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Research Article

Perturbation of Solitary Waves and Shock Waves with Surface Tension

Lakhveer Kaur[1], Yakup Yildirim[2,3*], Abdullahi Rashid Adem[4], Luminita Moraru[5,6], Anjan Biswas[7,8,9]

[1] Department of Mathematics, Jaypee Institute of Information Technology, Noida, 201304, India
[2] Department of Computer Engineering, Biruni University, Istanbul, 34010, Turkey
[3] Mathematics Research Center, Near East University, 99138, Nicosia, Cyprus
[4] Department of Mathematical Sciences, University of South Africa, UNISA, 0003, South Africa
[5] Department of Chemistry, Physics and Environment, Faculty of Sciences and Environment, Dunarea de Jos University of Galati, 47 Domneasca Street, 800008, Romania
[6] Department of Physics, Sefako Makgatho Health Sciences University, Medunsa, 0204, South Africa
[7] Department of Mathematics and Physics, Grambling State University, Grambling, LA, 71245-2715, USA
[8] Department of Applied Sciences, Cross-Border Faculty of Humanities, Economics and Engineering, Dunarea de Jos University of Galati, 111 Domneasca Street, Galati, 800201, Romania
[9] Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Medunsa, 0204, South Africa
E-mail: yyildirim@biruni.edu.tr

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Abstract: This manuscript is designed with an extensive aim to investigate solitary waves in shallow water with surface tension. The governing model is the perturbed sixth-order Boussinesq equation, which incorporates higher-order dispersion effects and perturbative terms that influence wave dynamics. The G'/G-expansion procedure is employed to systematically retrieve exact solitary wave solutions, providing a diverse set of wave structures that depend on the interplay between dispersion, nonlinearity, and perturbative effects. The study further establishes the necessary parameter constraints for the existence of such solitary waves, ensuring the physical viability of the obtained solutions. Additionally, a detailed analysis of the influence of perturbation terms on the soliton characteristics is provided, revealing novel behaviors and stability conditions that were previously unexplored. These findings contribute to a deeper understanding of wave propagation in shallow water systems, with potential applications in engineering and fluid dynamics.

Keywords: solitary waves, surface tension, perturbation

MSC: 35C08, 76B25

1. Introduction

The dynamical implications of nonlinear partial differential equations (NLPDEs) are very well conceived from diverse scientific fields, such as aeronautics, mechanics, nonlinear optics, oceanography, and plasma physics. Their widespread influence has been seen in each and every discipline of research and innovation. The rapidly growing interest among scientific community is to decipher exact solutions to nonlinear partial differential equations (NLPDEs), as they are essential to comprehending both their mathematical and real-world applications. Each of these equations, which have important practical ramifications, is derived from several mathematical and physical models.

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Researchers in mathematical physics and engineering have been fascinated by the study of nonlinear phenomena in recent years, as they have shown intriguing properties with a broad range of applications. As a focal topic of study, NLPDEs provide important insights into the intricacies of several physical phenomena. These formulas are very helpful in explaining complex processes in a variety of fields, including atmospheric science, fluid mechanics, plasma waves, optical fiber communications, and soliton theory. The investigation of NLPDEs provides pathways for the comprehension and control of many systems, opening doors for creative responses to pressing problems in a variety of fields. Since there are no standrad basic techniques that can be used to evaluate all of these NLPDEs equations, in most of instances, these equations are difficult to solve analytically, leading to substantial individual analysis of each of these equations. The two primary categories of obtaining solutions to NLPDEs are numerical or analytical. In recent years, there have seen significant progress, leading to the development of a number of reliable and efficient mathematical procedures for getting exact solutions for nonlinear equations. These mathematical procedures include bilinear method [1, 2], inverse scattering transformation (IST) [3, 4], symmetry reductions [5, 6], variable separation approaches [7, 8], Bäcklund and Darboux transformations [9, 10], G'/G-expansion method [11], the tanh method [12], the Jacobi elliptic function method [13], the Exp-function method [14, 15], the homogeneous balance method [16] etc.

The process of the shallow-water wave (SWW) has been described by the basic Boussinessq equation (BE) [17]. This equation takes into account a number of waves and shallow water effects, including diffraction, refraction, shoaling, weak nonlinearity, and shoaling, playing significant role in fluid dynamics along with many other spheres of physics, such as one-dimensional nonlinear lattice waves, vibrations in a nonlinear string, ion sound waves in plasma, and propagation of long waves in shallow water [18–23].

