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Well-Posedness of the Second-Order SDEs Describing an *N*-Particle System Interacting via Coulomb Interaction

Rong Yang
Faculty of Science, Beijing University of Technology, Ping Le Yuan100, Chaoyang District, Beijing, 100124, China
Email: ysihan2010@163.com

Abstract: This paper concerns a second-order N interacting stochastic particle system with singular potential for any dimension $n \geq 2$. By some estimates of total energy of the system, we prove that there is no collision among particles almost surely in any finite time interval, then the well-posedness of this interacting particle system can be established.
Keywords: second-order SDEs, N-body system, singular interaction kernel, well-posedness

1. Introduction

In this work, we consider the N indistinguishable point stochastic particles. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a filtered probability space, endowed with a sequence of independent n-dimensional standard Brownian motions $\{(B_t^i)_{t\geq 0}\}_{i=1}^N$. Denote by $(X_t^i, V_t^i) \in \mathbb{R}^n \times \mathbb{R}^n$ for any $t \geq 0$ the position and velocity of particle number i. We consider the following second-order stochastic differential equations (SDEs) describing an interacting N-particle system:

$$\begin{cases} dX_t^i = V_t^i dt, \\ dV_t^i = \dfrac{1}{N}\displaystyle\sum_{j\neq i}^{N} F(X_t^i - X_t^j)dt + \sqrt{2}\, dB_t^i,\ i = 1, \cdots, N, \end{cases} \tag{1}$$

with the initial data $\{(X_0^i, V_0^i)\}_{i=1}^N$. The interaction force F is taken to be the Coulomb interaction, which is described by the following singular potential,

$$F(x) = -\nabla\Phi(x) = \frac{C^* x}{|x|^n}, \forall x \in \mathbb{R}^n \setminus \{0\}, n \geq 2;$$

$$\Phi(x) = \begin{cases} \dfrac{C_n}{|x|^{n-2}}, & \text{if } n \geq 3, \\ -\dfrac{1}{2\pi}\ln|x|, & \text{if } n = 2, \end{cases} \tag{2}$$

where $C^* = \dfrac{\Gamma(n/2)}{2\pi^{n/2}}$, $C_n = \dfrac{1}{n(n-2)\alpha_n}$, $\alpha_n = \dfrac{\pi^{n/2}}{\Gamma(n/2+1)}$, i.e. α_n is the volume of n-dimensional unit ball. $F(x)$ corresponds to the electrostatic (repulsive) interaction of charged particles in a plasma. Thus (1) describes classical Coulomb dynamics. We refer readers to [1] for the original modelings.

(1) is significant in connection with the nonlinear SDEs:

$$\begin{cases} dX_t = V_t dt, \\ dV_t = \displaystyle\int_{\mathbb{R}^n} F(X_t - y)\rho(t, y)dydt + \sqrt{2}\, dB_t, \end{cases} \tag{3}$$

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and (3) is related to the Vlasov-Poisson-Fokker-Planck (VPFP) equations:

$$\begin{cases} \partial_t f + v\cdot\nabla_x f + E(t,x)\cdot\nabla_v f - \Delta_v f = 0, \\ E(t,x) = \int_{\mathbb{R}^n} \rho(t,y)F(x-y)dy, \\ \rho(t,x) = \int_{\mathbb{R}^n} f(t,x,v)dv, \end{cases} \tag{4}$$

where ρ is the spatial density, $f(t, x, v)$ is the distribution function in time, position and velocity, and the initial density f_0 is given by the common distribution of the independent and identically distributed (i.i.d.) initial data $\{(X_0^i, V_0^i)\}_{i=1}^N$.

In fact, when $N \to \infty$, the empirical measure of the system (1) tending towards the unique solution of (4) is called the propagation of chaos property or mean-field limit[2]. Many authors have worked on this for the last a few years. Bolley, Canizo and Carrilo[3] rigorously proved the mean-field to the particle system with only locally Lipschitz interacting force. Jabin and Wang[4] rigorously justified the mean-filed limit and propagation of chaos for VPFP system with bounded forces by using a relative entropy method. Propagation of chaos for the VPFP equation with singular forces by a polynomial cutoff has been studied in [5-9]. To our knowledge, so far the propagation of chaos for the VPFP equation without cutoff seems to be a difficult problem. Our results in this paper will be significant to further proof the propagation of chaos for the VPFP equation without cutoff.

Considering (1), if there exist two particles $(X_t^i)_{t\geq 0}$ and $(X_t^j)_{t\geq 0}$ colliding with each other for some finite time t, i.e. $X_t^i = X_t^j$, then $F(X_t^i - X_t^j) = \infty$, which means that the solution to (1) breaks up in finite time. In [10], Jabin showed that the singularity never occur for (1) without randomness coming from the noise. The stochastic problem we consider here is the existence and uniqueness of a solution to (1) and we show that this blow up will not happen by a probabilistic method[11].

Let $f \in \mathbf{P}((\mathbb{R}^{2n})^N)$ be the joint distribution of $(\mathbb{R}^{2n})^N$-valued random variable $(X, V) = (X_1, V_1, \ldots, X_N, V_N)$. If f has a density $\rho \in L^1((\mathbb{R}^{2n})^N)$, we introduce the normalized entropy and partial Fisher information:

$$H_N(f) := \frac{1}{N}\int_{\mathbb{R}^{2Nn}} \rho \ln \rho dxdv,\ \tilde{I}_N(f) := \int_{\mathbb{R}^{2Nn}} \frac{|\nabla_v \rho|^2}{\rho} dxdv.$$

If f has no density, we simply put $H_N(f) = +\infty$ and $\tilde{I}_N(f) = +\infty$. In this paper, we also use $H_N(\rho)$ to present $H_N(f)$ and $\tilde{I}_N(\rho)$ to $\tilde{I}_N(f)$.

The total energy of the N-interacting particle system (1) is defined by

$$E_t^N := \frac{1}{2N}\sum_{i=1}^N |V_t^i|^2 + \frac{1}{2N^2}\sum_{\substack{i,j=1\\ i\neq j}}^N \Phi(X_t^i - X_t^j), \tag{5}$$

with the initial total energy is

$$E_0^N := \frac{1}{2N}\sum_{i=1}^N |V_0^i|^2 + \frac{1}{2N^2}\sum_{\substack{i,j=1\\ i\neq j}}^N \Phi(X_0^i - X_0^j). \tag{6}$$

Next, we state the main result about the well-posedness for the particle system (1).

