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Research Article

# The Modified Log-Logistic Distribution: Properties and Inference with Real-Life Data Applications

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**Abstract:** This paper introduces a new three-parameter extension of the log-logistic distribution called the generalized Kavya-Manoharan log-logistic distribution. The proposed distribution extends the Kavya-Manoharan log-logistic model, and its density exhibits symmetric, right-skewed, reversed-J, and left-skewed shapes. Its hazard function can provide non-monotonic and monotonic shapes, which makes it particularly valuable for accurately modeling complex time-to-event data. Several distributional properties are addressed. The introduced model parameters are estimated via eight frequentist estimation approaches. Moreover, extensive simulations are obtained to explore the performance of the proposed methods of estimations. Finally, the performance of the proposed distribution is investigated using three real-lifetime datasets from medicine and reliability engineering. The analyzed data illustrated that the proposed distribution provided an adequate fit as compared to other competing log-logistic distributions.

***Keywords***: generalized KM transformation, log-logistic distribution, order statistics, maximum product spacing estimators, entropy

**MSC:** 47N30, 62J05, 62J20

## 1. Introduction

The log-logistic (LL) distribution is an essential probability model characterized by a heavy tail defined by scale and shape parameters. The LL distribution has several applications in various fields, such as engineering, economics, survival analysis, actuarial science, and social sciences. However, due to the symmetry exhibited by the LL model, its performance may deteriorate in cases where the hazard rate exhibits heavy tailing or skewness. Consequently, there has been a growing inclination towards enhancing the baseline LL distribution by adding an extra shape parameter to the parent distribution or utilizing alternative generalization techniques.

Researchers have suggested many generalizations of the LL distribution to improve its flexibility in analyzing lifetime data. Some notable LL extensions include the Kumaraswamy-LL [1], Marshall-Olkin LL [2], Zografos-Balakrishnan LL [3], beta-LL [4], McDonald LL [5], additive-Weibull LL (AWLL) [6], odd Lomax LL [7], Weibull


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generalized LL (WGLL) [8], alpha-power LL (APLL) [9], extended LL (ExLL) [10], extended odd Weibull LL (EOWLL) [11], generalized odd LL BXII (GOLLBXII) [12], exponentiated alpha-power log-logistic [13], skew LL [14], and cubic transmuted LL [15] distributions.

This paper presents a novel extension of the LL distribution called the generalized Kavya-Manoharan log-logistic (GKMLL) distribution. The new GKMLL distribution is constructed by incorporating the classical LL model into the generalized Kavya-Manoharan (GKM-G) family [16]. The GKMLL distribution shows significant patterns in failure rates, making it especially valuable for modeling time-to-event data in comparison to classical well-known models. The primary motivation of the paper is two-fold. First, it aims to present the GKMLL model as a new version of the LL distribution, which exhibits several desirable features including:

• The GKMLL distribution is an extension of existing LL probability distributions that introduces greater flexibility by incorporating an additional shape parameter. This enhancement allows the distribution to better capture the characteristics of real-world data, particularly when the data exhibits skewness-either negative (left-skewed) or positive (right-skewed). Furthermore, the GKMLL model allows for more kurtosis flexibility than the standard LL model.

• The GKMLL hazard rate function (HRF) exhibits upside-down bathtub, decreasing, reversed-J, and modified bathtub shapes. Its density exhibits a reversed-J shape, left-skewed, symmetric, right-skewed, and unimodal shapes. This adaptability makes it particularly valuable for accurately modeling complex time-to-event data in medical research and reliability engineering fields where traditional distributions may fall short.

• Moreover, the GKMLL model can analyze various medical research and reliability engineering datasets. It provides closer fit for three real-lifetime data, highlighting its superior adaptability as compared to alternative LL models.

Second, the GKMLL parameters are estimated utilizing various estimation methods: the least-squares estimators (LSEs), Anderson-Darling estimators (ADEs), maximum likelihood estimators (MLEs), weighted least-squares estimators (WLSEs), right-tail ADEs (RADEs), maximum product of spacing estimators (MPSEs), percentiles estimators (PCEs) and Cramér-von Mises estimators (CRVMEs). The proposed estimators are compared through extensive simulations to determine the best estimation approach to estimate the GKMLL parameters based on partial and overall ranks.

The rest of the article is outlined in six sections as follows. In Section 2, we define the GKMLL distribution. In Section 3, we derive some characteristics of the GKMLL model. Eight estimation methods are presented in Section 4. A simulation study is presented in Section 5. Three real-life data applications are used to show the usefulness of the GKMLL distribution in Section 6. Some final conclusions are given in Section 7.

## 2. The GKMLL distribution

In this section, we define the three-parameter GKMLL model. The GKMLL distribution is constructed based on the GKM-G family, which is proposed by Mahran et al. [16]. Let $G(t; v)$ and $g(t; v)$ denote the cumulative distribution function (CDF) and probability density function (PDF) of a baseline model with parameter vector $v$, then the CDF of the GKM-G family reduces to

$$F(t;\ \beta,\ \boldsymbol{v}) = \varphi^{\beta}\{1-\exp[-G(t;\ \boldsymbol{v})]\}^{\beta},\ t\in\Re,\ \beta>0, \tag{1}$$

where $\varphi = \exp(1)/[\exp(1) - 1]$.

The PDF of the GKM-G family has the form

$$f(t;\ \beta,\ \boldsymbol{v}) = \varphi^{\beta}\beta g(t;\ \boldsymbol{v})\exp[-G(t;\ \boldsymbol{v})]\{1-\exp[-G(t;\ \boldsymbol{v})]\}^{\beta-1}$$

The HRF of the GKM-G family follows as

$$h(t;\ \beta,\ \boldsymbol{v}) = \frac{\varphi^{\beta} \beta g(t;\ \boldsymbol{v}) \exp[-G(t;\ \boldsymbol{v})]\{1-\exp[-G(t;\ \boldsymbol{v})]\}^{\beta-1}}{1-\varphi^{\beta}\{1-\exp[-G(t;\ \boldsymbol{v})]\}^{\beta}}.$$

The PDF and CDF of the LL distribution are given by $g(t;\ \alpha,\ \theta) = \alpha\theta^{-\alpha}t^{\alpha-1}\left[1+\left(\frac{t}{\theta}\right)^{\alpha}\right]^{-2}$ and $G(t;\ \alpha,\ \theta) = 1-\left[1+\left(\frac{t}{\theta}\right)^{\alpha}\right]^{-1}$, $\theta,\ \alpha > 0$.

By inserting the CDF of the LL model in (1), the CDF of the GKMLL distribution follows as

$$F(t;\ \boldsymbol{\psi}) = \varphi^{\beta}\left(1-\exp\left\{\left[1+\left(\frac{t}{\theta}\right)^{\alpha}\right]^{-1}-1\right\}\right)^{\beta},\ t>0,\ \alpha,\ \theta,\ \beta>0,$$

where $\psi = (\alpha, \theta, \beta)$.

The corresponding PDF takes the form

$$f(t;\ \boldsymbol{\psi}) = \varphi^{\beta}\beta\alpha\theta^{-\alpha}t^{\alpha-1}\left[1+\left(\frac{t}{\theta}\right)^{\alpha}\right]^{-2}\exp\left\{\left[1+\left(\frac{t}{\theta}\right)^{\alpha}\right]^{-1}-1\right\}$$

$$\times\left(1-\exp\left\{\left[1+\left(\frac{t}{\theta}\right)^{\alpha}\right]^{-1}-1\right\}\right)^{\beta-1} \tag{2}$$

Therefore, a random variable ($rv$) with PDF (2) is denoted by $T$~GKMLL($\beta$, $\alpha$, $\theta$). For $\beta = 1$, the KMLL model follows as a special case from the GKMLL distribution.

The HRF of the GKMLL distribution is given by

$$h(t;\ \boldsymbol{\psi}) = \frac{\varphi^{\beta}\beta\alpha t^{\alpha-1}\exp\left\{\left[1+\left(\frac{t}{\theta}\right)^{\alpha}\right]^{-1}-1\right\}\left(1-\exp\left\{\left[1+\left(\frac{t}{\theta}\right)^{\alpha}\right]^{-1}-1\right\}\right)^{\beta-1}}{\theta^{\alpha}\left[1+\left(\frac{t}{\theta}\right)^{\alpha}\right]^{2}\left[1-\varphi^{\beta}\left(1-\exp\left\{\left[1+\left(\frac{t}{\theta}\right)^{\alpha}\right]^{-1}-1\right\}\right)^{\beta}\right]}.$$

Some shapes of the density and failure rate functions of the GKMLL distribution are displayed in Figures 1 and 2. The PDF plots show that the GKMLL distribution provides symmetric, right-skewed, reversed-J, and left-skewed densities. Furthermore, the plots illustrate that the GKMLL distribution can provide non-monotonic and monotonic failure rates, including reversed-J shaped, modified bathtub, decreasing, and unimodal shapes.

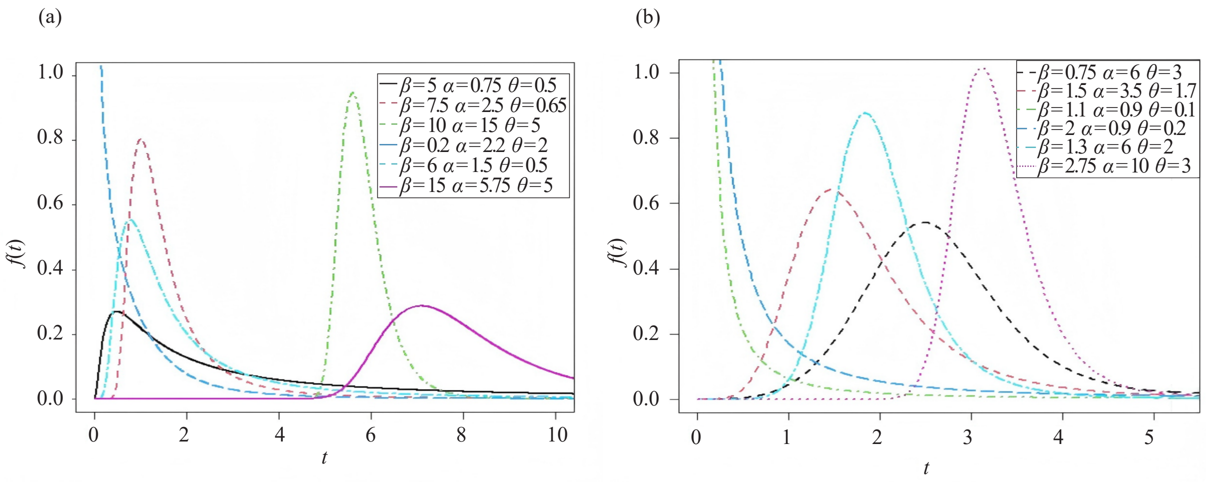


**Figure 1.** Plots of the GKMLL density for different parametric values

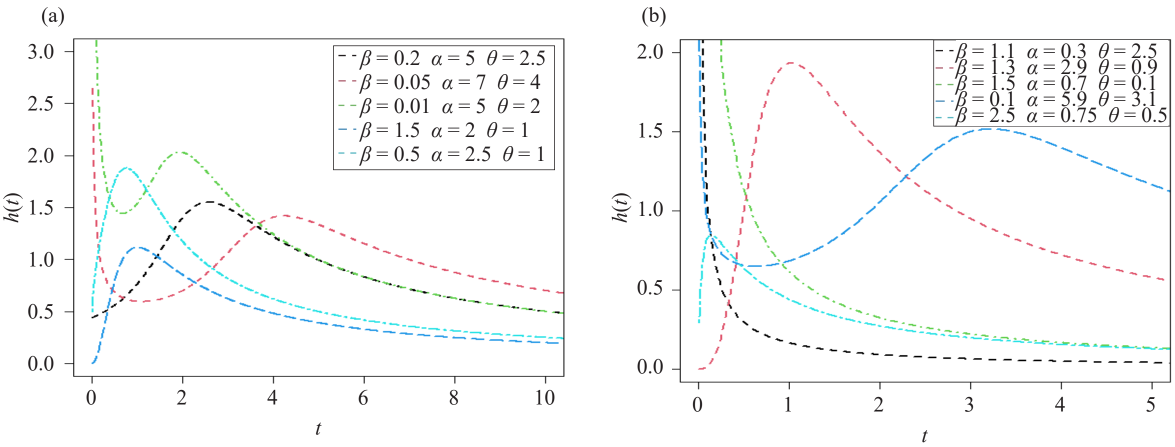


**Figure 2.** Plots of the GKMLL failure rate for different parametric values

## 3. Mathematical properties

This section explores some properties of the GKMLL model including mixture representation for its density, quantile function (QF), moments, moment generating function (MGF), entropy, and order statistics.

### 3.1 *Linear representation*

We provide a useful linear representation of the PDF of the GKMLL distribution. Mahran et al. [16] introduced a linear representation of the GKM-G density as follows

$$f(t) = \sum\nolimits_{k=0}^{\infty} a_k h_k(t), \tag{3}$$

where $a_k = \sum_{j=0}^{\infty} \varphi^{\beta} \frac{(-1)^{j+k} j^k}{k!} \binom{\beta}{j}$ and $h_k(t) = kg(t)G(t)^{k-1}$ is the exp-G density with power parameter $k > 0$. For the LL distribution, Equation (3) can be rewritten as follows

$$f(t)=\sum_{m=0}^{\infty} b_m \alpha\theta^{-\alpha} t^{\alpha-1}\left[1+\left(\frac{t}{\theta}\right)^{\alpha}\right]^{-m-2}, \tag{4}$$

where $b_m=\sum_{j,\,k=0}^{\infty}\varphi^{\beta}\frac{(-1)^{j+k+m}j^{k}}{(k-1)!}\binom{\beta}{j}\binom{k-1}{m}$.

Some useful essential mathematical properties, such as the MGF, the $r$th moments, entropy, and order statistics can be provided based on Equation (4).

## 3.2 *Quantile and related measures*

The QF of the GKMLL distribution reduces to

$$Q(u)=\theta\left\{-\frac{\log\left[1+\log\left(1-\left(\varphi^{-\beta}u\right)^{\frac{1}{\beta}}\right)\right]}{1+\log\left[1+\log\left(1-\left(\varphi^{-\beta}u\right)^{\frac{1}{\beta}}\right)\right]}\right\}^{\frac{1}{\alpha}},\ 0<u<1.$$

The Bowley skewness (BS) [17] as one of the earliest skewness (SK) measures is given by $S=\left[Q\left(\frac{3}{4}\right)+Q\left(\frac{1}{4}\right)-2Q\left(\frac{1}{2}\right)\right]/\left[Q\left(\frac{3}{4}\right)-Q\left(\frac{1}{4}\right)\right]$. The Moors kurtosis (MK) [18] depends on octiles and it is defined by $K=\left[Q\left(\frac{3}{8}\right)-Q\left(\frac{1}{8}\right)+Q\left(\frac{7}{8}\right)-Q\left(\frac{5}{8}\right)\right]/\left[Q\left(\frac{6}{8}\right)-Q\left(\frac{2}{8}\right)\right]$. The BS and MK of the GKMLL distribution for selected choices of $\alpha$ and $\theta$ as function of $\beta$ are showed in Figure 3. We take $\theta=0.5$ to plot the BS and MK. The plots of the two measures show that the shapes of the GKMLL distribution have significance dependence on the values of $\alpha$ and $\beta$. Further, the GKMLL distribution can be used to model positive and negative skewness data sets.

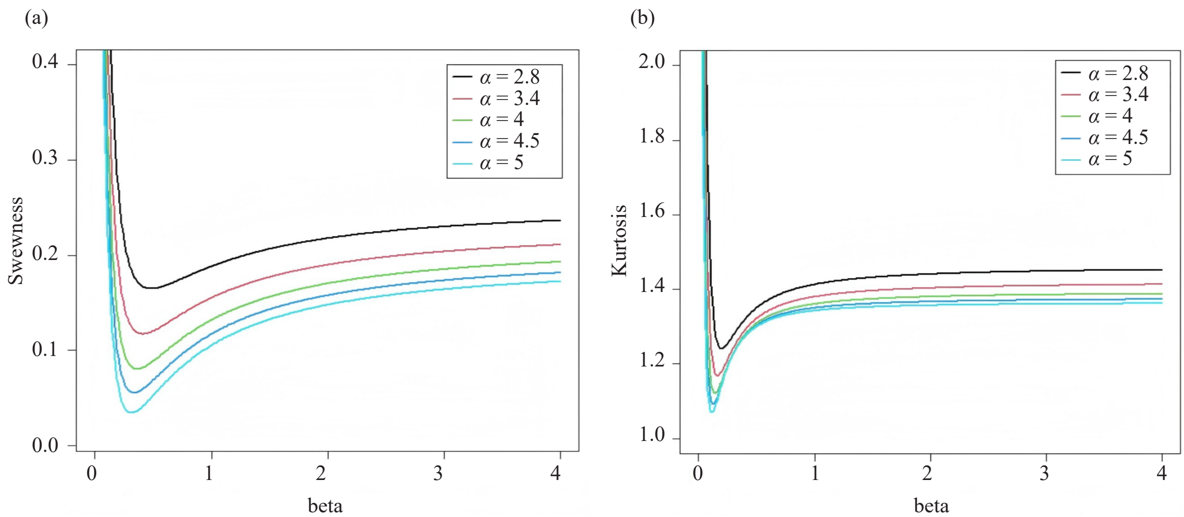


**Figure 3.** Plots for the BS and MK of the GKMLL distribution for some parameter values

### 3.3 *Moments*

We provide the MGF of the th non-central moments of the GKMLL model as shown in the following theorem.

**Theorem 1** The MGF and th non-central moments of the GKMLL modelare given, respectively, by

$$M_T(s) = \sum_{m,\, l\, =\, 0}^{\infty} b_m \left(\frac{(s\theta)^l}{l!}\right) B\left(\frac{\alpha + l}{\alpha},\ \frac{\alpha(m+1) - l}{\alpha}\right)$$

and

$$\mu_r' = \sum_{m\, =\, 0}^{\infty} b_m \theta^r B\left(\frac{\alpha + r}{\alpha},\ \frac{\alpha(m+1) - r}{\alpha}\right),$$

where $B(.,\,.)$ is the beta function.

**Proof.** Using Equation (4), the MGF of the GKMLL can be expressed as

$$M_T(s) = \sum_{m=0}^{\infty} b_m \int_0^{\infty} e^{st} \alpha \theta^{-\alpha} t^{\alpha - 1} \left[1 + \left(\frac{t}{\theta}\right)^{\alpha}\right]^{-m-2} dx,$$

Using the MacLaurin series, the last equation follows as

$$M_T(s) = \sum_{m=0}^{\infty} b_m \left[\int_0^{\infty} \alpha \theta^{-\alpha} t^{\alpha-1} \left[1 + \left(\frac{t}{\theta}\right)^{\alpha}\right]^{-m-2} dt\right.$$

$$+ \int_0^{\infty} s\alpha \theta^{-\alpha} t^{\alpha} \left[1 + \left(\frac{t}{\theta}\right)^{\alpha}\right]^{-m-2} dt$$

$$\left. + \int_0^{\infty} \left(\frac{s^2}{2!}\right) \alpha \theta^{-\alpha} t^{\alpha+1} \left[1 + \left(\frac{t}{\theta}\right)^{\alpha}\right]^{-m-2} dt + \int_0^{\infty} \left(\frac{s^3}{3!}\right) \alpha \theta^{-\alpha} t^{\alpha+2} \left[1 + \left(\frac{t}{\theta}\right)^{\alpha}\right]^{-m-2} dt + \ldots\right],$$

Let $p = \left(\frac{t}{\theta}\right)^{\alpha}$, so the MGF reduces to

$$M_T(s) = \sum_{m=0}^{\infty} b_m \left[\int_0^{\infty} [1 + p]^{-m-2} dp\right.$$

$$\left. + \int_0^{\infty} (s\theta) p^{\alpha^{-1}} [1 + p]^{-m-2} dp + \int_0^{\infty} \frac{(s\theta)^2}{2!} p^{2\alpha^{-1}} [1 + p]^{-m-2} dp + \int_0^{\infty} \frac{(s\theta)^3}{3!} p^{3\alpha^{-1}} [1 + p]^{-m-2} dp + \ldots\right],$$

Applying the beta function, we get

$$M_T(s) = \sum_{m=0}^{\infty} b_m \left[B(1,\ m+1) + (s\theta) B\left(\alpha^{-1} + 1,\ m - \alpha^{-1} + 1\right) + \frac{(s\theta)^2}{2!} B\left(2\alpha^{-1} + 1,\ m - 2\alpha^{-1} + 1\right)\right.$$

$$+\frac{(s\theta)^3}{3!}B\left(3\alpha^{-1}+1,\ m-3\alpha^{-1}+1\right)+\ldots\Bigg]$$

Therefore, the MGF of the GKMLL becomes

$$M_T(s)=\sum_{m,\ l=0}^{\infty}b_m\left(\frac{(s\theta)^l}{l!}\right)B\left(\frac{\alpha+l}{\alpha},\ \frac{\alpha(m+1)-l}{\alpha}\right).$$

The $r$th moments of the GKMLL model can be obtained by differentiating its MGF for $r$ times with respect to $s$ at $t$ = 0. Then, we have

$$\mu_r'=\sum_{m=0}^{\infty}b_m\theta^r B\left(\frac{\alpha+r}{\alpha},\ \frac{\alpha(m+1)-r}{\alpha}\right),\ r=1,\ 2,\ 3,\ \ldots \tag{5}$$

In addition, the first four moments can be obtained from Equation (5), for $r$ = 1, 2, 3, 4. The mean ($\mu_T$) and variance ( $\sigma_T^2$ ) of the GKMLL distribution are given by

$$\mu_T=\sum_{m=0}^{\infty}b_m\theta B\left(\alpha^{-1}+1,\ m-\alpha^{-1}+1\right)$$

and

$$\sigma_T^2=\sum_{m=0}^{\infty}b_m\left\{\theta^2B\left(2\alpha^{-1}+1,\ m-2\alpha^{-1}+1\right)-\left[\theta B\left(\alpha^{-1}+1,\ m-\alpha^{-1}+1\right)\right]^2\right\}.$$

Table 1 provides the values of $\mu_T$ for different values of $\theta$, $\beta$ and $\alpha$, which are obtained based on numerical integration (NUI) and summation (SUM) formula at truncated $H$ terms, where $H$ is the truncated terms of the summation. It is noted that the summation in (5) converges to the NUI of $\mu_T$ for different values of $\theta$, $\beta$ and $\alpha$ with the increase of $H$. Furthermore, Table 2 provides the $\mu_T$, $\sigma_T^2$ , SK ($\psi_1$), and kurtosis ($\psi_2$) of the GKMLL distribution for different values of $\theta$, $\beta$ and $\alpha$. The results in this section are obtained by $R$ software.

## 3.4 *Entropies*

Entropy is a major measure in risk assessment and reliability analysis. It measures the variation of the uncertainty for a random variable. For a non-negative *rv* $T$, the generalized entropy [19, 20] of order ($\rho$, $\mu$) is given by

$$G_\rho^\mu(t)=\frac{1}{\mu-\rho}\log\left[\int_0^\infty f^{\mu+\rho-1}(t)dt\right],\ \mu\neq\rho,\ \mu-1<\rho<\mu$$

For the GKMLL distribution, the generalized entropy has the form

$$G_\rho^\mu(t)=\frac{\theta}{\mu-\rho}\log\left\{\sum_{m=0}^{\infty}b_m\left(\frac{\alpha}{\theta^\alpha}\right)^{\mu+\rho-2}B\left(\left(\frac{\alpha-1}{\alpha}\right)\gamma+\frac{1}{\alpha},\ \gamma\left[(m+2)-\left(\frac{\alpha-1}{\alpha}\right)\right]-\frac{1}{\alpha}\right)\right\}, \tag{6}$$

where $\gamma=(\mu+\rho-1)$.