Even though this situation has been the subject of countless models, but none of them have taken surface tension into account. Boussinesq equation (BE) is one of the models that also explains the passage of waves in shallow water. A few decades ago, Daripa analyzed even though this situation has been the subject of countless models, none of them have taken surface tension into account. The Boussinesq equation (BE) is one of the models that also explains the passage of waves in shallow water. A few decades ago, Daripa analyzed BE again, including the surface tension effect. leading to formation of the sixth-order BE (6BE) using first principles [24]. It has been demonstrated that solutions to the 6BE exist and are unique [25–28]. Additionally, without using conserved values, conservation laws for this model have been examined [29].

The 6BE in its dimensionless form has been perused as follows:

$$q_{tt}-k^2q_{xx}+c\left(q^{2n}\right)_{xx}+a_1q_{xxxx}+a_2q_{xxtt}+b_1q_{xxxxxx}+b_2q_{xxxxtt}=0. \tag{1}$$

The independent variables in this case are the spatial and temporal variables, denoted by x and t respectively and wave form is indicated by the dependent variable $q(x, t)$. The wave operator is represented by the first two terms, along with k as wave number. The parameter n, represents the generic power law parameter, providing a more generalized flavor to considered model, c signifies nonlinearity coefficient. Several many times BE has been investigated with very specific parameter $n=1$. The coefficients of the fourth- and sixth-order dispersion terms are thus shown by the coefficients a_j and b_j for $j=1, 2$. The coefficients of b_j is originated from the surface tension effect.

The 6BE Model with perturbation terms has been portrayed as:

$$\begin{aligned}&q_{tt}-k^2q_{xx}+c\left(q^{2n}\right)_{xx}+a_1q_{xxxx}+a_2q_{xxtt}+b_1q_{xxxxxx}+b_2q_{xxxxtt}\\&=\theta q_xq_{xx}+\delta q^mq_x+\Lambda qq_{xxx}+\nu qq_xq_{xx}+\xi q_xq_{xxx}+\psi q_{xxxxx}+\kappa qq_{xxxxx}.\end{aligned} \tag{2}$$

Here, the higher spatial dispersion is represented by the coefficient of ψ, whilst the higher order non-linear dispersion is represented by the coefficient of δ and m is considered as a positive integer. The other remaining coefficients are found in the context Whitham hierarchy [24–30].

Several characteristics of the 6BE with and without perturbation terms are covered in this current work. Yet, because to its intricate structure, it is quite challenging to analyze this model. With the aid of the ansatz method and the G'/G-expansion method, this model has been studied with a generic appeal for deriving its analytic solutions, considering some particular values of m, n and hence resulting into diverse variety of analytic solutions and then complemented by a thorough brief discussion in concluding remarks section.

2. Governing model

The 6BE in its dimensionless form is written as

$$q_{tt}-k^2q_{xx}+c(q^{2n})_{xx}+a_1q_{xxxx}+a_2q_{xxtt}+b_1q_{xxxxxx}+b_2q_{xxxxtt}=0, \tag{3}$$

with $q=q(x,\,t)$.

Utilizing $\tau=-Vt+x$, along with $q(x,\,t)=F(\tau)$, we have recovered the following ordinary differential equation from the equation (3):

$$\begin{aligned}&(V^2b_2+b_1)F'''''' + (a_2V^2+a_1)F'''' + c(4F(\tau)^{2n-2}n^2-2F(\tau)^{2n-2}n)(F')^2\\&+2ncF(\tau)^{2n-1}F''+(-k^2+V^2)F''=0.\end{aligned} \tag{4}$$

Integrating equation (4) with respect to τ twice, obtained ordinary differential equation (ODE) is written as follows:

$$(V^2b_2+b_1)F''''+(a_2V^2+a_1)F''+cF(\tau)^{2n}+(-k^2+V^2)F=0. \tag{5}$$

Next, equation (5) has been examined with the help of G'/G-expansion method for $n=1$, $n=3/2$, $n=5/2$ to establish the new analytic solutions as in subsequent sections. In this context, it is worth mentioning that an attempt to recover the complete spectrum of solutions by Lie symmetry analysis was made in 2023. However the research-incapacitated author, Bansal, failed to integrate it [29]. The current paper successfully recovers this full spectrum of solutions.