Theorem 1.1 For any $n \geq 2$, $N \geq 2$ and $T > 0$, given a sequence of independent n-dimensional Brownian motions $\{(B_t^i)_{t\geq 0}\}_{i=1}^N$ and the initial data $\{(X_0^i, V_0^i)\}_{i=1}^N$ with a joint distribution f_0^N satisfying $f_0^N \in L^1(\mathbb{R}^{2Nn}, (1+|x|^2+|v|^2)dxdv)$. Assume that the initial total energy satisfies $\mathbb{E}[E_0^N] < +\infty$ and the initial data $X_0^i \neq X_0^j$ almost surely (a.s.) for all $i \neq j$, then a.s. $X_t^i \neq X_t^j$ for all $t \in [0, T]$, $i \neq j$, and hence there exists a unique global strong solution to (1). Moreover, the entropy satisfies

$$H_N(f_t^N)+\frac{1}{N}\int_0^t \tilde{I}_N(f_t^N)ds \le H_N(f_0^N) \text{ for any } n\ge 2 \text{ and } t\in[0,T], \tag{7}$$

where $f_t^N \in \mathbf{P}(\mathbb{R}^{2Nn})$ is the joint time marginal distribution of $\{(X_t^i, V_t^i)\}_{i=1}^N$. The total energy satisfies

$$\mathbb{E}[E_t^N] \le \begin{cases} \mathbb{E}[E_0^N]+2t+C, & \text{for } n=2 \text{ and } t\in[0,T], \\ \mathbb{E}[E_0^N]+nt, & \text{for } n\ge 3 \text{ and } t\in[0,T], \end{cases} \tag{8}$$

and the second moments satisfy

$$\mathbb{E}\Big[\sum_{i=1}^N \sup_{t\in[0,T]} |X_t^i|^2\Big] \le C, \quad \sup_{t\in[0,T]} \mathbb{E}\Big[\sum_{i=1}^N |V_t^i|^2\Big] \le C, \tag{9}$$

where C are constants depending on n, N, $\sum_{i=1}^N \mathbb{E}[|X_0^i|^2]$, $\mathbb{E}[|E_0^N|]$ and T.

Comments: This paper is some kind of adaptation of the work[11-12]. They show the propagation of chaos of some first order particle system with the singular potential without cutoff. The proof is thus sometimes very similar to those in [11-12], but there are some differences due to the fact that the SDEs in this article is second-order. We thus have to deal with some additional terms and find a good prior estimate for the total energy (5), which is more interesting and difficult than the first order SDEs. Since we just get the estimate of partial Fisher information (7), then the propagation of (1) is still very difficult and we will study this problem in future research.

The structure of this article is as following. In section 2, we give a regularized particle system to approximate (1) and provide uniform estimates on the entropy, total energy, second moments and stopping time. In section 3, we present a detailed proof of Theorem 1.1.

2. Regularization for the system (1) and the uniform estimates

Notice that the interacting force F in (1) is singular, so we will regularize F by a blob function firstly. Then we consider (1) with this regularised force and derive some important estimates. We directly recall the following lemma stated in [12], which collects some useful properties of the regularized force and potential.

Lemma 2.1 Suppose $J(x) \in C^2(\mathbb{R}^n)$, $J(x) \subset B(0, 1)$, $J(x) = J(|x|)$, $J(x) \ge 0$ and $\int_{\mathbb{R}^n} J(x)dx = 1$. *Let* .Let $x) = \frac{1}{\varepsilon^d} J(\frac{x}{\varepsilon})$ and $\Phi_\varepsilon(x) = J_\varepsilon * \Phi(x)$ for $x\in\mathbb{R}^n$, $F_\varepsilon(x) = -\nabla\Phi_\varepsilon(x)$, then $F_\varepsilon(x)\in C^1(\mathbb{R}^n)$, $\nabla\cdot F_\varepsilon(x) = J_\varepsilon(x)$ and

(i) $F_\varepsilon(0)=0$ and $F_\varepsilon(x) = F(x)g\big(\frac{|x|}{\varepsilon}\big)$ for any $x\ne 0$, where $g(r) = \frac{1}{C^*}\int_0^r J(s)s^{n-1}ds$, $C^* = \frac{\Gamma(n/2)}{2\pi^{n/2}}$, $n\ge 2$ and $g(r)=1$ for $r\ge 1$;

(ii) $|F_\varepsilon(x)| \le \min\big\{\frac{C|x|}{\varepsilon^n}, |F(x)|\big\}$ and $|\nabla F_\varepsilon(x)| \le \frac{C}{\varepsilon^n}$;

(iii) when $n\ge 3$, $\Phi_\varepsilon(x) = \Phi(x)$ for any $|x|\ge\varepsilon>0$; when $n=2$ and $0<\varepsilon\le 1$, $\Phi_\varepsilon(x) = \Phi(x)+\Phi_\varepsilon(1)$ for any $|x|>\varepsilon$, in fact, for small enough ε,

$$\Phi_\varepsilon(x) \ge -\frac{1}{2\pi}|x|, \text{ for any } x\in\mathbb{R}^2, \tag{10}$$

$$\Phi_\varepsilon(\varepsilon) \to +\infty, \text{ as } \varepsilon\to 0^+ \text{ for } n\ge 2. \tag{11}$$

Proof of (iii) We just need to prove (10). Let $r = |x|$. By the proof of (i), one knows that

$$r^{n-1}\partial_r\Phi_\varepsilon(r) = -\int_0^r J_\varepsilon(s)s^{n-1}ds = -\int_0^{\frac{r}{\varepsilon}} J(s)s^{n-1}ds = -C^* g\Big(\frac{r}{\varepsilon}\Big). \tag{12}$$

Then for any $r < \varepsilon \le 1$, integrating the above equality, one has

$$\Phi_\varepsilon(r) - \Phi_\varepsilon(1) = \int_1^r \partial_s \Phi_\varepsilon(s) ds = -C^* \int_1^r \frac{g(\frac{s}{\varepsilon})}{s} ds \ge -\frac{1}{2\pi}\int_1^r \frac{1}{s} ds = -\frac{1}{2\pi} \ln r \text{ for } n = 2. \tag{13}$$

Combining the fact $\Phi_\varepsilon(x) = \Phi(x) + \Phi_\varepsilon(1)$ for any $|x| \ge \varepsilon$, since $\Phi_\varepsilon(1) \to 0$ as $\varepsilon \to 0$, then (10) holds for small enough ε.

In this article, we take a blob function $J(x) \ge 0$ satisfying the conditions of Lemma 2.1. Next, we regularize the force term by this blob function $J(x)$, then F_ε is bounded Lipschitz by (ii) of Lemma 2.1. We consider the following N interacting particle system via the regularized force:

$$\begin{cases} dX_t^{i,\varepsilon} = V_t^{i,\varepsilon} dt, \\ dV_t^{i,\varepsilon} = \dfrac{1}{N}\displaystyle\sum_{j \ne i}^{N} F_\varepsilon(X_t^{i,\varepsilon} - X_t^{j,\varepsilon}) dt + \sqrt{2} dB_t^i, \; i = 1, \cdots, N, \end{cases} \tag{14}$$

with the initial data $\{(X_0^i, V_0^i)\}_{i=1}^N$. For any $\varepsilon > 0$, if $\sum_{i=1}^N \mathbb{E}[|X_0^i|^2 + |V_0^i|^2] < +\infty$, then there exists a unique global strong solution $\{(X_t^{i,\varepsilon}, V_t^{i,\varepsilon})_{t \ge 0}\}_{i=1}^N$ to (14) by the standard theorem of SDE[13].

