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Research Article

Intrinsic Cheeger Energy for the Intrinsically Lipschitz Constants

Daniela Di Donato

Department of Mathematics, University of Pavia, Via Adolfo Ferrata, 5, Pavia, 27100, Italy
E-mail: daniela.didonato@unitn.it

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Abstract: Recently, in metric spaces, Le Donne and the author have introduced the so-called intrinsically Lipschitz sections. The main aim of this note is to adapt Cheeger theory for the classical Lipschitz constants, in our new context. To be more precise, we define the intrinsic Cheeger energy from $L^2(Y, \mathbb{R}^s)$ to $[0, +\infty]$, where $(Y, d_Y, \mathfrak{m})$ is a metric measure space and we characterize it in terms of a suitable notion of relaxed slope. In order to get this result, in a more general context, we have established some properties of the intrinsically Lipschitz constants like the Leibniz formula, the product formula and the upper semicontinuity of the asymptotic intrinsically Lipschitz constant.

***Keywords*:** Cheeger energy, Lipschitz graphs, metric spaces

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1. Introduction

In [1], Le Donne and the author have introduced and studied the so-called intrinsically Lipschitz sections in metric spaces which generalize the notion of Lipschitz maps in the particolar context of Carnot groups. The purpose of this note is to introduce and give some properties of the Cheeger energy [2] in our intrinsic context. The main result is Theorem 3. A good ingredient was finding the Leibniz formula which represents the first step in [2] in order to get Rademacher type theorem. Hence, the long-term goal is to obtain a Rademacher type theorem á la Cheeger [2] for the intrinsically Lipschitz sections.

This project fits in an active line of research [3–5] on Rademacher type theorems, but it also studies connections to different notions and mathematical areas such as Cheeger energy, Sobolev spaces and Optimal transport theory. The reader can see [6–16].

In metric measure spaces, there are different approaches to define the Cheeger energy; one of these is represented by the theory of relaxed slopes. In this project, we define intrinsic Cheeger energy (see Definition 11) and then characterize it in terms of a suitable relaxed slope (see Theorem 3). Here we use a similar technique exploited in [17, Lemma 4.3]. A basic difference with the classical case is given by the fact that we use a *non usual* Leibniz formula for the intrinsic Lipschitz constants. Differently from the usual case, we cannot obtain a result without additional condition on the fibers of π and we get a non linear formula (see (7)). On the other hand, we did not expect the classic formula to work because we proved the Lipschitz constant for φ and for $\lambda\varphi$ with $\lambda \neq 0$ to be the same (see Proposition 1). Yet, it is not possible to get the usual Sobolev space using the intrinsic slope because we don't have homogeneity property. Moreover, it is

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interesting to underline that our corresponding triangle inequality can be true with a small constant between 1/2 and 1 in particular cases (cf. (12) for $c \in [1, 2)$).

Thanks to the study of the Leibniz formula, we find a suitable setting in order that

$$\varphi, \psi \text{ are intrinsic Lipschitz} \quad \Longrightarrow \quad \varphi + \psi \text{ is intrinsic Lipschitz.}$$

To prove this result (see Theorem 1), we use basic mathematical tools. However, our result does not include Carnot groups since the projection map is not linear in this specific setting. On the other hand, it is not too restrictive to ask the linearity of a projection map.

The rest of the paper is organized as follows. In Section 2, we recall the definition of the intrinsically Lipschitz sections and we present some properties which we will use later. In Section 3, we provide some basic properties of the intrinsic Lipschitz constants as the upper semicontinuity of the asymptotic intrinsically Lipschitz constant (see Proposition 2) and the product formula (see Proposition 3). Section 4 states the sum of two intrinsic Lipschitz sections is the same and the Leibniz formula is proven. In Section 5 there is the main result of this paper (see Theorem 3). Here, we define the intrinsic Cheeger energy and we give a characterization of it in terms of an appropriate notion of intrinsically relaxed slope.

2. Intrinsically Lipschitz sections

In [1], we give the following notion.

Definition 1 Let (X, d) be a metric space, Y be a topological space and $\pi\colon X \to Y$ be a quotient map, i.e., it is continuous, open, and surjective. We say that a map $\varphi\colon Y \to X$ is a section of π if

$$\pi \circ \varphi = \mathrm{id}_Y. \tag{1}$$

Moreover, we say that a map $\varphi\colon Y \to X$ is an intrinsically Lipschitz section of π with constant L, with $L \in [1, \infty)$, if in addition

$$d(\varphi(y_1), \varphi(y_2)) \leq L d(\varphi(y_1), \pi^{-1}(y_2)), \quad \text{for all } y_1, y_2 \in Y. \tag{2}$$

Here d denotes the distance on X, and, as usual, for a subset $A \subset X$ and a point $x \in X$, we have $d(x, A) := \inf\{d(x, a) \colon a \in A\}$.

A first observation is that we study a sort of biLipschitz condition; indeed, since $\varphi(y_2) \in \pi^{-1}(y_2)$ it holds

$$d(\varphi(y_1), \pi^{-1}(y_2)) \leq d(\varphi(y_1), \varphi(y_2)) \leq L d(\varphi(y_1), \pi^{-1}(y_2)), \quad \text{for all } y_1, y_2 \in Y.$$

In fact, we underline that, in the case π is a Lipschitz quotient or submetry [18, 19], being intrinsically Lipschitz is equivalent to biLipschitz embedding, see Proposition 2.4 in [1]. Moreover, since φ is injective by (1), the class of Lipschitz sections does not include the constant maps. In [1] we analyze the notion of intrinsically Lipschitz sections given some basic properties like Ahlfors-David regularity, etc.; on the other hand in [20] we studied the intrinsic Hopf-Lax formula and their properties; moreover, in [21] and in [22] we consider the Hölder and quasi-isometric case.

However, we recall some examples of linear sections and intrinsically Lipschitz sections.

1. Let the general linear group $X = GL(n, \mathbb{R})$ or $X = GL(n, \mathbb{C})$ of degree n which is the set of $n \times n$ invertible matrices, together with the operation of ordinary matrix multiplication. We consider $Y = \mathbb{R}^* = GL(n, \mathbb{R})/SL(n, \mathbb{R})$ or $Y = \mathbb{C}^* = GL(n, \mathbb{C})/SL(n, \mathbb{C})$ where the special linear group $SL(n, \mathbb{R})$ (or $SL(n, \mathbb{C})$) is the subgroup of $GL(n, \mathbb{R})$ (or $GL(n, \mathbb{C})$) consisting of matrices with determinant of 1. Here the linear map $\pi = det$: $GL(n, \mathbb{R}) \to \mathbb{R}^*$ is a surjective homomorphism where $Ker(\pi) = SL(n, \mathbb{R})$.

2. Let $X = GL(n, \mathbb{R})$ as above and $Y = GL(n, \mathbb{R})/O(n, \mathbb{R})$ where $O(n, \mathbb{R})$ is the orthogonal group in dimension n. Recall that Y is diffeomorphic to the space of upper-triangular matrices with positive entries on the diagonal, the natural map π: $X \to Y$ is linear.

3. Let $X = \mathbb{R}^2$, $Y = \mathbb{R}$ endowed with the Euclidean distance and π: $\mathbb{R}^2 \to \mathbb{R}$ defined as $\pi((x_1, x_2)) := x_1 + x_2$ for any $(x_1, x_2) \in \mathbb{R}^2$. An easy example of sections of π is the following one: let φ: $\mathbb{R} \to \mathbb{R}^2$ given by

$$\varphi(y) = (by + af(y), (1-b)y - af(y)), \quad \forall y \in \mathbb{R}, \text{ for all } y_1, y_2 \in Y,$$

where $a, b \in \mathbb{R}$ and f: $\mathbb{R} \to \mathbb{R}$ is a continuous map.