2.1 *n = 1*

Firstly, we consider $n=1$ for (5) and then equation (5) is rewritten as follows:

$$(V^2b_2+b_1)F''''+(a_2V^2+a_1)F''+cF(\tau)^2+(-k^2+V^2)F=0. \tag{6}$$

Homogeneous balance among highest order linear and most nonlinear derivative containing terms in equation (6) gives $m=4$ and hence generate following solution structure:

$$F(\tau) = A_0 + A_1\left(\frac{G'(\tau)}{G(\tau)}\right) + A_2\left(\frac{G'(\tau)}{G(\tau)}\right)^2 + A_3\left(\frac{G'(\tau)}{G(\tau)}\right)^3 + A_4\left(\frac{G'(\tau)}{G(\tau)}\right)^4, \tag{7}$$

with A_i, $i = 0, 1, 2, 3, 4$ are constants, to be determined in the mean process of calculations and $G = G(\tau)$ following the auxiliary equation

$$G'' + \lambda G' + \mu G = 0. \tag{8}$$

Using (5) in equation (4), along with (8), we have furnished following two cases to exploring variety of solutions of equation (3):

Case-I

$$V = \pm\frac{\sqrt{-210c\lambda^4 A_4 + 1{,}680c\lambda^2\mu A_4 - 3{,}360c\mu^2 A_4 + 4{,}900k^2}}{70},$$

$$A_0 = \mu^2 A_4,\ \ A_1 = 2\lambda\,\mu A_4,\ \ A_2 = \lambda^2 A_4 + 2\mu A_4,\ \ A_3 = 2\lambda A_4,$$

$$a_1 = \frac{3c\lambda^4 A_4 a_2}{70} - \frac{12c\lambda^2\mu A_4 a_2}{35} + \frac{24c\mu^2 A_4 a_2}{35} + \frac{13c\lambda^2 A_4}{840} - \frac{13c\mu A_4}{210} - k^2 a_2,$$

$$b_1 = \frac{3c\lambda^4 A_4 b_2}{70} - \frac{12c\lambda^2\mu A_4 b_2}{35} + \frac{24c\mu^2 A_4 b_2}{35} - k^2 b_2 - \frac{cA_4}{840}, \tag{9}$$

equipped with A_4 as free parameters.

As a result, with the help of (9) and using (7) along with $\tau = -Vt + x$, $F(\tau) = q(x, t)$, we have obtained following solutions structures for equation (3), by utilizing the solutions of (8):

For the instance $\lambda^2 - 4\mu > 0$

$$q(x, t) = \frac{A_4\left(\lambda^2 - 4\mu\right)^2 (w_1 - w_2)^2 (w_1 + w_2)^2}{16\left(w_2 \sinh\left(\frac{1}{2}(-Vt + x)\sqrt{\lambda^2 - 4\mu}\right) + w_1 \cosh\left(\frac{1}{2}(-Vt + x)\sqrt{\lambda^2 - 4\mu}\right)\right)^4}. \tag{10}$$

In (10), for $w_1 = 0$, $w_2 \neq 0$, one recovers singular solitary waves while for $w_2 = 0$, $w_1 \neq 0$, one obtains solitary waves.

For the instance $\lambda^2 - 4\mu < 0$

$$q(x, t) = \frac{A_4\left(\lambda^2 - 4\mu\right)^2 \left(w_1^2 + w_2^2\right)^2}{16\left(w_2 \sin\left(\frac{1}{2}(-Vt + x)\sqrt{-\lambda^2 + 4\mu}\right) + w_1 \cos\left(\frac{1}{2}(-Vt + x)\sqrt{-\lambda^2 + 4\mu}\right)\right)^4}, \tag{11}$$

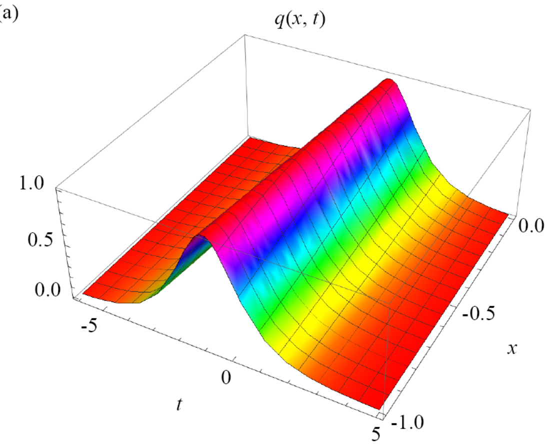
For the instance $\lambda^2 - 4\mu = 0$

$$q(x, t) = \frac{A_4 {w_1}^4}{((-Vt + x) w_1 + w_2)^4}, \tag{12}$$

along with A_4, a_2, b_2, k, c, w_1, w_2 as free parameters.