For $\mu$ = 1, Equation (6) reduces to the Rényi entropy [17] of order $\rho>0$, $\rho\neq1$. Then, the Rényi entropy of the GKMLL distribution follows as

$$R_\rho(t)=\frac{\theta}{1-\rho}\log\left\{\left(\sum_{m=0}^{\infty}b_m\left(\frac{\alpha}{\theta^\alpha}\right)^{\rho-1}B\left(\left(\frac{\alpha-1}{\alpha}\right)\rho+\frac{1}{\alpha},\ \rho\left[(m+2)-\left(\frac{\alpha-1}{\alpha}\right)\right]-\frac{1}{\alpha}\right)\right)\right\}.$$

The Shannon entropy of the GKMLL model follows for $\mu = 1$, and $\rho \to 1$ in Equation (6).

## 3.5 *Order statistics*

For an ordered random sample of size $n\left(T_{1:l},\ T_{2:l},\ T_{3:l},\ \ldots,\ T_{l:l}\right)$, the density of the $s$th order statistic, say, $T_{s:n}$, is given by

$$f_{T_{s:l}}(t)=\frac{f(t)}{B(s,\ l-s+1)}\sum_{i=1}^{l-s}(-1)^i\binom{l-s}{i}F(t)^{s+i-1},$$

Using the PDF and CDF of the GKMLL distribution, the density of the th order statistic, $T_{s:n}$ reduces to

$$f_{T_{s:l}}(t)=\sum_{r=0}^{\infty}\tau_r t^{\alpha-1}\left[1+\left(\frac{t}{\theta}\right)^\alpha\right]^{-(m+r+2)}, \tag{7}$$

where

$$\tau_r=\frac{\alpha\theta^{-\alpha}}{B(s,\ l-s+1)}\sum_{i=0}^{l-s}\sum_{m=0}^{\infty}\sum_{h=0}^{\beta(s+i-1)}\frac{b_m s^r}{r!}\binom{l-s}{i}\binom{\beta(s+i-1)}{h}(-1)^{i+h}\varphi^{\beta(s+i-1)},$$

and $b_m=\sum_{j,\,k=0}^{\infty}\varphi^\beta\frac{(-1)^{j+k+m}j^k}{(k-1)!}\binom{\beta}{j}\binom{k-1}{m}$.

For $s = 1$, Equation (7) reduced to the smallest order statistics $X_{1:i}$ of the GKMLL distribution with density

$$f_{T_{1:l}}(t)=\sum_{m,\ r=0}^{\infty}\pi_{m,\ r}t^{\alpha-1}\left[1+\left(\frac{t}{\theta}\right)^\alpha\right]^{-m-r-2},$$

where

$$\pi_{m,\ r}=l\alpha\theta^{-\alpha}\sum_{i=0}^{l-1}\sum_{h=0}^{\beta i}\frac{b_m}{r!}\binom{l-1}{i}\binom{\beta i}{h}(-1)^{i+h}\varphi^{\beta i}.$$

The largest order statistics $X_{i:i}$ density of the GKMLL distribution can be produced by substituting $s = l$, in Equation (7) as follows

$$f_{T_{l:l}}(t)=\sum_{m,\ r=0}^{\infty}\sigma_{m,\ r}t^{\alpha-1}\left[1+\left(\frac{t}{\theta}\right)^\alpha\right]^{-m-r-2},$$

where

$$\sigma_{m,r} = \alpha\theta^{-\alpha}\sum_{h=0}^{\beta(l-1)}\frac{b_{ml^{r+1}}}{r!}\binom{\beta(l-1)}{h}(-1)^h\varphi^{\beta(l-1)}.$$

**Table 1.** The values of $\mu_T$ based on the SUM and NUI formula for some values of $\theta$, $\beta$ and $\alpha$ at truncated $L$ terms

| $\theta$ | $\beta$ | $\alpha$ | $L$ | Summation | Numerical integration |
|---|---|---|---|---|---|
| 0.25 | 2 | 4.5 | 10 | 0.29613 | 0.29608 |
| | | | 20 | 0.29608 | |
| | | | 50 | 0.29608 | |
| | | 6 | 10 | 0.28051 | 0.28047 |
| | | | 20 | 0.28047 | |
| | | | 50 | 0.28047 | |
| | 3 | 4.5 | 10 | 0.32178 | 0.32854 |
| | | | 20 | 0.32854 | |
| | | | 50 | 0.32854 | |
| | | 6 | 10 | 0.29789 | 0.30352 |
| | | | 20 | 0.30352 | |
| | | | 50 | 0.30352 | |
| 0.5 | 2 | 4.5 | 10 | 0.59227 | 0.59216 |
| | | | 20 | 0.59216 | |
| | | | 50 | 0.59216 | |
| | | 6 | 10 | 0.56102 | 0.56093 |
| | | | 20 | 0.56093 | |
| | | | 50 | 0.56093 | |
| | 3 | 4.5 | 10 | 0.64357 | 0.65708 |
| | | | 20 | 0.65708 | |
| | | | 50 | 0.65708 | |
| | | 6 | 10 | 0.59579 | 0.60703 |
| | | | 20 | 0.60703 | |
| | | | 50 | 0.60703 | |
| 1.5 | 2 | 4.5 | 10 | 1.77681 | 1.77648 |
| | | | 20 | 1.77648 | |
| | | | 50 | 1.77648 | |
| | | 6 | 10 | 1.68307 | 1.68279 |
| | | | 20 | 1.68279 | |
| | | | 50 | 1.68279 | |
| | 3 | 4.5 | 10 | 1.93071 | 1.97124 |
| | | | 20 | 1.97124 | |
| | | | 50 | 1.97124 | |
| | | 6 | 10 | 1.78736 | 1.82110 |
| | | | 20 | 1.82110 | |
| | | | 50 | 1.82110 | |

**Table 2.** Some measures of the GKMLL distribution for different values of $\theta$, $\beta$ and $\alpha$

| $\theta$ | $\beta$ | $\alpha$ | $\mu_T$ | $\sigma_T^2$ | $\psi_1$ | $\psi_2$ |
|---|---|---|---|---|---|---|
| 0.25 | 2 | 4.5 | 0.29608 | 0.01389 | 3.65707 | 79.34371 |
| | | 6 | 0.28047 | 0.00615 | 2.29949 | 19.32936 |
| | | 8 | 0.27077 | 0.00299 | 1.69732 | 10.95562 |
| | | 10 | 0.26569 | 0.00177 | 1.41724 | 8.46578 |
| 0.5 | 3 | 4.5 | 0.65708 | 0.06372 | 3.85840 | 85.82370 |
| | | 6 | 0.60703 | 0.02662 | 2.47631 | 21.09784 |
| | | 8 | 0.57492 | 0.01234 | 1.87088 | 12.00477 |
| | | 10 | 0.55756 | 0.00711 | 1.59241 | 9.27703 |
| 1.5 | 2 | 4.5 | 1.77648 | 0.49995 | 3.65704 | 79.34401 |
| | | 6 | 1.68279 | 0.22153 | 2.29949 | 19.32936 |
| | | 8 | 1.62464 | 0.10752 | 1.69732 | 10.95561 |
| | | 10 | 1.59411 | 0.06369 | 1.41724 | 8.46578 |
| 2 | 3 | 4.5 | 2.62832 | 1.01958 | 3.85840 | 85.82374 |
| | | 6 | 2.42813 | 0.42587 | 2.47631 | 21.09784 |
| | | 8 | 2.29968 | 0.19747 | 1.87088 | 12.00477 |
| | | 10 | 2.23026 | 0.11373 | 1.59241 | 9.27704 |
| 3 | 2 | 4.5 | 3.55296 | 1.99982 | 3.65704 | 79.34399 |
| | | 6 | 3.36558 | 0.88611 | 2.29949 | 19.32936 |
| | | 8 | 3.24928 | 0.43010 | 1.69732 | 10.95561 |
| | | 10 | 3.18823 | 0.25477 | 1.41724 | 8.46578 |
| 5 | 3 | 4.5 | 6.57080 | 6.37240 | 3.85840 | 85.82375 |
| | | 6 | 6.07033 | 2.66168 | 2.47631 | 21.09784 |
| | | 8 | 5.74920 | 1.23417 | 1.87088 | 12.00477 |
| | | 10 | 5.57564 | 0.71082 | 1.59241 | 9.27704 |

# 4. Estimation approaches

The estimation of the GKMLL parameters is discussed using eight different estimators. Suppose, $t_1$, ..., $t_l$ is a random sample from the GKMLL($\psi$), of size $l$ with corresponding $t_{1:l} < t_{2:l} < ... < t_{l:l}$ order statistics sample, where $\psi = (\beta, \alpha, \theta)$.

The MLEs of the GKMLL parameters are obtained by maximizing the following log-likelihood function

$$\ell(\boldsymbol{\psi}) = l\log\left(\beta\alpha\varphi^{\beta}\theta^{-\alpha}\right) + (\alpha-1)\sum_{s=1}^{l}\log\left(t_s\right) - 2\sum_{s=1}^{l}\log E_s + \sum_{s=1}^{l}\left(E_s^{-1}-1\right) + (\beta-1)\sum_{s=1}^{l}\log\left(1-\exp\left(E_s^{-1}-1\right)\right),$$

where $E_s = 1+\left(\frac{t_s}{\theta}\right)^{\alpha}$.

The score functions are given by

$$\frac{\partial\ell(\boldsymbol{\psi})}{\partial\beta} = l\log\varphi + \frac{l}{\beta} + \sum_{s=1}^{l}\log\left(1-\exp\left(E_s^{-1}-1\right)\right) = 0,$$

$$\frac{\partial\ell(\boldsymbol{\psi})}{\partial\alpha} = \frac{l}{\alpha} - l\log\theta + \sum_{s=1}^{l}\log t_s - \sum_{s=1}^{l}\left(\frac{t_s}{\theta}\right)^{\alpha}\log\left(\frac{t_s}{\theta}\right)E_s^{-1}\left(2+E_s^{-1}\right)$$

$$+(\beta-1)\sum_{s=1}^{l}\left(E_s\right)^{-2}\left(\frac{t_s}{\theta}\right)^{\alpha}\log\left(\frac{t_s}{\theta}\right)\frac{\exp\left(E_s^{-1}-1\right)}{1-\exp\left(E_s^{-1}-1\right)} = 0$$

and

$$\frac{\partial\ell(\boldsymbol{\psi})}{\partial\theta} = \frac{-l\alpha}{\theta} + \alpha(\beta-1)\sum_{i=1}^{l}\left(E_s\right)^{-2}\left(\frac{t_s}{\theta}\right)^{\alpha-1}\left(\frac{t_s}{\theta^2}\right)\frac{\exp\left(E_s^{-1}-1\right)}{1-\exp\left(E_s^{-1}-1\right)} + \alpha\sum_{i=1}^{l}\left(\frac{t_s}{\theta}\right)^{\alpha}\frac{2+E_s^{-1}}{\theta E_s} = 0.$$

The previous equations can be solved numerically using different statistical programs such as *R*, Mathematica, and SAS to produce MLEs of GKMLL parameters.

The LSEs and WLSEs [21, 22] of GKMLL parameters are determined by minimizing

$$LW(\boldsymbol{\psi}) = \sum\nolimits_{s=1}^{l} B_s\left\{\varphi^{\beta}\left[1-\exp\left(\xi_{s:l}-1\right)\right]^{\beta} - \frac{s}{l+1}\right\}^2, \tag{8}$$

where $\xi_{s:l} = \left[1+\left(\frac{t_{s:l}}{\theta}\right)^{\alpha}\right]^{-1}$.

If $B_s = 1$, Equation (8) reduces to the LSEs and if $B_s = (l+1)^2(l+2)(s(l-s+1))^{-1}$, then Equation (10) gives the WLSEs of GKMLL parameters.

Moreover, these estimators are obtained by solving the following non-linear equations simultaneously by iterative techniques

$$\frac{\partial LW(\boldsymbol{\psi})}{\partial\beta} = \sum_{s=1}^{l} B_s\left\{\varphi^{\beta}\left[1-\exp\left(\xi_{s:l}-1\right)\right]^{\beta} - \frac{s}{l+1}\right\}\aleph_1\left(t_{s:l};\,\boldsymbol{\psi}\right) = 0,$$

$$\frac{\partial LW(\boldsymbol{\psi})}{\partial\alpha} = \sum_{s=1}^{l} B_s\left\{\varphi^{\beta}\left[1-\exp\left(\xi_{s:l}-1\right)\right]^{\beta} - \frac{s}{l+1}\right\}\aleph_2\left(t_{s:l};\,\boldsymbol{\psi}\right) = 0$$

and

$$\frac{\partial LW(\boldsymbol{\psi})}{\partial \theta} = \sum_{s=1}^{l} B_s \left\{ \varphi^{\beta} \left[1 - \exp\left(\xi_{s:l} - 1\right)\right]^{\beta} - \frac{s}{l+1} \right\} \aleph_3\left(t_{s:l};\ \boldsymbol{\psi}\right) = 0,$$

where

$$\aleph_1\left(t_{s:l};\ \boldsymbol{\psi}\right) = \frac{\partial F\left(t_{s:l};\ \boldsymbol{\psi}\right)}{\partial \beta} = \varphi^{\beta} \left[1 - \exp\left(\xi_{s:l} - 1\right)\right]^{\beta} \log\left[\varphi\left(1 - \exp\left(\xi_{s:l} - 1\right)\right)\right]$$

$$\aleph_2\left(t_{s:l};\ \boldsymbol{\psi}\right) = \frac{\partial F\left(t_{s:l};\ \boldsymbol{\psi}\right)}{\partial \alpha} = \beta\varphi^{\beta-1} \left(\frac{t_{s:l}}{\theta}\right)^{\alpha} \log\left(\frac{t_{s:l}}{\theta}\right)\left(E_{s:l}\right)^{-2} \exp\left(\xi_{s:l} - 1\right)\left[1 - \exp\left(\xi_{s:l} - 1\right)\right]^{\beta-1}, \tag{9}$$

and

$$\aleph_3\left(t_{s:l};\ \boldsymbol{\psi}\right) = \frac{\partial F\left(t_{s:l};\ \boldsymbol{\psi}\right)}{\partial \theta} = -\alpha\beta\theta^{-1}\varphi^{\beta-1} \left(\frac{t_{s:l}}{\theta}\right)^{\alpha} \left(E_{s:l}\right)^{-2} \exp\left(\xi_{s:l} - 1\right)\left[1 - \exp\left(\xi_{s:l} - 1\right)\right]^{\beta-1}$$

where $E_{s:l} = 1 + \left(\frac{t_{s:l}}{\theta}\right)^{\alpha}$.

The MPSEs [23] are a good alternative to the MLEs. The uniform spacing of the GKMLL distribution can be defined as

$$d_s(\boldsymbol{\psi}) = F\left(t_{s:l};\ \boldsymbol{\psi}\right) - F\left(t_{s-1:l};\ \boldsymbol{\psi}\right),\ s = 1,\ 2,\ \ldots,\ l+1.$$

Since

$$F\left(t_0;\ \boldsymbol{\psi}\right) = 0,\ F\left(t_{l+1};\ \boldsymbol{\psi}\right) = 1,\ \text{and } \sum_{s=1}^{l+1} d_s(\boldsymbol{\psi}).$$

The MPSEs of the GKMLL parameters are given by maximizing the following function

$$M(\boldsymbol{\psi}) = (l+1)^{-1} \sum_{s=1}^{l+1} \ln d_s(\boldsymbol{\psi}).$$

Therefore, the MPSEs of the GKMLL parameters can be derived by solving the following non-linear equations

$$\frac{\partial M(\boldsymbol{\psi})}{\partial \beta} = (l+1)^{-1} \sum_{s=1}^{l+1} d_s(\boldsymbol{\psi})^{-1} \left[\aleph_1\left(t_{s:l};\ \boldsymbol{\psi}\right) - \aleph_1\left(t_{s-1:l};\ \boldsymbol{\psi}\right)\right] = 0$$

$$\frac{\partial M(\boldsymbol{\psi})}{\partial \alpha} = (l+1)^{-1} \sum_{s=1}^{l+1} d_s(\boldsymbol{\psi})^{-1} \left[\aleph_2\left(t_{s:l};\ \boldsymbol{\psi}\right) - \aleph_2\left(t_{s-1:l};\ \boldsymbol{\psi}\right)\right] = 0$$

and

$$\frac{\partial M(\boldsymbol{\psi})}{\partial \theta} = (l+1)^{-1} \sum_{s=1}^{l+1} d_s(\boldsymbol{\psi})^{-1} \left[\aleph_3\left(t_{s:l};\ \boldsymbol{\psi}\right) - \aleph_3\left(t_{s-1:l};\ \boldsymbol{\psi}\right)\right] = 0$$

Since equations $\aleph_{\varepsilon}\left(t_{s-1:l};\ \boldsymbol{\psi}\right),\ \varepsilon = 1,\ 2,\ 3$ are given in Equation (9), and by replacing $s$ with $(s-1)$.

The PCEs are obtained by equating sample with population percentile points. The PCEs of the GKMLL parameters are given by minimizing

$$P(\boldsymbol{\psi}) = \sum_{s=1}^{l}\left[t_{s:l} - \theta\left\{\frac{-\log\left[1+\log\left(1-\left(\varphi^{-\beta}v_s\right)^{\frac{1}{\beta}}\right)\right]}{1+\log\left[1+\log\left(1-\left(\varphi^{-\beta}v_s\right)^{\frac{1}{\beta}}\right)\right]}\right\}^{\frac{1}{\alpha}}\right]^2,$$

with respect to GKMLL parameters, where $v_s = \left(\frac{s}{l+1}\right)$ be an unbiased estimator of the CDF of th order statistics.

Also, the PCEs of the three-parameter GKMLL model are determined by solving the following equations

$$\frac{\partial P(\boldsymbol{\psi})}{\partial \beta} = \sum_{s=1}^{l}\left[t_{s:l} - \theta\left\{\frac{-\log\left[1+\log\left(\vartheta_{\beta}\right)\right]}{1+\log\left[1+\log\left(\vartheta_{\beta}\right)\right]}\right\}^{\frac{1}{\alpha}}\right]^2 \omega_{\beta} = 0,$$

$$\frac{\partial P(\boldsymbol{\psi})}{\partial \alpha} = \sum_{s=1}^{l}\left[t_{s:l} - \theta\left\{\frac{-\log\left[1+\log\left(\vartheta_{\beta}\right)\right]}{1+\log\left[1+\log\left(\vartheta_{\beta}\right)\right]}\right\}^{\frac{1}{\alpha}}\right]^2 \omega_{\alpha} = 0$$

and

$$\frac{\partial P(\boldsymbol{\psi})}{\partial \theta} = \sum_{s=1}^{l}\left[t_{s:l} - \theta\left\{\frac{-\log\left[1+\log\left(\vartheta_{\beta}\right)\right]}{1+\log\left[1+\log\left(\vartheta_{\beta}\right)\right]}\right\}^{\frac{1}{\alpha}}\right]^2 \omega_{\theta} = 0,$$

where

$$\vartheta_{\beta} = \left(1-\left(\varphi^{-\beta}v_s\right)^{\frac{1}{\beta}}\right),\ \frac{\partial \vartheta_{\beta}}{\partial \beta} = -\left[\beta^{-1}\left(\log\varphi + \beta^{-1}\log\left(\varphi^{-\beta}v_s\right)\right)\right]$$

and

$$\omega_{\beta} = -\left(\frac{\theta}{\alpha}\right)\left\{\frac{-\log\left[1+\log\left(\vartheta_{\beta}\right)\right]}{1+\log\left[1+\log\left(\vartheta_{\beta}\right)\right]}\right\}^{\frac{1}{\alpha}-1}\left\{\frac{\frac{\left(\partial\vartheta_{\beta}/\beta\right)}{\left(\vartheta_{\beta}\right)\left[1+\log\left(\vartheta_{\beta}\right)\right]}}{1+\log\left[1+\log\left(\vartheta_{\beta}\right)\right]}\left[1+\frac{\log\left[1+\log\left(\vartheta_{\beta}\right)\right]}{1+\log\left[1+\log\left(\vartheta_{\beta}\right)\right]}\right]\right\},$$

$$\omega_\alpha = \left(\frac{\theta}{\alpha^2}\right)\left\{\frac{-\log\left[1+\log\left(\vartheta_\beta\right)\right]}{1+\log\left[1+\log\left(\vartheta_\beta\right)\right]}\right\}^{\frac{1}{\alpha}} \log\left\{\frac{-\log\left[1+\log\left(\vartheta_\beta\right)\right]}{1+\log\left[1+\log\left(\vartheta_\beta\right)\right]}\right\},$$

$$\omega_\theta = -\left\{\frac{-\log\left[1+\log\left(\vartheta_\beta\right)\right]}{1+\log\left[1+\log\left(\vartheta_\beta\right)\right]}\right\}^{\frac{1}{\alpha}}.$$

The CRVMEs of the GKMLL parameters are found by minimizing

$$C(\boldsymbol{\psi}) = (12l)^{-1} + \sum_{s=1}^{l}\left\{\varphi^\beta\left[1-\exp\left(\xi_{s:l}-1\right)\right]^\beta - \frac{2s-1}{2l}\right\},$$

with respect to $\psi$, as follows:

$$\frac{\partial C(\boldsymbol{\psi})}{\partial\beta} = \sum_{s=1}^{l}\frac{\aleph_1\left(t_{s:l};\,\boldsymbol{\psi}\right)}{F\left(t_{s:l};\,\boldsymbol{\psi}\right)}\left\{\varphi^\beta\left[1-\exp\left(\xi_{s:l}-1\right)\right]^\beta - \frac{2s-1}{2l}\right\} = 0,$$

$$\frac{\partial C(\boldsymbol{\psi})}{\partial\alpha} = \sum_{s=1}^{l}\frac{\aleph_2\left(t_{s:l};\,\boldsymbol{\psi}\right)}{F\left(t_{s:l};\,\boldsymbol{\psi}\right)}\left\{\varphi^\beta\left[1-\exp\left(\xi_{s:l}-1\right)\right]^\beta - \frac{2s-1}{2l}\right\} = 0$$

and

$$\frac{\partial C(\boldsymbol{\psi})}{\partial\theta} = \sum_{s=1}^{l}\frac{\aleph_3\left(t_{s:l};\,\boldsymbol{\psi}\right)}{F\left(t_{s:l};\,\boldsymbol{\psi}\right)}\left\{\varphi^\beta\left[1-\exp\left(\xi_{s:l}-1\right)\right]^\beta - \frac{2s-1}{2l}\right\} = 0.$$

The ADEs of the GKMLL parameters can be obtained by making the following equation as minimum as possible with respect to the parameters

$$AD(\boldsymbol{\psi}) = -\left(l + l^{-1}\sum_{s=1}^{l}(2s-1)\Delta_s\right),$$

where

$$\Delta_s = \beta\log\left\{\varphi\left[1-\exp\left(\xi_{s:l}-1\right)\right]\right\} + \log\left\{1-\varphi^\beta\left[1-\exp\left(\xi_{s:l}-1\right)\right]^\beta\right\}.$$

Similarly, The RADEs can be manipulated to estimate GKMLL parameters by minimizing

$$RAD(\boldsymbol{\psi}) = \frac{1}{2}l - 2\sum_{s=1}^{l}\varphi^\beta\left[1-\exp\left(\xi_{s:l}-1\right)\right]^\beta - l^{-1}\sum_{s=1}^{l}(2s-1)\log\left\{1-\varphi^\beta\left[1-\exp\left(\xi_{s:l}-1\right)\right]^\beta\right\},$$

with respect to the parameters.

## 5. Simulation analysis

This section explores the performance of the addressed estimation methods of the GKMLL parameters using a simulation study. We generate 5,000 samples from the GKMLL distribution for some sample sizes $n$ = {20, 50, 100, 200, 400} and different parametric values of $\beta$ = (0.75, 2.75), $\alpha$ = (0.5, 2) and $\theta$ = (0.67, 1.5). The average values of the biases (BIAS), mean square errors (MSE), and mean relative errors (MRE) are calculated for each estimate to explore the performance of studied estimators.