4. Let $X = \mathbb{R}^{2\kappa}$, $Y = \mathbb{R}$ endowed with the Euclidean distance and π: $\mathbb{R}^{2\kappa} \to \mathbb{R}$ defined as $\pi((x_1, \ldots, x_{2\kappa})) := x_1 + \cdots + x_{2\kappa}$ for any $(x_1, \ldots, x_{2\kappa}) \in \mathbb{R}^{2\kappa}$. An easy example of sections of π is the following one: let φ: $\mathbb{R} \to \mathbb{R}^{2\kappa}$ given by

$$\varphi(y) = (y + a_1 f_1(y), -a_1 f_1(y), a_2 f_2(y), -a_2 f_2(y), \ldots, a_\kappa f_\kappa(y), -a_\kappa f_\kappa(y)), \quad \forall y \in \mathbb{R},$$

where $a_i \in \mathbb{R}$ and f_i: $\mathbb{R} \to \mathbb{R}$ are continuous maps for any $i = 1, \ldots, \kappa$.

5. Regarding examples of intrinsically Lipschitz sections the reader can see [23, Example 4.58].

Finally, we present a result which will be used later.

Proposition 1 (Proposition 3.1-3.2 [21]) Let π: $X \to Y$ be a linear and quotient map between normed spaces.

1. The set of all sections is a vector space over $\mathbb{R}$ or $\mathbb{C}$.

2. If φ: $Y \to X$ is an intrinsically Lipschitz section of π, then for any $\lambda \in \mathbb{R} - \{0\}$ the section $\lambda\varphi$ is also intrinsic Lipschitz for $1/\lambda\pi$ with the same Lipschitz constant.

3. Intrinsic Lipschitz constants

In this section we studies Lipschitz constants. Here, we give new results such as the product formula and the upper semicontinuity of the asymptotic intrinsically Lipschitz constant. We underline the study done in [20, 24, 25] where we analyze the notion of intrinsic Hopf-Lax formula and its connection with the Hamilton-Jacobi type equation.

Definition 2 Let φ: $Y \to X$ be a section of π. Then we define

$$ILS(\varphi) := \sup_{\substack{y_1, y_2 \in Y \\ y_1 \neq y_2}} \frac{d(\varphi(y_1), \varphi(y_2))}{d(\varphi(y_1), \pi^{-1}(y_2))} \in [0, \infty]$$

and

$$ILS(Y, X, \pi) := \{\varphi: Y \to X : \varphi \text{ is a continuous section of } \pi \text{ and } ILS(\varphi) < \infty\},$$

$$ILS_b(Y, X, \pi) := \{\varphi \in ILS(Y, X, \pi) : \text{spt}(\varphi) \text{ is bounded}\},$$

$$ILS_0(Y, X, \pi) := \{\varphi \in ILS(Y, X, \pi) : \text{spt}(\varphi) \text{ is compact}\}.$$

To simplify, we will write $ILS(Y, X)$ instead of $ILS(Y, X, \pi)$.

Definition 3 Let $\varphi: Y \to X$ be a section of π. Then we define the local intrinsically Lipschitz constant (also called intrinsic slope) of φ the map $Ils(\varphi): Y \to [0, +\infty)$ defined as

$$Ils(\varphi)(z) := \limsup_{y \to z} \frac{d(\varphi(y), \varphi(z))}{d(\varphi(y), \pi^{-1}(z))},$$

if $z \in Y$ is an accumulation point; and $Ils(\varphi)(z) := 0$ otherwise.

Definition 4 Let $\varphi: Y \to X$ be a section of π. Then we define the asymptotic intrinsically Lipschitz constant of φ the map $Ils_a(\varphi): Y \to [0, +\infty)$ given by

$$Ils_a(\varphi)(z) := \limsup_{y_1, y_2 \to z} \frac{d(\varphi(y_1), \varphi(y_2))}{d(\varphi(y_1), \pi^{-1}(y_2))},$$

if $z \in Y$ is an accumulation point and $Ils(\varphi)(z) := 0$ otherwise.

Remark 1 Notice that by $\varphi(y_2) \in \pi^{-1}(y_2)$, it is trivial that $d(\varphi(y_1), \pi^{-1}(y_2)) \le d(\varphi(y_1), \varphi(y_2))$ and so $Ils(\varphi) \ge 1$. Moreover, it holds

$$Ils(\varphi) \le Ils_a(\varphi) \le ILS(\varphi).$$

Following Proposition 2.1.11 in [26], we prove the upper semicontinuity of the asymptotic intrinsically Lipschitz constant.

Proposition 2 Let $\varphi: Y \to X$ be a continuous section of a quotient map $\pi: X \to Y$ between two metric spaces. Then, the map $y \mapsto Ils_a(\varphi)(y)$ is upper semicontinuous with respect to the metric topology of Y, i.e.,

$$\limsup_{z \to y} Ils_a(\varphi)(z) \le Ils_a(\varphi)(y), \quad \forall y \in Y.$$

Proof. Fix $y \in Y$. We want to prove that if $(y_n)_n$ is a sequence in Y such that $d_Y(y_n, y) \to 0$, then

$$\limsup_{n \to \infty} Ils_a(\varphi)(y_n) \le Ils_a(\varphi)(y).$$

By definition, it holds

$$Ils_a(\varphi)(y_n) = \inf_{\varepsilon>0} \sup_{\substack{z_1,\, z_2 \in B(y_n,\, \varepsilon) \\ z_1 \neq z_2}} \frac{d(\varphi(z_1),\, \varphi(z_2))}{d(\varphi(z_1),\, \pi^{-1}(z_2))}.$$

Hence, fix $\varepsilon > 0$ we can show that

$$B\left(y_n,\, \frac{\varepsilon}{3}\right) \subset B(y,\, \varepsilon), \qquad \text{for } n \text{ big enough.}$$

Indeed, since y_n converges to y we have that there is $N \in \mathbb{N}$ such that for any $n \geq N$ it holds $d(y_n,\, y) < \frac{\varepsilon}{3}$; on the other hand, if $z \in B(y_n,\, \frac{\varepsilon}{3})$ with $n \geq N$ we deduce that

$$d(z,\, y) \leq d(z,\, y_n) + d(y_n,\, y) < \varepsilon,$$

i.e., $z \in B(y,\, \varepsilon)$. As a consequence, for such $n \in \mathbb{N}$ it follows that

$$Ils_a(\varphi)(y_n) \leq \sup_{\substack{z_1,\, z_2 \in B(y_n,\, \frac{\varepsilon}{3}) \\ z_1 \neq z_2}} \frac{d(\varphi(z_1),\, \varphi(z_2))}{d(\varphi(z_1),\, \pi^{-1}(z_2))} \leq \sup_{\substack{z_1,\, z_2 \in B(y,\, \varepsilon) \\ z_1 \neq z_2}} \frac{d(\varphi(z_1),\, \varphi(z_2))}{d(\varphi(z_1),\, \pi^{-1}(z_2))}.$$

Eventually, taking the limsup in n and then the infimum in ε, we obtain the thesis. □

It is not possible to get the same statement for the intrinsically slope but we can define its semicontinuous envelope as follows for $X = \mathbb{R}$.

Definition 5 Let $f\colon Y \to \mathbb{R}$. We denote by f^* (f_* respectively) the upper semicontinuous (lower semicontinuous) envelope of f, namely:

$$f^*(y) \colon = \inf\{g(y) \colon\, g \geq f,\, g(.) \text{ upper semicontinuous}\},$$

$$f_*(y) \colon = \inf\{g(y) \colon\, g \geq f,\, g(.) \text{ lower semicontinuous}\}.$$

Remark 2 Obviously if $\varphi(.)$ is already upper (lower) semicontinuous, its envelope coincides with φ itself. Moreover,

$$ILS(\varphi) \geq Ils_a(\varphi)(y) \geq Ils(\varphi)^*(y).$$

The first inequality comes from the definition of $ILS(\varphi)$, while the second one is due to the fact that the asymptotic Lipschitz constant of a function is upper semicontinuous (see Proposition 2) and bigger or equal than the slope.