In next subsections, we start from this regularized system (14) to obtain the uniform estimates of the second moments, entropy, total energy and the stopping time. Notice that the sign of F and Φ are crucially important for our estimates of the stopping time.

2.1 Uniform estimates of (14)

Here the total energy of (14) is defined by

$$E_t^{N,\varepsilon} = \frac{1}{2N}\sum_{i=1}^N |V_t^{i,\varepsilon}|^2 + \frac{1}{2N^2}\sum_{\substack{i,j=1 \\ i \ne j}}^N \Phi_\varepsilon(X_t^{i,\varepsilon} - X_t^{j,\varepsilon}), \tag{15}$$

with the same initial total energy defined by (6).

Lemma 2.2 Suppose $\sum_{i=1}^N \mathbb{E}[|X_0^i|^2 + |V_0^i|^2] < +\infty$, let $\{(X_t^{i,\varepsilon}, V_t^{i,\varepsilon})_{t \ge 0}\}_{i=1}^N$ be the unique strong solution to (14) and $(f_t^{N,\varepsilon})_{t \ge 0}$ be its joint time marginal density function. We have the uniform estimates for entropy:

$$H_N(f_t^{N,\varepsilon}) + \frac{1}{N}\int_0^t \tilde{I}_N(f_s^{N,\varepsilon}) ds \le H_N(f_0^N) \text{ for } n \ge 2 \text{ and } t \ge 0, \tag{16}$$

and the total energy satisfies

$$\mathbb{E}[E_t^{N,\varepsilon}] = \mathbb{E}[E_0^N] + nt \text{ for } n \ge 2 \text{ and } t \ge 0, \tag{17}$$

where E_0^N is defined by (6), and the second moments satisfy

$$\mathbb{E}\Big[\sum_{i=1}^N \sup_{t \in [0,T]} |X_t^{i,\varepsilon}|^2\Big] \le C, \quad \sup_{t \in [0,T]} \mathbb{E}\Big[\sum_{i=1}^N |V_t^{i,\varepsilon}|^2\Big] \le C, \tag{18}$$

where C are constants depending on n, N, $\sum_{i=1}^N \mathbb{E}[|X_0^i|^2]$, $\mathbb{E}[|E_0^N|]$ and T.

Proof. Entropy estimate: Denote by $\mathbf{X}_t^{N,\varepsilon} = (X_t^{1,\varepsilon}, \cdots, X_t^{N,\varepsilon})$ and $\mathbf{V}_t^{N,\varepsilon} = (V_t^{1,\varepsilon}, \cdots, V_t^{N,\varepsilon})$. For any $\varphi \in C_b^2(\mathbb{R}^{2Nn})$, applying the Itô's formula, one deduces that

$$\varphi(\mathbf{X}_t^{N,\varepsilon},\mathbf{V}_t^{N,\varepsilon})=\varphi(\mathbf{X}_0^{N,\varepsilon},\mathbf{V}_0^{N,\varepsilon})+\frac{1}{N}\sum_{\substack{i,j=1\\ i\neq j}}^{N}\int_0^t F_\varepsilon(X_s^{i,\varepsilon}-X_s^{j,\varepsilon})\cdot\nabla_{v_i}\varphi(\mathbf{X}_s^{N,\varepsilon},\mathbf{V}_s^{N,\varepsilon})ds$$

$$+\sum_{i=1}^{N}\int_0^t\Big(V_s^{i,\varepsilon}\cdot\nabla_{x_i}\varphi(\mathbf{X}_s^{N,\varepsilon},\mathbf{V}_s^{N,\varepsilon})+\triangle_{v_i}\varphi(\mathbf{X}_s^{N,\varepsilon},\mathbf{V}_s^{N,\varepsilon})\Big)ds$$

$$+\sqrt{2}\sum_{i=1}^{N}\int_0^t\nabla_{v_i}\varphi(\mathbf{X}_s^{N,\varepsilon},\mathbf{V}_s^{N,\varepsilon})\cdot dB_s^i. \tag{19}$$

Taking expectation of (19), one has

$$\int_{\mathbb{R}^{2Nn}}\varphi f_t^{N,\varepsilon}\,dXdV=\int_{\mathbb{R}^{2Nn}}\varphi f_0^N\,dXdV+\frac{1}{N}\sum_{\substack{i,j=1\\ i\neq j}}^{N}\int_0^t\int_{\mathbb{R}^{2Nn}}\nabla_{v_i}\varphi\cdot F_\varepsilon(x_i-x_j)f_s^{N,\varepsilon}\,dXdVds$$

$$+\sum_{i=1}^{N}\int_0^t\int_{\mathbb{R}^{2Nn}}(v_i\cdot\nabla_{x_i}\varphi+\triangle_{v_i}\varphi)f_s^{N,\varepsilon}\,dXdVds, \tag{20}$$

i.e., $f^{N,\varepsilon}(t, X, V)$ is a positive weak solution to

$$\partial_t f_t^{N,\varepsilon}+V\cdot\nabla_X f_t^{N,\varepsilon}+N\nabla_V f_t^{N,\varepsilon}\cdot\nabla_X\Phi^{N,\varepsilon}=\triangle_V f_t^{N,\varepsilon},\ t>0,$$

with the initial data f_0^N, where $\Phi^{N,\varepsilon}(x)=\frac{1}{2N^2}\sum_{\substack{i,j=1\\ i\neq j}}^{N}\Phi_\varepsilon(x_i-x_j)$.