Tables 3-10 show the BIAS, MSE, and MRE of the MLEs, LSEs, WLSEs, MPSEs, PCEs, CVMEs, ADEs, and RTADEs. Table 11 reports the partial and overall ranks of the addressed estimators. It is noted that, as the sample size increases, the MSE, BIAS, and MRE decay toward zero. Furthermore, the values of the MSE tend to zero, illustrating that all methods are asymptotically unbiased. The values in Tables 3-10 reveal that the eight estimation methods perform very well.

Table 11 shows that the MPSEs outperform all other estimators with overall scores of 49. Hence, the results confirm the superiority of the MPSEs for estimating the GKMLL parameters.

## 6. Real-life data analysis

In this section, we demonstrate the significance and flexibility of the GKMLL distribution using three real-life datasets. The first dataset is studied by Lee and Wang [24], and it refers to the remission times (in months) for 128 bladder cancer patients. The data are: 2.09, 3.48, 6.94, 0.08, 4.87, 23.63, 8.66, 13.11, 3.52, 0.20, 2.23, 25.74, 4.98, 9.02, 13.29, 6.97, 2.26, 3.57, 0.40, 7.09, 5.06, 9.22, 13.80, 3.64, 0.50, 0.81, 2.46, 2.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 5.32, 2.62, 3.82, 12.07, 7.32, 14.77, 32.15, 10.06, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 17.14, 36.66, 4.26, 15.96, 4.23, 1.05, 2.69, 8.65, 5.41, 10.75, 16.62, 7.62, 1.19, 2.75, 43.01, 11.25, 7.63, 5.41, 17.12, 1.26, 46.12, 2.83, 5.49, 4.33, 7.66, 3.36, 21.73, 22.69, 6.93, 4.50, 12.63, 2.07, 8.37, 79.05, 2.87, 5.62, 1.35, 11.64, 17.36, 7.87, 3.02, 4.34, 1.40, 7.93, 6.25, 5.71, 6.76, 12.02, 11.79, 18.10, 1.46, 2.02, 3.31, 4.51, 4.40, 5.85, 8.26, 6.54, 8.53, 12.03, 11.98, 19.13, 1.76, 20.28, 2.02, 3.36, 3.25.

The second dataset is comprised of 100 observations. The dataset refers to breaking stress of carbon fibers, which are measured in Gba [25]. The data are: 0.98, 5.56, 5.08, 0.39, 1.57, 3.19, 4.90, 2.93, 2.85, 2.77, 2.76, 1.73, 2.48, 3.68, 1.08, 3.22, 3.75, 3.22, 3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.40, 3.15, 2.67,3.31, 2.81, 2.56, 2.17, 4.91, 1.59, 1.18, 2.48, 2.03, 1.69, 2.43, 3.39, 3.56, 2.83, 3.68, 2.00, 3.51, 0.85, 1.61, 3.28, 2.95, 2.81, 3.15, 1.92, 1.84, 1.22, 2.17, 1.61, 2.12, 3.09, 2.97, 4.20, 2.35, 1.41, 1.59, 1.12, 1.69, 2.79, 1.89, 1.87, 3.39, 3.33, 2.55, 3.68, 3.19, 1.71, 1.25, 4.70, 2.88, 2.96, 2.55, 2.59, 2.97, 1.57, 2.17, 4.38, 2.03, 2.82, 2.53, 3.31, 2.38, 1.36, 0.81, 1.17, 1.84, 1.80, 2.05, 3.65.

The third dataset refers to Kevlar 49/epoxy strands (with pressure at 90%) failure times [26]. The data are: 0.01, 0.01, 0.02, 0.02, 0.02, 0.03, 0.03, 0.04, 0.05, 0.06, 0.07, 0.07, 0.08, 0.09, 0.09, 0.10, 0.10, 0.11, 0.11, 0.12, 0.13, 0.18 ,0.19, 0.20, 0.23, 0.24, 0.24, 0.29, 0.34, 0.35, 0.36, 0.38, 0.40, 0.42, 0.43, 0.52, 0.54, 0.56, 0.60, 0.60, 0.63, 0.65, 0.67, 0.68, 0.72, 0.72, 0.72, 0.73, 0.79, 0.79, 0.80, 0.80, 0.83, 0.85, 0.90, 0.92, 0.95, 0.99, 1.00, 1.01, 1.02, 1.03, 1.05, 1.10, 1.10, 1.11, 1.15, 1.18, 1.20, 1.29, 1.31, 1.33, 1.34, 1.40, 1.43, 1.45, 1.50, 1.51, 1.52, 1.53, 1.54, 1.54, 1.55, 1.58, 1.60, 1.63, 1.64, 1.80 ,1.80, 1.81, 2.02, 2.05, 2.14, 2.17, 2.33, 3.03, 3.03, 3.34, 4.20, 4.69, 7.89.

The goodness-of-fit measures are calculated to check the performance of GKMLL distribution and other versions of the LL distribution including the AWLL distribution [6], EOWLL distribution [11], WGLL distribution [8], GOLLBXII distribution [12], ExLL distribution [11], APLL distribution [9], and LL distribution. These measures include the Akaike information criteria (AIC), consistent AIC (CAIC), Bayesian information criteria (BIC), Hannan-Quinn information criterion (HQIC), Cramer-von Mises ($W^*$), Anderson-Darling ($A^*$) statistics, minus log-likelihood (-$\mathfrak{L}$), and Kolmogorov-Smirnov (KS) statistic and its associated $p$-value (KS $p$-value). The MLEs are calculated by using different statistical programs such as Mathematica, Mathcad, and R (optim function), among others.

The total time on test (TTT) plots for the three datasets are presented in Figure 4. The TTT plots indicate that bladder cancer data has an inverted bathtub HR shape, while carbon fibers are characterized by increasing HRF, and finally, failure times data has a modified bathtub HR shape. Figure 5 provides the HRF plots of the GKMLL model based on the estimates obtained from the three datasets. These HRF plots are consistent with the TTT plots and confirm that the GKMLL model is suitable for fitting the three datasets. The ML estimates the parameters of the competing models, and their standard errors (SEs) are reported in Tables 12-14, for the three datasets. Tables 15-17 present the goodness-of-fit measures of the competing models for the three datasets. The values in these tables show the superior fit of the GKMLL distribution over other LL extensions.

**Table 3.** Simulation results of eight different estimators for $\beta = 0.75$, $\alpha = 0.5$, $\theta = 0.67$

| $n$ | Est. | Est. Par. | MLEs | LSEs | WLSEs | CRVMEs | MPSEs | PCEs | ADEs | RADEs |
|---|---|---|---|---|---|---|---|---|---|---|
| 20 | \|BIAS\| | $\hat{\beta}$ | 0.44805{4} | 0.46606{5} | 0.40387{3} | 0.47616{6} | 0.39067{2} | 2.44726{8} | 0.37975{1} | 0.48266{7} |
| | | $\hat{\alpha}$ | 0.14364{5} | 0.15331{7} | 0.13623{4} | 0.14834{6} | 0.13404{3} | 0.65763{8} | 0.13143{1} | 0.13160{2} |
| | | $\hat{\theta}$ | 0.66572{4} | 0.66928{8} | 0.66508{3} | 0.66627{6} | 0.66622{5} | 0.64430{1} | 0.64661{2} | 0.66871{7} |
| | MSE | $\hat{\beta}$ | 0.20075{4} | 0.21722{5} | 0.16311{3} | 0.22673{6} | 0.15262{2} | 5.98910{8} | 0.14421{1} | 0.23296{7} |
| | | $\hat{\alpha}$ | 0.02063{5} | 0.02350{7} | 0.01856{4} | 0.02200{6} | 0.01797{3} | 0.43248{8} | 0.01727{1} | 0.01732{2} |
| | | $\hat{\theta}$ | 0.44319{4} | 0.44793{8} | 0.44233{3} | 0.44392{6} | 0.44384{5} | 0.41513{1} | 0.41811{2} | 0.44717{7} |
| | MRE | $\hat{\beta}$ | 0.5974{4} | 0.62142{5} | 0.53849{3} | 0.63489{6} | 0.52089{2} | 0.88991{8} | 0.50633{1} | 0.64355{7} |
| | | $\hat{\alpha}$ | 0.28728{5} | 0.30662{7} | 0.27246{4} | 0.29667{6} | 0.26808{3} | 0.32881{8} | 0.26286{1} | 0.26320{2} |
| | | $\hat{\theta}$ | 0.99362{4} | 0.99892{8} | 0.99265{3} | 0.99444{6} | 0.99435{5} | 0.96165{1} | 0.96509{2} | 0.99807{7} |
| | | $\sum RANKS$ | 39{4} | 60{8} | 30{2.5} | 54{7} | 30{2.5} | 51{6} | 12{1} | 48{5} |
| 50 | BIAS | $\hat{\beta}$ | 0.25007{3} | 0.29937{5} | 0.24733{2} | 0.31057{6} | 0.22757{1} | 2.37250{8} | 0.25066{4} | 0.31836{7} |
| | | $\hat{\alpha}$ | 0.08437{3} | 0.09749{7} | 0.08281{2} | 0.09732{6} | 0.07884{1} | 0.49530{8} | 0.08464{4} | 0.08707{5} |
| | | $\hat{\theta}$ | 0.51229{1} | 0.59472{6} | 0.52254{3} | 0.57940{5} | 0.51959{2} | 0.63231{8} | 0.53069{4} | 0.60133{7} |
| | MSE | $\hat{\beta}$ | 0.06254{3} | 0.08962{5} | 0.06117{2} | 0.09645{6} | 0.05179{1} | 5.62873{8} | 0.06283{4} | 0.10135{7} |
| | | $\hat{\alpha}$ | 0.00712{3} | 0.00951{7} | 0.00686{2} | 0.00947{6} | 0.00622{1} | 0.24532{8} | 0.00716{4} | 0.00758{5} |
| | | $\hat{\theta}$ | 0.26244{1} | 0.35369{6} | 0.27305{3} | 0.33571{5} | 0.26998{2} | 0.39982{8} | 0.28164{4} | 0.3616{7} |
| | MRE | $\hat{\beta}$ | 0.33343{3} | 0.39916{5} | 0.32978{2} | 0.41409{6} | 0.30342{1} | 0.86273{8} | 0.33422{4} | 0.42447{7} |
| | | $\hat{\alpha}$ | 0.16874{3} | 0.19499{7} | 0.16562{2} | 0.19465{6} | 0.15768{1} | 0.24765{8} | 0.16927{4} | 0.17413{5} |
| | | $\hat{\theta}$ | 0.76461{1} | 0.88764{6} | 0.77991{3} | 0.86478{5} | 0.77551{2} | 0.94375{8} | 0.79208{4} | 0.89751{7} |
| | | $\sum RANKS$ | 21{2.5} | 54{6} | 21{2.5} | 51{5} | 12{1} | 72{8} | 36{4} | 57{7} |
| 100 | BIAS | $\hat{\beta}$ | 0.17337{3} | 0.21780{6} | 0.17020{2} | 0.22097{7} | 0.15701{1} | 2.39904{8} | 0.17686{4} | 0.21457{5} |
| | | $\hat{\alpha}$ | 0.05580{3} | 0.06830{6} | 0.05577{2} | 0.06885{7} | 0.05374{1} | 0.41721{8} | 0.05725{4} | 0.06044{5} |
| | | $\hat{\theta}$ | 0.39296{2} | 0.46927{7} | 0.39857{3} | 0.45336{5} | 0.38016{1} | 0.62525{8} | 0.40082{4} | 0.46343{6} |
| | MSE | $\hat{\beta}$ | 0.03006{3} | 0.04744{6} | 0.02897{2} | 0.04883{7} | 0.02465{1} | 5.75540{8} | 0.03128{4} | 0.04604{5} |
| | | $\hat{\alpha}$ | 0.00311{2.5} | 0.00466{6} | 0.00311{2.5} | 0.00474{7} | 0.00289{1} | 0.17406{8} | 0.00328{4} | 0.00365{5} |
| | | $\hat{\theta}$ | 0.15441{2} | 0.22022{7} | 0.15886{3} | 0.20554{5} | 0.14452{1} | 0.39094{8} | 0.16066{4} | 0.21477{6} |
| | MRE | $\hat{\beta}$ | 0.23116{3} | 0.29040{6} | 0.22693{2} | 0.29462{7} | 0.20935{1} | 0.87238{8} | 0.23582{4} | 0.2861{5} |
| | | $\hat{\alpha}$ | 0.11161{3} | 0.13660{6} | 0.11155{2} | 0.13770{7} | 0.10749{1} | 0.20860{8} | 0.11449{4} | 0.12089{5} |
| | | $\hat{\theta}$ | 0.58650{2} | 0.70041{7} | 0.59488{3} | 0.67666{5} | 0.56741{1} | 0.93322{8} | 0.59824{4} | 0.69169{6} |
| | | $\sum RANKS$ | 23.5{3} | 57{6.5} | 21.5{2} | 57{6.5} | 9{1} | 72{8} | 36{4} | 48{5} |

**Table 3.** (cont.)

| $n$ | Est. | Est. Par. | MLEs | LSEs | WLSEs | CRVMEs | MPSEs | PCEs | ADEs | RADEs |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $\hat{\beta}$ | 0.11921{2} | 0.15274{6} | 0.12118{3} | 0.15801{7} | 0.11128{1} | 2.36301{8} | 0.12134{4} | 0.15004{5} |
| | BIAS | $\hat{\alpha}$ | 0.03944{2} | 0.04894{6} | 0.03986{4} | 0.04966{7} | 0.0378{1} | 0.35089{8} | 0.03953{3} | 0.04230{5} |
| | | $\hat{\theta}$ | 0.28268{2} | 0.35098{6} | 0.28501{3} | 0.35253{7} | 0.27401{1} | 0.61586{8} | 0.29546{4} | 0.35009{5} |
| | | $\hat{\beta}$ | 0.01421{2} | 0.02333{6} | 0.01469{3} | 0.02497{7} | 0.01238{1} | 5.58381{8} | 0.01472{4} | 0.02251{5} |
| 200 | MSE | $\hat{\alpha}$ | 0.00156{2.5} | 0.00239{6} | 0.00159{4} | 0.00247{7} | 0.00143{1} | 0.12312{8} | 0.00156{2.5} | 0.00179{5} |
| | | $\hat{\theta}$ | 0.07991{2} | 0.12319{6} | 0.08123{3} | 0.12428{7} | 0.07508{1} | 0.37928{8} | 0.08730{4} | 0.12256{5} |
| | | $\hat{\beta}$ | 0.15895{2} | 0.20365{6} | 0.16158{3} | 0.21068{7} | 0.14838{1} | 0.85928{8} | 0.16179{4} | 0.20005{5} |
| | MRE | $\hat{\alpha}$ | 0.07887{2} | 0.09787{6} | 0.07972{4} | 0.09932{7} | 0.0756{1} | 0.17545{8} | 0.07905{3} | 0.08460{5} |
| | | $\hat{\theta}$ | 0.42191{2} | 0.52386{6} | 0.42539{3} | 0.52616{7} | 0.40896{1} | 0.91919{8} | 0.44099{4} | 0.52252{5} |
| | | $\sum RANKS$ | 18.5{2} | 54{6} | 30{3} | 63{7} | 9{1} | 72{8} | 32.5{4} | 45.0{5} |
| | | $\hat{\beta}$ | 0.08419{2} | 0.10545{5} | 0.08572{3} | 0.10981{7} | 0.07707{1} | 2.51646{8} | 0.08962{4} | 0.10567{6} |
| | BIAS | $\hat{\alpha}$ | 0.02771{2} | 0.03356{6} | 0.02858{3} | 0.03361{7} | 0.02586{1} | 0.30410{8} | 0.02884{4} | 0.03016{5} |
| | | $\hat{\theta}$ | 0.20338{1} | 0.25009{5} | 0.20668{3} | 0.25785{7} | 0.20635{2} | 0.60619{8} | 0.21717{4} | 0.25239{6} |
| | | $\hat{\beta}$ | 0.00709{2} | 0.01112{5} | 0.00735{3} | 0.01206{7} | 0.00594{1} | 6.33269{8} | 0.00803{4} | 0.01117{6} |
| 400 | MSE | $\hat{\alpha}$ | 0.00077{2} | 0.00113{6.5} | 0.00082{3} | 0.00113{6.5} | 0.00067{1} | 0.09247{8} | 0.00083{4} | 0.00091{5} |
| | | $\hat{\theta}$ | 0.04136{1} | 0.06255{5} | 0.04272{3} | 0.06649{7} | 0.04258{2} | 0.36747{8} | 0.04716{4} | 0.06370{6} |
| | | $\hat{\beta}$ | 0.11225{2} | 0.14061{5} | 0.11429{3} | 0.14641{7} | 0.10276{1} | 0.91508{8} | 0.11950{4} | 0.14089{6} |
| | MRE | $\hat{\alpha}$ | 0.05542{2} | 0.06711{6} | 0.05716{3} | 0.06723{7} | 0.05173{1} | 0.15205{8} | 0.05768{4} | 0.06033{5} |
| | | $\hat{\theta}$ | 0.30355{1} | 0.37327{5} | 0.30848{3} | 0.38486{7} | 0.30798{2} | 0.90477{8} | 0.32413{4} | 0.37671{6} |
| | | $\sum RANKS$ | 15{2} | 48.5{5} | 27{3} | 62.5{7} | 12{1} | 72{8} | 36{4} | 51{6} |

**Table 4.** Simulation results of eight different estimators for $\beta = 0.75$, $\alpha = 0.5$, $\theta = 1.5$

| $n$ | Est. | Est. Par. | MLEs | LSEs | WLSEs | CRVMEs | MPSEs | PCEs | ADEs | RADEs |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $\hat{\beta}$ | 0.44162{5} | 0.45673{6} | 0.3954{3} | 0.47100{7} | 0.37463{1} | 0.43726{4} | 0.38769{2} | 0.49604{8} |
| | \|BIAS\| | $\hat{\alpha}$ | 0.13354{4} | 0.15011{7} | 0.13183{2} | 0.14886{6} | 0.13270{3} | 0.26809{8} | 0.12864{1} | 0.13839{5} |
| | | $\hat{\theta}$ | 1.47733{4} | 1.49877{8} | 1.47140{3} | 1.49190{6} | 1.48698{5} | 1.35153{1} | 1.45209{2} | 1.49742{7} |
| | | $\hat{\beta}$ | 0.19503{5} | 0.20860{6} | 0.15634{3} | 0.22184{7} | 0.14035{1} | 0.19119{4} | 0.15031{2} | 0.24606{8} |
| 20 | MSE | $\hat{\alpha}$ | 0.01783{4} | 0.02253{7} | 0.01738{2} | 0.02216{6} | 0.01761{3} | 0.07187{8} | 0.01655{1} | 0.01915{5} |
| | | $\hat{\theta}$ | 2.18251{4} | 2.24631{8} | 2.16502{3} | 2.22577{6} | 2.21112{5} | 1.82665{1} | 2.10858{2} | 2.24227{7} |
| | | $\hat{\beta}$ | 0.58883{5} | 0.60897{6} | 0.52720{3} | 0.62800{7} | 0.49951{1} | 0.58301{4} | 0.51692{2} | 0.66139{8} |
| | MRE | $\hat{\alpha}$ | 0.26708{4} | 0.30022{7} | 0.26366{2} | 0.29773{6} | 0.26540{3} | 0.53617{8} | 0.25727{1} | 0.27678{5} |
| | | $\hat{\theta}$ | 0.98489{4} | 0.99918{8} | 0.98093{3} | 0.99460{6} | 0.99132{5} | 0.90102{1} | 0.96806{2} | 0.99828{7} |
| | | $\sum RANKS$ | 39{4.5} | 63{8} | 24{2} | 57{6} | 27{3} | 39{4.5} | 15{1} | 60{7} |

**Table 4.** (cont.)

| $n$ | Est. | Est. Par. | MLEs | LSEs | WLSEs | CRVMEs | MPSEs | PCEs | ADEs | RADEs |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $\hat{\beta}$ | 0.24612{3} | 0.29527{5} | 0.23259{2} | 0.30691{7} | 0.22709{1} | 0.46869{8} | 0.24645{4} | 0.30246{6} |
| | BIAS | $\hat{\alpha}$ | 0.08201{4} | 0.09663{6} | 0.07955{2} | 0.09834{7} | 0.07917{1} | 0.23168{8} | 0.08191{3} | 0.08817{5} |
| | | $\hat{\theta}$ | 1.15973{2} | 1.32348{6} | 1.12050{1} | 1.30486{5} | 1.16152{3} | 1.34705{8} | 1.17097{4} | 1.34406{7} |
| | | $\hat{\beta}$ | 0.06057{3} | 0.08719{5} | 0.05410{2} | 0.09419{7} | 0.05157{1} | 0.21967{8} | 0.06074{4} | 0.09148{6} |
| 50 | MSE | $\hat{\alpha}$ | 0.00673{4} | 0.00934{6} | 0.00633{2} | 0.00967{7} | 0.00627{1} | 0.05368{8} | 0.00671{3} | 0.00777{5} |
| | | $\hat{\theta}$ | 1.34497{2} | 1.75159{6} | 1.25551{1} | 1.70267{5} | 1.34914{3} | 1.81454{8} | 1.37117{4} | 1.80650{7} |
| | | $\hat{\beta}$ | 0.32816{3} | 0.39370{5} | 0.31012{2} | 0.40921{7} | 0.30278{1} | 0.62492{8} | 0.3286{4} | 0.40328{6} |
| | MRE | $\hat{\alpha}$ | 0.16403{4} | 0.19327{6} | 0.15910{2} | 0.19667{7} | 0.15834{1} | 0.46336{8} | 0.16382{3} | 0.17634{5} |
| | | $\hat{\theta}$ | 0.77315{2} | 0.88232{6} | 0.74700{1} | 0.86991{5} | 0.77435{3} | 0.89803{8} | 0.78065{4} | 0.89604{7} |
| | | $\sum RANKS$ | 27{3} | 51{5} | 15{1.5} | 57{7} | 15{1.5} | 72{8} | 33{4} | 54{6} |
| | | $\hat{\beta}$ | 0.17291{3} | 0.21340{5} | 0.16630{2} | 0.21454{6} | 0.15159{1} | 0.43509{8} | 0.17803{4} | 0.21941{7} |
| | BIAS | $\hat{\alpha}$ | 0.05439{2} | 0.06645{6} | 0.05553{3} | 0.06884{7} | 0.05391{1} | 0.20023{8} | 0.05833{4} | 0.06166{5} |
| | | $\hat{\theta}$ | 0.87160{3} | 1.03998{6} | 0.83089{1} | 1.03598{5} | 0.84583{2} | 1.36877{8} | 0.88557{4} | 1.06747{7} |
| | | $\hat{\beta}$ | 0.02990{3} | 0.04554{5} | 0.02766{2} | 0.04603{6} | 0.02298{1} | 0.18930{8} | 0.03169{4} | 0.04814{7} |
| 100 | MSE | $\hat{\alpha}$ | 0.00296{2} | 0.00442{6} | 0.00308{3} | 0.00474{7} | 0.00291{1} | 0.04009{8} | 0.00340{4} | 0.00380{5} |
| | | $\hat{\theta}$ | 0.75970{3} | 1.08156{6} | 0.69039{1} | 1.07326{5} | 0.71544{2} | 1.87354{8} | 0.78424{4} | 1.13950{7} |
| | | $\hat{\beta}$ | 0.23054{3} | 0.28453{5} | 0.22173{2} | 0.28605{6} | 0.20212{1} | 0.58012{8} | 0.23737{4} | 0.29254{7} |
| | MRE | $\hat{\alpha}$ | 0.10878{2} | 0.13290{6} | 0.11106{3} | 0.13769{7} | 0.10782{1} | 0.40045{8} | 0.11665{4} | 0.12332{5} |
| | | $\hat{\theta}$ | 0.58107{3} | 0.69332{6} | 0.55393{1} | 0.69066{5} | 0.56389{2} | 0.91252{8} | 0.59038{4} | 0.71165{7} |
| | | $\sum RANKS$ | 24{3} | 51{5} | 18{2} | 54{6} | 12{1} | 72{8} | 36{4} | 57{7} |
| | | $\hat{\beta}$ | 0.11946{3} | 0.15286{6} | 0.11270{2} | 0.15544{7} | 0.10406{1} | 0.43063{8} | 0.12174{4} | 0.15193{5} |
| | BIAS | $\hat{\alpha}$ | 0.03810{2} | 0.04960{7} | 0.03851{3} | 0.04951{6} | 0.03762{1} | 0.18274{8} | 0.04033{4} | 0.04195{5} |
| | | $\hat{\theta}$ | 0.62012{3} | 0.79150{5} | 0.57381{1} | 0.79293{6} | 0.60339{2} | 1.34895{8} | 0.65080{4} | 0.79752{7} |
| | | $\hat{\beta}$ | 0.01427{3} | 0.02337{6} | 0.01270{2} | 0.02416{7} | 0.01083{1} | 0.18544{8} | 0.01482{4} | 0.02308{5} |
| 200 | MSE | $\hat{\alpha}$ | 0.00145{2} | 0.00246{7} | 0.00148{3} | 0.00245{6} | 0.00142{1} | 0.03340{8} | 0.00163{4} | 0.00176{5} |
| | | $\hat{\theta}$ | 0.38455{3} | 0.62648{5} | 0.32926{1} | 0.62873{6} | 0.36408{2} | 1.81966{8} | 0.42354{4} | 0.63604{7} |
| | | $\hat{\beta}$ | 0.15928{3} | 0.20382{6} | 0.15026{2} | 0.20725{7} | 0.13875{1} | 0.57417{8} | 0.16232{4} | 0.20258{5} |
| | MRE | $\hat{\alpha}$ | 0.07620{2} | 0.09920{7} | 0.07701{3} | 0.09902{6} | 0.07524{1} | 0.36549{8} | 0.08066{4} | 0.08389{5} |
| | | $\hat{\theta}$ | 0.41342{3} | 0.52767{5} | 0.38254{1} | 0.52862{6} | 0.40226{2} | 0.89930{8} | 0.43387{4} | 0.53168{7} |
| | | $\sum RANKS$ | 24{3} | 54{6} | 18{2} | 57{7} | 12{1} | 72{8} | 36{4} | 51{5} |