3.1 *Product for the intrinsic constants*

Here, we give a condition in order to get a formula for the product of asymptotic intrinsically Lipschitz constants. It is easy to see that the same statement holds for the intrinsic slope.

Proposition 3 (Product of asymptotic intrinsically Lipschitz constants) Let $Y \subset \mathbb{R}^s$ and $\pi: \mathbb{R}^s \to Y$ be a linear and quotient map. Assume also that φ and ψ are intrinsically L-Lipschitz sections of π bounded by M such that

$$\frac{\min\{d(\varphi(z), \pi^{-1}(y)), d(\psi(z), \pi^{-1}(y))\}}{d(\varphi(z)\psi(z), \pi^{-1}(y))} \leq k, \quad \forall z, y \in Y. \tag{3}$$

Then

$$Ils_a(\varphi\psi)(y) \leq MkIls_a(\varphi)(y) + MkIls_a(\psi)(y), \quad \forall y \in Y. \tag{4}$$

Proof. Fix $\varepsilon > 0$ and $y \in Y$. Notice that for every $z_1, z_2 \in B(y, \varepsilon)$ with $z_1 \neq z_2$ we have

$$\begin{aligned} d(\varphi(z_1)\psi(z_1), \varphi(z_2)\psi(z_2)) &\leq d(\varphi(z_1)\psi(z_1), \varphi(z_1)\psi(z_2)) + d(\varphi(z_1)\psi(z_2), \varphi(z_2)\psi(z_2)) \\ &\leq Md(\psi(z_1), \psi(z_2)) + Md(\varphi(z_1), \varphi(z_2)). \end{aligned}$$

Hence, dividing for $d(\varphi(z_1)\psi(z_1), \pi^{-1}(z_2))$ we obtain

$$\begin{aligned} &\frac{d(\varphi(z_1)\psi(z_1), \varphi(z_2)\psi(z_2))}{d(\varphi(z_1)\psi(z_1), \pi^{-1}(z_2))} \\ \leq& M\frac{d(\varphi(z_2), \varphi(z_1))}{d(\varphi(z_1), \pi^{-1}(z_2))}\frac{d(\varphi(z_1), \pi^{-1}(z_2))}{d(\varphi(z_1)\psi(z_1), \pi^{-1}(z_2))} + M\frac{d(\psi(z_2), \psi(z_1))}{d(\psi(z_1), \pi^{-1}(z_2))}\frac{d(\psi(z_1), \pi^{-1}(z_2))}{d(\varphi(z_1)\psi(z_1), \pi^{-1}(z_2))} \\ \leq& Mk\frac{d(\varphi(z_2), \varphi(z_1))}{d(\varphi(z_1), \pi^{-1}(z_2))} + Mk\frac{d(\psi(z_2), \psi(z_1))}{d(\psi(z_1), \pi^{-1}(z_2))}, \end{aligned}$$

where in the second equality we used the hypothesis (3).

Now, taking the supremum in $z_1, z_2 \in B(y, \varepsilon)$ and then the infimum in ε, we obtain the thesis (4). □

4. Sum of intrinsically Lipschitz sections

4.1 *Sum of intrinsically Lipschitz sections is the same*

In this section, we prove that the sum of Lipschitz section is also Lipschitz. This is relevant essentially because, in the case of the Carnot groups, it is not known whether the sum of two intrinsic Lipschitz maps is still Lipschitz. However, we ask that π is linear and in Carnot groups this fact is not true.

Theorem 1 Let $Y \subset \mathbb{R}^s$ and $\pi: \mathbb{R}^s \to Y$ be a linear and quotient map. Then, the sum of two intrinsically Lipschitz sections is also Lipschitz.

Proof. The thesis follows using basic mathematical tools. Indeed, for any $y_1, y_2 \in Y$ with $y_1 \neq y_2$ the inequality

$$d(\eta(y_1), \eta(y_2)) \leq Ld(\eta(y_1), 1/2\pi^{-1}(y_2)), \tag{5}$$

can be described in the following way: suppose that the trivial case $d(\eta(y_1), \eta(y_2)) \neq d(\eta(y_1), 1/2\pi^{-1}(y_2))$ is not true; then, we have a right triangle where

1. $\eta(y_1), \eta(y_2)$ are two vertices;
2. $d(\eta(y_1), \eta(y_2))$ is the hypotenuse;
3. either $d(\eta(y_1), 1/2\pi^{-1}(y_2))$ is a cathetus or it is a larger segment of the cathetus with vertex $\eta(y_1)$;
4. $L = 1/\sin\alpha$ where $\alpha \in (0, \pi/2)$ is the angle between the two fix sides: the hypotenuse and the other cathetus (i.e., the segment which connects $\eta(y_2)$ and the point of minimal distant belongs to the fiber).

The important point now is that $\alpha \neq 0$ and so there is a sufficiently large and finite constant L which satisfies (5). □

At this point, we want to obtain the Leibniz formula for the intrinsic slope using an additional suitable condition on the fibers.

Theorem 2 (Leibniz formula) Let $Y \subset \mathbb{R}^s$ and $\pi\colon \mathbb{R}^s \to Y$ be a linear and quotient map. Assume also that φ and ψ are intrinsically L-Lipschitz sections of π such that there is $c \geq 1$ satisfying

$$d(\pi^{-1}(y), \pi^{-1}(z)) \geq \frac{1}{c} d(f(y), \pi^{-1}(z)), \quad \forall y, z \in Y, \tag{6}$$

for $f = \varphi, \psi$. Then, denoting $\eta = \alpha\varphi + \beta\psi$ the map $Y \to \mathbb{R}^s$ with $\alpha, \beta \in \mathbb{R} - \{0\}$, we have that

$$Ils(\eta)(y) \leq c/2(Ils(\varphi)(y) + Ils(\psi)(y)), \quad \forall y \in Y. \tag{7}$$

We need the following lemma which is true in more general setting.

Lemma 1 Let X and Y be normed spaces and $\pi\colon X \to Y$ be a linear and quotient map. Then

$$|\lambda| d(\pi^{-1}(y_1), \pi^{-1}(y_2)) = d((1/\lambda\pi)^{-1}(y_1), (1/\lambda\pi)^{-1}(y_2)), \quad \forall y_1, y_2 \in Y, \forall \lambda \in \mathbb{R} - \{0\}. \tag{8}$$

Proof. Fix $y_1, y_2 \in Y$ such that $y_1 \neq y_2$. If $d(\pi^{-1}(y_1), \pi^{-1}(y_2)) = d(a, b)$ for some $a \in \pi^{-1}(y_1)$ and $b \in \pi^{-1}(y_2)$, then a similar computation as in [22, Theorem 2.10] we deduce that

$$\lambda a \in (1/\lambda\pi)^{-1}(y_1) \quad \text{and} \quad \lambda b \in (1/\lambda\pi)^{-1}(y_2),$$

and, consequently,

$$d(\pi^{-1}(y_1), \pi^{-1}(y_2)) = \|a - b\| = \frac{1}{|\lambda|}\|\lambda a - \lambda b\| \geq \frac{1}{|\lambda|} d((1/\lambda\pi)^{-1}(y_1), (1/\lambda\pi)^{-1}(y_2)). \tag{9}$$

On the other hand, if $d((1/\lambda\pi)^{-1}(y_1), (1/\lambda\pi)^{-1}(y_2)) = d(s, t)$ for some $s \in (1/\lambda\pi)^{-1}(y_1)$ and $t \in (1/\lambda\pi)^{-1}(y_2)$, then

$$\frac{1}{\lambda} s \in \pi^{-1}(y_1) \quad \text{and} \quad \frac{1}{\lambda} t \in \pi^{-1}(y_2),$$

and so

$$d((1/\lambda\pi)^{-1}(y_1),\,(1/\lambda\pi)^{-1}(y_2)) = \|s-t\| = |\lambda|\left\|\frac{1}{\lambda}s-\frac{1}{\lambda}t\right\| \geq |\lambda|d(\pi^{-1}(y_1),\,\pi^{-1}(y_2)). \tag{10}$$

Hence, putting together (9) and (10), we obtain (8). □

Proof of Theorem 2 Notice that since $Ils(\varphi) = Ils(\lambda\varphi)$ for any $\lambda \neq 0$ (see Proposition 1 (2)), it is sufficient to prove (7) for $\eta = \varphi+\psi$.

