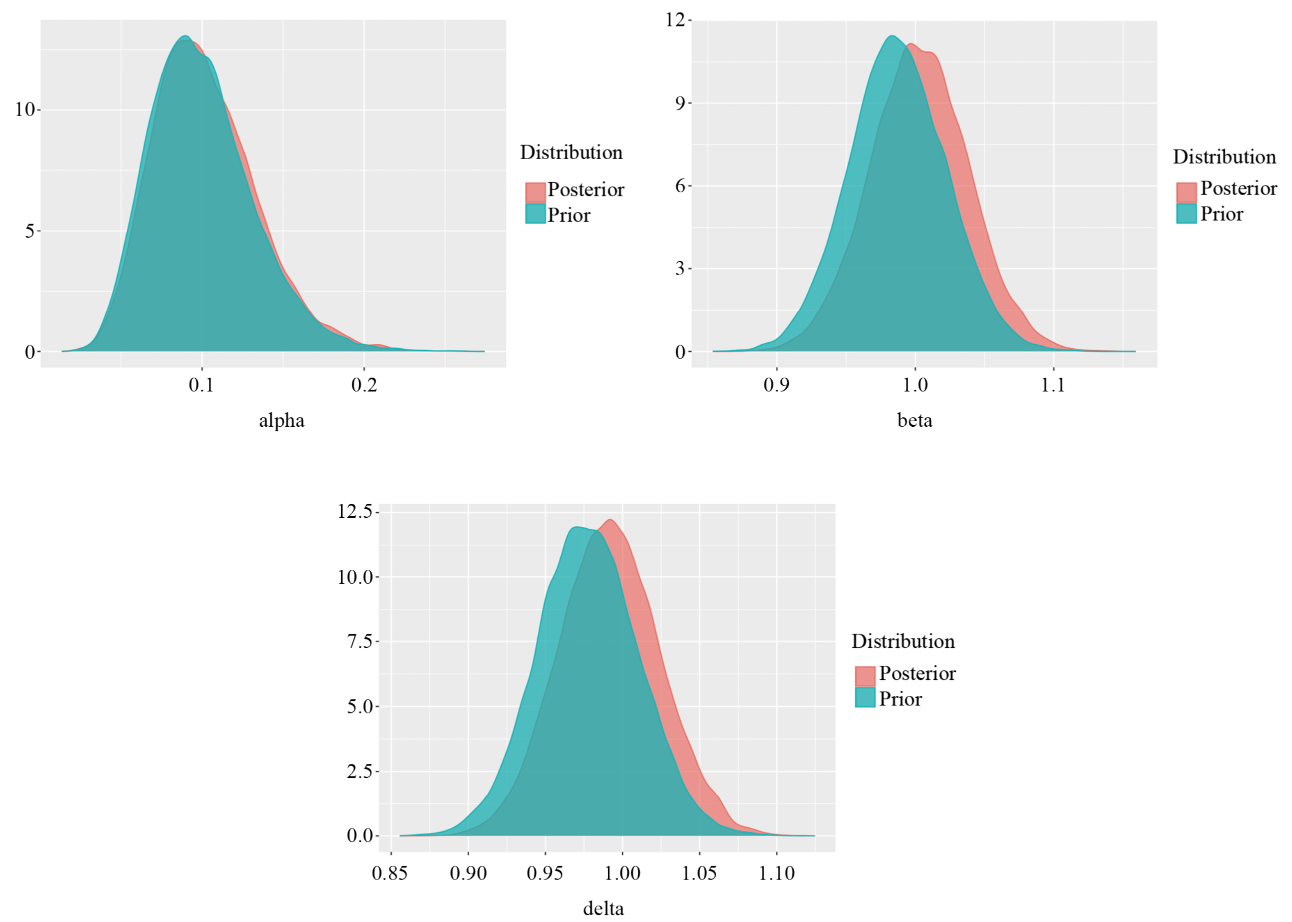


Figure 4. Prior and posterior density plots of α, β, and δ

8.1 *Sampling information*

The results presented in Table 9 were obtained through the application of the STAN probabilistic programming language to our dataset. STAN employs the HMC algorithm, a powerful method for Bayesian inference, to estimate the posterior distributions of model parameters [15, 16, 25].