Denoted by $\tilde{f}_0^N$ the approximated initial data, one can get the classical solution $f^{N,\varepsilon}(t, X, V)$ (without relabeling) to (20). Then the entropy can be estimated by

$$\frac{d}{dt}H_N(f_t^{N,\varepsilon})=\frac{1}{N}\int_{\mathbb{R}^{2Nn}}(1+\ln f_t^{N,\varepsilon})\partial_t f_t^{N,\varepsilon}\,dXdV$$

$$=\frac{1}{N}\int_{\mathbb{R}^{2Nn}}\nabla_X(1+\ln f_t^{N,\varepsilon})\cdot Vf_t^{N,\varepsilon}\,dXdV+\int_{\mathbb{R}^{2Nn}}\nabla_V(1+\ln f_t^{N,\varepsilon})\cdot\Big(f_t^{N,\varepsilon}\nabla_X\Phi^{N,\varepsilon}\Big)dXdV$$

$$-\frac{1}{N}\int_{\mathbb{R}^{2Nn}}\nabla_V(1+\ln f_t^{N,\varepsilon})\cdot\nabla_V f_t^{N,\varepsilon}\,dXdV$$

$$=\frac{1}{N}\int_{\mathbb{R}^{2Nn}}\nabla_X f_t^{N,\varepsilon}\cdot V\,dXdV+\int_{\mathbb{R}^{2Nn}}\nabla_V f_t^{N,\varepsilon}\cdot\nabla_X\Phi^{N,\varepsilon}\,dXdV-\frac{1}{N}\int_{\mathbb{R}^{2Nn}}\frac{|\nabla_V f_t^{N,\varepsilon}|^2}{f_t^{N,\varepsilon}}dXdV$$

$$=-\frac{1}{N}\int_{\mathbb{R}^{2Nn}}\frac{|\nabla_V f_t^{N,\varepsilon}|^2}{f_t^{N,\varepsilon}}dXdV. \tag{21}$$

Integrating the above equality, one has

$$H_N(f_t^{N,\varepsilon}) + \frac{1}{N}\int_0^t \int_{\mathbb{R}^{2Nn}} \frac{|\nabla_V f_s^{N,\varepsilon}|^2}{f_s^{N,\varepsilon}} dXdVds = H_N(\tilde{f}_0^N), \tag{22}$$

thus (16) is obtained by taking the limit of initial data.

Total energy estimate: To prove (17), we just need to show that

$$E_t^{N,\varepsilon} = E_0^N + nt + M_t, \tag{23}$$

where

$$M_t := \frac{\sqrt{2}}{N}\sum_{i=1}^N \int_0^t V_s^{i,\varepsilon} \cdot dB_s^i, \tag{24}$$

and $(M_t)_{t\geq 0}$ is a martingale w.r.t. the filtration generated by Brownian motions $\{(B_t^i)_{t\geq 0}\}_{i=1}^N$.

Using the Itô's formula and the fact $F_\varepsilon(x) = F(x)g\left(\frac{|x|}{\varepsilon}\right) = -F(-x)g\left(\frac{|-x|}{\varepsilon}\right) = -F_\varepsilon(-x)$ on $|x| > 0$, one has

$$\begin{aligned}
dE_t^{N,\varepsilon} &= -\frac{1}{2N^2}\sum_{\substack{i,j=1\\ i\neq j}}^N F_\varepsilon(X_t^{i,\varepsilon} - X_t^{j,\varepsilon})\cdot(dX_t^{i,\varepsilon} - dX_t^{j,\varepsilon}) + \frac{1}{N}\sum_{i=1}^N V_t^{i,\varepsilon}\cdot dV_t^{i,\varepsilon} + \frac{1}{N}\sum_{i=1}^N n\,dt \\
&= -\frac{1}{2N^2}\sum_{\substack{i,j=1\\ i\neq j}}^N F_\varepsilon(X_t^{i,\varepsilon} - X_t^{j,\varepsilon})\cdot(V_t^{i,\varepsilon} - V_t^{j,\varepsilon})dt + n\,dt + \frac{1}{N}\sum_{i=1}^N V_t^{i,\varepsilon}\cdot\left(\frac{1}{N}\sum_{j\neq i}^N F_\varepsilon(X_t^{i,\varepsilon} - X_t^{j,\varepsilon})dt + \sqrt{2}\,dB_t^i\right) \\
&= -\frac{1}{2N^2}\sum_{\substack{i,j=1\\ i\neq j}}^N F_\varepsilon(X_t^{i,\varepsilon} - X_t^{j,\varepsilon})\cdot(V_t^{i,\varepsilon} - V_t^{j,\varepsilon})dt + n\,dt \\
&\quad + \frac{1}{2N^2}\sum_{\substack{i,j=1\\ i\neq j}}^N (V_t^{i,\varepsilon} - V_t^{j,\varepsilon})\cdot F_\varepsilon(X_t^{i,\varepsilon} - X_t^{j,\varepsilon})dt + \frac{\sqrt{2}}{N}\sum_{i=1}^N V_t^{i,\varepsilon}\cdot dB_t^i \\
&= n\,dt + \frac{\sqrt{2}}{N}\sum_{i=1}^N V_t^{i,\varepsilon}\cdot dB_t^i.
\end{aligned} \tag{25}$$

Then

$$E_t^{N,\varepsilon} = E_0^N + nt + \frac{\sqrt{2}}{N}\sum_{i=1}^N \int_0^t V_s^{i,\varepsilon}\cdot dB_s^i, \tag{26}$$

which means that (23) holds true. Taking expectation of (23), one obtains (17) immediately.

Second moment estimates: From (14) and by Hölder inequality, for any $t \in [0, T]$, one has

$$\sum_{i=1}^{N}\sup_{s\in[0,t]}|X_s^{i,\varepsilon}|^2 \le 2\sum_{i=1}^{N}|X_0^i|^2 + 2\sum_{i=1}^{N}\sup_{s\in[0,t]}\left|\int_0^s V_r^{i,\varepsilon}\,dr\right|^2$$

$$\le 2\sum_{i=1}^{N}|X_0^i|^2 + 2t\int_0^t\Big(\sum_{i=1}^{N}|V_s^{i,\varepsilon}|^2\Big)ds. \tag{27}$$

Combining (15) and (23), one has

$$\frac{1}{2N}\sum_{i=1}^{N}|V_t^{i,\varepsilon}|^2 = E_0^N + nt + M_t - \frac{1}{2N^2}\sum_{\substack{i,j=1\\ i\ne j}}^{N}\Phi_\varepsilon(X_t^{i,\varepsilon} - X_t^{j,\varepsilon}). \tag{28}$$

Next, we split the proof into two cases.