We split the proof in two steps. In the first step we prove that η is intrinsic Lipschitz and in the second one we show (7) following similarly to [27, Lemma 3.2].

(1). The thesis follows from Theorem 1.

(2). Fix $\bar{y} \in Y$ and $r > 0$. Suppose we are given $\varepsilon > 0$. For any $y \in B(\bar{y},\,r)$ we have

$$\frac{d(\varphi(\bar{y}),\,\varphi(y))}{d(\varphi(\bar{y}),\,\pi^{-1}(y))} \leq Ils(\varphi)(\bar{y})+\varepsilon \qquad \text{and} \qquad \frac{d(\psi(\bar{y}),\,\psi(y))}{d(\psi(\bar{y}),\,\pi^{-1}(y))} \leq Ils(\psi)(\bar{y})+\varepsilon. \tag{11}$$

On the other hand, thanks to Theorem 1, we find out that $z \in B(\bar{y},\,r)$ so that

$$Ils(\varphi+\psi)(\bar{y}) \leq \frac{d(\varphi(\bar{y})+\psi(\bar{y}),\,\varphi(z)+\psi(z))}{d(\varphi(\bar{y})+\psi(\bar{y}),\,(1/2\pi)^{-1}(z))}+\varepsilon,$$

and so

$$\begin{aligned}
Ils(\varphi+\psi)(\bar{y}) &\leq \frac{d(\varphi(\bar{y}),\,\varphi(z))+d(\psi(\bar{y}),\,\psi(z))}{d(\varphi(\bar{y})+\psi(\bar{y}),\,(1/2\pi)^{-1}(z))}+\varepsilon \\
&\leq \frac{d(\varphi(\bar{y}),\,\varphi(z))+d(\psi(\bar{y}),\,\psi(z))}{d((1/2\pi)^{-1}(\bar{y}),\,(1/2\pi)^{-1}(z))}+\varepsilon \\
&= \frac{d(\varphi(\bar{y}),\,\varphi(z))+d(\psi(\bar{y}),\,\psi(z))}{2d(\pi^{-1}(\bar{y}),\,\pi^{-1}(z))}+\varepsilon \\
&\leq \frac{c}{2}\frac{d(\varphi(\bar{y}),\,\varphi(z))}{d(\varphi(\bar{y}),\,\pi^{-1}(z))}+\frac{c}{2}\frac{d(\psi(\bar{y}),\,\psi(z))}{d(\psi(\bar{y}),\,\pi^{-1}(z))}+\varepsilon \\
&\leq \frac{c}{2}(Ils(\varphi)(\bar{y})+\varepsilon)+\frac{c}{2}(Ils(\psi)(\bar{y})+\varepsilon)+\varepsilon,
\end{aligned}$$

where in the first equality we used the triangle inequality, and in the second one we used the simple fact $d((1/2\pi)^{-1}(\bar{y}),\,(1/2\pi)^{-1}(z)) \leq d(\varphi(\bar{y})+\psi(\bar{y}),\,(1/2\pi)^{-1}(z))$, noting that $\varphi(\bar{y})+\psi(\bar{y}) \in (1/2\pi)^{-1}(\bar{y})$. In the first equality we used Lemma 1 for $\lambda = 2$ and in the last two inequalities we used (6) and (11).

Hence, by the arbitrariness of ε, the proof is completed. □

Proposition 4 Under the same assumption of Theorem 2, denoting $\eta = \alpha\varphi+\beta\psi$ with $\alpha,\,\beta \in \mathbb{R}-\{0\}$ the map $Y \to \mathbb{R}^s$ we have that

$$Ils_a(\eta)(y) \leq c/2(Ils_a(\varphi)(y) + Ils_a(\psi)(y)), \quad \forall y \in Y. \tag{12}$$

Proof. The proof is equal to Theorem 2, noting that for any $z_1, z_2 \in B(\bar{y}, r)$ we have

$$\frac{d(\varphi(z_1), \varphi(z_2))}{d(\varphi(z_1), \pi^{-1}(z_2))} \leq Ils_a(\varphi)(\bar{y}) + \varepsilon,$$

and we can find $z_1, z_2 \in B(\bar{y}, r)$ so that

$$Ils_a(\varphi + \psi)(\bar{y}) \leq \frac{d(\varphi(z_1) + \psi(z_1), \varphi(z_2) + \psi(z_2))}{d(\varphi(z_1) + \psi(z_1), (1/2\pi)^{-1}(z_2))} + \varepsilon.$$

□

Remark 3 Let X and Y be normed spaces and $\pi: X \to Y$ be a linear and quotient map. If $\varphi, \psi: Y \to X$ are intrinsic Lipschitz sections satisfying (6), then

$$\begin{aligned} & d(\alpha\varphi(y) + \beta\psi(y), (1/\lambda\pi)^{-1}(z)) \\ \leq & 2c \max\{|\alpha| d((1/\alpha\pi)^{-1}(y), (1/\alpha\pi)^{-1}(z)), |\beta| d((1/\beta\pi)^{-1}(y), (1/\beta\pi)^{-1}(z))\}, \end{aligned}$$

for every $y, z \in Y$ with $\alpha, \beta \in \mathbb{R} - \{0\}$ such that $\alpha + \beta = \lambda$. In particular, when $\alpha = \beta = 1/2,$ we have

$$d(\varphi(y) + \psi(y), (1/2\pi)^{-1}(z)) \leq 2c\, d((1/2\pi)^{-1}(y), (1/2\pi)^{-1}(z)),$$

for every $y, z \in Y$.

Indeed, fix $y, z \in Y$ such that $y \neq z$ and let $a \in (1/\alpha\pi)^{-1}(z)$ and $b \in (1/\beta\pi)^{-1}(z)$ such that $d(\alpha\varphi(y), (1/\alpha\pi)^{-1}(z)) = d(\alpha\varphi(y), a)$ and $d(\beta\psi(y), (1/\beta\pi)^{-1}(z)) = d(\beta\psi(y), b)$. It is easy to see that $a + b \in (1/\lambda\pi)^{-1}(z)$ and so

$$\begin{aligned} & d(\alpha\varphi(y) + \beta\psi(y), (1/\lambda\pi)^{-1}(z)) \\ \leq & \|\alpha\varphi(y) - a\| + \|\beta\psi(y) - b\| \\ \leq & 2\max\{d(\alpha\varphi(y), (1/\alpha\pi)^{-1}(z)), d(\beta\psi(y), (1/\beta\pi)^{-1}(z))\} \\ \leq & 2c \max\{|\alpha| d((1/\alpha\pi)^{-1}(y), (1/\alpha\pi)^{-1}(z)), |\beta| d((1/\beta\pi)^{-1}(y), (1/\beta\pi)^{-1}(z))\}. \end{aligned}$$

Notice that in the last inequality we used the fact

$$d((1/\alpha\pi)^{-1}(y),\,(1/\alpha\pi)^{-1}(z)) = |\alpha| d(\pi^{-1}(y),\,\pi^{-1}(z)) \geq \frac{|\alpha|}{c} d(\varphi(y),\,\pi^{-1}(z))$$

$$= \frac{1}{c} d(\alpha\varphi(y),\,(1/\alpha\pi)^{-1}(z)),$$

where in the first equality we used Lemma 1 and in the first inequality we used (6).