• Acceptance Statistics: The acceptance statistic, representing the proportion of proposed samples that are accepted during the Markov chain Monte Carlo (MCMC) sampling process, indicates the efficiency of the sampler in exploring the parameter space. A high acceptance rate suggests that the sampler is effectively traversing the space of possible parameter values.

• Step Size: The step size parameter regulates the magnitude of the proposed changes to the parameter values during each iteration of the MCMC algorithm. An optimal step size as observed in Table 9 ensures efficient exploration of the parameter space while minimizing the autocorrelation between samples (See Figure 5).

• Tree Depth and Leapfrog Steps: In the context of STAN's implementation of the HMC algorithm, the tree depth and the number of leapfrog steps are related to the algorithm's ability to construct trajectories through the parameter space. A deeper tree and a larger number of leapfrog steps allow for more complex trajectories, potentially improving the exploration of the posterior distribution.

• Divergent Transitions: Divergent transitions indicate no potential issues with the sampler's performance, such as inadequate step size or insufficient adaptation of other tuning parameters.

• Energy: The energy value, derived from the Hamiltonian function, serves as a diagnostic measure of the overall goodness-of-fit of the model to the data. Deviations from expected energy levels may suggest problems with model convergence or discrepancies between the model and the observed data.

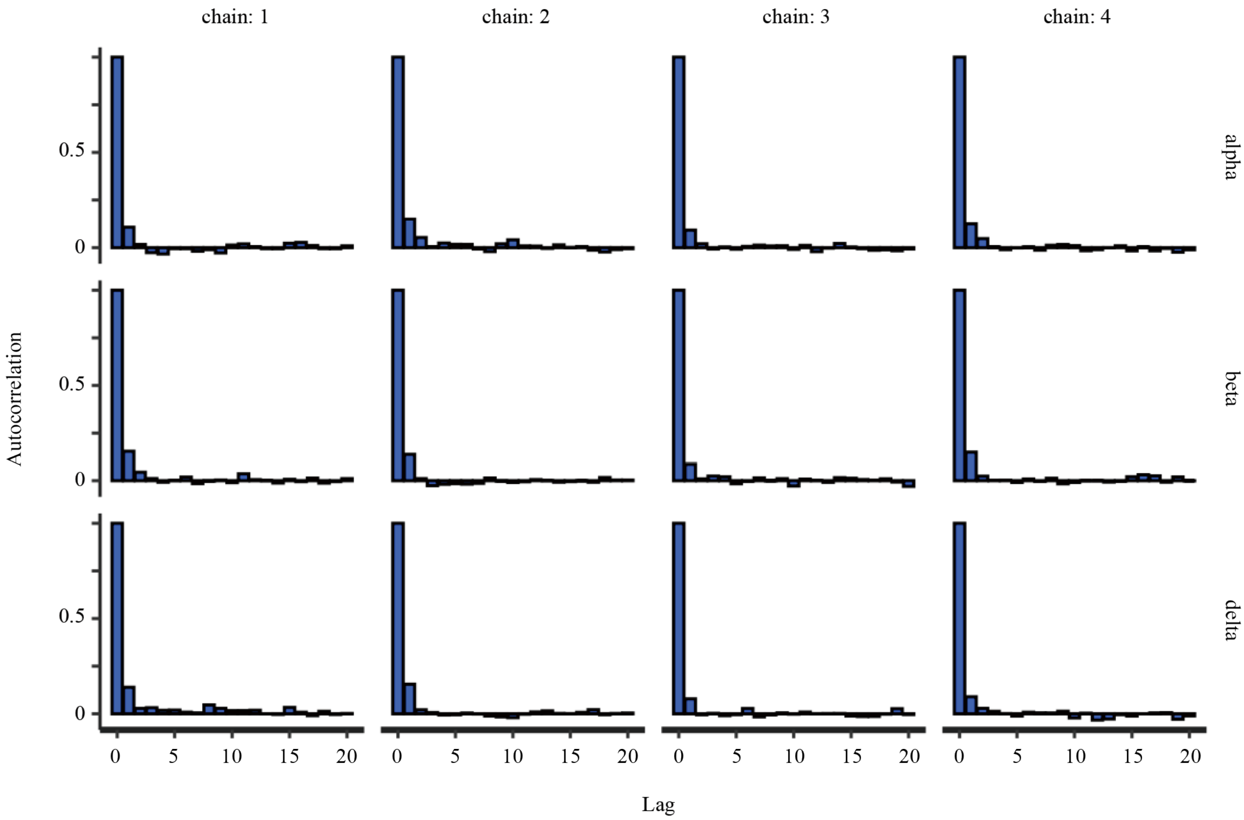


Figure 5. Auto-correlation plot for all chains for the parameters α, β, and δ