Case 1 ($n = 2$**):** By (10) in Lemma 2.1, one has the fact that $\Phi_\varepsilon(x) \ge -\frac{1}{2\pi}|x|$ for any $x \in \mathbb{R}^2$ and small enough ε, then

$$\frac{1}{2N}\sum_{i=1}^{N}|V_t^{i,\varepsilon}|^2 \le E_0^N + nt + M_t + \frac{1}{4\pi N^2}\sum_{\substack{i,j=1\\ i\ne j}}^{N}|X_t^{i,\varepsilon} - X_t^{j,\varepsilon}|$$

$$\le E_0^N + nt + M_t + \frac{1}{4\pi N^2}\sum_{\substack{i,j=1\\ i\ne j}}^{N}(|X_t^{i,\varepsilon}|^2 + |X_t^{j,\varepsilon}|^2 + 2)$$

$$\le E_0^N + nt + M_t + \frac{1}{2\pi N}\sum_{i=1}^{N}|X_t^{i,\varepsilon}|^2 + \frac{1}{2\pi}. \tag{29}$$

Plugging (29) into (27), one has

$$\sum_{i=1}^{N}\sup_{s\in[0,t]}|X_s^{i,\varepsilon}|^2 \le 2\sum_{i=1}^{N}|X_0^i|^2 + 4N(E_0^N + \frac{1}{2\pi})t^2 + 2Nnt^3 + 4Nt\int_0^t M_s\,ds + \frac{2t}{\pi}\int_0^t(\sum_{i=1}^{N}\sup_{r\in[0,s]}|X_r^{i,\varepsilon}|^2)ds. \tag{30}$$

Taking expectation of (30) and by Gronwall's lemma, one has

$$\mathbb{E}\Big[\sum_{i=1}^{N}\sup_{t\in[0,T]}|X_t^{i,\varepsilon}|^2\Big] \le C, \tag{31}$$

where C is a constant depending on n, N, $\mathbb{E}[\sum_{i=1}^{N}|X_0^i|^2]$, $\mathbb{E}[|E_0^N|]$ and T. Taking expectation of (29) and combining with (31), we have

$$\mathbb{E}\Big[\frac{1}{2N}\sum_{i=1}^{N}|V_t^{i,\varepsilon}|^2\Big] \le \mathbb{E}[E_0^N] + nt + \frac{C}{2\pi N} + \frac{1}{2\pi}. \tag{32}$$

Combining (31) and (32), one obtains (19) for $n = 2$.

Case 2 ($n \ge 3$**):** Since

$$\mathbb{E}[E_t^{N,\varepsilon}] = \mathbb{E}[\frac{1}{2N}\sum_{i=1}^{N}|V_t^{i,\varepsilon}|^2 + \frac{1}{2N^2}\sum_{\substack{i,j=1\\ i\neq j}}^{N}\Phi_\varepsilon(X_t^{i,\varepsilon} - X_t^{j,\varepsilon})], \tag{33}$$

where $\Phi_\varepsilon(x)$ is positive, recalling (17), then one has

$$\mathbb{E}[\frac{1}{2N}\sum_{i=1}^{N}|V_t^{i,\varepsilon}|^2] \le \mathbb{E}[E_0^N] + nt. \tag{34}$$

Combining (34) and (27), one has

$$\mathbb{E}\Big[\sum_{i=1}^{N}\sup_{t\in[0,T]}|X_t^{i,\varepsilon}|^2\Big] \le 2\mathbb{E}[\sum_{i=1}^{N}|X_0^i|^2] + 2NT^2(2\mathbb{E}[E_0^N] + nT). \tag{35}$$

Combining (34) and (35), one obtains (19) for $n \ge 3$ immediately.

Remark 2.1 Notice that, if the initial data $\{(X_0^i, V_0^i)\}_{i=1}^N$ are i.i.d. with the common density $f_0(x, v)$, then $\mathbb{E}[|E_0^N|]$ can be controlled easily. In fact,

$$\mathbb{E}[|E_0^N|] \le \frac{1}{2N}\sum_{i=1}^{N}\mathbb{E}[|V_0^i|^2] + \mathbb{E}[|\Phi_0^N|].$$

For $n \ge 3$, one has

$$\mathbb{E}[|\Phi_0^N|] = \langle \rho_0^N, \Phi^N\rangle \le \int_{\mathbb{R}^{2n}}\rho_0(x)\rho_0(y)\Phi(x-y)dxdy \le C(n)\|\rho_0\|^2_{L^{\frac{2n}{n+2}}}, \tag{36}$$

where $C(n) = \frac{1}{n(n-2)\pi}\{\frac{\Gamma(n)}{\Gamma(\frac{n}{2})}\}^{\frac{2}{n}}$, $\rho_0 = \int_{\mathbb{R}^n} f_0(x,v)dv$, and the last inequality comes from the Hardy-Littlewood-Sobolev inequality.

For $n = 2$, one has

$$\mathbb{E}[|\Phi_0^N|] \le \frac{1}{4\pi}\int_{\mathbb{R}^4}\rho_0(x)\rho_0(y)|\ln|x-y||dxdy,$$

then using the logarithmic Hardy-Littlewood-Sobolev inequality (see [14]), one has

$$H_1(\rho_0) + 2\int_{\mathbb{R}^4}\rho_0(x)\rho_0(y)\ln|x-y|dxdy \ge -1 - \ln\pi. \tag{37}$$

On the other hand,

$$\int_{\mathbb{R}^4}\rho_0(x)\rho_0(y)\ln|x-y|dxdy \le \int_{\mathbb{R}^4}\rho_0(x)\rho_0(y)(x^2+y^2)dxdy = 2m_2(\rho_0). \tag{38}$$

Combining (37) and (38), one knows that $\mathbb{E}[|\Phi_0^N|]$ can be controlled by $H_1(\rho_0)$ and $m_2(\rho_0)$.

2.2 Estimate of the stopping time

We adapt the techniques of [11-12, 15] to prove the following estimate, which is the key step for proving the well-posedness of (1).

Lemma 2.3 For any $n \geq 2$, let $N \geq 2$ and $T > 0$, consider a sequence of independent n-dimensional Brownian motions and the initial data $\{(B_t^i)_{t\geq 0}\}_{i=1}^N$ and the initial data $\{(X_0^i, V_0^i)\}_{i=1}^N$ with a joint distribution f_0^N satisfying $f_0^N \in L^1(\mathbb{R}^{2Nn}, (1+|x|^2+|v|^2)dxdv)$. Assume that the initial total energy satisfies $\mathbb{E}[E_0^N] < +\infty$ and the initial data $X_0^i \neq X_0^j$ almost surely (a.s.) for all $i \neq j$. Let $\{(X_t^{i,\varepsilon}, V_t^{i,\varepsilon})_{t\geq 0}\}_{i=1}^N$ be the unique strong solution to (14) with the initial data $\{(X_0^i, V_0^i)\}_{i=1}^N$ and the stopping time is defined by

$$\tau_\varepsilon = \inf\{t \geq 0 : \min_{i\neq j} |X_t^{i,\varepsilon} - X_t^{j,\varepsilon}| \leq \varepsilon\} \wedge 2T, \tag{39}$$

then we have

$$\lim_{\varepsilon\to 0} \mathbb{P}(\tau_\varepsilon \leq T) = 0. \tag{40}$$

Proof. Define the potential function as

$$\Phi_t^{N,\varepsilon} = \frac{1}{2N^2} \sum_{\substack{i,j=1 \\ i\neq j}}^N \Phi_\varepsilon (X_t^{i,\varepsilon} - X_t^{j,\varepsilon}). \tag{41}$$

From (23), one knows that

$$\Phi_t^{N,\varepsilon} = E_0^N + nt + M_t - \frac{1}{2N} \sum_{i=1}^N |V_t^{i,\varepsilon}|^2, \tag{42}$$

where M_t is a martingale. Observe that

$$\{\tau_\varepsilon \leq T\} \subset \Big\{ \sup_{t\in[0,T]} \Phi_{t\wedge\tau_\varepsilon}^{N,\varepsilon} \geq \Phi_{\tau_\varepsilon}^{N,\varepsilon} \Big\}. \tag{43}$$

Next, we split into two steps to prove (40) based on the above fact.