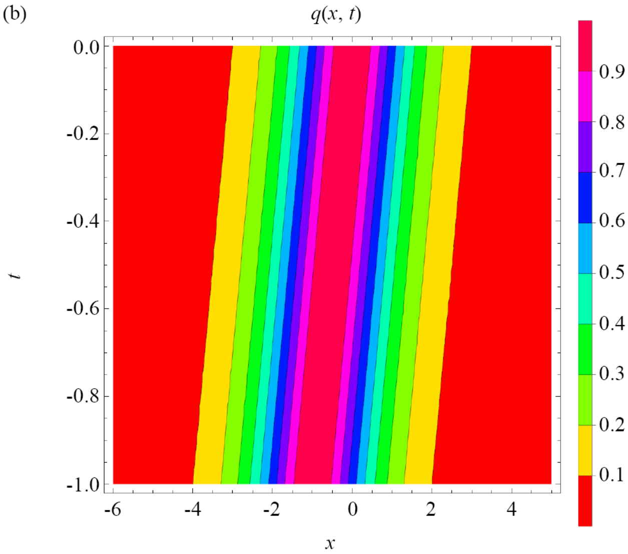
We analyze the solitary wave (10) and shock wave (14) as depicted in Figures 1 and 2. Each figure illustrates the wave dynamics, with surface plots, contour plots, and 2D plots providing detailed insights into the behavior of the waves. The parameters are addressed constant across all figures as $w_1 = 1$, $\mu = 1$, $\lambda = 2.1$, $A_4 = 1$, $c = 1$, and $k = 1$.

(a)



Surface plot

(b)



Contour plot

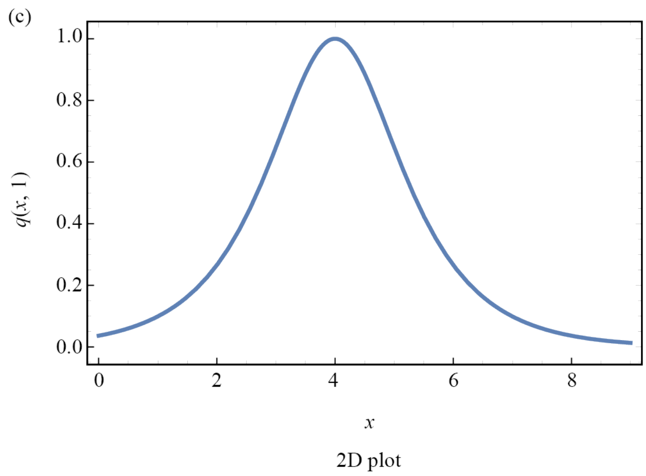
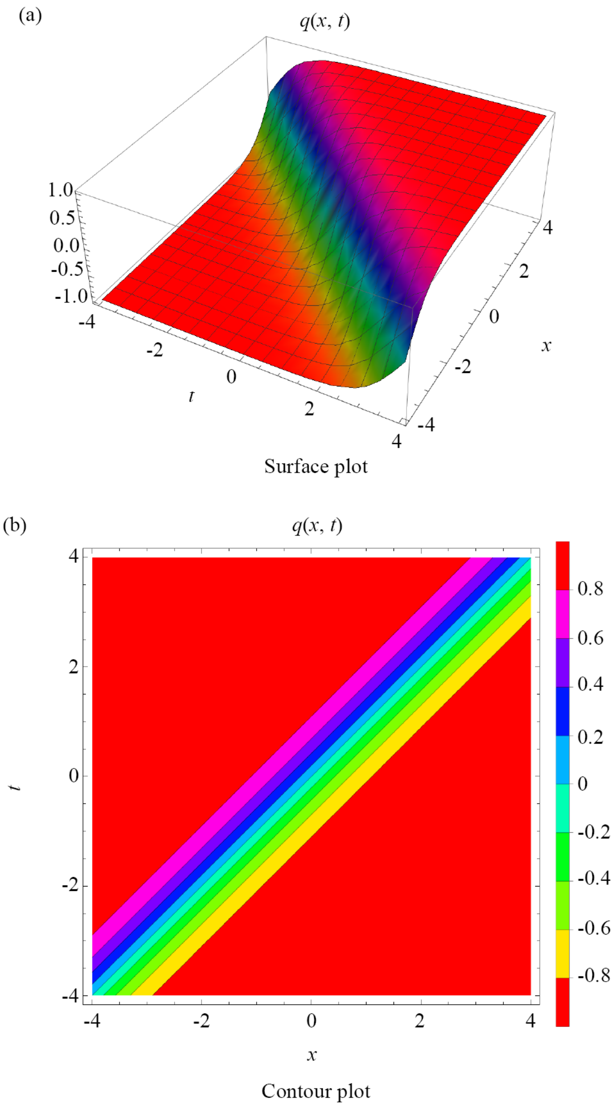
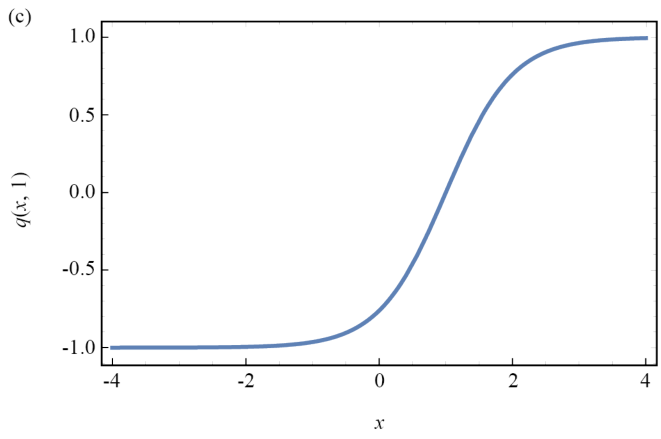