Overall, the results demonstrate the favorable performance of the STAN sampler in exploring the posterior distribution of model parameters. The high acceptance rate and absence of divergent transitions indicate efficient sampling, while reasonable values of step size, tree depth, and leapfrog steps suggest effective exploration of the parameter space. The consistency of energy levels across chains further supports the validity of the sampling process. These diagnostic tools assess the efficiency and convergence properties of the MCMC sampler [26]. The well-mixing trace plots (See Figure 6) for model parameters α, β, and δ demonstrate favorable convergence and low autocorrelation (See Figure 5) implies that the HMC algorithm is effectively navigating efficiently exploring the posterior landscape, leading to faster resulting in more rapid convergence. These characteristics indicate that the HMC sampling process has successfully captured the posterior distribution of the model parameters, providing reliable estimates for inference and interpretation. The absence of systematic patterns or trends in the trace plots further supports the validity of the sampling procedure, enhancing confidence in the reliability of the statistical analyses and conclusions drawn from the study.

Table 9. Sampling information statistics using HMC and NUTS

	accept_stat	stepsize	treedepth	n_leapfrog	divergent	energy
All chains	0.9850	0.3486	3.1739	9.9256	0.0000	1,772.5561
Chain 1	0.9824	0.3771	3.0664	8.9564	0.0000	1,772.5262
Chain 2	0.9845	0.3621	3.1254	9.6836	0.0000	1,772.5618
Chain 3	0.9920	0.2588	3.4992	12.6024	0.0000	1,772.5831
Chain 4	0.9810	0.3963	3.0046	8.4600	0.0000	1,772.5532