Step 1 We prove that for any $R > 0$ and small enough ε,

$$\mathbb{P}(\tau_\varepsilon \leq T) \leq \frac{C}{R} + \mathbb{P}\Big(\inf_{t\in[0,T]} M_{t\wedge\tau_\varepsilon} > -R, \sup_{t\in[0,T]} M_{t\wedge\tau_\varepsilon} \geq \frac{\Phi_\varepsilon(\varepsilon)}{2N^2} - R \Big), \tag{44}$$

where C is a constant depending only on n, N, $\mathbb{E}[\sum_{i=1}^N |X_0^i|^2]$, $\mathbb{E}[|E_0^N|]$ and T, which will be proved by dividing into the following two cases.

Case 1 ($n \geq 3$)**:** Recalling (41) and the definition of τ_ε, since $\Phi_\varepsilon(x) \geq 0$, then $\{\tau_\varepsilon \leq T\} \subset \{\Phi_{\tau_\varepsilon}^{N,\varepsilon} \geq \frac{\Phi_\varepsilon(\varepsilon)}{2N^2}\}$. Combining with (43), one has

$$\mathbb{P}(\tau_\varepsilon \leq T) \leq \mathbb{P}\Big(\sup_{t\in[0,T]} \Phi_{t\wedge\tau_\varepsilon}^{N,\varepsilon} \geq \frac{\Phi_\varepsilon(\varepsilon)}{2N^2} \Big) =: I_1. \tag{45}$$

Recalling (42), one also has

$$0 < \Phi_{t\wedge\tau_\varepsilon}^{N,\varepsilon} \leq E_0^N + n(t\wedge\tau_\varepsilon) + M_{t\wedge\tau_\varepsilon}. \tag{46}$$

Combining (45) and (46), we obtain that

$$I_1 \le \mathbb{P}\big(\sup_{t\in[0,T]} M_{t\wedge\tau_\varepsilon} \ge \frac{\Phi_\varepsilon(\varepsilon)}{2N^2} - E_0^N - nT\big)$$

$$= \mathbb{P}\Big(\inf_{t\in[0,T]} M_{t\wedge\tau_\varepsilon} \ge -E_0^N - nT,\ \sup_{t\in[0,T]} M_{t\wedge\tau_\varepsilon} \ge \frac{\Phi_\varepsilon(\varepsilon)}{2N^2} - E_0^N - nT\Big). \tag{47}$$

Then for any $R > 0$,

$$I_1 \le \mathbb{P}(-E_0^N - nT \le -R) + \mathbb{P}\Big(\inf_{t\in[0,T]} M_{t\wedge\tau_\varepsilon} > -R,\ \sup_{t\in[0,T]} M_{t\wedge\tau_\varepsilon} \ge \frac{\Phi_\varepsilon(\varepsilon)}{2N^2} - R\Big). \tag{48}$$

Thanks to the Markov's inequality, and combining with (45), we derive that

$$\mathbb{P}(\tau_\varepsilon \le T) \le \frac{\mathbb{E}[|E_0^N|] + nT}{R} + \mathbb{P}\Big(\inf_{t\in[0,T]} M_{t\wedge\tau_\varepsilon} > -R,\ \sup_{t\in[0,T]} M_{t\wedge\tau_\varepsilon} \ge \frac{\Phi_\varepsilon(\varepsilon)}{2N^2} - R\Big), \tag{49}$$

which implies that (44) holds for $n \ge 3$.

Case 2 (n = 2): Reusing the fact $\Phi_\varepsilon(x) \ge -\frac{1}{2\pi}|x|$ for any $x \in \mathbb{R}^2$ by (iii) in Lemma 2.1, for small enough ε, one deduces that

$$\Phi_{\tau_\varepsilon}^{N,\varepsilon} \ge \frac{\Phi_\varepsilon(\varepsilon)}{2N^2} - \frac{1}{4N^2\pi}\sum_{\substack{i,j=1\\ i\ne j}}^{N} |X_{\tau_\varepsilon}^{i,\varepsilon} - X_{\tau_\varepsilon}^{j,\varepsilon}|$$

$$\ge \frac{\Phi_\varepsilon(\varepsilon)}{2N^2} - \frac{1}{2N\pi}\sum_{i=1}^{N} |X_{\tau_\varepsilon}^{i,\varepsilon}|$$

$$\ge \frac{\Phi_\varepsilon(\varepsilon)}{2N^2} - \frac{1}{2N\pi}\sum_{i=1}^{N} \sup_{t\in[0,T]} |X_t^{i,\varepsilon}|, \text{ if } \tau_\varepsilon \le T. \tag{50}$$

Combining with (43), it arrives that

$$\mathbb{P}(\tau_\varepsilon \le T) \le \mathbb{P}\Big(\sup_{t\in[0,T]} \Phi_{t\wedge\tau_\varepsilon}^{N,\varepsilon} \ge \frac{\Phi_\varepsilon(\varepsilon)}{2N^2} - \frac{1}{2N\pi}\sum_{i=1}^{N} \sup_{t\in[0,T]} |X_t^{i,\varepsilon}|\Big) =: I_2. \tag{51}$$

On other hand, using (42), one also has

$$-\frac{1}{2N\pi}\sum_{i=1}^{N} |X_{t\wedge\tau_\varepsilon}^{i,\varepsilon}| < \Phi_{t\wedge\tau_\varepsilon}^{N,\varepsilon} \le E_0^N + n(t\wedge\tau_\varepsilon) + M_{t\wedge\tau_\varepsilon}, \tag{52}$$

for small enough ε. Define $Y := E_0^N + nT + \frac{1}{2N\pi}\sum_{i=1}^{N} \sup_{t\in[0,T]} |X_t^{i,\varepsilon}|$. Collecting (51) and (52), one obtains that

$$I_2 \leq \mathbb{P}\Big(\sup_{t\in[0,T]} M_{t\wedge\tau_\varepsilon} \geq \frac{\Phi_\varepsilon(\varepsilon)}{2N^2} - E_0^N - nT - \frac{1}{2N\pi}\sum_{i=1}^{N}\sup_{t\in[0,T]}|X_t^{i,\varepsilon}|\Big)$$

$$= \mathbb{P}\Big(\inf_{t\in[0,T]} M_{t\wedge\tau_\varepsilon} \geq -E_0^N - nT - \inf_{t\in[0,T]}\Big\{\frac{1}{2N\pi}\sum_{i=1}^{N}|X_{t\wedge\tau_\varepsilon}^{i,\varepsilon}|\Big\},\ \sup_{t\in[0,T]} M_{t\wedge\tau_\varepsilon} \geq \frac{\Phi_\varepsilon(\varepsilon)}{2N^2} - Y\Big). \tag{53}$$