**Table 4.** (cont.)

| $n$ | Est. | Est. Par. | MLEs | LSEs | WLSEs | CRVMEs | MPSEs | PCEs | ADEs | RADEs |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $\hat{\beta}$ | 0.08145{3} | 0.10834{6} | 0.07765{2} | 0.10996{7} | 0.06527{1} | 0.42848{8} | 0.08406{4} | 0.10115{5} |
| | BIAS | $\hat{\alpha}$ | 0.02696{3} | 0.03427{7} | 0.02667{2} | 0.03386{6} | 0.02580{1} | 0.17010{8} | 0.02815{4} | 0.02969{5} |
| | | $\hat{\theta}$ | 0.43756{3} | 0.56945{6} | 0.41546{2} | 0.57343{7} | 0.01833{1} | 1.32070{8} | 0.45841{4} | 0.52871{5} |
| | | $\hat{\beta}$ | 0.00663{3} | 0.01174{6} | 0.00603{2} | 0.01209{7} | 0.00426{1} | 0.18359{8} | 0.00707{4} | 0.01023{5} |
| 400 | MSE | $\hat{\alpha}$ | 0.00073{3} | 0.00117{7} | 0.00071{2} | 0.00115{6} | 0.00067{1} | 0.02893{8} | 0.00079{4} | 0.00088{5} |
| | | $\hat{\theta}$ | 0.19146{3} | 0.32427{6} | 0.17261{2} | 0.32882{7} | 0.00034{1} | 1.74424{8} | 0.21014{4} | 0.27954{5} |
| | | $\hat{\beta}$ | 0.10860{3} | 0.14446{6} | 0.10354{2} | 0.14661{7} | 0.08703{1} | 0.57130{8} | 0.11208{4} | 0.13487{5} |
| | MRE | $\hat{\alpha}$ | 0.05393{3} | 0.06854{7} | 0.05334{2} | 0.06773{6} | 0.05160{1} | 0.34020{8} | 0.05631{4} | 0.05937{5} |
| | | $\hat{\theta}$ | 0.29171{3} | 0.37963{6} | 0.27698{2} | 0.38228{7} | 0.01222{1} | 0.88047{8} | 0.30561{4} | 0.35247{5} |
| | | $\sum RANKS$ | 27{3} | 57{6} | 18{2} | 60{7} | 9{1} | 72{8} | 36{4} | 45{5} |

**Table 5.** Simulation results of eight different estimators for $\beta = 0.75$, $\alpha = 2$, $\theta = 0.67$

| $n$ | Est. | Est. Par. | MLEs | LSEs | WLSEs | CRVMEs | MPSEs | PCEs | ADEs | RADEs |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $\hat{\beta}$ | 0.44938{4} | 0.46774{6} | 0.40426{3} | 0.46510{5} | 0.37879{1} | 0.57838{8} | 0.38593{2} | 0.48677{7} |
| | \|BIAS\| | $\hat{\alpha}$ | 0.56476{5} | 0.60835{7} | 0.54987{4} | 0.60611{6} | 0.52740{3} | 0.79577{8} | 0.51197{1} | 0.52580{2} |
| | | $\hat{\theta}$ | 0.26686{2} | 0.30267{6} | 0.27509{4} | 0.28314{5} | 0.27067{3} | 0.56556{8} | 0.23727{1} | 0.31111{7} |
| | | $\hat{\beta}$ | 0.20194{4} | 0.21878{6} | 0.16342{3} | 0.21632{5} | 0.14348{1} | 0.33452{8} | 0.14894{2} | 0.23695{7} |
| 20 | MSE | $\hat{\alpha}$ | 0.31895{5} | 0.37009{7} | 0.30236{4} | 0.36736{6} | 0.27815{3} | 0.63325{8} | 0.26212{1} | 0.27646{2} |
| | | $\hat{\theta}$ | 0.07122{2} | 0.09161{6} | 0.07567{4} | 0.08017{5} | 0.07326{3} | 0.31985{8} | 0.05630{1} | 0.09679{7} |
| | | $\hat{\beta}$ | 0.59918{4} | 0.62366{6} | 0.53901{3} | 0.62014{5} | 0.50505{1} | 0.77117{8} | 0.51457{2} | 0.64903{7} |
| | MRE | $\hat{\alpha}$ | 0.28238{5} | 0.30417{7} | 0.27494{4} | 0.30305{6} | 0.26370{3} | 0.39789{8} | 0.25599{1} | 0.26290{2} |
| | | $\hat{\theta}$ | 0.39830{2} | 0.45174{6} | 0.41057{4} | 0.42260{5} | 0.40398{3} | 0.84411{8} | 0.35413{1} | 0.46435{7} |
| | | $\sum RANKS$ | 33{3.5} | 57{7} | 33{3.5} | 48{5.5} | 21{2} | 72{8} | 12{1} | 48{5.5} |
| | | $\hat{\beta}$ | 0.25606{3} | 0.30208{6} | 0.26116{4} | 0.29675{5} | 0.23256{1} | 0.49177{8} | 0.25207{2} | 0.31408{7} |
| | BIAS | $\hat{\alpha}$ | 0.33208{3} | 0.39151{7} | 0.34687{4} | 0.38613{6} | 0.32425{1} | 0.57611{8} | 0.33049{2} | 0.35422{5} |
| | | $\hat{\theta}$ | 0.15912{3} | 0.18921{6} | 0.16704{4} | 0.18234{5} | 0.15387{1} | 0.44449{8} | 0.15905{2} | 0.19722{7} |
| | | $\hat{\beta}$ | 0.06557{3} | 0.09125{6} | 0.06821{4} | 0.08806{5} | 0.05408{1} | 0.24184{8} | 0.06354{2} | 0.09864{7} |
| 50 | MSE | $\hat{\alpha}$ | 0.11028{3} | 0.15328{7} | 0.12032{4} | 0.14909{6} | 0.10514{1} | 0.33191{8} | 0.10923{2} | 0.12547{5} |
| | | $\hat{\theta}$ | 0.02532{3} | 0.03580{6} | 0.02790{4} | 0.03325{5} | 0.02368{1} | 0.19757{8} | 0.02530{2} | 0.03889{7} |
| | | $\hat{\beta}$ | 0.34141{3} | 0.40277{6} | 0.34822{4} | 0.39567{5} | 0.31008{1} | 0.65569{8} | 0.33609{2} | 0.41877{7} |
| | MRE | $\hat{\alpha}$ | 0.16604{3} | 0.19576{7} | 0.17343{4} | 0.19306{6} | 0.16213{1} | 0.28806{8} | 0.16525{2} | 0.17711{5} |
| | | $\hat{\theta}$ | 0.23749{3} | 0.28241{6} | 0.24931{4} | 0.27216{5} | 0.22966{1} | 0.66341{8} | 0.23739{2} | 0.29435{7} |
| | | $\sum RANKS$ | 27{3} | 57{6.5} | 36{4} | 48{5} | 9{1} | 72{8} | 18{2} | 57{6.5} |

**Table 5.** (cont.)

| $n$ | Est. | Est. Par. | MLEs | LSEs | WLSEs | CRVMEs | MPSEs | PCEs | ADEs | RADEs |
|---|---|---|---|---|---|---|---|---|---|---|
| 100 | BIAS | $\hat{\beta}$ | 0.16844{2} | 0.21339{5} | 0.18209{4} | 0.21507{6} | 0.16581{1} | 0.47933{8} | 0.17596{3} | 0.22177{7} |
| | | $\hat{\alpha}$ | 0.22129{1} | 0.26959{6} | 0.23744{4} | 0.27581{7} | 0.22660{2} | 0.47274{8} | 0.22813{3} | 0.24442{5} |
| | | $\hat{\theta}$ | 0.10637{1} | 0.12984{5} | 0.11177{4} | 0.13146{6} | 0.10691{2} | 0.40713{8} | 0.10818{3} | 0.13970{7} |
| | MSE | $\hat{\beta}$ | 0.02837{2} | 0.04554{5} | 0.03315{4} | 0.04625{6} | 0.02749{1} | 0.22976{8} | 0.03096{3} | 0.04918{7} |
| | | $\hat{\alpha}$ | 0.04897{1} | 0.07268{6} | 0.05638{4} | 0.07607{7} | 0.05135{2} | 0.22348{8} | 0.05204{3} | 0.05974{5} |
| | | $\hat{\theta}$ | 0.01131{1} | 0.01686{5} | 0.01249{4} | 0.01728{6} | 0.01143{2} | 0.16576{8} | 0.01170{3} | 0.01952{7} |
| | MRE | $\hat{\beta}$ | 0.22459{2} | 0.28452{5} | 0.24278{4} | 0.28676{6} | 0.22108{1} | 0.63911{8} | 0.23462{3} | 0.29570{7} |
| | | $\hat{\alpha}$ | 0.11065{1} | 0.13480{6} | 0.11872{4} | 0.13790{7} | 0.11330{2} | 0.23637{8} | 0.11406{3} | 0.12221{5} |
| | | $\hat{\theta}$ | 0.15876{1} | 0.19378{5} | 0.16682{4} | 0.19621{6} | 0.15957{2} | 0.60766{8} | 0.16146{3} | 0.20851{7} |
| | | $\sum RANKS$ | 12{1} | 48{5} | 36{4} | 57{6.5} | 15{2} | 72{8} | 27{3} | 57{6.5} |
| 200 | BIAS | $\hat{\beta}$ | 0.12208{2} | 0.15698{6} | 0.12868{4} | 0.15130{5} | 0.11106{1} | 0.45508{8} | 0.12688{3} | 0.16401{7} |
| | | $\hat{\alpha}$ | 0.16428{3} | 0.19845{7} | 0.16873{4} | 0.19378{6} | 0.14914{1} | 0.39519{8} | 0.16266{2} | 0.17522{5} |
| | | $\hat{\theta}$ | 0.07458{2} | 0.09412{6} | 0.08094{4} | 0.09314{5} | 0.07238{1} | 0.37465{8} | 0.07889{3} | 0.09974{7} |
| | MSE | $\hat{\beta}$ | 0.01490{2} | 0.02464{6} | 0.01656{4} | 0.02289{5} | 0.01233{1} | 0.20710{8} | 0.01610{3} | 0.02690{7} |
| | | $\hat{\alpha}$ | 0.02699{3} | 0.03938{7} | 0.02847{4} | 0.03755{6} | 0.02224{1} | 0.15618{8} | 0.02646{2} | 0.03070{5} |
| | | $\hat{\theta}$ | 0.00556{2} | 0.00886{6} | 0.00655{4} | 0.00867{5} | 0.00524{1} | 0.14036{8} | 0.00622{3} | 0.00995{7} |
| | MRE | $\hat{\beta}$ | 0.16278{2} | 0.20931{6} | 0.17157{4} | 0.20173{5} | 0.14808{1} | 0.60677{8} | 0.16917{3} | 0.21867{7} |
| | | $\hat{\alpha}$ | 0.08214{3} | 0.09922{7} | 0.08437{4} | 0.09689{6} | 0.07457{1} | 0.19759{8} | 0.08133{2} | 0.08761{5} |
| | | $\hat{\theta}$ | 0.11132{2} | 0.14048{6} | 0.12080{4} | 0.13901{5} | 0.10803{1} | 0.55917{8} | 0.11775{3} | 0.14887{7} |
| | | $\sum RANKS$ | 21{2} | 57{6.5} | 36{4} | 48{5} | 9{1} | 72{8} | 24{3} | 57{6.5} |
| 400 | BIAS | $\hat{\beta}$ | 0.08427{2} | 0.10896{5} | 0.08957{4} | 0.11127{6} | 0.08149{1} | 0.44859{8} | 0.08899{3} | 0.11513{7} |
| | | $\hat{\alpha}$ | 0.11273{2} | 0.13697{6} | 0.11639{4} | 0.14122{7} | 0.10462{1} | 0.34473{8} | 0.11391{3} | 0.12526{5} |
| | | $\hat{\theta}$ | 0.05304{1} | 0.06488{5} | 0.05442{3} | 0.06805{6} | 0.05328{2} | 0.36491{8} | 0.05528{4} | 0.06990{7} |
| | MSE | $\hat{\beta}$ | 0.00710{2} | 0.01187{5} | 0.00802{4} | 0.01238{6} | 0.00664{1} | 0.20124{8} | 0.00792{3} | 0.01325{7} |
| | | $\hat{\alpha}$ | 0.01271{2} | 0.01876{6} | 0.01355{4} | 0.01994{7} | 0.01094{1} | 0.11884{8} | 0.01298{3} | 0.01569{5} |
| | | $\hat{\theta}$ | 0.00281{1} | 0.00421{5} | 0.00296{3} | 0.00463{6} | 0.00284{2} | 0.13316{8} | 0.00306{4} | 0.00489{7} |
| | MRE | $\hat{\beta}$ | 0.11236{2} | 0.14528{5} | 0.11943{4} | 0.14836{6} | 0.10865{1} | 0.59812{8} | 0.11866{3} | 0.15350{7} |
| | | $\hat{\alpha}$ | 0.05637{2} | 0.06849{6} | 0.05820{4} | 0.07061{7} | 0.05231{1} | 0.17237{8} | 0.05696{3} | 0.06263{5} |
| | | $\hat{\theta}$ | 0.07917{1} | 0.09684{5} | 0.08122{3} | 0.10157{6} | 0.07952{2} | 0.54464{8} | 0.08251{4} | 0.10433{7} |
| | | $\sum RANKS$ | 15{2} | 48{5} | 33{4} | 57{6.5} | 12{1} | 72{8} | 30{3} | 57{6.5} |

**Table 6.** Simulation results of eight different estimators for $\beta = 0.75$, $\alpha = 2$, $\theta = 1.5$

| $n$ | Est. | Est. Par. | MLEs | LSEs | WLSEs | CRVMEs | MPSEs | PCEs | ADEs | RADEs |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $\hat{\beta}$ | 0.42906{4} | 0.46219{5} | 0.41089{3} | 0.47442{6} | 0.39423{2} | 0.58057{8} | 0.38929{1} | 0.49759{7} |
| | \|BIAS\| | $\hat{\alpha}$ | 0.53429{2} | 0.61280{6} | 0.55860{5} | 0.61337{7} | 0.54864{4} | 0.79080{8} | 0.52099{1} | 0.54519{3} |
| | | $\hat{\theta}$ | 0.59329{2} | 0.67365{6} | 0.60474{4} | 0.62962{5} | 0.59740{3} | 1.22929{8} | 0.54015{1} | 0.71056{7} |
| | | $\hat{\beta}$ | 0.18409{4} | 0.21362{5} | 0.16883{3} | 0.22507{6} | 0.15541{2} | 0.33706{8} | 0.15155{1} | 0.24760{7} |
| 20 | MSE | $\hat{\alpha}$ | 0.28547{2} | 0.37553{6} | 0.31204{5} | 0.37623{7} | 0.30101{4} | 0.62536{8} | 0.27143{1} | 0.29724{3} |
| | | $\hat{\theta}$ | 0.35200{2} | 0.45380{6} | 0.36572{4} | 0.39642{5} | 0.35689{3} | 1.51115{8} | 0.29176{1} | 0.50489{7} |
| | | $\hat{\beta}$ | 0.57208{4} | 0.61625{5} | 0.54786{3} | 0.63255{6} | 0.52564{2} | 0.77409{8} | 0.51905{1} | 0.66346{7} |
| | MRE | $\hat{\alpha}$ | 0.26714{2} | 0.30640{6} | 0.27930{5} | 0.30669{7} | 0.27432{4} | 0.39540{8} | 0.26050{1} | 0.27260{3} |
| | | $\hat{\theta}$ | 0.39553{2} | 0.44910{6} | 0.40316{4} | 0.41975{5} | 0.39827{3} | 0.81953{8} | 0.36010{1} | 0.47371{7} |
| | | $\sum RANKS$ | 24{2} | 51{5.5} | 36{4} | 54{7} | 27{3} | 72{8} | 9{1} | 51{5.5} |
| | | $\hat{\beta}$ | 0.25627{4} | 0.30102{5} | 0.25430{3} | 0.30518{6} | 0.22729{1} | 0.50710{8} | 0.24994{2} | 0.32164{7} |
| | BIAS | $\hat{\alpha}$ | 0.32936{2} | 0.39463{7} | 0.33585{4} | 0.38875{6} | 0.31556{1} | 0.57806{8} | 0.33340{3} | 0.35119{5} |
| | | $\hat{\theta}$ | 0.34965{3} | 0.41520{5} | 0.35833{4} | 0.42036{6} | 0.34741{2} | 0.98788{8} | 0.34177{1} | 0.45039{7} |
| | | $\hat{\beta}$ | 0.06567{4} | 0.09061{5} | 0.06467{3} | 0.09313{6} | 0.05166{1} | 0.25715{8} | 0.06247{2} | 0.10345{7} |
| 50 | MSE | $\hat{\alpha}$ | 0.10848{2} | 0.15574{7} | 0.11279{4} | 0.15113{6} | 0.09958{1} | 0.33416{8} | 0.11116{3} | 0.12334{5} |
| | | $\hat{\theta}$ | 0.12225{3} | 0.17239{5} | 0.12840{4} | 0.17670{6} | 0.12069{2} | 0.97590{8} | 0.11681{1} | 0.20285{7} |
| | | $\hat{\beta}$ | 0.34169{4} | 0.40136{5} | 0.33906{3} | 0.40690{6} | 0.30306{1} | 0.67613{8} | 0.33325{2} | 0.42885{7} |
| | MRE | $\hat{\alpha}$ | 0.16468{2} | 0.19732{7} | 0.16792{4} | 0.19438{6} | 0.15778{1} | 0.28903{8} | 0.16670{3} | 0.17560{5} |
| | | $\hat{\theta}$ | 0.23310{3} | 0.27680{5} | 0.23889{4} | 0.28024{6} | 0.23160{2} | 0.65859{8} | 0.22785{1} | 0.30026{7} |
| | | $\sum RANKS$ | 27{3} | 51{5} | 33{4} | 54{6} | 12{1} | 72{8} | 18{2} | 57{7} |
| | | $\hat{\beta}$ | 0.16865{2} | 0.21234{5} | 0.18232{4} | 0.21804{6} | 0.16231{1} | 0.48019{8} | 0.17889{3} | 0.22210{7} |
| | BIAS | $\hat{\alpha}$ | 0.22392{2} | 0.27326{7} | 0.24026{4} | 0.26871{6} | 0.21739{1} | 0.47918{8} | 0.23465{3} | 0.24903{5} |
| | | $\hat{\theta}$ | 0.23415{2} | 0.29299{5} | 0.25724{4} | 0.30287{6} | 0.22977{1} | 0.90603{8} | 0.25219{3} | 0.30620{7} |
| | | $\hat{\beta}$ | 0.02844{2} | 0.04509{5} | 0.03324{4} | 0.04754{6} | 0.02634{1} | 0.23058{8} | 0.03200{3} | 0.04933{7} |
| 100 | MSE | $\hat{\alpha}$ | 0.05014{2} | 0.07467{7} | 0.05772{4} | 0.07220{6} | 0.04726{1} | 0.22961{8} | 0.05506{3} | 0.06202{5} |
| | | $\hat{\theta}$ | 0.05483{2} | 0.08584{5} | 0.06617{4} | 0.09173{6} | 0.05279{1} | 0.82089{8} | 0.06360{3} | 0.09376{7} |
| | | $\hat{\beta}$ | 0.22487{2} | 0.28312{5} | 0.24309{4} | 0.29072{6} | 0.21641{1} | 0.64025{8} | 0.23852{3} | 0.29613{7} |
| | MRE | $\hat{\alpha}$ | 0.11196{2} | 0.13663{7} | 0.12013{4} | 0.13435{6} | 0.10869{1} | 0.23959{8} | 0.11733{3} | 0.12452{5} |
| | | $\hat{\theta}$ | 0.15610{2} | 0.19533{5} | 0.17149{4} | 0.20191{6} | 0.15318{1} | 0.60402{8} | 0.16813{3} | 0.20413{7} |
| | | $\sum RANKS$ | 18.0{2} | 51{5} | 36{4} | 54{6} | 9{1} | 72{8} | 27{3} | 57{7} |