We conclude this section noting that the set of all sections satisfying the condition (6) is convex.

Proposition 5 Let $Y \subset \mathbb{R}^s$ and $\pi\colon \mathbb{R}^s \to Y$ be a linear and quotient map. Then the set of all intrinsically L-Lipschitz sections φ and ψ of π satisfying (6) for some $c \geq 1$ is a convex set.

We need the following lemma.

Lemma 2 Let X and Y be normed spaces and $\pi\colon X \to Y$ be a linear and quotient map. Then

$$|\lambda| d(\varphi(y_1),\,\pi^{-1}(y_2)) = d(\lambda\varphi(y_1),\,(1/\lambda\pi)^{-1}(y_2)), \quad \forall y_1,\,y_2 \in Y, \forall \lambda \in \mathbb{R} - \{0\}. \tag{13}$$

Proof. Fix $y_1,\,y_2 \in Y,$ and $\lambda \in \mathbb{R} - \{0\}$. We consider $a \in \pi^{-1}(y_2)$ such that $d(\varphi(y_1),\,\pi^{-1}(y_2)) = d(\varphi(y_1),\,a)$. By $\lambda a \in (1/\lambda\pi)^{-1}(y_2)$ we deduce that

$$d(\varphi(y_1),\,\pi^{-1}(y_2)) = \|\varphi(y_1) - a\| = \frac{1}{|\lambda|}\|\lambda\varphi(y_1) - \lambda a\| \geq \frac{1}{|\lambda|} d(\lambda\varphi(y_1),\,(1/\lambda\pi)^{-1}(y_2)).$$

Moreover, if we consider $s \in (1/\lambda\pi)^{-1}(y_2)$ such that $d(\lambda\varphi(y_1),\,(1/\lambda\pi)^{-1}(y_2)) = d(\lambda\varphi(y_1),\,s)$ and recall that $1/\lambda s \in \pi^{-1}(y_2),$ then

$$d(\lambda\varphi(y_1),\,(1/\lambda\pi)^{-1}(y_2)) = \|\lambda\varphi(y_1) - s\| = |\lambda|\|\varphi(y_1) - 1/\lambda s\| \geq |\lambda| d(\varphi(y_1),\,\pi^{-1}(y_2)).$$

Putting together the last two inequalities, we get (13). □

Proof of Proposition 5 Fix $t \in [0,\,1]$ and $\varphi,\,\psi$. We want to prove that the section $t\varphi + (1-t)\psi$ satisfies (6), i.e.,

$$c\, d(\pi^{-1}(y),\,\pi^{-1}(z)) \geq d\left(t\varphi(y) + (1-t)\psi(y),\,\pi^{-1}(z)\right), \quad \forall y,\,z \in Y.$$

Fix $y,\,z \in Y$ such that $y \neq z$ and let $a \in (1/t\pi)^{-1}(z)$ and $b \in (1/(1-t)\pi)^{-1}(z)$ such that $d\left(t\varphi(y),\,(1/t\pi)^{-1}(z)\right) = d\left(t\varphi(y),\,a\right)$ and $d\left((1-t)\psi(y),\,(1/(1-t)\pi)^{-1}(z)\right) = d\left((1-t)\psi(y),\,b\right)$. Notice that $a+b \in \pi^{-1}(z),$ we have that

$$\begin{aligned} d\left(t\varphi(y) + (1-t)\psi(y),\,\pi^{-1}(z)\right) &\leq \|t\varphi(y) - a\| + \|(1-t)\psi(y) - b\| \\ &= d\left(t\varphi(y),\,(1/t\pi)^{-1}(z)\right) + d\left((1-t)\psi(y),\,(1/(1-t)\pi)^{-1}(z)\right) \\ &= t d(\varphi(y),\,\pi^{-1}(z)) + (1-t) d(\psi(y),\,\pi^{-1}(z)) \end{aligned}$$

$$\leq cd(\pi^{-1}(y), \pi^{-1}(z)),$$

where in the second equality we used Lemma 2 and in the last inequality we used (6). □

5. Intrinsic Cheeger energy of a metric measure space

In this section we use the same notation of the last one, except for the constant c that will appear in (6). Here

$$c \geq 2.$$

This assumption guarantees us that the constant appearing in Proposition 4 and so in Leibniz formula is greater or equal to one. Here we use $L^q(Y, \mathfrak{m})$ and $L^q(Y, \mathbb{R}^s, \mathfrak{m})$ in order to consider the maps $Y \to \mathbb{R}$ and $Y \to \mathbb{R}^s$, respectively.

5.1 *The spaces $L^q(Y, \mathbb{R}^s, \mathfrak{m})$*

It is unusual to consider the space $L^q(Y, \mathbb{R}^s, \mathfrak{m})$; however, here it is fundamental because, in our intrinsic context, if $X = \mathbb{R}$ then $Y \subset \mathbb{R}$ and so we get the trivial case since π is also injective. Hence, we must study the case $X = \mathbb{R}^s$ and $Y \subset \mathbb{R}^s$ with $s > 1$. Because of this, we define the space $L^q(Y, \mathbb{R}^s, \mathfrak{m})$ for $q \in [1, +\infty)$. In a natural way, we ask that every component of a map $\psi = (\psi_1, \ldots, \psi_s)\colon Y \to \mathbb{R}^s$ belongs to $L^q(Y, \mathfrak{m})$ (see, for instance, [28]). More precisely,

Definition 6 Let $\pi\colon \mathbb{R}^s \to Y$ be a quotient map. We define the set

$$L^q(Y, \mathbb{R}^s, \mathfrak{m}) = \{\psi = (\psi_1, \ldots, \psi_s)\colon Y \to \mathbb{R}^s\colon\ \psi \text{ is a section of } \pi,$$

$$\text{and } \psi_i \in L^q(Y, \mathfrak{m}) \text{ for any } i = 1, \ldots, s\},$$

and we endow it with the following norm

$$\|\psi\|_q = \sum_{i=1}^{s} |\psi_i|_q,$$

where $|.|_q$ is the usual norm in $L^q(Y, \mathfrak{m})$.

We give a short proof regarding the fact that $\|.\|_q$ is a norm.

1. $\|\psi\|_q \geq 0$ by definition of $|.|_q$;
2. $\|\psi\|_q = 0$ if and only if $|\psi_i|_q = 0$ for every $i = 1, \ldots, s$ and so iff $\psi \equiv 0$;
3. By $|\lambda\psi_i|_q = |\lambda||\psi_i|_q$ for any $\lambda \in \mathbb{R}$, we have that $\|\lambda\psi\|_q = |\lambda|\|\psi\|_q$ for any $\lambda \in \mathbb{R}$;
4. For any $\psi, \eta \in L^q(Y, \mathbb{R}^s, \mathfrak{m})$, we get

$$\|\psi + \eta\|_q \leq |\psi_1|_q + |\eta_1|_q + \cdots + |\psi_s|_q + |\eta_s|_q = \|\psi\|_q + \|\eta\|_q,$$

i.e., the triangle inequality holds.