Combining (53) and (51), one has

$$\mathbb{P}(\tau_\varepsilon \leq T) \leq \mathbb{P}\Big(\inf_{t\in[0,T]} M_{t\wedge\tau_\varepsilon} \geq -E_0^N - nT - \inf_{t\in[0,T]}\Big\{\frac{1}{2N\pi}\sum_{i=1}^{N}|X_{t\wedge\tau_\varepsilon}^{i,\varepsilon}|\Big\},\ \sup_{t\in[0,T]} M_{t\wedge\tau_\varepsilon} \geq \frac{\Phi_\varepsilon(\varepsilon)}{2N^2} - Y\Big)$$

$$\leq \mathbb{P}\Big(\inf_{t\in[0,T]} M_{t\wedge\tau_\varepsilon} \geq -Y,\ \sup_{t\in[0,T]} M_{t\wedge\tau_\varepsilon} \geq \frac{\Phi_\varepsilon(\varepsilon)}{2N^2} - Y\Big)$$

$$\leq \mathbb{P}(-Y \leq -R) + \mathbb{P}\Big(\inf_{t\in[0,T]} M_{t\wedge\tau_\varepsilon} > -R,\ \sup_{t\in[0,T]} M_{t\wedge\tau_\varepsilon} \geq \frac{\Phi_\varepsilon(\varepsilon)}{2N^2} - R\Big). \tag{54}$$

Thanks to the Markov's inequality, the first term on the right hand of (54) is estimated by

$$\mathbb{P}(-Y \leq -R) \leq \frac{\mathbb{E}\Big[\Big|E_0^N + nT + \frac{1}{2N\pi}\sum_{i=1}^{N}\sup_{t\in[0,T]}|X_t^{i,\varepsilon}|\Big|\Big]}{R}$$

$$\leq \frac{\mathbb{E}[|E_0^N|] + \frac{1}{2N\pi}\sum_{i=1}^{N}\mathbb{E}[\sup_{t\in[0,T]}|X_t^{i,\varepsilon}|] + nT}{R}. \tag{55}$$

Simple computation shows that

$$\mathbb{E}[\sup_{t\in[0,T]}|X_t^{i,\varepsilon}|] \leq \Big\{\mathbb{E}[\sup_{t\in[0,T]}|X_t^{i,\varepsilon}|^2]\Big\}^{\frac{1}{2}} \leq \Big\{\mathbb{E}\Big[\sum_{i=1}^{N}\sup_{t\in[0,T]}|X_t^{i,\varepsilon}|^2\Big]\Big\}^{\frac{1}{2}}. \tag{56}$$

Recalling (31), one obtains

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|X_t^{i,\varepsilon}|\Big] \leq C. \tag{57}$$

Combining (57) and (55), we have

$$\mathbb{P}(-Y \leq -R) \leq \frac{\mathbb{E}[|E_0^N|] + C}{R}. \tag{58}$$

Plugging (58) into (54), we achieve

$$\mathbb{P}(\tau_\varepsilon \le T) \le \frac{\mathbb{E}[|E_0^N|]+C}{R} + \mathbb{P}\Big(\inf_{t\in[0,T]} M_{t\wedge\tau_\varepsilon} > -R, \sup_{t\in[0,T]} M_{t\wedge\tau_\varepsilon} \ge \frac{\Phi_\varepsilon(\varepsilon)}{2N^2} - R\Big), \tag{59}$$

i.e. (44) also holds for $n = 2$.

Step 2 We first deal with the second term on the right hand in (44) and then prove (40).

Applying Doobs inequality for martingales[16], we have

$$\mathbb{P}\Big(\inf_{t\in[0,T]} M_{t\wedge\tau_\varepsilon} > -R, \sup_{t\in[0,T]} M_{t\wedge\tau_\varepsilon} \ge \frac{\Phi_\varepsilon(\varepsilon)}{2N^2} - R\Big) \le \mathbb{P}\Big(\sup_{t\in[0,T]} M_{t\wedge\tau_\varepsilon} \ge \frac{\Phi_\varepsilon(\varepsilon)}{2N^2} - R\Big)$$

$$\le \frac{2N^2\big(\mathbb{E}[\sup_{t\in[0,T]} |M_{t\wedge\tau_\varepsilon}|^2]\big)^{\frac{1}{2}}}{\Phi_\varepsilon(\varepsilon) - 2N^2 R} \le \frac{4N^2\big(\mathbb{E}[|M_{T\wedge\tau_\varepsilon}|^2]\big)^{\frac{1}{2}}}{\Phi_\varepsilon(\varepsilon) - 2N^2 R}.$$

for small enough ε such that $\frac{\Phi_\varepsilon(\varepsilon)}{2N^2} > R$.

Recalling (18) and (24), by Itô isometry, it directly has

$$\mathbb{E}[|M_{T\wedge\tau_\varepsilon}|^2] \le C,$$

where C are constants depending on n, N, $\sum_{i=1}^N \mathbb{E}[|X_0^i|^2]$, $\mathbb{E}[|E_0^N|]$ and T.

Therefore, we derive that

$$\mathbb{P}\Big(\inf_{t\in[0,T]} M_{t\wedge\tau_\varepsilon} > -R, \sup_{t\in[0,T]} M_{t\wedge\tau_\varepsilon} \ge \frac{\Phi_\varepsilon(\varepsilon)}{2N^2} - R\Big) \le \frac{C}{\Phi_\varepsilon(\varepsilon) - 2N^2 R}. \tag{60}$$

Combining (44) and (60) together, we have

$$\mathbb{P}(\tau_\varepsilon \le T) \le \frac{C}{R} + \frac{C}{\Phi_\varepsilon(\varepsilon) - 2N^2 R}. \tag{61}$$

Taking $R = \Phi_\varepsilon^{\frac{1}{2}}(\varepsilon)$ and then letting $\varepsilon \to 0^+$ in (61), the conclusion immediately follows from the fact that $\Phi_\varepsilon(\varepsilon) \xrightarrow{\varepsilon\to 0^+} +\infty$ by (iv) in Lemma 2.1.

3. Proof of Theorem 1.1

Proof of Theorem 1.1: We first show that there is no collision among particles almost surely in finite time, and there exists a unique global strong solution to the N-particle system (1).