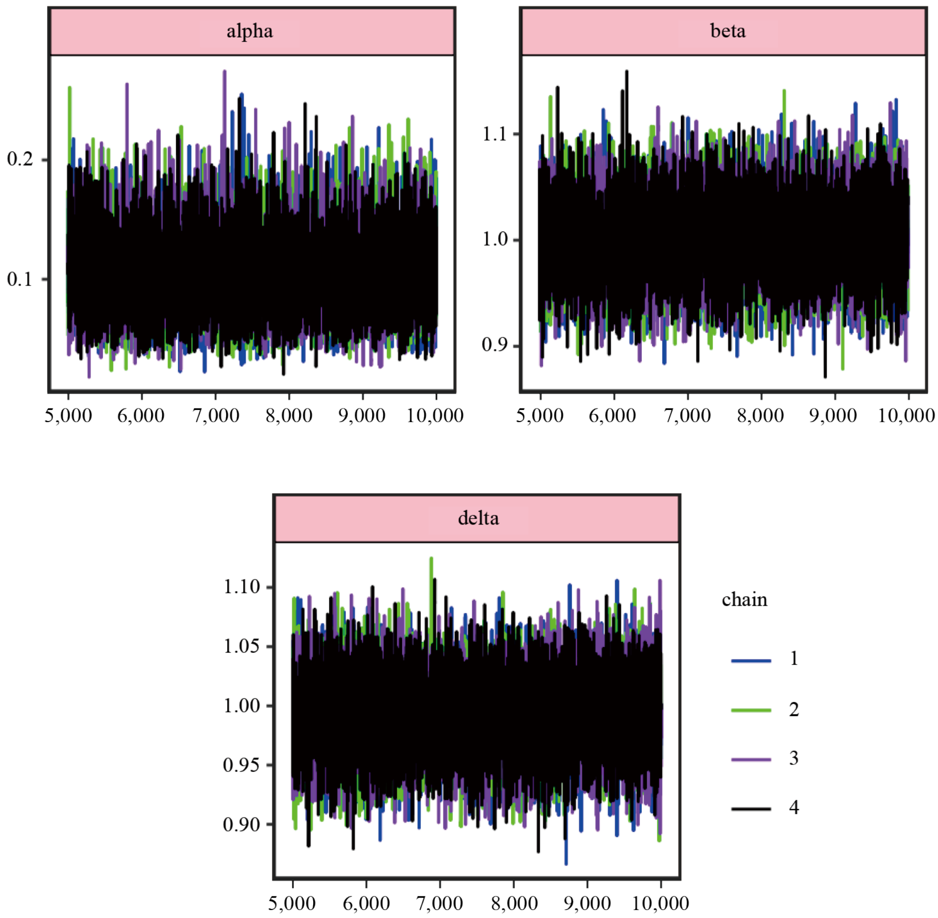


Figure 6. Trace plots for the model parameters α, β, and δ

8.2 *Posterior analysis*

The posterior summary statistics presented in Table 10 provide a comprehensive characterization of the posterior distribution of model parameters α, β, and δ. These statistics offer valuable insights into the central tendency, variability, and uncertainty associated with each parameter, facilitating robust inference and interpretation of the model. The high effective sample sizes and close-to-unity $\hat{R}$ values indicate reliable estimation and convergence of the MCMC sampling process, enhancing confidence in the validity of the statistical analyses and conclusions drawn from the study. We have also constructed the highest posterior density (HPD) and credible interval (CI) for a 95% confidence level displayed in Table 11.

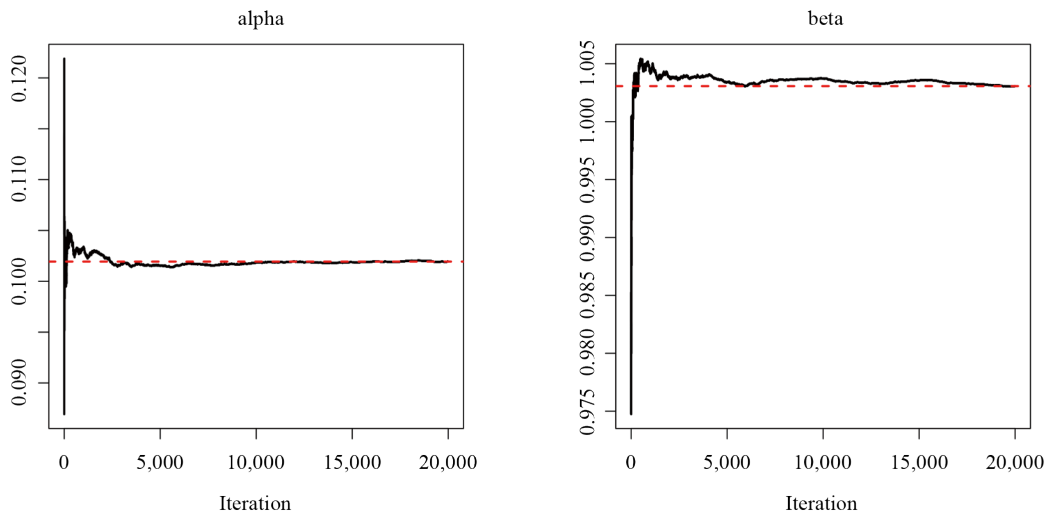
Table 10. Posterior summary statistics

Parameter	mean	se_mean	sd	2.50%	50%	97.50%	n_eff	Rhat
alpha	0.1019	0.0003	0.0317	0.0492	0.0986	0.1735	15,454	1.0000
beta	1.0030	0.0003	0.0355	0.9334	1.0028	1.0740	15,185	1.0003
delta	0.9917	0.0003	0.0325	0.9290	0.9914	1.0571	15,300	1.0004
lp__	-1,771.0700	0.0100	1.2200	-1,774.19	-1,770.75	-1,769.68	8,113	1.0002