Figure 1. Exploring the features of a solitary wave





2D plot

Figure 2. Exploring the features of a shock wave

Case-II

$$V = \frac{\sqrt{210c\lambda^4 A_4 - 1{,}680c\lambda^2\mu A_4 + 3{,}360c\mu^2 A_4 + 4{,}900k^2}}{70},$$

$$A_0 = -\frac{A_4\left(3\lambda^4 - 24\lambda^2\mu - 22\mu^2\right)}{70},$$

$$A_1 = 2\lambda\mu A_4, \quad A_2 = \lambda^2 A_4 + 2\mu A_4, \quad A_3 = 2\lambda A_4, \quad A_4 = A_4,$$

$$a_1 = -\frac{3c\lambda^4 A_4 a_2}{70} + \frac{12c\lambda^2\mu A_4 a_2}{35} - \frac{24c\mu^2 A_4 a_2}{35} + \frac{13c\lambda^2 A_4}{840} - \frac{13c\mu A_4}{210} - k^2 a_2,$$

$$b_1 = -\frac{3c\lambda^4 A_4 b_2}{70} + \frac{12c\lambda^2\mu A_4 b_2}{35} - \frac{24c\mu^2 A_4 b_2}{35} - k^2 b_2 - \frac{cA_4}{840}, \tag{13}$$

equipped with A_4 as free parameter.

As a result, with the help of (13) and using (7) along with $\tau = -Vt + x$, $F(\tau) = q(x, t)$, we have obtained following solutions structures for equation (3):

For the instance $\lambda^2 - 4\mu > 0$

$$q(x, t) = A_4\phi_1{}^4 + 2\phi_1{}^3\lambda A_4 + \phi_1{}^2\left(\lambda^2 A_4 + 2\mu A_4\right) + 2\phi_1\lambda\mu A_4 - \frac{A_4}{70}\left(3\lambda^4 - 24\lambda^2\mu - 22\mu^2\right), \tag{14}$$

where

$$\phi_1=\frac{\sqrt{\lambda^2-4\mu}\left(w_1\sinh\left(\frac{1}{2}(-Vt+x)\sqrt{\lambda^2-4\mu}\right)+w_2\cosh\left(\frac{1}{2}(-Vt+x)\sqrt{\lambda^2-4\mu}\right)\right)}{2w_2\sinh\left(\frac{1}{2}(-Vt+x)\sqrt{\lambda^2-4\mu}\right)+2w_1\cosh\left(\frac{1}{2}(-Vt+x)\sqrt{\lambda^2-4\mu}\right)}-\frac{\lambda}{2}.$$