**Table 6.** (cont.)

| $n$ | Est. | Est. Par. | MLEs | LSEs | WLSEs | CRVMEs | MPSEs | PCEs | ADEs | RADEs |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $\hat{\beta}$ | 0.11724{2} | 0.15330{5} | 0.12987{4} | 0.15602{6} | 0.11439{1} | 0.46752{8} | 0.12641{3} | 0.15819{7} |
| | BIAS | $\hat{\alpha}$ | 0.15541{2} | 0.19474{6} | 0.16395{4} | 0.19508{7} | 0.15373{1} | 0.41732{8} | 0.16309{3} | 0.17372{5} |
| | | $\hat{\theta}$ | 0.16557{2} | 0.21091{5} | 0.18216{4} | 0.21096{6} | 0.16222{1} | 0.85692{8} | 0.17501{3} | 0.21581{7} |
| | | $\hat{\beta}$ | 0.01374{2} | 0.02350{5} | 0.01687{4} | 0.02434{6} | 0.01309{1} | 0.21857{8} | 0.01598{3} | 0.02502{7} |
| 200 | MSE | $\hat{\alpha}$ | 0.02415{2} | 0.03792{6} | 0.02688{4} | 0.03806{7} | 0.02363{1} | 0.17416{8} | 0.02660{3} | 0.03018{5} |
| | | $\hat{\theta}$ | 0.02741{2} | 0.04448{5} | 0.03318{4} | 0.04450{6} | 0.02632{1} | 0.73432{8} | 0.03063{3} | 0.04658{7} |
| | | $\hat{\beta}$ | 0.15631{2} | 0.20440{5} | 0.17315{4} | 0.20803{6} | 0.15252{1} | 0.62336{8} | 0.16854{3} | 0.21092{7} |
| | MRE | $\hat{\alpha}$ | 0.07770{2} | 0.09737{6} | 0.08197{4} | 0.09754{7} | 0.07686{1} | 0.20866{8} | 0.08154{3} | 0.08686{5} |
| | | $\hat{\theta}$ | 0.11038{2} | 0.14061{5} | 0.12144{4} | 0.14064{6} | 0.10815{1} | 0.57128{8} | 0.11667{3} | 0.14388{7} |
| | | $\sum RANKS$ | 18{2} | 48{5} | 36{4} | 57{6.5} | 9{1} | 72{8} | 27{3} | 57{6.5} |
| | | $\hat{\beta}$ | 0.08474{2} | 0.11052{6} | 0.08948{4} | 0.11022{5} | 0.08174{1} | 0.41621{8} | 0.08923{3} | 0.11353{7} |
| | BIAS | $\hat{\alpha}$ | 0.11235{2} | 0.13674{6} | 0.11364{3} | 0.13868{7} | 0.10914{1} | 0.33532{8} | 0.11468{4} | 0.12120{5} |
| | | $\hat{\theta}$ | 0.11599{2} | 0.15200{6} | 0.12736{4} | 0.14572{5} | 0.11376{1} | 0.74445{8} | 0.12465{3} | 0.15534{7} |
| | | $\hat{\beta}$ | 0.00718{2} | 0.01222{6} | 0.00801{4} | 0.01215{5} | 0.00668{1} | 0.17323{8} | 0.00796{3} | 0.01289{7} |
| 400 | MSE | $\hat{\alpha}$ | 0.01262{2} | 0.01870{6} | 0.01291{3} | 0.01923{7} | 0.01191{1} | 0.11244{8} | 0.01315{4} | 0.01469{5} |
| | | $\hat{\theta}$ | 0.01345{2} | 0.02310{6} | 0.01622{4} | 0.02124{5} | 0.01294{1} | 0.55421{8} | 0.01554{3} | 0.02413{7} |
| | | $\hat{\beta}$ | 0.11299{2} | 0.14736{6} | 0.11931{4} | 0.14695{5} | 0.10899{1} | 0.55494{8} | 0.11897{3} | 0.15137{7} |
| | MRE | $\hat{\alpha}$ | 0.05617{2} | 0.06837{6} | 0.05682{3} | 0.06934{7} | 0.05457{1} | 0.16766{8} | 0.05734{4} | 0.06060{5} |
| | | $\hat{\theta}$ | 0.07733{2} | 0.10133{6} | 0.08491{4} | 0.09715{5} | 0.07584{1} | 0.49630{8} | 0.08310{3} | 0.10356{7} |
| | | $\sum RANKS$ | 18{2} | 54{6} | 33{4} | 51{5} | 9{1} | 72{8} | 30{3} | 57{7} |

**Table 7.** Simulation results of eight different estimators for $\beta = 2.75$, $\alpha = 0.5$, $\theta = 0.67$

| $n$ | Est. | Est. Par. | MLEs | LSEs | WLSEs | CRVMEs | MPSEs | PCEs | ADEs | RADEs |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $\hat{\beta}$ | 2.64153{8} | 2.25648{5} | 2.09198{4} | 2.33980{6} | 1.99359{1} | 2.04498{2} | 2.05114{3} | 2.46340{7} |
| | \|BIAS\| | $\hat{\alpha}$ | 0.09194{1} | 0.10941{5} | 0.10301{4} | 0.11683{7} | 0.09699{2} | 0.25218{8} | 0.09736{3} | 0.11173{6} |
| | | $\hat{\theta}$ | 0.66997{8} | 0.66974{3} | 0.66979{4} | 0.66966{2} | 0.66994{7} | 0.63964{1} | 0.66980{5} | 0.66986{6} |
| | | $\hat{\beta}$ | 6.97767{8} | 5.09170{5} | 4.37638{4} | 5.47467{6} | 3.97439{1} | 4.18193{2} | 4.20718{3} | 6.06832{7} |
| 20 | MSE | $\hat{\alpha}$ | 0.00845{1} | 0.01197{5} | 0.01061{4} | 0.01365{7} | 0.00941{2} | 0.06360{8} | 0.00948{3} | 0.01248{6} |
| | | $\hat{\theta}$ | 0.44886{8} | 0.44855{3} | 0.44862{4} | 0.44844{2} | 0.44881{7} | 0.40914{1} | 0.44864{5} | 0.44871{6} |
| | | $\hat{\beta}$ | 0.96056{8} | 0.82054{5} | 0.76072{4} | 0.85084{6} | 0.72494{1} | 0.74363{2} | 0.74587{3} | 0.89578{7} |
| | MRE | $\hat{\alpha}$ | 0.18388{1} | 0.21881{5} | 0.20602{4} | 0.23367{7} | 0.19398{2} | 0.50436{8} | 0.19472{3} | 0.22346{6} |
| | | $\hat{\theta}$ | 0.99996{8} | 0.99961{3} | 0.99969{4} | 0.99949{2} | 0.99990{7} | 0.95469{1} | 0.99971{5} | 0.99979{6} |
| | | $\sum RANKS$ | 51{7} | 39{5} | 36{4} | 45{6} | 30{1} | 33{2.5} | 33{2.5} | 57{8} |

**Table 7.** (cont.)

| $n$ | Est. | Est. Par. | MLEs | LSEs | WLSEs | CRVMEs | MPSEs | PCEs | ADEs | RADEs |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $\hat{\beta}$ | 1.55748{4} | 1.83382{5} | 1.50209{2} | 1.87975{6} | 1.33840{1} | 2.10345{8} | 1.53872{3} | 1.99573{7} |
| | BIAS | $\hat{\alpha}$ | 0.06001{1} | 0.07382{7} | 0.06264{2} | 0.07183{6} | 0.06292{3} | 0.21886{8} | 0.06410{4} | 0.07016{5} |
| | | $\hat{\theta}$ | 0.66949{8} | 0.66877{6} | 0.66831{3} | 0.66859{4} | 0.66532{2} | 0.63522{1} | 0.66867{5} | 0.66920{7} |
| | | $\hat{\beta}$ | 2.42575{4} | 3.36288{5} | 2.25626{2} | 3.53347{6} | 1.79131{1} | 4.42451{8} | 2.36765{3} | 3.98294{7} |
| 50 | MSE | $\hat{\alpha}$ | 0.00360{1} | 0.00545{7} | 0.00392{2} | 0.00516{6} | 0.00396{3} | 0.04790{8} | 0.00411{4} | 0.00492{5} |
| | | $\hat{\theta}$ | 0.44822{8} | 0.44726{6} | 0.44663{3} | 0.44702{4} | 0.44265{2} | 0.40351{1} | 0.44712{5} | 0.44783{7} |
| | | $\hat{\beta}$ | 0.56636{4} | 0.66684{5} | 0.54621{2} | 0.68355{6} | 0.48669{1} | 0.76489{8} | 0.55953{3} | 0.72572{7} |
| | MRE | $\hat{\alpha}$ | 0.12001{1} | 0.14764{7} | 0.12528{2} | 0.14367{6} | 0.12584{3} | 0.43773{8} | 0.12820{4} | 0.14032{5} |
| | | $\hat{\theta}$ | 0.99924{8} | 0.99817{6} | 0.99747{3} | 0.99790{4} | 0.99302{2} | 0.94809{1} | 0.99802{5} | 0.99881{7} |
| | | $\sum RANKS$ | 39{4} | 54{7} | 21{2} | 48{5} | 18{1} | 51{6} | 36{3} | 57{8} |
| | | $\hat{\beta}$ | 1.12674{3} | 1.49640{6} | 1.12077{2} | 1.48966{5} | 0.96032{1} | 2.16606{8} | 1.19167{4} | 1.61035{7} |
| | BIAS | $\hat{\alpha}$ | 0.04503{2} | 0.05457{6} | 0.04616{3} | 0.05464{7} | 0.04454{1} | 0.19827{8} | 0.04626{4} | 0.05148{5} |
| | | $\hat{\theta}$ | 0.61440{2} | 0.66580{6} | 0.62582{3} | 0.66612{7} | 0.56960{1} | 0.63434{5} | 0.63328{4} | 0.66763{8} |
| | | $\hat{\beta}$ | 1.26954{3} | 2.23921{6} | 1.25612{2} | 2.21909{5} | 0.92221{1} | 4.69182{8} | 1.42007{4} | 2.59324{7} |
| 100 | MSE | $\hat{\alpha}$ | 0.00203{2} | 0.00298{6} | 0.00213{3} | 0.00299{7} | 0.00198{1} | 0.03931{8} | 0.00214{4} | 0.00265{5} |
| | | $\hat{\theta}$ | 0.37748{2} | 0.44328{6} | 0.39165{3} | 0.44372{7} | 0.32445{1} | 0.40239{5} | 0.40105{4} | 0.44572{8} |
| | | $\hat{\beta}$ | 0.40972{3} | 0.54415{6} | 0.40755{2} | 0.54170{5} | 0.34921{1} | 0.78766{8} | 0.43333{4} | 0.58558{7} |
| | MRE | $\hat{\alpha}$ | 0.09005{2} | 0.10913{6} | 0.09233{3} | 0.10927{7} | 0.08908{1} | 0.39654{8} | 0.09252{4} | 0.10296{5} |
| | | $\hat{\theta}$ | 0.91701{2} | 0.99372{6} | 0.93406{3} | 0.99421{7} | 0.85015{1} | 0.94678{5} | 0.94520{4} | 0.99646{8} |
| | | $\sum RANKS$ | 21{2} | 54{5} | 24{3} | 57{6} | 9{1} | 63{8} | 36{4} | 60{7} |
| | | $\hat{\beta}$ | 0.80039{2} | 1.15145{6} | 0.80254{3} | 1.14572{5} | 0.10457{1} | 2.21651{8} | 0.86725{4} | 1.27905{7} |
| | BIAS | $\hat{\alpha}$ | 0.03248{2} | 0.03944{6} | 0.03372{3} | 0.03986{7} | 0.02866{1} | 0.19788{8} | 0.03433{4} | 0.03844{5} |
| | | $\hat{\theta}$ | 0.47184{2} | 0.62716{5} | 0.48355{3} | 0.62746{6} | 0.15901{1} | 0.63970{7} | 0.50685{4} | 0.65363{8} |
| | | $\hat{\beta}$ | 0.64062{2} | 1.32584{6} | 0.64407{3} | 1.31268{5} | 0.01093{1} | 4.91294{8} | 0.75213{4} | 1.63596{7} |
| 200 | MSE | $\hat{\alpha}$ | 0.00106{2} | 0.00156{6} | 0.00114{3} | 0.00159{7} | 0.00082{1} | 0.03916{8} | 0.00118{4} | 0.00148{5} |
| | | $\hat{\theta}$ | 0.22263{2} | 0.39332{5} | 0.23382{3} | 0.39370{6} | 0.02528{1} | 0.40921{7} | 0.25690{4} | 0.42723{8} |
| | | $\hat{\beta}$ | 0.29105{2} | 0.41871{6} | 0.29183{3} | 0.41663{5} | 0.03802{1} | 0.80601{8} | 0.31536{4} | 0.46511{7} |
| | MRE | $\hat{\alpha}$ | 0.06496{2} | 0.07888{6} | 0.06745{3} | 0.07973{7} | 0.05732{1} | 0.39577{8} | 0.06866{4} | 0.07689{5} |
| | | $\hat{\theta}$ | 0.70423{2} | 0.93605{5} | 0.72171{3} | 0.93650{6} | 0.23733{1} | 0.95477{7} | 0.75650{4} | 0.97556{8} |
| | | $\sum RANKS$ | 18{2} | 51{5} | 27{3} | 54{6} | 9{1} | 69{8} | 36{4} | 60{7} |

**Table 7.** (cont.)

| $n$ | Est. | Est. Par. | MLEs | LSEs | WLSEs | CRVMEs | MPSEs | PCEs | ADEs | RADEs |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $\hat{\beta}$ | 0.55160{3} | 0.82931{5} | 0.52890{2} | 0.85586{6} | 0.04107{1} | 2.39111{8} | 0.62955{4} | 0.94368{7} |
| | BIAS | $\hat{\alpha}$ | 0.02256{2} | 0.02914{7} | 0.02304{3} | 0.02853{6} | 0.01733{1} | 0.20416{8} | 0.02331{4} | 0.02777{5} |
| | | $\hat{\theta}$ | 0.34604{3} | 0.49476{5} | 0.33707{2} | 0.50442{6} | 0.06425{1} | 0.64741{8} | 0.38704{4} | 0.54130{7} |
| | | $\hat{\beta}$ | 0.30427{3} | 0.68776{5} | 0.27973{2} | 0.73249{6} | 0.00169{1} | 5.71740{8} | 0.39633{4} | 0.89054{7} |
| 400 | MSE | $\hat{\alpha}$ | 0.00051{2} | 0.00085{7} | 0.00053{3} | 0.00081{6} | 0.00030{1} | 0.04168{8} | 0.00054{4} | 0.00077{5} |
| | | $\hat{\theta}$ | 0.11975{3} | 0.24479{5} | 0.11362{2} | 0.25444{6} | 0.00413{1} | 0.41914{8} | 0.14980{4} | 0.29301{7} |
| | | $\hat{\beta}$ | 0.20058{3} | 0.30157{5} | 0.19233{2} | 0.31122{6} | 0.01494{1} | 0.86949{8} | 0.22893{4} | 0.34316{7} |
| | MRE | $\hat{\alpha}$ | 0.04511{2} | 0.05828{7} | 0.04609{3} | 0.05707{6} | 0.03466{1} | 0.40831{8} | 0.04661{4} | 0.05555{5} |
| | | $\hat{\theta}$ | 0.51648{3} | 0.73845{5} | 0.50309{2} | 0.75287{6} | 0.09589{1} | 0.96628{8} | 0.57767{4} | 0.80791{7} |
| | | $\sum RANKS$ | 24{3} | 51{5} | 21{2} | 54{6} | 9{1} | 72{8} | 36{4} | 57{7} |

**Table 8.** Simulation results of eight different estimators for $\beta = 2.75$, $\alpha = 0.5$, $\theta = 1.5$

| $n$ | Est. | Est. Par. | MLEs | LSEs | WLSEs | CRVMEs | MPSEs | PCEs | ADEs | RADEs |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $\hat{\beta}$ | 3.48234{8} | 2.23256{5} | 2.13330{4} | 2.31571{6} | 1.86026{1} | 1.96962{2} | 2.02251{3} | 2.44492{7} |
| | \|BIAS\| | $\hat{\alpha}$ | 0.08779{1} | 0.11077{6} | 0.10578{4} | 0.11609{7} | 0.09656{2} | 0.25163{8} | 0.09703{3} | 0.10642{5} |
| | | $\hat{\theta}$ | 1.49994{7} | 1.49923{2} | 1.4996{06} | 1.49896{1} | 1.49952{3} | 1.57402{8} | 1.49958{5} | 1.49956{4} |
| | | $\hat{\beta}$ | 12.12672{8} | 4.98434{5} | 4.55095{4} | 5.36253{6} | 3.46059{1} | 3.87939{2} | 4.09056{3} | 5.97765{7} |
| 20 | MSE | $\hat{\alpha}$ | 0.00771{1} | 0.01227{6} | 0.01119{4} | 0.01348{7} | 0.00932{2} | 0.06332{8} | 0.00942{3} | 0.01133{5} |
| | | $\hat{\theta}$ | 2.24981{7} | 2.24770{2} | 2.24880{6} | 2.24688{1} | 2.24857{3} | 2.47753{8} | 2.24875{5} | 2.24869{4} |
| | | $\hat{\beta}$ | 1.26631{8} | 0.81184{5} | 0.77574{4} | 0.84208{6} | 0.67646{1} | 0.71622{2} | 0.73546{3} | 0.88906{7} |
| | MRE | $\hat{\alpha}$ | 0.17558{1} | 0.22154{6} | 0.21157{4} | 0.23218{7} | 0.19313{2} | 0.50326{8} | 0.19407{3} | 0.21285{5} |
| | | $\hat{\theta}$ | 0.99996{7} | 0.99949{2} | 0.99973{6} | 0.99931{1} | 0.99968{3} | 1.04934{8} | 0.99972{5} | 0.99971{4} |
| | | $\sum RANKS$ | 48{6.5} | 39{3} | 42{4.5} | 42{4.5} | 18{1} | 54{8} | 33{2} | 48{6.5} |
| | | $\hat{\beta}$ | 1.63011{4} | 1.82611{5} | 1.59594{3} | 1.89649{6} | 1.23985{1} | 2.07735{8} | 1.53191{2} | 2.05525{7} |
| | BIAS | $\hat{\alpha}$ | 0.06158{2} | 0.07168{6} | 0.06683{4} | 0.07338{7} | 0.06022{1} | 0.21958{8} | 0.06549{3} | 0.06881{5} |
| | | $\hat{\theta}$ | 1.49925{8} | 1.49729{5} | 1.49737{6} | 1.49667{4} | 1.47269{2} | 1.45143{1} | 1.49662{3} | 1.49795{7} |
| | | $\hat{\beta}$ | 2.65727{4} | 3.33469{5} | 2.54703{3} | 3.59669{6} | 1.53722{1} | 4.31539{8} | 2.34676{2} | 4.22406{7} |
| 50 | MSE | $\hat{\alpha}$ | 0.00379{2} | 0.00514{6} | 0.00447{4} | 0.00538{7} | 0.00363{1} | 0.04821{8} | 0.00429{3} | 0.00473{5} |
| | | $\hat{\theta}$ | 2.24775{8} | 2.24188{5} | 2.24212{6} | 2.24001{4} | 2.16881{2} | 2.10665{1} | 2.23987{3} | 2.24385{7} |
| | | $\hat{\beta}$ | 0.59277{4} | 0.66404{5} | 0.58034{3} | 0.68963{6} | 0.45085{1} | 0.7554{08} | 0.55706{2} | 0.74736{7} |
| | MRE | $\hat{\alpha}$ | 0.12315{2} | 0.14336{6} | 0.13365{4} | 0.14675{7} | 0.12043{1} | 0.43916{8} | 0.13097{3} | 0.13762{5} |
| | | $\hat{\theta}$ | 0.99950{8} | 0.99819{5} | 0.99825{6} | 0.99778{4} | 0.98179{2} | 0.96762{1} | 0.99775{3} | 0.99863{7} |
| | | $\sum RANKS$ | 42{4} | 48{5} | 39{3} | 51{6.5} | 12{1} | 51{6.5} | 24{2} | 57{8} |

**Table 8.** (cont.)

| $n$ | Est. | Est. Par. | MLEs | LSEs | WLSEs | CRVMEs | MPSEs | PCEs | ADEs | RADEs |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $\hat{\beta}$ | 1.13533{2} | 1.49870{5} | 1.20080{4} | 1.50242{6} | 0.35642{1} | 2.22868{8} | 1.17518{3} | 1.60461{7} |
| | BIAS | $\hat{\alpha}$ | 0.04480{2} | 0.05452{7} | 0.04767{4} | 0.05381{6} | 0.04059{1} | 0.21202{8} | 0.04716{3} | 0.05055{5} |
| | | $\hat{\theta}$ | 1.36905{2} | 1.49096{7} | 1.44972{5} | 1.49035{6} | 0.24427{1} | 1.41759{4} | 1.41582{3} | 1.49412{8} |
| | | $\hat{\beta}$ | 1.28897{2} | 2.24611{5} | 1.44191{4} | 2.25726{6} | 0.12703{1} | 4.96701{8} | 1.38104{3} | 2.57476{7} |
| 100 | MSE | $\hat{\alpha}$ | 0.00201{2} | 0.00297{7} | 0.00227{4} | 0.00290{6} | 0.00165{1} | 0.04495{8} | 0.00222{3} | 0.00256{5} |
| | | $\hat{\theta}$ | 1.87430{2} | 2.22297{7} | 2.10168{5} | 2.22116{6} | 0.05967{1} | 2.00958{4} | 2.00455{3} | 2.23240{8} |
| | | $\hat{\beta}$ | 0.41285{2} | 0.54498{5} | 0.43665{4} | 0.54633{6} | 0.12961{1} | 0.81043{8} | 0.42734{3} | 0.58349{7} |
| | MRE | $\hat{\alpha}$ | 0.08959{2} | 0.10904{7} | 0.09534{4} | 0.10761{6} | 0.08118{1} | 0.42404{8} | 0.09433{3} | 0.10111{5} |
| | | $\hat{\theta}$ | 0.91270{2} | 0.99398{7} | 0.96648{5} | 0.99357{6} | 0.16285{1} | 0.94506{4} | 0.94388{3} | 0.99608{8} |
| | | $\sum RANKS$ | 18{2} | 57{6} | 39{4} | 54{5} | 9{1} | 60{7.5} | 27{3} | 60{7.5} |
| | | $\hat{\beta}$ | 0.78682{2} | 1.14165{5} | 0.88392{4} | 1.16496{6} | 0.14712{1} | 2.34227{8} | 0.87133{3} | 1.26495{7} |
| | BIAS | $\hat{\alpha}$ | 0.03224{2} | 0.03978{6} | 0.03354{3} | 0.04082{7} | 0.02674{1} | 0.20880{8} | 0.03361{4} | 0.03755{5} |
| | | $\hat{\theta}$ | 1.02634{2} | 1.38688{5} | 1.18348{4} | 1.41165{6} | 0.10209{1} | 1.41490{7} | 1.14431{3} | 1.46127{8} |
| | | $\hat{\beta}$ | 0.61908{2} | 1.30336{5} | 0.78131{4} | 1.35714{6} | 0.02164{1} | 5.48624{8} | 0.75922{3} | 1.60009{7} |
| 200 | MSE | $\hat{\alpha}$ | 0.00104{2} | 0.00158{6} | 0.00112{3} | 0.00167{7} | 0.00071{1} | 0.04360{8} | 0.00113{4} | 0.00141{5} |
| | | $\hat{\theta}$ | 1.05338{2} | 1.92344{5} | 1.40063{4} | 1.99276{6} | 0.01042{1} | 2.00194{7} | 1.30944{3} | 2.13531{8} |
| | | $\hat{\beta}$ | 0.28612{2} | 0.41514{5} | 0.32142{4} | 0.42362{6} | 0.05350{1} | 0.85174{8} | 0.31685{3} | 0.45998{7} |
| | MRE | $\hat{\alpha}$ | 0.06449{2} | 0.07957{6} | 0.06707{3} | 0.08163{7} | 0.05347{1} | 0.41759{8} | 0.06723{4} | 0.07510{5} |
| | | $\hat{\theta}$ | 0.68423{2} | 0.92459{5} | 0.78899{4} | 0.9411{6} | 0.06806{1} | 0.94327{7} | 0.76287{3} | 0.97418{8} |
| | | $\sum RANKS$ | 18{2} | 48{5} | 33{4} | 57{6} | 9{1} | 69{8} | 30{3} | 60{7} |
| | | $\hat{\beta}$ | 0.55264{2} | 0.85360{6} | 0.61850{3} | 0.80650{5} | 0.07793{1} | 2.42350{8} | 0.65406{4} | 0.94629{7} |
| | BIAS | $\hat{\alpha}$ | 0.02144{2} | 0.02946{7} | 0.02358{3} | 0.02750{6} | 0.01549{1} | 0.20641{8} | 0.02374{4} | 0.02711{5} |
| | | $\hat{\theta}$ | 0.75628{2} | 1.13602{6} | 0.85756{3} | 1.10882{5} | 0.05503{1} | 1.42167{8} | 0.88385{4} | 1.20951{7} |
| | | $\hat{\beta}$ | 0.30541{2} | 0.72864{6} | 0.38254{3} | 0.65044{5} | 0.00607{1} | 5.87337{8} | 0.42779{4} | 0.89546{7} |
| 400 | MSE | $\hat{\alpha}$ | 0.00046{2} | 0.00087{7} | 0.00056{3.5} | 0.00076{6} | 0.00024{1} | 0.04260{8} | 0.00056{3.5} | 0.00074{5} |
| | | $\hat{\theta}$ | 0.57195{2} | 1.29053{6} | 0.73541{3} | 1.22949{5} | 0.00303{1} | 2.02114{8} | 0.78119{4} | 1.46292{7} |
| | | $\hat{\beta}$ | 0.20096{2} | 0.31040{6} | 0.22491{3} | 0.29327{5} | 0.02834{1} | 0.88127{8} | 0.23784{4} | 0.34410{7} |
| | MRE | $\hat{\alpha}$ | 0.04287{2} | 0.05893{7} | 0.04716{3} | 0.05500{6} | 0.03097{1} | 0.41281{8} | 0.04749{4} | 0.05422{5} |
| | | $\hat{\theta}$ | 0.50418{2} | 0.75734{6} | 0.57171{3} | 0.73922{5} | 0.03668{1} | 0.94778{8} | 0.58923{4} | 0.80634{7} |
| | | $\sum RANKS$ | 18{2} | 57{6.5} | 27.5{3} | 48{5} | 9{1} | 72{8} | 35.5{4} | 57{6.5} |