Table 11. 95% HPD and CI for the model parameters

Parameter	HPD	CI
α	(0.0446 0.1652)	(0.0492 0.1735)
β	(0.9351 1.0755)	(0.9334 1.0740)
δ	(0.9288 1.0565)	(0.9290 1.0571)
lp__	(-1,773.434 -1,769.582)	(-1,774.19 -1,770.75)

The ergodic mean plots and histograms of model parameters α, β, and δ demonstrate favorable characteristics of the posterior distribution obtained from MCMC sampling. The convergence of the ergodic mean plots to their respective means indicates successful convergence of the Markov chains (See Figure 7), while the normal-shaped histograms (See Figure 8) suggest that the posterior distributions are symmetric and bell-shaped. These findings provide evidence of reliable estimation and convergence of the MCMC sampling process, enhancing confidence in the validity of the statistical analyses and conclusions drawn from the study.



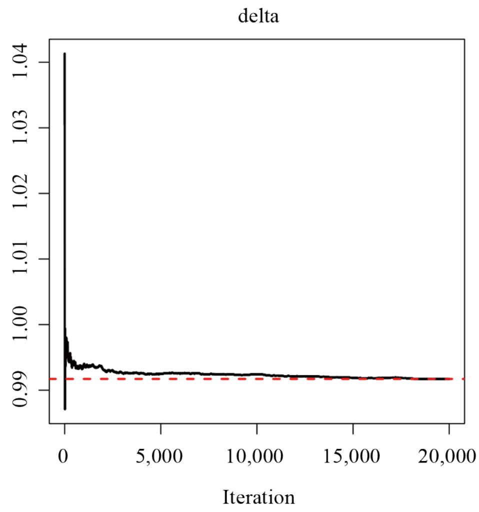


Figure 7. Ergodic mean plots for α, β, and δ

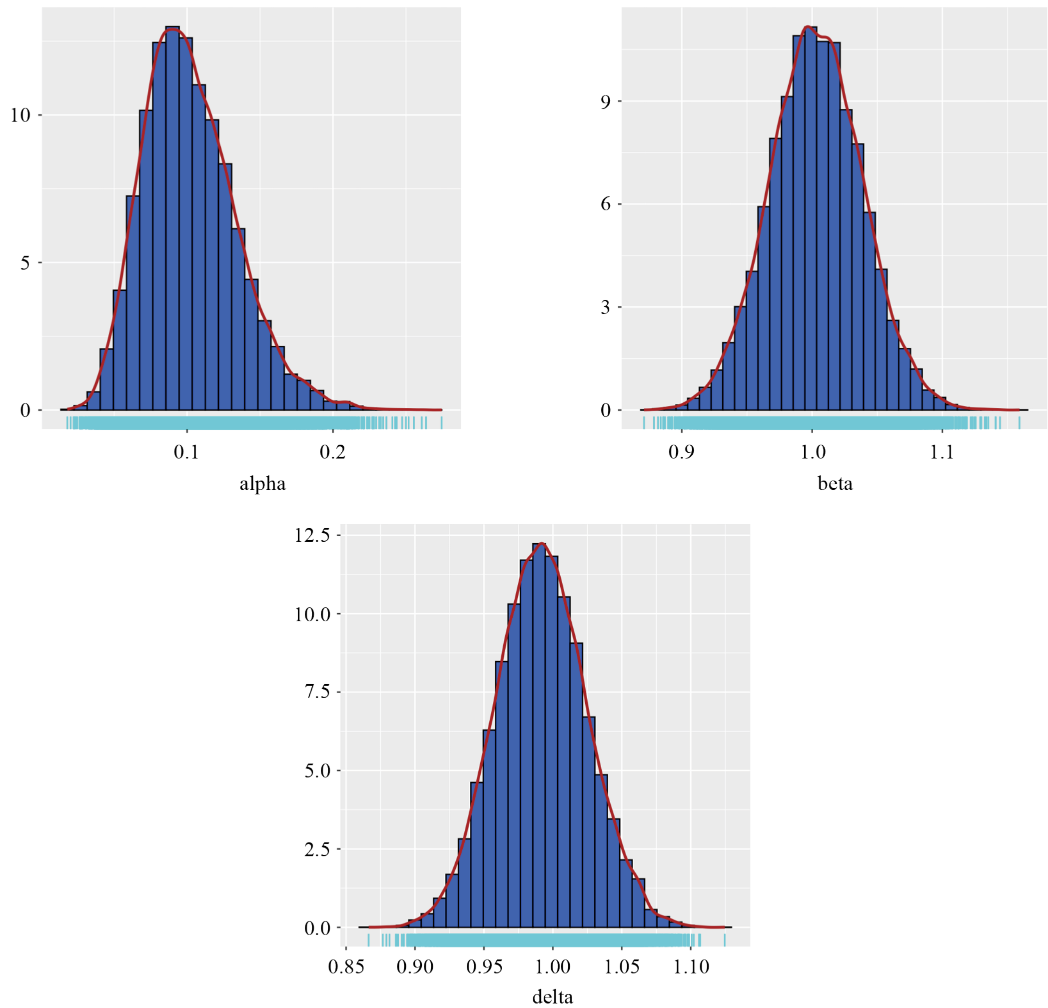


Figure 8. Histograms of posterior distributions for parameters α, β, and δ

8.3 *Posterior predictive check (PPC)*

PPCs are an essential part of Bayesian model validation, allowing us to assess the adequacy of our model in capturing the observed data. By comparing the observed data to simulated data generated from the posterior predictive distribution, PPCs help us evaluate whether our model adequately represents the underlying process that generated the data.