Notice that it is possible to define other norms in $L^q(Y, \mathbb{R}^s, \mathfrak{m})$ as

1. $\|\psi\|'_q := \max_{i=1, \dots, s} |\psi_i|_q$;
2. $\|\psi\|''_q := \sqrt{\sum_{i=1}^s |\psi_i|_q^2}$.

However, in this paper, we only use the convergence property of this class of maps and so we give the following definition.

Definition 7 Let $(\psi_n)_{n\in\mathbb{N}} := (\psi_{1,n}, \dots, \psi_{s,n})_{n\in\mathbb{N}} \subset L^q(Y, \mathbb{R}^s, \mathfrak{m})$ and $\psi := (\psi_1, \dots, \psi_s) \in L^q(Y, \mathbb{R}^s, \mathfrak{m})$. We define the convergence of the sequence $(\psi_n)_{n\in\mathbb{N}}$ on $L^q(Y, \mathbb{R}^s, \mathfrak{m})$ as the convergence of every component $\psi_{i,n}$, i.e.,

$$\psi_{i,n} \to \psi_i, \qquad \text{in } L^q(Y, \mathfrak{m}),$$

for any $i = 1, \dots, s$ as $n \to \infty$. We denote it as

$$\psi_n \to \psi \ \text{ in } L^q(Y, \mathbb{R}^s, \mathfrak{m}),$$

as $n \to \infty$.

Actually, the convergence is up to a no zero constant but this is not important because it appears when we use a *non usual* Leibniz formula and the intrinsically Lipschitz constants of ψ and of $\alpha\psi$ (with $\alpha \neq 0$) are the same and so they do not depend on α (see Proposition 1 (2)).

5.2 *Intrinsic relaxed slope*

Definition 8 (Intrinsic relaxed slope) Let $q \in (1, +\infty)$ and $\varphi \in L^q(Y, \mathbb{R}^s, \mathfrak{m})$ be a section of $\pi: \mathbb{R}^s \to Y$ with $Y \subset \mathbb{R}^s$. We say that a non negative map $G \in L^q(Y, \mathfrak{m})$ is an intrinsically relaxed slope of φ if there are $H_1, H_2 \in L^q(Y, \mathfrak{m})$ such that

1. $0 \le H_1(y) \le G(y)$ for a.e. $y \in Y$.
2. There is a sequence $(\varphi_h)_h \subset \mathbb{L}$ such that

$$\varphi_h \to \varphi \ \text{ in } L^q(Y, \mathbb{R}^s, \mathfrak{m}) \quad \text{ and } \quad Ils_a(\varphi_h)(.) \rightharpoonup H_2 \ \text{ in } L^q(Y, \mathfrak{m}).$$

3. $0 \le H_1(y) \le H_2(y)$ for a.e. $y \in Y$.

Note that, by definition, H_2 is also an intrinsic relaxed slope of φ with the same H_1 of G and itself.

The main difference with respect to the usual definition of relaxed slope is that we have $H_1 \neq H_2$. This is because our Leibniz formula is non linear.

Definition 9 (Set of intrinsic q-relaxed slopes) Let $q \in (1, +\infty)$ and $\varphi \in L^q(Y, \mathfrak{m})$ be a section of π. We denote by $R^q(\varphi)$ the set:

$$R^q(\varphi) := \{G \in L^q(Y, \mathfrak{m}) : \ G(\cdot) \text{ is an intrinsic relaxed slope of } \varphi\}.$$

An important point is that there is always a minimal element inside the class $R^q(\varphi)$ even in a pointwise sense, when it is not empty.

Proposition 6 (Locality and Minimality) Let $q \in (1, +\infty)$ and $\varphi \in L^q(Y, \mathbb{R}^s, \mathfrak{m})$ be a section of π. Then, the following properties hold:

1. $R^q(\varphi)$ is closed and convex subset of $L^q(Y, \mathfrak{m})$.

2. If $G_1, G_2 \in R^q(\varphi)$, then $\min\{G_1, G_2\} \in R^q(\varphi)$.

As a consequence, the $L^q(Y, \mathfrak{m})$-minimal element in the class $R^q(\varphi)$ is well defined and we denote it by $|D\varphi|_q$. Moreover,

$$|D\varphi|_q \leq G, \quad \mathfrak{m}-a.e. \text{ in } Y, \quad \forall G \in R^q(\varphi).$$

Proof. Proposition 4 and 1 easily imply that $R^q(\varphi)$ is a closed and convex family of sections. Indeed, let $\{G_h\}_h \subset R^q(\varphi)$ such that $G_h \to G$ in $L^q(Y, \mathfrak{m})$. In order to prove the closed property, we want to show that $G \in R^q(\varphi)$. By definition, we are able to find $(H_h^1)_h, (H_h^2)_h \subset L^q(Y, \mathfrak{m})$ such that

1. $0 \leq H_h^1(y) \leq G_h(y)$ for a.e. $y \in Y$.

2. There is a sequence $(\varphi_{h,n})_{h,n} \subset \mathbb{L}$ such that

$$\varphi_{h,n} \to_n \varphi \text{ in } L^q(Y, \mathbb{R}^s, \mathfrak{m}) \quad \text{and} \quad Ils_a(\varphi_{h,n})(\cdot) \rightharpoonup_n H_h^2 \text{ in } L^q(Y, \mathfrak{m}).$$

3. $1 \leq H_h^1(y) \leq H_h^2(y)$, for a.e. $y \in Y$.

Up to a subsequence, using the reflexivity of $L^q(Y, \mathfrak{m})$ (see [28, Theorem 4.10]), we suppose $H_h^i \rightharpoonup H^i \in L^q(Y, \mathfrak{m})$ with $i = 1, 2;$ then using a diagonal argument we find:

$$\varphi_{h,n(h)} \to_h \varphi \text{ in } L^q(Y, \mathbb{R}^s, \mathfrak{m}) \quad \text{and} \quad Ils_a(\varphi_{h,n(h)})(\cdot) \rightharpoonup_h H^2 \text{ in } L^q(Y, \mathfrak{m}).$$

On the other hand, $H^1(y) \leq G(y)$ and $H^1(y) \leq H^2(y)$ for a.e. $y \in Y$. This means that $G \in R^q(\varphi)$, as desired.

Regarding the convexity of $R^q(\varphi)$, let $G_1, G_2 \in R^q(\varphi)$. We would like to show

$$tG_1 + (1-t)G_2 \in R^q(\varphi) \quad \forall t \in [0, 1].$$

Fix $t \in [0, 1]$. By definition, we have $H_1, H_2, (\varphi_h)_h$ for G_1 and $F_1, F_2, (\psi_h)_h$ for G_2 as in Definition 8.

Hence, it is easy to see that

$$\tilde{H}_1 := tH_1 + (1-t)F_1$$

and so the first point holds. Regarding the second point in Definition 8, we set

$$\tilde{\varphi}_h = t\varphi_h + (1-t)\psi_h \qquad \text{and} \qquad \tilde{H}_2 = \frac{c}{2}(H_2 + F_2).$$

It is trivial that $\tilde{\varphi}_h \to \varphi$ and, by Proposition 4 and 1 (2), $Ils_a(\tilde{\varphi}_h) \rightharpoonup \tilde{H}^2$ in $L^q(Y, \mathfrak{m})$. Moreover, by Lemma 5, $(\tilde{\varphi}_h)_h \subset \mathbb{L}$.

Finally, by definition of $\tilde{H}_1, \tilde{H}_2$ and recall that $c \geq 2$, we obtain the last point and, as a consequence, we deduce the convexity of $R^q(\varphi)$.