**Table 9.** Simulation results of eight different estimators for $\beta = 2.75$, $\alpha = 2$, $\theta = 0.67$

| $n$ | Est. | Est. Par. | MLEs | LSEs | WLSEs | CRVMEs | MPSEs | PCEs | ADEs | RADEs |
|---|---|---|---|---|---|---|---|---|---|---|
| 20 | \|BIAS\| | $\hat{\beta}$ | 2.87148{8} | 2.28886{4} | 2.14573{3} | 2.31264{5} | 2.03953{1} | 2.50653{7} | 2.095{2} | 2.50123{6} |
| | | $\hat{\alpha}$ | 0.36811{1} | 0.46176{7} | 0.4227{4} | 0.45224{6} | 0.39324{2} | 0.65425{8} | 0.39582{3} | 0.434{5} |
| | | $\hat{\theta}$ | 0.61045{7} | 0.57121{5} | 0.54852{3} | 0.56373{4} | 0.52707{1} | 0.64449{8} | 0.52806{2} | 0.59419{6} |
| | MSE | $\hat{\beta}$ | 8.24549{8} | 5.23887{4} | 4.60417{3} | 5.34832{5} | 4.15967{1} | 6.28269{7} | 4.38903{2} | 6.25616{6} |
| | | $\hat{\alpha}$ | 0.1355{1} | 0.21322{7} | 0.17868{4} | 0.20452{6} | 0.15464{2} | 0.42805{8} | 0.15668{3} | 0.18836{5} |
| | | $\hat{\theta}$ | 0.37266{7} | 0.32628{5} | 0.30087{3} | 0.31779{4} | 0.27781{1} | 0.41536{8} | 0.27885{2} | 0.35307{6} |
| | MRE | $\hat{\beta}$ | 1.04417{8} | 0.83231{4} | 0.78027{3} | 0.84096{5} | 0.74165{1} | 0.91147{7} | 0.76182{2} | 0.90954{6} |
| | | $\hat{\alpha}$ | 0.18405{1} | 0.23088{7} | 0.21135{4} | 0.22612{6} | 0.19662{2} | 0.32713{8} | 0.19791{3} | 0.217{5} |
| | | $\hat{\theta}$ | 0.91113{7} | 0.85255{5} | 0.81868{3} | 0.84138{4} | 0.78668{1} | 0.96192{8} | 0.78815{2} | 0.88686{6} |
| | | $\sum RANKS$ | 48{5.5} | 48{5.5} | 30{3} | 45{4} | 12{1} | 69{8} | 21{2} | 51{7} |
| 50 | BIAS | $\hat{\beta}$ | 1.64454{4} | 1.85477{6} | 1.58464{3} | 1.85467{5} | 1.37147{1} | 2.37322{8} | 1.52974{2} | 1.99045{7} |
| | | $\hat{\alpha}$ | 0.24329{1} | 0.30112{7} | 0.26364{4} | 0.29724{6} | 0.24861{2} | 0.50846{8} | 0.26215{3} | 0.27987{5} |
| | | $\hat{\theta}$ | 0.31641{3} | 0.41524{5} | 0.32962{4} | 0.4155{6} | 0.28285{1} | 0.63377{8} | 0.30227{2} | 0.48827{7} |
| | MSE | $\hat{\beta}$ | 2.70451{4} | 3.44015{6} | 2.51107{3} | 3.4398{5} | 1.88093{1} | 5.63215{8} | 2.34011{2} | 3.96188{7} |
| | | $\hat{\alpha}$ | 0.05919{1} | 0.09067{7} | 0.0695{4} | 0.08835{6} | 0.06181{2} | 0.25853{8} | 0.06872{3} | 0.07833{5} |
| | | $\hat{\theta}$ | 0.10011{3} | 0.17243{5} | 0.10865{4} | 0.17264{6} | 0.08{1} | 0.40166{8} | 0.09137{2} | 0.23841{7} |
| | MRE | $\hat{\beta}$ | 0.59801{4} | 0.67446{6} | 0.57623{3} | 0.67443{5} | 0.49872{1} | 0.86299{8} | 0.55627{2} | 0.7238{7} |
| | | $\hat{\alpha}$ | 0.12164{1} | 0.15056{7} | 0.13182{4} | 0.14862{6} | 0.12431{2} | 0.25423{8} | 0.13107{3} | 0.13993{5} |
| | | $\hat{\theta}$ | 0.47225{3} | 0.61976{5} | 0.49197{4} | 0.62015{6} | 0.42216{1} | 0.94592{8} | 0.45115{2} | 0.72876{7} |
| | | $\sum RANKS$ | 24{3} | 54{6} | 33{4} | 51{5} | 12{1} | 72{8} | 21{2} | 57{7} |
| 100 | BIAS | $\hat{\beta}$ | 1.1315{2} | 1.4671{5} | 1.17812{4} | 1.49862{6} | 0.98744{1} | 2.40205{8} | 1.17151{3} | 1.62479{7} |
| | | $\hat{\alpha}$ | 0.17984{2} | 0.22024{7} | 0.18836{3} | 0.2197{6} | 0.17772{1} | 0.41277{8} | 0.18961{4} | 0.20521{5} |
| | | $\hat{\theta}$ | 0.1915{2} | 0.28261{5} | 0.21443{4} | 0.29204{6} | 0.18124{1} | 0.62265{8} | 0.21085{3} | 0.31746{7} |
| | MSE | $\hat{\beta}$ | 1.2803{2} | 2.15238{5} | 1.38796{4} | 2.24587{6} | 0.97504{1} | 5.76986{8} | 1.37244{3} | 2.63995{7} |
| | | $\hat{\alpha}$ | 0.03234{2} | 0.04851{7} | 0.03548{3} | 0.04827{6} | 0.03158{1} | 0.17038{8} | 0.03595{4} | 0.04211{5} |
| | | $\hat{\theta}$ | 0.03667{2} | 0.07987{5} | 0.04598{4} | 0.08529{6} | 0.03285{1} | 0.38769{8} | 0.04446{3} | 0.10078{7} |
| | MRE | $\hat{\beta}$ | 0.41146{2} | 0.53349{5} | 0.42841{4} | 0.54495{6} | 0.35907{1} | 0.87347{8} | 0.426{3} | 0.59083{7} |
| | | $\hat{\alpha}$ | 0.08992{2} | 0.11012{7} | 0.09418{3} | 0.10985{6} | 0.08886{1} | 0.20638{8} | 0.09481{4} | 0.10261{5} |
| | | $\hat{\theta}$ | 0.28583{2} | 0.4218{5} | 0.32004{4} | 0.43589{6} | 0.27051{1} | 0.92932{8} | 0.31471{3} | 0.47381{7} |
| | | $\sum RANKS$ | 18{2} | 51{5} | 33{4} | 54{6} | 9{1} | 72{8} | 30{3} | 57{7} |

**Table 9.** (cont.)

| $n$ | Est. | Est. Par. | MLEs | LSEs | WLSEs | CRVMEs | MPSEs | PCEs | ADEs | RADEs |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $\hat{\beta}$ | 0.79703{2} | 1.16488{6} | 0.89157{4} | 1.12918{5} | 0.58107{1} | 2.48122{8} | 0.88825{3} | 1.31088{7} |
| | BIAS | $\hat{\alpha}$ | 0.12916{2} | 0.16398{7} | 0.13710{4} | 0.15787{6} | 0.11987{1} | 0.34533{8} | 0.13370{3} | 0.15731{5} |
| | | $\hat{\theta}$ | 0.13172{2} | 0.20397{6} | 0.15265{4} | 0.19494{5} | 0.08622{1} | 0.61442{8} | 0.14890{3} | 0.23363{7} |
| | | $\hat{\beta}$ | 0.63525{2} | 1.35695{6} | 0.79490{4} | 1.27504{5} | 0.33764{1} | 6.15645{8} | 0.78899{3} | 1.71841{7} |
| 200 | MSE | $\hat{\alpha}$ | 0.01668{2} | 0.02689{7} | 0.01880{4} | 0.02492{6} | 0.01437{1} | 0.11925{8} | 0.01787{3} | 0.02475{5} |
| | | $\hat{\theta}$ | 0.01735{2} | 0.04160{6} | 0.02330{4} | 0.03800{5} | 0.00743{1} | 0.37751{8} | 0.02217{3} | 0.05458{7} |
| | | $\hat{\beta}$ | 0.28983{2} | 0.42359{6} | 0.32421{4} | 0.41061{5} | 0.21130{1} | 0.90226{8} | 0.32300{3} | 0.47668{7} |
| | MRE | $\hat{\alpha}$ | 0.06458{2} | 0.08199{7} | 0.06855{4} | 0.07893{6} | 0.05994{1} | 0.17266{8} | 0.06685{3} | 0.07865{5} |
| | | $\hat{\theta}$ | 0.19660{2} | 0.30444{6} | 0.22783{4} | 0.29095{5} | 0.12869{1} | 0.91705{8} | 0.22224{3} | 0.34871{7} |
| | | $\sum RANKS$ | 18{2} | 57{6.5} | 36{4} | 48{5} | 9{1} | 72{8} | 27{3} | 57{6.5} |
| | | $\hat{\beta}$ | 0.56052{2} | 0.85239{6} | 0.63031{3} | 0.82937{5} | 0.01596{1} | 3.46102{8} | 0.64853{4} | 0.94606{7} |
| | BIAS | $\hat{\alpha}$ | 0.08617{2} | 0.11626{7} | 0.09367{3} | 0.11492{6} | 0.07626{1} | 0.30696{8} | 0.09447{4} | 0.10726{5} |
| | | $\hat{\theta}$ | 0.09090{2} | 0.14109{6} | 0.10280{3} | 0.13740{5} | 0.02291{1} | 0.60680{8} | 0.10598{4} | 0.15594{7} |
| | | $\hat{\beta}$ | 0.31418{2} | 0.72657{6} | 0.39729{3} | 0.68785{5} | 0.00025{1} | 11.97882{8} | 0.42059{4} | 0.89503{7} |
| 400 | MSE | $\hat{\alpha}$ | 0.00743{2} | 0.01352{7} | 0.00877{3} | 0.01321{6} | 0.00582{1} | 0.09422{8} | 0.00892{4} | 0.01151{5} |
| | | $\hat{\theta}$ | 0.00826{2} | 0.01991{6} | 0.01057{3} | 0.01888{5} | 0.00052{1} | 0.36821{8} | 0.01123{4} | 0.02432{7} |
| | | $\hat{\beta}$ | 0.20382{2} | 0.30996{6} | 0.22920{3} | 0.30159{5} | 0.00580{1} | 1.25855{8} | 0.23583{4} | 0.34402{7} |
| | MRE | $\hat{\alpha}$ | 0.04309{2} | 0.05813{7} | 0.04684{3} | 0.05746{6} | 0.03813{1} | 0.15348{8} | 0.04723{4} | 0.05363{5} |
| | | $\hat{\theta}$ | 0.13567{2} | 0.21059{6} | 0.15344{3} | 0.20508{5} | 0.03419{1} | 0.90567{8} | 0.15819{4} | 0.23275{7} |
| | | $\sum RANKS$ | 18{2} | 57{6.5} | 27{3} | 48{5} | 9{1} | 72{8} | 36{4} | 57{6.5} |

**Table 10.** Simulation results of eight different estimators for $\beta = 2.75$, $\alpha = 2$, $\theta = 1.5$

| $n$ | Est. | Est. Par. | MLEs | LSEs | WLSEs | CRVMEs | MPSEs | PCEs | ADEs | RADEs |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $\hat{\beta}$ | 3.01331{8} | 2.31553{4} | 2.07515{3} | 2.36979{5} | 2.02820{1} | 2.46525{6} | 2.06262{2} | 2.49269{7} |
| | \|BIAS\| | $\hat{\alpha}$ | 0.36438{1} | 0.45589{7} | 0.39677{3} | 0.44265{6} | 0.39796{4} | 0.67015{8} | 0.38589{2} | 0.43484{5} |
| | | $\hat{\theta}$ | 1.36890{7} | 1.27190{5} | 1.21218{3} | 1.25910{4} | 1.16663{2} | 1.44234{8} | 1.16528{1} | 1.32874{6} |
| | | $\hat{\beta}$ | 9.08007{8} | 5.36166{4} | 4.30625{3} | 5.61588{5} | 4.11359{1} | 6.07745{6} | 4.25441{2} | 6.21351{7} |
| 20 | MSE | $\hat{\alpha}$ | 0.13277{1} | 0.20784{7} | 0.15742{3} | 0.19594{6} | 0.15837{4} | 0.44911{8} | 0.14891{2} | 0.18909{5} |
| | | $\hat{\theta}$ | 1.87388{7} | 1.61774{5} | 1.46937{3} | 1.58533{4} | 1.36102{2} | 2.08035{8} | 1.35787{1} | 1.76555{6} |
| | | $\hat{\beta}$ | 1.09575{8} | 0.84201{4} | 0.75460{3} | 0.86174{5} | 0.73753{1} | 0.89645{6} | 0.75004{2} | 0.90643{7} |
| | MRE | $\hat{\alpha}$ | 0.18219{1} | 0.22795{7} | 0.19838{3} | 0.22132{6} | 0.19898{4} | 0.33508{8} | 0.19294{2} | 0.21742{5} |
| | | $\hat{\theta}$ | 0.91260{7} | 0.84794{5} | 0.80812{3} | 0.83940{4} | 0.77775{2} | 0.96156{8} | 0.77685{1} | 0.88583{6} |
| | | $\sum RANKS$ | 48{5.5} | 48{5.5} | 27{3} | 45{4} | 21{2} | 66{8} | 15{1} | 54{7} |

**Table 10.** (cont.)

| $n$ | Est. | Est. Par. | MLEs | LSEs | WLSEs | CRVMEs | MPSEs | PCEs | ADEs | RADEs |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $\hat{\beta}$ | 1.62726{4} | 1.87435{6} | 1.55256{3} | 1.86632{5} | 1.37480{1} | 2.38135{8} | 1.55114{2} | 2.00102{7} |
| | BIAS | $\hat{\alpha}$ | 0.23710{1} | 0.29581{6} | 0.26630{4} | 0.29717{7} | 0.25734{2} | 0.49787{8} | 0.25832{3} | 0.28315{5} |
| | | $\hat{\theta}$ | 0.67621{2} | 0.95483{5} | 0.72375{4} | 0.96734{6} | 0.64355{1} | 1.41194{8} | 0.67788{3} | 1.09896{7} |
| | | $\hat{\beta}$ | 2.64798{4} | 3.51320{6} | 2.41043{3} | 3.48314{5} | 1.89007{1} | 5.67081{8} | 2.40602{2} | 4.00408{7} |
| 50 | MSE | $\hat{\alpha}$ | 0.05621{1} | 0.08750{6} | 0.07092{4} | 0.08831{7} | 0.06622{2} | 0.24788{8} | 0.06673{3} | 0.08018{5} |
| | | $\hat{\theta}$ | 0.45726{2} | 0.91171{5} | 0.52381{4} | 0.93576{6} | 0.41416{1} | 1.99356{8} | 0.45952{3} | 1.20772{7} |
| | | $\hat{\beta}$ | 0.59173{4} | 0.68158{6} | 0.56457{3} | 0.67866{5} | 0.49993{1} | 0.86594{8} | 0.56405{2} | 0.72764{7} |
| | MRE | $\hat{\alpha}$ | 0.11855{1} | 0.14790{6} | 0.13315{4} | 0.14858{7} | 0.12867{2} | 0.24894{8} | 0.12916{3} | 0.14158{5} |
| | | $\hat{\theta}$ | 0.45081{2} | 0.63656{5} | 0.48250{4} | 0.64490{6} | 0.42903{1} | 0.94129{8} | 0.45192{3} | 0.73264{7} |
| | | $\sum RANKS$ | 21{2} | 51{5} | 33{4} | 54{6} | 12{1} | 72{8} | 24{3} | 57{7} |
| | | $\hat{\beta}$ | 1.14520{2} | 1.48953{5} | 1.17044{3} | 1.50095{6} | 0.98324{1} | 2.34754{8} | 1.19687{4} | 1.64130{7} |
| | BIAS | $\hat{\alpha}$ | 0.17782{2} | 0.22151{7} | 0.18627{3} | 0.22147{6} | 0.17643{1} | 0.40808{8} | 0.19085{4} | 0.20961{5} |
| | | $\hat{\theta}$ | 0.43698{2} | 0.66033{6} | 0.47212{3} | 0.65413{5} | 0.40319{1} | 1.38904{8} | 0.47742{4} | 0.74446{7} |
| | | $\hat{\beta}$ | 1.31148{2} | 2.21869{5} | 1.36993{3} | 2.25284{6} | 0.96677{1} | 5.51095{8} | 1.43251{4} | 2.69387{7} |
| 100 | MSE | $\hat{\alpha}$ | 0.03162{2} | 0.04906{7} | 0.03470{3} | 0.04905{6} | 0.03113{1} | 0.16653{8} | 0.03642{4} | 0.04394{5} |
| | | $\hat{\theta}$ | 0.19095{2} | 0.43603{6} | 0.22290{3} | 0.42788{5} | 0.16256{1} | 1.92944{8} | 0.22793{4} | 0.55422{7} |
| | | $\hat{\beta}$ | 0.41644{2} | 0.54165{5} | 0.42561{3} | 0.54580{6} | 0.35754{1} | 0.85365{8} | 0.43523{4} | 0.59684{7} |
| | MRE | $\hat{\alpha}$ | 0.08891{2} | 0.11075{7} | 0.09314{3} | 0.11074{6} | 0.08821{1} | 0.20404{8} | 0.09542{4} | 0.10480{5} |
| | | $\hat{\theta}$ | 0.29132{2} | 0.44022{6} | 0.31475{3} | 0.43609{5} | 0.26879{1} | 0.92603{8} | 0.31828{4} | 0.49631{7} |
| | | $\sum RANKS$ | 18{2} | 54{6} | 27{3} | 51{5} | 9{1} | 72{8} | 36{4} | 57{7} |
| | | $\hat{\beta}$ | 0.80117{2} | 1.13729{5} | 0.81869{3} | 1.16811{6} | 0.09240{1} | 2.39985{8} | 0.88534{4} | 1.26065{7} |
| | BIAS | $\hat{\alpha}$ | 0.12559{2} | 0.15678{6} | 0.12826{3} | 0.16010{7} | 0.11618{1} | 0.34430{8} | 0.13589{4} | 0.15474{5} |
| | | $\hat{\theta}$ | 0.29194{2} | 0.45079{6} | 0.31492{3} | 0.44833{5} | 0.15284{1} | 1.36604{8} | 0.33329{4} | 0.50590{7} |
| | | $\hat{\beta}$ | 0.64188{2} | 1.29344{5} | 0.67025{3} | 1.36449{6} | 0.00854{1} | 5.75928{8} | 0.78382{4} | 1.58924{7} |
| 200 | MSE | $\hat{\alpha}$ | 0.01577{2} | 0.02458{6} | 0.01645{3} | 0.02563{7} | 0.01350{1} | 0.11854{8} | 0.01847{4} | 0.02395{5} |
| | | $\hat{\theta}$ | 0.08523{2} | 0.20321{6} | 0.09917{3} | 0.20100{5} | 0.02336{1} | 1.86605{8} | 0.11108{4} | 0.25593{7} |
| | | $\hat{\beta}$ | 0.29134{2} | 0.41356{5} | 0.29770{3} | 0.42477{6} | 0.03360{1} | 0.87267{8} | 0.32194{4} | 0.45842{7} |
| | MRE | $\hat{\alpha}$ | 0.06280{2} | 0.07839{6} | 0.06413{3} | 0.08005{7} | 0.05809{1} | 0.17215{8} | 0.06795{4} | 0.07737{5} |
| | | $\hat{\theta}$ | 0.19463{2} | 0.30053{6} | 0.20994{3} | 0.29889{5} | 0.10189{1} | 0.91069{8} | 0.22219{4} | 0.33727{7} |
| | | $\sum RANKS$ | 18{2} | 51{5} | 27{3} | 54{6} | 9{1} | 72{8} | 36{4} | 57{7} |

**Table 10.** (cont.)

| $n$ | Est. | Est. Par. | MLEs | LSEs | WLSEs | CRVMEs | MPSEs | PCEs | ADEs | RADEs |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $\hat{\beta}$ | 0.57533{2} | 0.83145{5} | 0.58097{3} | 0.86320{6} | 0.02261{1} | 2.60818{8} | 0.64620{4} | 0.91190{7} |
| | BIAS | $\hat{\alpha}$ | 0.08999{2} | 0.11144{6} | 0.09447{4} | 0.11528{7} | 0.07797{1} | 0.30020{8} | 0.09445{3} | 0.10902{5} |
| | | $\hat{\theta}$ | 0.20673{2} | 0.31376{5} | 0.21376{3} | 0.31548{6} | 0.04851{1} | 1.34641{8} | 0.23879{4} | 0.33728{7} |
| | | $\hat{\beta}$ | 0.33100{2} | 0.69131{5} | 0.33753{3} | 0.74512{6} | 0.00051{1} | 6.80263{8} | 0.41758{4} | 0.83155{7} |
| 400 | MSE | $\hat{\alpha}$ | 0.00810{2} | 0.01242{6} | 0.00892{3.5} | 0.01329{7} | 0.00608{1} | 0.09012{8} | 0.00892{3.5} | 0.01188{5} |
| | | $\hat{\theta}$ | 0.04274{2} | 0.09844{5} | 0.04569{3} | 0.09953{6} | 0.00235{1} | 1.81282{8} | 0.05702{4} | 0.11376{7} |
| | | $\hat{\beta}$ | 0.20921{2} | 0.30235{5} | 0.21126{3} | 0.31389{6} | 0.00822{1} | 0.94843{8} | 0.23498{4} | 0.33160{7} |
| | MRE | $\hat{\alpha}$ | 0.04499{2} | 0.05572{6} | 0.04723{3.5} | 0.05764{7} | 0.03898{1} | 0.15010{8} | 0.04723{3.5} | 0.05451{5} |
| | | $\hat{\theta}$ | 0.13782{2} | 0.20917{5} | 0.142500{3} | 0.21032{6} | 0.03234{1} | 0.89761{8} | 0.15920{4} | 0.22485{7} |
| | | $\sum RANKS$ | 18{2} | 48{5} | 29{3} | 57{6.5} | 9{1} | 72{8} | 34{4} | 57{6.5} |