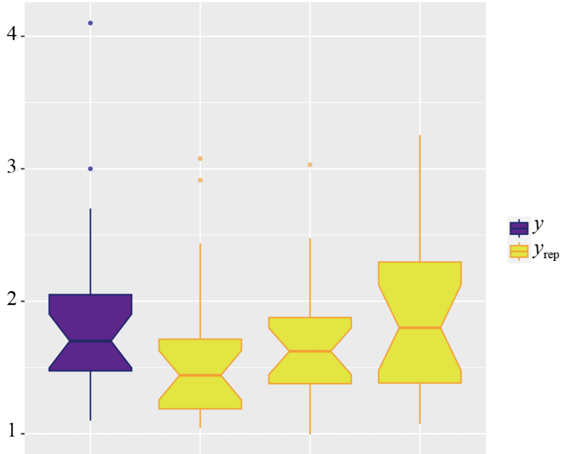


Figure 9. Box plots for the observed data y and predicted data y_{rep}

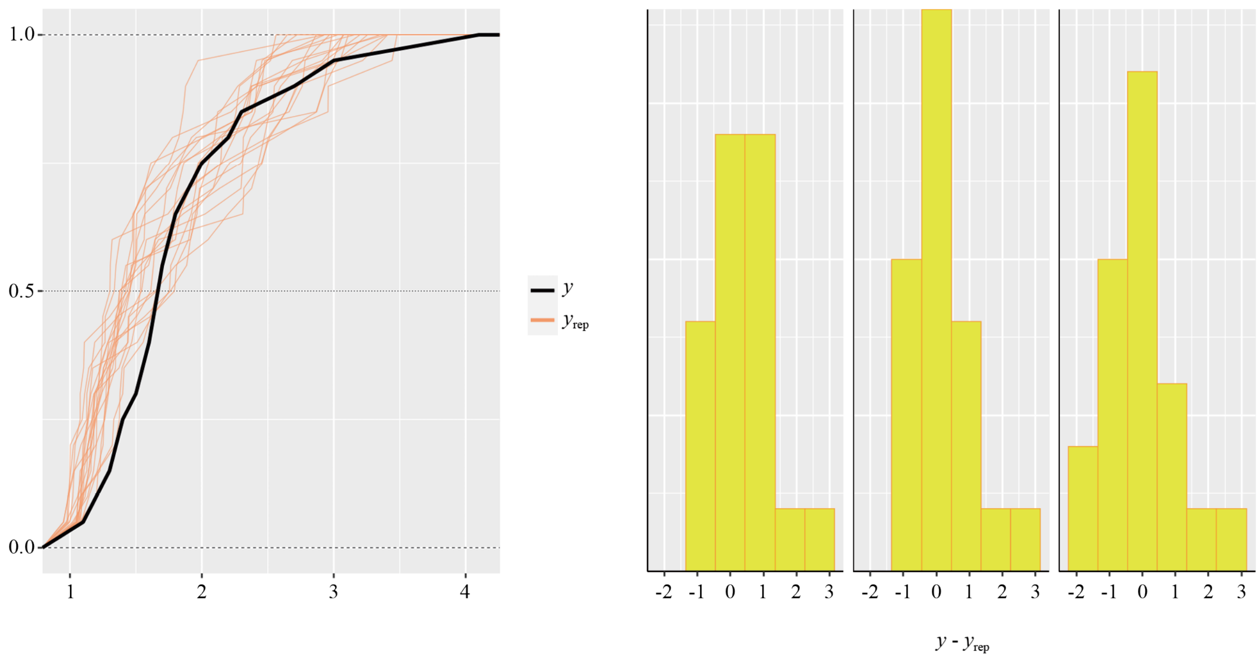


Figure 10. Fitted CDF and Histograms of error of posterior prediction

The box plots presented in Figure 9 provide a visual assessment of the fit between the observed and predicted data obtained from our Bayesian model. By comparing these distributions, we can evaluate the model's ability to accurately represent the observed data and identify potential areas for refinement or further investigation. Besides the PPC box plots, Figure 10 also displays the CDF and error histograms of the posterior prediction. Overall, PPCs offer valuable insights into the validity and robustness of our Bayesian modeling approach, aiding in the interpretation and confidence in the results obtained.

9. Conclusion

This research has presented and thoroughly examined the characteristics of the "New Exponentiated Inverse Weibull" (NEIW) distribution. It offers diverse applications in reliability engineering, survival analysis, and various other domains. Our research entailed a comprehensive analysis of several statistical characteristics, conducted simulation studies on maximum likelihood estimators (MLEs), and performed estimation and assessment of model parameters through real-world data sets.

Key Findings and Contributions:

• Model Construction: We have proposed the NEIW distribution as a flexible and robust alternative to existing probability distributions, offering a more comprehensive framework for modeling complex data sets with asymmetric and heavy-tailed characteristics.