Now, we show the second point of this statement. Let $G_1, G_2 \in R^q(\varphi)$. We want to show that $\min\{G_1(y), G_2(y)\} \in R^q(\varphi)$. We consider $H_1, H_2, (\varphi_h)_h$ for G_1 and $F_1, F_2, (\psi_h)_h$ for G_2 as in Definition 8. Here, we choose

$$\tilde{H}_1(y) = \begin{cases} H_1(y) & \text{for } y \in Y \text{ such that } \min\{G_1(y), G_2(y)\} = G_1(y), \\ \\ F_1(y) & \text{for } y \in Y \text{ such that } \min\{G_1(y), G_2(y)\} = G_2(y). \end{cases}$$

and so the first point in Definition 8 holds. Moreover, set

$$\tilde{\varphi}_h = \frac{1}{2}\varphi_h + \frac{1}{2}\psi_h \qquad \text{and} \qquad \tilde{H}_2 = \frac{c}{2}(H_2 + F_2).$$

By Lemma 5, $(\tilde{\varphi}_h)_h \subset \mathbb{L}$ and using Proposition 4 and 1 (2) we obtain the last two points in Definition 8.

Let's now prove the last part of the statement: since $R^q(\varphi)$ is a closed and convex subset of $L^q(Y, \mathfrak{m})$, it turns out to be weakly closed. Therefore, weak lower semicontinuity and coercivity of the $L^q(Y, \mathfrak{m})$ norm implies the existence of a minimal element in the weakly closed subset $R^q(\varphi) \subset L^q(Y, \mathfrak{m})$. Note that the pointwise minimality condition follows directly by absurd: suppose, it is not true, that is, we can find a $G \in R^q(\varphi)$ such that:

$$G < |D\varphi|_q \text{ on a set of } \mathfrak{m}\text{-positive measure.}$$

But then we can define: $\tilde{G} := \min\{G, |D\varphi|_q\}$ and by the first part of the statement, it follows $\tilde{G} \in R^q(\varphi)$. This gives a contradiction to the minimality of $|D\varphi|_q$. Hence, the proof of the statement is complete. □

Notice that the choice of a sequence $(\tilde{\varphi}_n)_n$ and $\tilde{H}_2$ in order to prove the convexity and the second point of last statement is quite independent to G but it strongly depends on the Leibniz formula. Unfortunately, at least using this approach, it does not seem possible to define a relationship between H_2 and G in order that, once the minimum relaxed slope has be defined, this is equal to its associated map H_2.

However, an easy consequence of Proposition 6 is the following relation between the minimal intrinsic relaxed slope of an intrinsically Lipschitz section and its Lipschitz constant.

Proposition 7 Let $\varphi \in \mathbb{L} \cap L^q(Y, \mathbb{R}^s, \mathfrak{m})$, then

$$|D\varphi|_q \leq Ils_a(\varphi)(y) \ \mathfrak{m} - a.e. \text{ in } Y.$$

Proof. The statement follows immediately from the fact that for an intrinsically Lipschitz and bounded section, its slope is a relaxed slope for φ. Indeed, one can take the constant approximation $(\varphi_h)_h := \varphi$ for any $h \in \mathbb{N}$, in the definition of relaxed slope. □

We conclude this section by giving a key notion in order to define intrinsic Cheeger energy.

Definition 10 (Minimal intrinsic q-relaxed slope of a section) Let $q \in (1, +\infty)$ and $\varphi \in L^q(Y, \mathbb{R}^s, \mathfrak{m})$ be a section of π. If $R^q(\varphi)$ is nonempty, which means that φ has, at least, one relaxed slope, then

$$|D\varphi|_q \text{ is called minimal intrinsic } q\text{-relaxed slope of } \varphi.$$

The following statement gives a condition in order to have a strong convergence of the minimal intrinsic q-relaxed slope.

Proposition 8 Let $q \in (1, +\infty)$ and $\varphi \in L^q(Y, \mathbb{R}^s, \mathfrak{m})$ be a section of π. Then, the following properties hold:

1. If $G \in R^q(\varphi)$ with H_2 as in Definition 8, then there is $(\varphi_n)_n \subset \mathbb{L}$ converging to φ in $L^q(Y, \mathbb{R}^s, \mathfrak{m})$ and $(G_n)_n \subset L^q(Y, \mathfrak{m})$ strongly convergent to H_2 in $L^q(Y, \mathfrak{m})$.

2. If $(G_n)_n \subset R^q(\varphi_n)$, $(\varphi_n)_n \subset \mathbb{L}$ such that $\varphi_n \rightharpoonup \varphi$ in $L^q(Y, \mathbb{R}^s, \mathfrak{m})$ and $G_n \rightharpoonup G$ in $L^q(Y, \mathfrak{m})$, then $G \in R^q(\varphi)$.

3. Let $K \subset Y$ be a compact subset and $\psi := \varphi_{|K}$. Then if $G \in R^q(\varphi)$ it holds

3.a $G_{|K} \in R^q(\psi)$;

3.b there are $H_1, H_2 \in L^q(K, \mathfrak{m})$ such that

- $0 \le H_1(y) \le G(y)$ and $0 \le H_1(y) \le H_2(y)$ for a.e. $y \in K$.
- There is a sequence $(\psi_h)_h \subset \mathbb{L}$ such that

$$\psi_h \to \psi \;\text{ in } L^q(K, \mathbb{R}^s, \mathfrak{m}) \quad \text{and} \quad Ils_a(\psi_h)(.) \to H_2 \;\text{ in } L^q(K, \mathfrak{m}).$$

Proof. The third point is trivial. For the other points, we follow Lemma 4.3 in [6].

(1). By definition, we know that there are $H_1, H_2 \in L^q(Y, \mathfrak{m})$ such that

- $0 \le H_1(y) \le G(y)$ and $0 \le H_1(y) \le H_2(y)$ for a.e. $y \in Y$.
- There is a sequence $(\varphi_h)_h \subset \mathbb{L}$ such that

$$\varphi_h \to \varphi \;\text{ in } L^q(Y, \mathbb{R}^s, \mathfrak{m}) \quad \text{and} \quad Ils_a(\varphi_h)(.) \rightharpoonup H_2 \;\text{ in } L^q(Y, \mathfrak{m}).$$

By Mazur's lemma (see Corollary 3.8 in [28]) we can find a sequence of convex combination Ils_h of $Ils_a(\varphi_h)$, starting from an index $h(n) \to \infty$ strongly convergent to H_2 in $L^q(Y, \mathfrak{m})$. The corresponding convex combinations of φ_h, that we shall denote by ψ_h, still converge to φ in $L^q(Y, \mathfrak{m})$ and belong to $\mathbb{L}$ (see Lemma 5).

(2). We need to prove that the set

$$R := \{(\varphi, G) \in L^q(Y, \mathbb{R}^s, \mathfrak{m}) \times L^q(Y, \mathfrak{m}) : \; \varphi \in \mathbb{L}, \; G \text{ is a relaxed slope of } \varphi\},$$

is weakly closed in $L^q(Y, \mathbb{R}^s, \mathfrak{m}) \times L^q(Y, \mathfrak{m})$. If we show that (2.a) R is closed, it is sufficient to prove that (2.b) R is strongly closed.

(2.a). Let $(\varphi_1, G_1), (\varphi_2, G_2) \in R$. We want that $(t\varphi_1 + (1-t)\varphi_2, tG_1 + (1-t)G_2) \in R$ for every $t \in [0, 1]$. This follows immediately from Proposition 6, Lemma 5 and from the fact L^q is convex.