**Table 11.** Partial and overall ranks of all estimation methods for various combinations of $\psi$

| $\psi^T$ | $n$ | MLEs | LSEs | WLSEs | CRVMEs | MPSEs | PCEs | ADEs | RADEs |
|---|---|---|---|---|---|---|---|---|---|
| | 20 | 4 | 8 | 2.5 | 7 | 2.5 | 6 | 1 | 5 |
| | 50 | 2.5 | 6 | 2.5 | 5 | 1 | 8 | 4 | 7 |
| ($\beta = 0.75$, $\alpha = 0.5$, $\theta = 0.67$) | 100 | 3 | 6.5 | 2 | 6.5 | 1 | 8 | 4 | 5 |
| | 200 | 2 | 6 | 3 | 7 | 1 | 8 | 4 | 5 |
| | 400 | 2 | 5 | 3 | 7 | 1 | 8 | 4 | 6 |
| | 20 | 4.5 | 8 | 2 | 6 | 3 | 4.5 | 1 | 7 |
| | 50 | 3 | 5 | 1.5 | 7 | 1.5 | 8 | 4 | 6 |
| ($\beta = 0.75$, $\alpha = 0.5$, $\theta = 1.5$) | 100 | 3 | 5 | 2 | 6 | 1 | 8 | 4 | 7 |
| | 200 | 3 | 6 | 2 | 7 | 1 | 8 | 4 | 5 |
| | 400 | 3 | 6 | 2 | 7 | 1 | 8 | 4 | 5 |
| | 20 | 3.5 | 7 | 3.5 | 5.5 | 2 | 8 | 1 | 5.5 |
| | 50 | 3 | 6.5 | 4 | 5 | 1 | 8 | 2 | 6.5 |
| ($\beta = 0.75$, $\alpha = 2$, $\theta = 0.67$) | 100 | 1 | 5 | 4 | 6.5 | 2 | 8 | 3 | 6.5 |
| | 200 | 2 | 6.5 | 4 | 5 | 1 | 8 | 3 | 6.5 |
| | 400 | 2 | 5 | 4 | 6.5 | 1 | 8 | 3 | 6.5 |

**Table 11.** (cont.)

| $\psi^T$ | $n$ | MLEs | LSEs | WLSEs | CRVMEs | MPSEs | PCEs | ADEs | RADEs |
|---|---|---|---|---|---|---|---|---|---|
| ($\beta = 0.75$, $\alpha = 2$, $\theta = 1.5$) | 20 | 2 | 5.5 | 4 | 7 | 3 | 8 | 1 | 5.5 |
| | 50 | 3 | 5 | 4 | 6 | 1 | 8 | 2 | 7 |
| | 100 | 2 | 5 | 4 | 6 | 1 | 8 | 3 | 7 |
| | 200 | 2 | 5 | 4 | 6.5 | 1 | 8 | 3 | 6.5 |
| | 400 | 2 | 6 | 4 | 5 | 1 | 8 | 3 | 7 |
| ($\beta = 2.75$, $\alpha = 0.5$, $\theta = 0.67$) | 20 | 7 | 5 | 4 | 6 | 1 | 2.5 | 2.5 | 8 |
| | 50 | 4 | 7 | 2 | 5 | 1 | 6 | 3 | 8 |
| | 100 | 2 | 5 | 3 | 6 | 1 | 8 | 4 | 7 |
| | 200 | 2 | 5 | 3 | 6 | 1 | 8 | 4 | 7 |
| | 400 | 3 | 5 | 2 | 6 | 1 | 8 | 4 | 7 |
| ($\beta = 0.75$, $\alpha = 0.5$, $\theta = 1.5$) | 20 | 6.5 | 3 | 4.5 | 4.5 | 1 | 8 | 2 | 6.5 |
| | 50 | 4 | 5 | 3 | 6.5 | 1 | 6.5 | 2 | 8 |
| | 100 | 2 | 6 | 4 | 5 | 1 | 7.5 | 3 | 7.5 |
| | 200 | 2 | 5 | 4 | 6 | 1 | 8 | 3 | 7 |
| | 400 | 2 | 6.5 | 3 | 5 | 1 | 8 | 4 | 6.5 |
| ($\beta = 2.75$, $\alpha = 0.5$, $\theta = 0.67$) | 20 | 5.5 | 5.5 | 3 | 4 | 1 | 8 | 2 | 7 |
| | 50 | 3 | 6 | 4 | 5 | 1 | 8 | 2 | 7 |
| | 100 | 2 | 5 | 4 | 6 | 1 | 8 | 3 | 7 |
| | 200 | 2 | 6.5 | 4 | 5 | 1 | 8 | 3 | 6.5 |
| | 400 | 2 | 6.5 | 3 | 5 | 1 | 8 | 4 | 6.5 |
| ($\beta = 2.75$, $\alpha = 0.5$, $\theta = 1.5$) | 20 | 5.5 | 5.5 | 3 | 4 | 2 | 8 | 1 | 7 |
| | 50 | 2 | 5 | 4 | 6 | 1 | 8 | 3 | 7 |
| | 100 | 2 | 6 | 3 | 5 | 1 | 8 | 4 | 7 |
| | 200 | 2 | 5 | 3 | 6 | 1 | 8 | 4 | 7 |
| | 400 | 2 | 5 | 3 | 6.5 | 1 | 8 | 4 | 6.5 |
| $\sum Ranks$ | | 115 | 226.5 | 128.5 | 233 | 49 | 305 | 119.5 | 263.5 |
| Overall Rank | | 2 | 5 | 4 | 6 | 1 | 8 | 3 | 7 |

Furthermore, visual comparisons, the fitted PDF, CDF, survival function (SF), and probability-probability (PP) plots of the GKMLL model are presented in Figures 6-8, for the three datasets. Finally, the GKMLL distribution has a close fit for the three datasets as shown numerically (Tables 15-17) and visually (Figures 6-8).

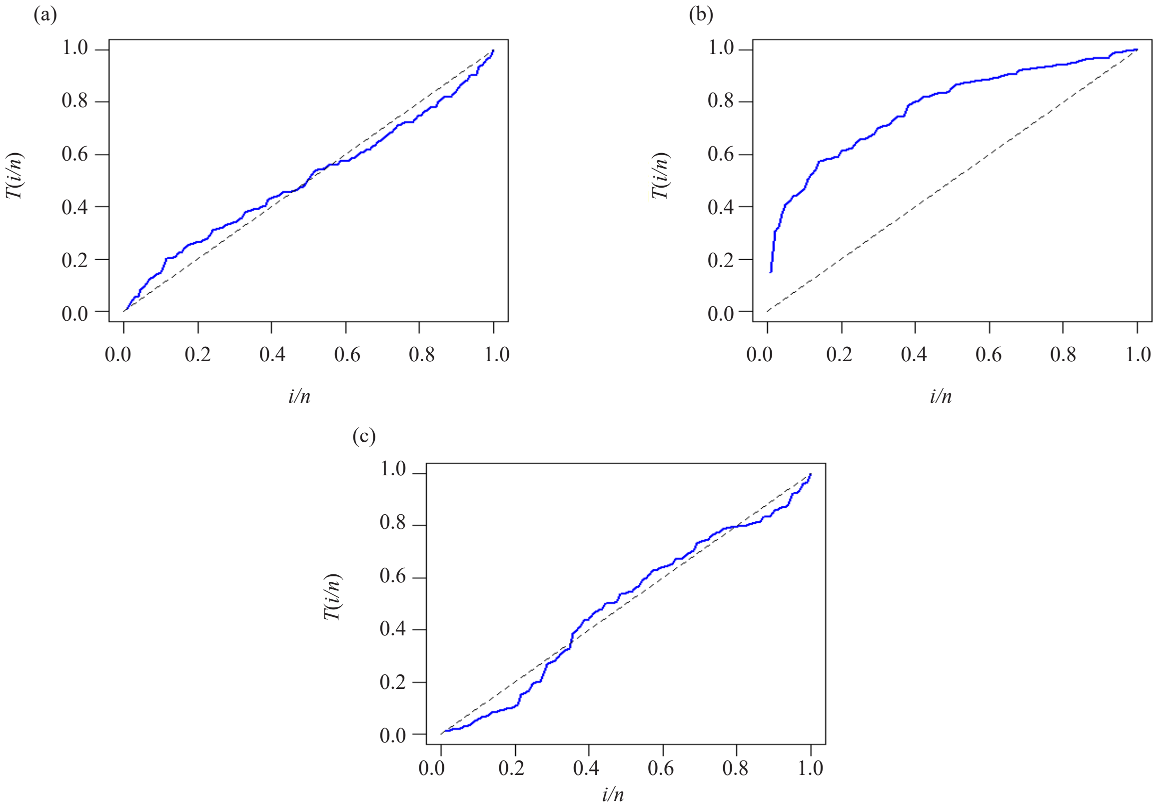


**Figure 4.** The TTT plots for bladder cancer (a), carbon fiber (b), and failure times (c) datasets

**Table 12.** Findings from the fitted distributions for bladder cancer data

| Distribution | MLE and SE | | | | | |
|---|---|---|---|---|---|---|
| GKMLL | $\hat{\beta} = 0.5976$ | $\hat{\alpha} = 2.1266$ | $\hat{\theta} = 11.4676$ | | | |
| | (0.1640) | (0.3160) | (2.1107) | | | |
| AWLL | $\hat{\alpha} = 1.3717$ | $\hat{\lambda} = 3.3618$ | $\hat{\alpha} = 0.0023$ | $\hat{b} = 0.7639$ | $\hat{c} = 0.3322$ | $\hat{d} = 0.7639$ |
| | (9.8485) | (66.6209) | (17.0986) | (6.5262) | (16.3328) | (5.4744) |
| EOWLL | $\hat{\alpha} = 0.2432$ | $\hat{\beta} = 0.4831$ | $\hat{\alpha} = 1.6178$ | $\hat{b} = 5.8700$ | | |
| | (2.7520) | (0.2260) | (8.8062) | (66.4190) | | |
| WGLL | $\hat{\alpha} = 2.6411$ | $\hat{\beta} = 0.1602$ | $\hat{\alpha} = 18.0212$ | $\hat{b} = 0.7447$ | | |
| | (0.9335) | (0.1908) | (88.3930) | (0.2956) | | |
| GOLLBXII | $\hat{\alpha} = 20.3271$ | $\hat{\beta} = 1.3391$ | $\hat{\alpha} = 0.0736$ | $\hat{b} = 1.1895$ | | |
| | (32.8233) | (2.5084) | (0.1272) | (1.9562) | | |
| ExLL | $\hat{\alpha} = 2.0701$ | $\hat{\beta} = 1.4276$ | $\hat{\lambda} = 72.1691$ | | | |
| | (0.9682) | (0.1779) | (59.0535) | | | |
| APLL | $\hat{\alpha} = 2.0976$ | $\hat{\alpha} = 4.9175$ | $\hat{b} = 1.7119$ | | | |
| | (6.8441) | (4.6265) | (0.1708) | | | |
| LL | $\hat{\beta} = 1.7252$ | $\hat{\lambda} = 22.5719$ | | | | |
| | (0.1279) | (6.3153) | | | | |

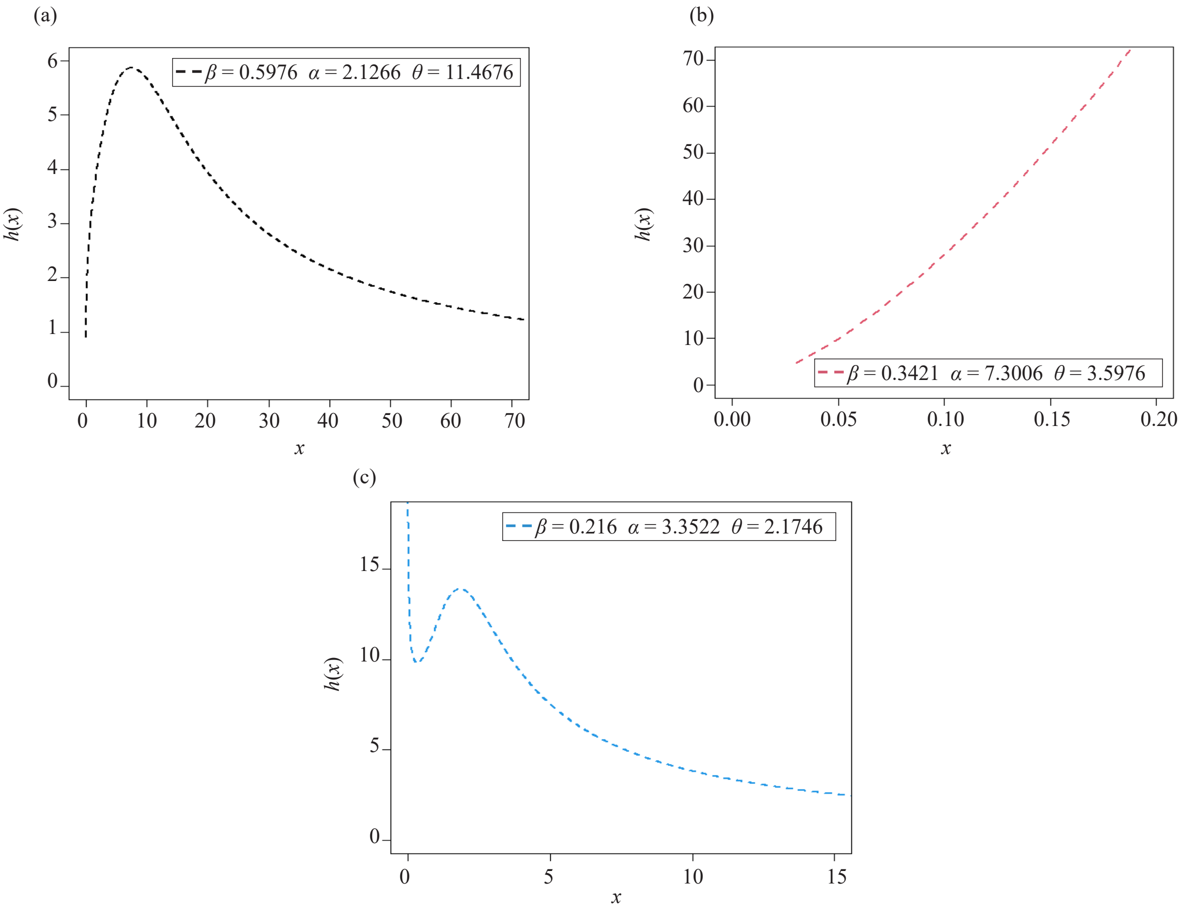


**Figure 5.** The HRF plots for bladder cancer (a), carbon fiber (b), and failure times (c) datasets

**Table 13.** Findings from the fitted distributions for carbon fibers data

| Distribution | MLE and SE | | | | | |
|---|---|---|---|---|---|---|
| GKMLL | $\hat{\beta} = 0.3421$ | $\hat{\alpha} = 7.3006$ | $\hat{\theta} = 3.5976$ | | | |
| | (0.10025) | (1.4618) | (0.2034) | | | |
| AWLL | $\hat{\alpha} = 2.0217$ | $\hat{\lambda} = 14.5887$ | $\hat{\alpha} = 73.9629$ | $\hat{b} = 1.3814$ | $\hat{c} = 13.3971$ | $\hat{d} = 1.3814$ |
| | (20.2800) | (796.2300) | (37,017.0206) | (13.8310) | (16,238.6984) | (13.9913) |
| EOWLL | $\hat{\alpha} = 3.8684$ | $\hat{\beta} = 0.1270$ | $\hat{\alpha} = 57.5254$ | $\hat{b} = 0.7816$ | | |
| | (327.6504) | (0.1847) | (19,744.0900) | (66.1995) | | |
| WGLL | $\hat{\alpha} = 3.9259$ | $\hat{\beta} = 0.4939$ | $\hat{\alpha} = 1.4351$ | $\hat{b} = 0.9319$ | | |
| | (11.7890) | (1.3299) | (32.1301) | (2.4779) | | |
| GOLLBXII | $\hat{\alpha} = 20.1293$ | $\hat{\beta} = 1.4565$ | $\hat{\alpha} = 0.1702$ | $\hat{b} = 1.2545$ | | |
| | (34.1731) | (3.0643) | (0.3420) | (2.2698) | | |
| ExLL | $\hat{\alpha} = 7.8128$ | $\hat{\beta} = 3.0250$ | $\hat{\lambda} = 1,450.3376$ | | | |

**Table 13.** (cont.)

| Distribution | MLE and SE | | |
|---|---|---|---|
| | (10.4898) | (0.3800) | (3,640.7970) |
| APLL | $\hat{\alpha}$ = 0.9999 | $\hat{\alpha}$ = 2.4982 | $\hat{b}$ = 4.1178 |
| | (9.7441) | (2.9579) | (0.3441) |
| LL | $\hat{\beta}$ = 4.1178 | $\hat{\lambda}$ = 43.3874 | |
| | (0.3441) | (16.0602) | |

**Table 14.** Findings from the fitted distributions for failure times data

| Distribution | MLE and SE | | | | | |
|---|---|---|---|---|---|---|
| GKMLL | $\hat{\beta}$ = 0.2160 | $\hat{\alpha}$ = 3.3522 | $\hat{\theta}$ = 2.1746 | | | |
| | (0.0563) | (0.6789) | (0.2541) | | | |
| AWLL | $\hat{\alpha}$ = 0.9526 | $\hat{\lambda}$ = 65.0784 | $\hat{\alpha}$ = 56.8453 | $\hat{b}$ = 1.3073 | $\hat{c}$ = 18.1082 | $\hat{d}$ = 0.8286 |
| | (9.7793) | (7,600.1172) | (8,264.9242) | (13.4292) | (1,670.1486) | (8.5086) |
| EOWLL | $\hat{\alpha}$ = 15.6874 | $\hat{\beta}$ = -0.0777 | $\hat{\alpha}$ = 8.6514 | $\hat{b}$ = 0.0566 | | |
| | (0.0491) | (0.0787) | (0.0490) | (0.0061) | | |
| WGLL | $\hat{\alpha}$ = 0.1593 | $\hat{\beta}$ = 1.2433 | $\hat{\alpha}$ = 0.9102 | $\hat{b}$ = 5.2939 | | |
| | (0.0215) | (0.2361) | (0.1266) | (0.3556) | | |
| GOLLBXII | $\hat{\alpha}$ = 29.6667 | $\hat{\theta}$ = 0.9942 | $\hat{\alpha}$ = 0.0428 | $\hat{b}$ = 1.0088 | | |
| | (84.1377) | (1.9664) | (0.13426) | (1.9825) | | |
| ExLL | $\hat{\alpha}$ = 63.9024 | $\hat{\beta}$ = 0.9355 | $\hat{\lambda}$ = 3,999.9920 | | | |
| | (480.0218) | (0.1024) | (60,428.4397) | | | |
| APLL | $\hat{\alpha}$ = 1.1094 | $\hat{\alpha}$ = 0.5991 | $\hat{b}$ = 1.2703 | | | |
| | (4.7304) | (1.0034) | (0.1080) | | | |
| LL | $\hat{\beta}$ = 1.2705 | $\hat{\lambda}$ = 0.5493 | | | | |
| | (0.1069) | (0.3615) | | | | |

**Table 15.** Adequacy measures for bladder cancer data

| Distribution | AIC | CAIC | BIC | HQIC | $W^*$ | $A^*$ | $-\mathfrak{L}$ | KS | KS *p*-value |
|---|---|---|---|---|---|---|---|---|---|
| GKMLL | 825.4644 | 825.6579 | 834.0205 | 828.9407 | 0.0166 | 0.1073 | 409.7322 | 0.0322 | 0.999355 |
| AWLL | 840.1738 | 840.8680 | 857.2859 | 847.1265 | 0.1314 | 0.7865 | 414.0869 | 0.0700 | 0.556997 |
| EOWLL | 827.4798 | 827.8050 | 838.8879 | 832.1150 | 0.0195 | 0.1308 | 409.7399 | 0.0351 | 0.997517 |

**Table 15.** (cont.)

| Distribution | AIC | CAIC | BIC | HQIC | $W^*$ | $A^*$ | $-\pounds$ | KS | KS $p$-value |
|---|---|---|---|---|---|---|---|---|---|
| WGLL | 829.7561 | 830.0813 | 841.1642 | 834.3913 | 0.0475 | 0.3140 | 410.8781 | 0.0482 | 0.927705 |
| GOLLBXII | 830.9997 | 831.3249 | 842.4078 | 835.6348 | 0.0438 | 0.3167 | 411.4998 | 0.0401 | 0.986312 |
| ExLL | 825.4798 | 825.6733 | 834.0359 | 828.9562 | 0.0195 | 0.1308 | 409.7399 | 0.0351 | 0.997519 |
| APLL | 828.8983 | 829.0919 | 837.4544 | 832.3747 | 0.0429 | 0.3099 | 411.4492 | 0.0400 | 0.986626 |
| LL | 826.9151 | 827.0111 | 832.6191 | 829.2327 | 0.0430 | 0.3111 | 411.4575 | 0.0399 | 0.987015 |

**Table 16.** Adequacy measures for carbon fibers data

| Distribution | AIC | CAIC | BIC | HQIC | $W^*$ | $A^*$ | $-\pounds$ | KS | KS $p$-value |
|---|---|---|---|---|---|---|---|---|---|
| GKMLL | 288.1315 | 288.3815 | 295.9470 | 291.2946 | 0.0458 | 0.2989 | 141.0658 | 0.0515 | 0.953602 |
| AWLL | 295.0603 | 295.9635 | 310.6913 | 301.3865 | 0.0620 | 0.4146 | 141.5302 | 0.0605 | 0.858013 |
| EOWLL | 290.5125 | 290.9335 | 300.9332 | 294.7299 | 0.0669 | 0.3886 | 141.2562 | 0.0648 | 0.795226 |
| WGLL | 290.7219 | 291.1430 | 301.1426 | 294.9393 | 0.0695 | 0.4138 | 141.3610 | 0.0636 | 0.813097 |
| GOLLBXII | 300.6704 | 301.0915 | 311.0911 | 304.8879 | 0.2402 | 1.2480 | 146.3352 | 0.0907 | 0.383090 |
| ExLL | 288.5125 | 288.7625 | 296.3280 | 291.6756 | 0.0670 | 0.3887 | 141.2562 | 0.0649 | 0.794009 |
| APLL | 298.5534 | 298.8034 | 306.3689 | 301.7165 | 0.2385 | 1.2385 | 146.2767 | 0.0903 | 0.388048 |
| LL | 296.5534 | 296.6771 | 301.7638 | 298.6621 | 0.2385 | 1.2385 | 146.2767 | 0.0903 | 0.388047 |

**Table 17.** Adequacy measures for failure times data

| Distribution | AIC | CAIC | BIC | HQIC | $W^*$ | $A^*$ | $-\pounds$ | KS | KS -value |
|---|---|---|---|---|---|---|---|---|---|
| GKMLL | 206.1537 | 206.4011 | 213.9990 | 209.3297 | 0.0739 | 0.4957 | 100.0768 | 0.0643 | 0.798427 |
| AWLL | 217.6292 | 218.5228 | 233.3199 | 223.9812 | 0.1602 | 0.9373 | 102.8146 | 0.0827 | 0.494896 |
| EOWLL | 214.8941 | 215.3107 | 225.3546 | 219.1288 | 0.1825 | 1.0447 | 103.4470 | 0.0955 | 0.315342 |
| WGLL | 212.9685 | 213.3852 | 223.4290 | 217.2032 | 0.1435 | 0.8559 | 102.4842 | 0.0822 | 0.502391 |
| GOLLBXII | 233.3602 | 233.7768 | 243.8206 | 237.5949 | 0.5652 | 3.0701 | 112.6801 | 0.1114 | 0.163298 |
| ExLL | 211.9813 | 212.2287 | 219.8266 | 215.1573 | 0.2031 | 1.1316 | 102.9906 | 0.0911 | 0.371171 |
| APLL | 231.3722 | 231.6196 | 239.2176 | 234.5482 | 0.5653 | 3.0707 | 112.6861 | 0.1113 | 0.163922 |
| LL | 229.3724 | 229.4948 | 234.6026 | 231.4898 | 0.5654 | 3.0709 | 112.6862 | 0.1113 | 0.163673 |