• Properties and Simulation Studies: Through rigorous theoretical analysis and simulation studies, we have elucidated several important properties of the NEIW distribution, including moments, hazard function behavior, and reliability measures. Additionally, our simulation studies have provided valuable insights into the performance of MLEs under different scenarios, highlighting the efficacy and reliability of parameter estimation techniques.

• Empirical Analysis: Utilizing real-world data sets, we have applied both classical and Bayesian approaches to estimate and analyze the parameters of the NEIW distribution. Our empirical analysis has demonstrated the applicability of the NEIW model in capturing the underlying data structure and extracting meaningful insights from diverse data sources. To augment the versatility of data analysis for practitioners, the Bayesian parameter estimation approach is introduced. For Bayesian analysis, we have used the Bayesian analysis software STAN which uses the HMC algorithm under NUTS.

Implications and Future Directions: The findings of this research have significant implications for practitioners and researchers in various fields, offering a powerful tool for analyzing and interpreting data with complex distributional characteristics. Future research endeavors may focus on further exploring the theoretical properties of the NEIW distribution, refining estimation techniques, and extending its applicability to new domains and applications.

In conclusion, our study contributes to the advancement of statistical modeling and analysis by introducing the NEIW distribution and providing a comprehensive framework for its utilization in practice. We anticipate that the insights gained from this research will stimulate further investigations and foster continued innovation in the field of probability distributions and statistical modeling.

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Conflict of interest

The authors declare no competing financial interest.

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