Combining with the fact $F_\varepsilon(x) = F(x)$ and $F(x)$ is Lipschitz continuous in the region of $|x| \ge \varepsilon$, we know that $\{(X_t^{i,\varepsilon}, V_t^{i,\varepsilon})_{t\ge 0}\}_{i=1}^N$ is the unique strong solution to the following equations:

$$\begin{cases} dX_t^{i,\varepsilon} = V_t^{i,\varepsilon}\,dt,\ 0 \le t \le \tau_\varepsilon; \\ dV_t^{i,\varepsilon} = \frac{1}{N}\sum_{j\ne i}^N F(X_t^{i,\varepsilon} - X_t^{j,\varepsilon})dt + \sqrt{2}\,dB_t^i,\ i = 1, \cdots, N, \end{cases} \tag{62}$$

with initial data $\{(X_0^i, V_0^i)\}_{i=1}^N$, i.e. the solution to (1) on $t \in [0, \tau_\varepsilon]$ is unique,

$$X_t^i(\omega) \equiv X_t^{i,\varepsilon}(\omega) \text{ for any } 0 \le t \le \tau_\varepsilon,\ i = 1, \cdots, N. \tag{63}$$

For any fixed $T > 0$, define $\tau(\omega, T) := \lim_{\varepsilon \to 0} \tau_\varepsilon(\omega) = \inf\{t \ge 0 : \exists i \ne j,\ |X_t^i - X_t^j| = 0\} \wedge 2T$. Since $\{\tau_\varepsilon(\omega)\}_{\varepsilon > 0}$ is a non-decreasing sequence with respect to ε, then (40) implies that

$$\mathbb{P}(\lim_{\varepsilon \to 0} \tau_\varepsilon > T) = \lim_{\varepsilon \to 0} \mathbb{P}(\tau_\varepsilon > T) = 1, \tag{64}$$

which means that $\tau(\omega, T) > T$ a.s.

Thus, from (63), we know that (X_t^i, V_t^i) is exactly the unique strong solution to (1) on $t \in [0, T]$. Furthermore, since T is arbitrary, we obtain that the explosion time

$$\tau := \inf\{t \ge 0 : \exists i \ne j, | X_t^i - X_t^j | = 0\} = \infty, \text{ a.s.}$$

which implies that there is no collision among N particles almost surely, and then the strong solution to (1) is global.

Finally, we show the uniform estimates. Combining (16) with the fact that the functionals H_N and $\tilde{I}_N$ both are lower semi-continuous with respect to weak convergence[17], one has

$$H_N(f_t^N) + \int_0^t \tilde{I}_N(f_s^N)ds \le \liminf_{\varepsilon \to 0} \left\{H_N(f_t^{N,\varepsilon}) + \int_0^t \tilde{I}_N(f_s^{N,\varepsilon})ds\right\} \le H_N(f_0^N) \text{ for any } n \ge 2, \tag{65}$$

which gives (7).

For $n \ge 3$, using the Fatou lemma and (17), one has

$$\mathbb{E}[E_t^N] \le \liminf_{\varepsilon \to 0} \mathbb{E}[E_t^{N,\varepsilon}] \le \mathbb{E}[E_0^N] + nt, \tag{66}$$

i.e. (8) holds for $n \ge 3$.

Otherwise, for $n = 2$, by (iii) in Lemma 2.1, one has the fact that $\Phi_\varepsilon(x) \ge -\frac{1}{2\pi}|x|$ for small enough ε. Combining with (23), one has

$$\frac{1}{2N}\sum_{i=1}^N |V_t^{i,\varepsilon}|^2 + \Big\{\Phi_t^{N,\varepsilon} + \frac{1}{4\pi N^2}\sum_{\substack{i,j=1\\ i\ne j}}^N |X_t^{i,\varepsilon} - X_t^{j,\varepsilon}|\Big\} = E_0^N + nt + M_t + \frac{1}{4\pi N^2}\sum_{\substack{i,j=1\\ i\ne j}}^N |X_t^{i,\varepsilon} - X_t^{j,\varepsilon}|, \tag{67}$$

with the fact $\Phi_t^{N,\varepsilon} + \frac{1}{4\pi N^2}\sum_{\substack{i,j=1\\ i\ne j}}^N |X_t^{i,\varepsilon} - X_t^{j,\varepsilon}| \ge 0$.

Taking expectation of (67), we have

$$\mathbb{E}\Big[\frac{1}{2N}\sum_{i=1}^N |V_t^{i,\varepsilon}|^2 + \Big\{\Phi_t^{N,\varepsilon} + \frac{1}{4\pi N^2}\sum_{\substack{i,j=1\\ i\ne j}}^N |X_t^{i,\varepsilon} - X_t^{j,\varepsilon}|\Big\}\Big] = \mathbb{E}[E_0^N] + nt + \mathbb{E}\Big[\frac{1}{4\pi N^2}\sum_{\substack{i,j=1\\ i\ne j}}^N |X_t^{i,\varepsilon} - X_t^{j,\varepsilon}|\Big]. \tag{68}$$

Then using the Fatou lemma and (18), one has

$$\mathbb{E}[E_t^N]+\mathbb{E}\Big[\frac{1}{4\pi N^2}\sum_{\substack{i,j=1\\ i\neq j}}^{N}|X_t^i-X_t^j|\Big]\leq \mathbb{E}[E_0^N]+nt+\liminf_{\varepsilon\to 0}\mathbb{E}\Big[\frac{1}{4\pi N^2}\sum_{\substack{i,j=1\\ i\neq j}}^{N}|X_t^{i,\varepsilon}-X_t^{j,\varepsilon}|\Big]$$

$$\leq \mathbb{E}[E_0^N]+nt+\liminf_{\varepsilon\to 0}\mathbb{E}\Big[\frac{1}{2\pi N}\sum_{i=1}^{N}|X_t^{i,\varepsilon}|^2+\frac{1}{2\pi}\Big]$$

$$\leq \mathbb{E}[E_0^N]+nt+C, \tag{69}$$

which means that (8) holds for $n = 2$.

Finally, one also has

$$\mathbb{E}\Big[\sum_{i=1}^{N}\sup_{t\in[0,T]}|X_t^i|^2\Big]\leq \liminf_{\varepsilon\to 0}\mathbb{E}\Big[\sum_{i=1}^{N}\sup_{t\in[0,T]}|X_t^{i,\varepsilon}|^2\Big]\leq C, \tag{70}$$

$$\sup_{t\in[0,T]}\mathbb{E}\Big[\sum_{i=1}^{N}|V_t^i|^2\Big]\leq \liminf_{\varepsilon\to 0}\sup_{t\in[0,T]}\mathbb{E}\Big[\sum_{i=1}^{N}|V_t^{i,\varepsilon}|^2\Big]\leq C, \tag{71}$$

which means that (9) holds. We have concluded the proof of Theorem 1.1 so far.

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Conflict of interest

The corresponding author states that there is no conflict of interest.

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