(2.b). If $(\varphi_n, G_n)_n \subset R$ strongly converges to (φ, G) in $L^q(Y, \mathbb{R}^s, \mathfrak{m}) \times L^q(Y, \mathfrak{m})$, we want to show that $(\varphi, G) \in R$. We can find sequences $(\psi_{n,h})_{n,h} \subset \mathbb{L} \cap L^q(Y, \mathfrak{m})$ and $G_{n,h} \subset L^q(Y, \mathfrak{m})$ such that

- $\psi_{n,h} \to \varphi_n$ in $L^q(Y, \mathbb{R}^s, \mathfrak{m})$ with $Ils_a(\psi_{n,h}) \rightharpoonup_h H_{2,n}$ in $L^q(Y, \mathfrak{m})$;
- $G_{n,h} \to H_{2,n}$ in $L^q(Y, \mathfrak{m})$ with $0 \le H_{1,n} \le G_n$ and $0 \le H_{1,n} \le H_{2,n}$ a.e. $y \in Y$.

Possibly extracting a suitable subsequence, we can assume that $H_{2,n} \rightharpoonup H_2$ in $L^q(Y, \mathfrak{m})$. Moreover, using the reflexivity of $L^q(Y, \mathfrak{m})$, we get that, up to subsequences, $H_{1,n} \rightharpoonup H_1$ in $L^q(Y, \mathfrak{m})$ (see, [28, Theorem 3.18]). Hence, by standard diagonal argument we can find an increasing sequence $h \mapsto n(h)$ such that $\psi_{n,h(n)} \to \varphi$ in $L^q(Y, \mathbb{R}^s, \mathfrak{m})$ and $Ils(\psi_{n,h(n)}) \rightharpoonup H_2$ in $L^q(Y, \mathfrak{m})$ with $0 \le H_1 \le G$ and $0 \le H_1 \le H_2$ a.e. $y \in Y$. Hence $(\varphi, G) \in R$, as desired. □

Remark 4 We do not know if it is possible to get Proposition 8 (3) without the condition of compactness of K. On the other hand, it is not possible to apply the proof of Theorem 2.2.7 in [26] because the inequality (2.2.12) does not work in our intrinsic context. The motivation is given by our *non usual* Leibniz formula (see Proposition 4).

5.3 *Intrinsic Cheeger energy: definition*

In this section we consider $(Y, d, \mathfrak{m})$ a metric measure space with $Y \subset \mathbb{R}^s$ and $\pi : \mathbb{R}^s \to Y$ a quotient and linear map. Set

$$\mathbb{L} := S \cap ILS_b(Y), \tag{14}$$

where S is the set of all intrinsically L-Lipschitz sections of π satisfying the inequality (6) for some $c \geq 2.$ Using Proposition 6 for $q = 2,$ we define the intrinsic Cheeger energy as follows.

Definition 11 (Intrinsic Cheeger energy) The intrinsic Cheeger energy is the functional

$$iCh_a : L^2(Y, \mathbb{R}^s, \mathfrak{m}) \to [0, +\infty]$$

defined as

$$iCh_a(\varphi) := \inf\left\{ \liminf_{n\to+\infty} \int_Y Ils_a^2(\varphi_n)(y)\, d\mathfrak{m}(y) : \, (\varphi_n)_n \subset \mathbb{L},\, \varphi_n \to \varphi \text{ in } L^2(Y, \mathbb{R}^s, \mathfrak{m}) \right\}.$$

The domain of the intrinsic Cheeger energy is the subset of $L^2(Y, \mathfrak{m})$ defined by:

$$Dom(iCh_a) := \left\{ \varphi \in L^2(Y, \mathbb{R}^s, \mathfrak{m}) : \, iCh_a(\varphi) < +\infty \right\}.$$

Another possibility could be to use the slope instead of the asymptotic Lipschitz constant, namely:

$$iCh_{Ils}(\varphi) := \inf\left\{ \liminf_{n\to+\infty} \int_Y Ils^2(\varphi_n)(y)\, d\mathfrak{m}(y) : \, (\varphi_n)_n \subset \mathbb{L},\, \varphi_n \to \varphi \text{ in } L^2(Y, \mathbb{R}^s, \mathfrak{m}) \right\}.$$

Remark 2 directly implies the easier inequality:

$$iCh_a(\varphi) \geq iCh_{Ils}(\varphi), \quad \forall \varphi \in Dom(iCh_a).$$

These two definitions of intrinsic Cheeger energy are built through a relaxation process; indeed, iCh_a is the relaxation of the functional

$$\Gamma_a(\varphi) := \begin{cases} \liminf_{n\to+\infty} \int_Y Ils_a^2(\varphi_n)(y)\, d\mathfrak{m}(y), & \text{if } \varphi \in \mathbb{L}, \\ \\ +\infty, & \text{if } \varphi \in L^2(Y, \mathbb{R}^s, \mathfrak{m}) - \mathbb{L}, \end{cases}$$

with respect to the $L^2(Y, \mathfrak{m})$ distance, in the sense that it is the biggest lower semicontinuous functional which is less or equal than Γ_a. In a similar way, iCh_{Ils} is the relaxation of the functional

$$\Gamma_{Ils}(\varphi) := \begin{cases} \liminf_{n\to+\infty} \int_Y Ils^2(\varphi_n)(y)\, d\mathfrak{m}(y), & \text{if } \varphi \in \mathbb{L}, \\ +\infty, & \text{if } \varphi \in L^2(Y, \mathbb{R}^s, \mathfrak{m}) - \mathbb{L}. \end{cases}$$

In the classical case, it is well known that the space of Lipschitz maps is dense in $L^2(Y)$ but in the intrinsic case we do not have this property. However, we recall that we get the trivial case when $s = 1$.

5.4 *Intrinsic Cheeger energy: equivalent condition*

The set $R^2(\varphi)$ given by Definition 9 represents the relaxation on the asymptotic Lipschitz constant instead of the above, where we do a relaxation on the functional Γ_a and it is the key notion in order to give the following result.

Theorem 3 Let $\varphi \in L^2(K, \mathbb{R}^s, \mathfrak{m})$ with $K \subset Y$ compact and suppose that $R^2(\varphi)$ is nonempty. Then, we have the following representation formula of the Cheeger energy:

$$iCh_a(\varphi) = \int_K H_2^2(y)\, d\mathfrak{m}(y),$$

where H_2 is given by Definition 8 for $|D\varphi|_2$.

Proof. We follow the last part of the proof in [26, Theorem 2.2.7]. Without loss of generality, we suppose $\varphi \in Dom(iCh_a)$. Hence, by definition of intrinsic Cheeger energy and Proposition 8 (3) applying to $|D\varphi|_2$, we have that

$$iCh_a(\varphi) \le \liminf_{n\to+\infty} \int_K Ils_a^2(\varphi_n)(y)\, d\mathfrak{m}(y) = \int_K H_2^2(y)\, d\mathfrak{m}(y),$$

and this provides the upper bound on iCh_a. On the other hand, let $(\psi_n)_n \subset \mathbb{L}$ be a sequence such that

- $\psi_n \to \varphi$ in $L^2(K, \mathbb{R}^s, \mathfrak{m})$;
- $\liminf_{n\to+\infty} \int_K Ils_a^2(\psi_n)(y)\, d\mathfrak{m}(y) < \infty$.

By reflexivity of L^2, we get that, up to a subsequence, $(Ils_a^2(\psi_n))_n$ converges weakly in $L^2(K, \mathfrak{m})$ and also strongly since K is compact. Finally, using definition of minimal intrinsic relaxed slope, we obtain

$$\liminf_{n\to+\infty} \int_K Ils_a^2(\psi_n)(y)\, d\mathfrak{m}(y) \ge \int_K H_2^2(y)\, d\mathfrak{m}(y).$$

Since $(\psi_n)_n$ is generic we end up with

$$iCh_a(\varphi) \ge \int_K H_2^2(y)\, d\mathfrak{m}(y),$$

and we are done. □

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Conflict of interest

The author declares no competing financial interest.

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