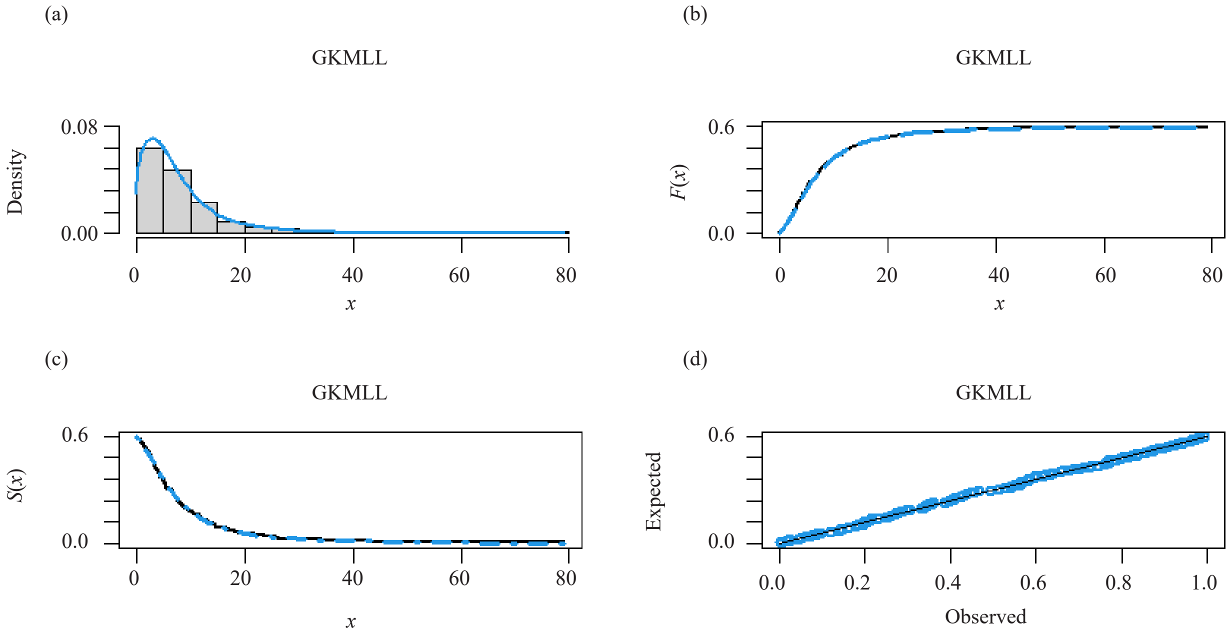


**Figure 6.** The fitted functions of the GKMLL distribution for bladder cancer data

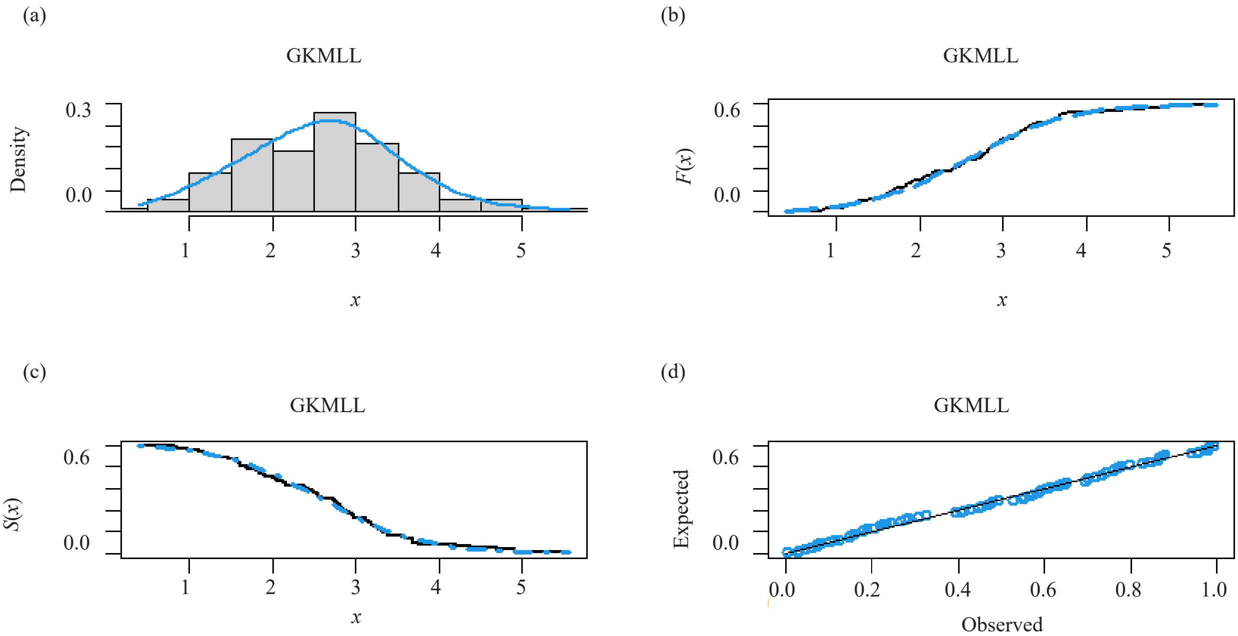


**Figure 7.** The fitted functions of the GKMLL distribution for carbon fibers data

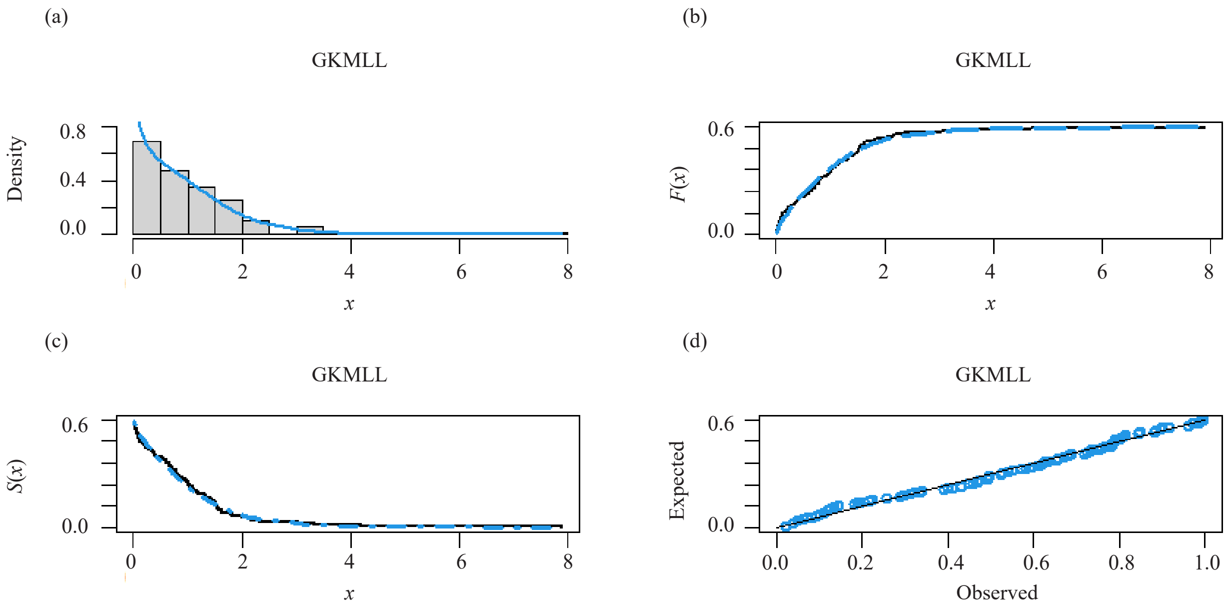


**Figure 8.** The fitted functions of the GKMLL distribution for failure times data

Furthermore, Figures 9-11 display the histograms of the three datasets along with the fitted densities of the GKMLL distribution and other competing distributions. The GKMLL distribution consistently provides a better fit to all three datasets compared to other log-logistic (LL) extensions. Additionally, the probability-probability (PP) plots for the three datasets, presented in Figures 12-14, further confirm that the GKMLL distribution offers a closer fit than the other distributions considered.

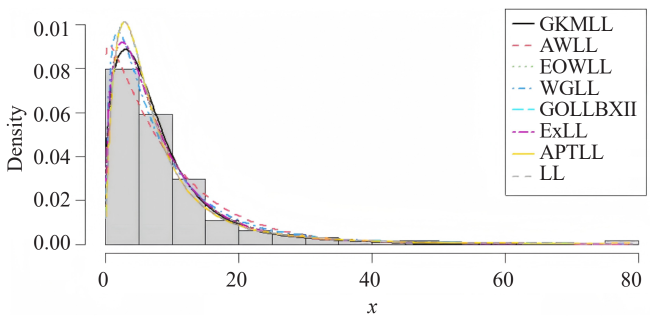


**Figure 9.** The histogram and fitted densities plot for bladder cancer data

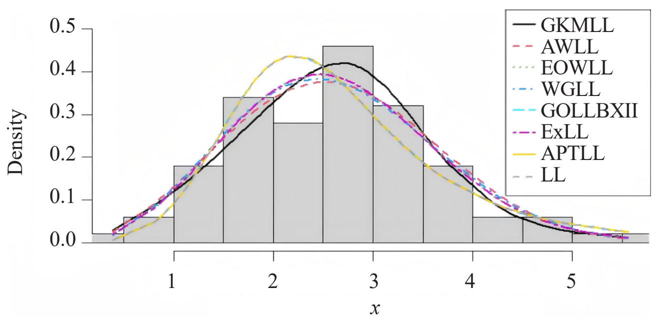


**Figure 10.** The histogram and fitted densities plot for carbon fiber data

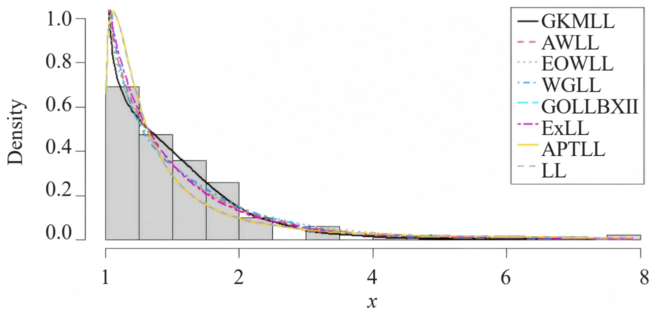


**Figure 11.** The histogram and fitted densities plot for failure times data

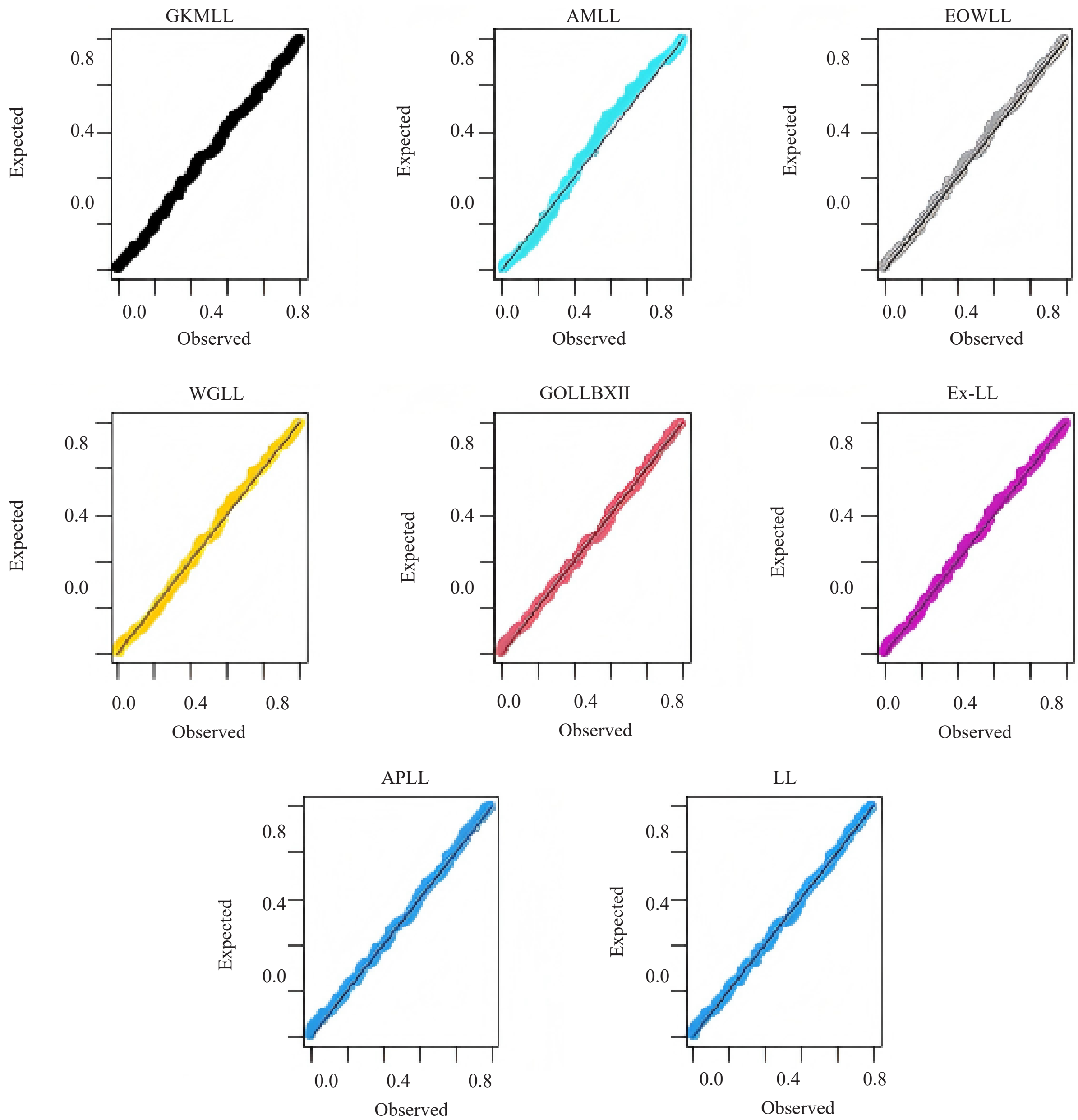


**Figure 12.** The PP plots of the fitted models for bladder cancer data

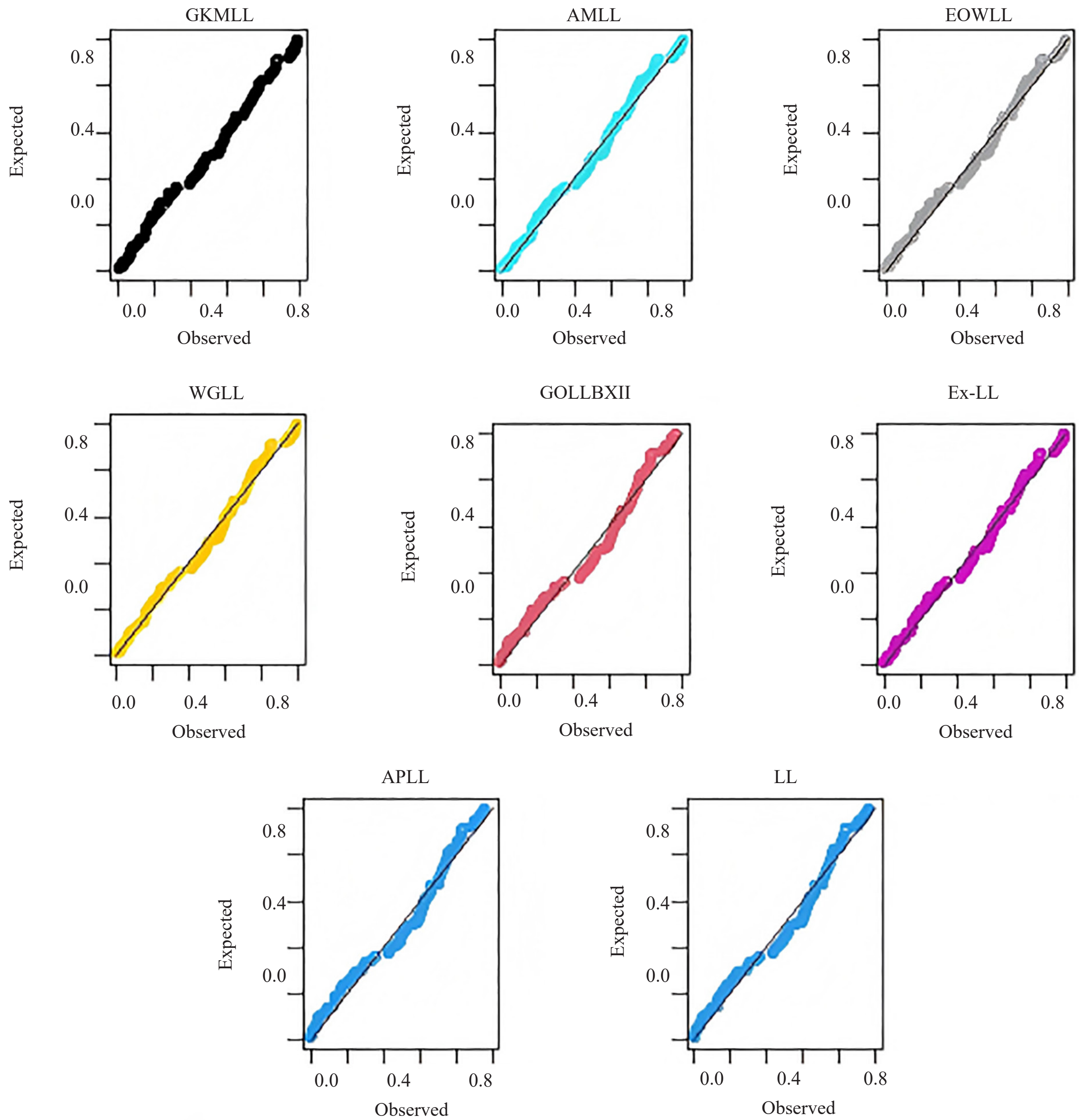


**Figure 13.** The PP plots of the fitted models for carbon fiber data

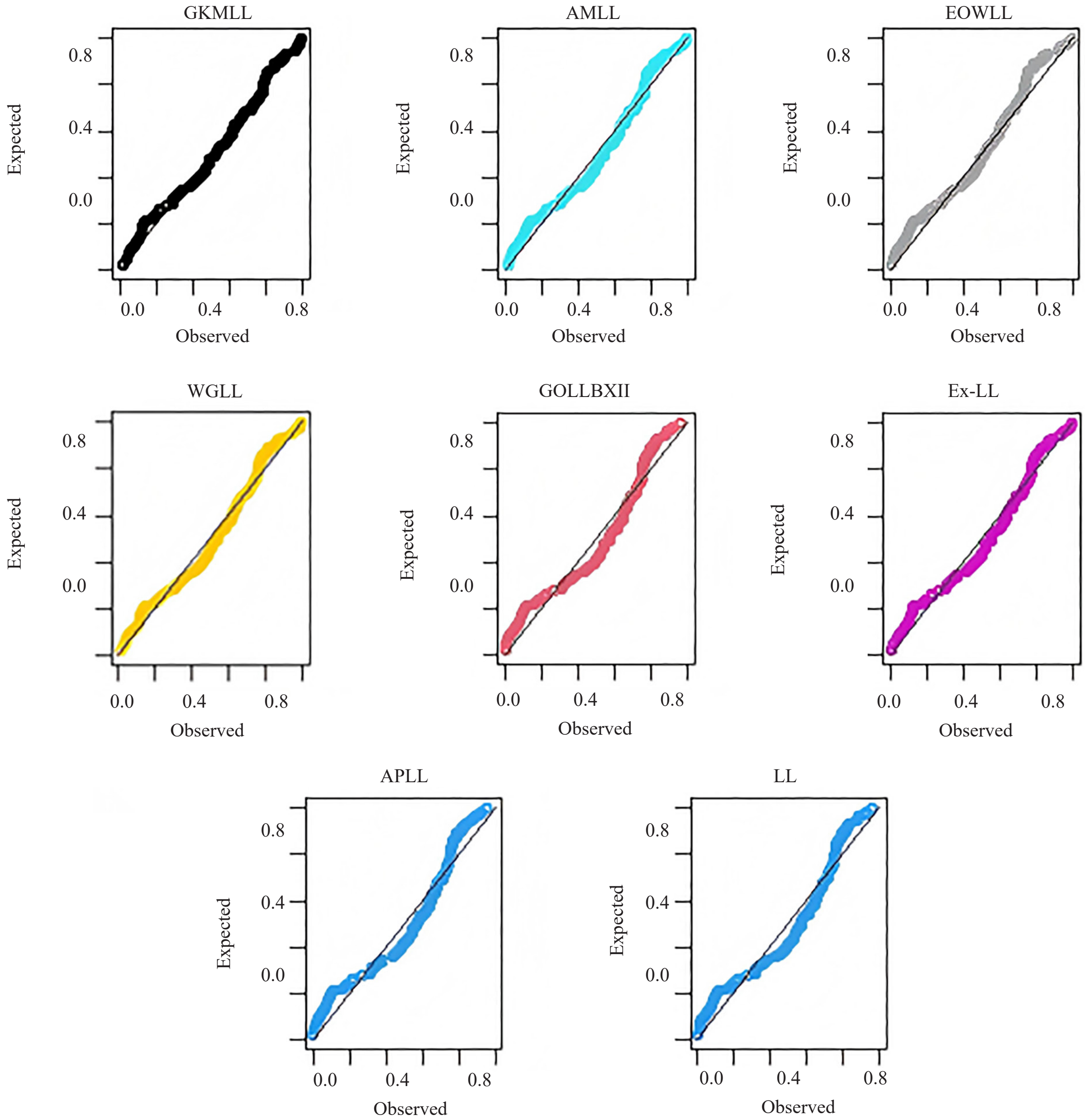


**Figure 14.** The PP plots of the fitted models for failure times data

# 7. Conclusions and future directions

This paper introduces the generalized Kavya-Manoharan log-logistic (GKMLL) distribution, a novel extension of the log-logistic model that offers a versatile and flexible approach to modeling various types of data. The GKMLL distribution stands out due to its ability to accommodate a wide range of distribution shapes, including symmetric, right-skewed, reversed-J, and left-skewed densities, making it applicable to diverse real-world datasets. Furthermore, it can model both non-monotonic and monotonic failure rates, which increases its adaptability in various statistical contexts.

The paper explores the mathematical properties of the GKMLL model and demonstrates its parameter estimation using eight different techniques. The results from simulation studies reveal that the maximum product of spacings approach outperforms all other estimators in accurately estimating the GKMLL parameters. This finding has practical implications for improving the precision of statistical modeling in real-world applications.

The GKMLL distribution is applied to three real-life datasets, where it shows superior fit compared to existing log-logistic distributions, further emphasizing its potential for enhanced data analysis. The practical relevance of the GKMLL model lies in its ability to improve modeling flexibility and accuracy, particularly in fields like survival analysis, where it can offer more reliable insights into failure rates and data behavior.

Future work aims to expand the applicability of the GKMLL distribution to areas beyond survival analysis, including large-scale datasets, and to refine computational methods for parameter estimation. These efforts will enhance the utility and robustness of the GKMLL model, making it a valuable tool for decision-making in various domains, particularly those that require nuanced statistical analysis. Additionally, several avenues for future research could further enhance its utility and applicability across various fields. Here are some potential extensions:

• Developing existing parameter estimation methods such as maximum likelihood or Bayesian approaches for censored data could improve the robustness and accuracy of the GKMLL model in these contexts.

• Additionally, exploring non-parametric or semi-parametric versions of the model could prove valuable.

• The application of Bayesian methods for both parameter estimation and model comparison represents another promising extension of the GKMLL framework.

• Lastly, addressing a discrete version of the GKMLL model would enable its use in modeling count data across a variety of applied fields.

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## Conflicts of interest

The authors declare no conflict of interest.

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