In (14), for $w_1=0$, $w_2\neq 0$, one recovers singular solitary waves while for $w_2=0$, $w_1\neq 0$, one obtains shock waves. For the instance $\lambda^2-4\mu<0$

$$q(x,t)=A_4{\phi_2}^4+2{\phi_2}^3\lambda A_4+{\phi_2}^2\left(\lambda^2A_4+2\mu A_4\right)+2\phi_2\lambda\mu A_4-\frac{A_4}{70}\left(3\lambda^4-24\lambda^2\mu-22\mu^2\right), \tag{15}$$

where

$$\phi_2=\frac{\sqrt{-\lambda^2+4\mu}\left(-w_1\sin\left(\frac{1}{2}(-Vt+x)\sqrt{-\lambda^2+4\mu}\right)+w_2\cos\left(\frac{1}{2}(-Vt+x)\sqrt{-\lambda^2+4\mu}\right)\right)}{2w_2\sin\left(\frac{1}{2}(-Vt+x)\sqrt{-\lambda^2+4\mu}\right)+2w_1\cos\left(\frac{1}{2}(-Vt+x)\sqrt{-\lambda^2+4\mu}\right)}-\frac{\lambda}{2}.$$

For the instance $\lambda^2-4\mu=0$

$$q(x,t)=\frac{A_4{w_1}^4}{((-Vt+x)w_1+w_2)^4}, \tag{16}$$

along with A_4, a_2, b_2, k, c, w_1, w_2 as free parameters.

2.2 *n* = 3/2

Here, we consider $n=\frac{3}{2}$ for (3) and then reduced ODE is rewritten as follows:

$$(V^2b_2+b_1)F''''+(a_2V^2+a_1)F''+cF(\tau)^3+(-k^2+V^2)F=0. \tag{17}$$

Homogeneous balance among highest order linear and most nonlinear derivative containing terms in equation (17) gives $m=2$ and hence generate following solution structure:

$$F(\tau)=A_0+A_1\left(\frac{G'(\tau)}{G(\tau)}\right)+A_2\left(\frac{G'(\tau)}{G(\tau)}\right)^2, \tag{18}$$

with A_i, $i=0$, 1, 2 are constants, to be determined in the mean process of calculations and $G(\tau)$ following the auxiliary equation (8). Using (18) in equation (17), along with (8), we have furnished following two cases to exploring variety of solutions of equation (3):

Case-I

$$c=-\frac{4(V^2-k^2)}{(\lambda^2-4\mu)A_1{}^2},$$

$$a_1=-\frac{V^2\lambda^2a_2-4V^2\mu a_2-2V^2+2k^2}{\lambda^2-4\mu},$$

$$A_0=\frac{1}{2}\lambda A_1,\ A_2=0,\ b_1=-V^2b_2, \tag{19}$$

equipped with A_1, V, a_2, b_2, k as free parameters.

As a result, with the help of (19) and using (18) along with $\tau=-Vt+x$, $F(\tau)=q(x,t)$, we have obtained following solutions structures for equation (3):

For the instance $\lambda^2-4\mu>0$

$$q(x,t)=\frac{1}{2}\frac{A_1\sqrt{\lambda^2-4\mu}\left(w_1\sinh\left(\frac{1}{2}(-Vt+x)\sqrt{\lambda^2-4\mu}\right)+w_2\cosh\left(\frac{1}{2}(-Vt+x)\sqrt{\lambda^2-4\mu}\right)\right)}{w_2\sinh\left(\frac{1}{2}(-Vt+x)\sqrt{\lambda^2-4\mu}\right)+w_1\cosh\left(\frac{1}{2}(-Vt+x)\sqrt{\lambda^2-4\mu}\right)}. \tag{20}$$

In (20), for $w_1=0$, $w_2\neq 0$, one recovers singular solitary waves while for $w_2=0$, $w_1\neq 0$, one obtains shock waves.

For the instance $\lambda^2-4\mu<0$

$$q(x,t)=-\frac{1}{2}\frac{A_1\sqrt{-\lambda^2+4\mu}\left(w_1\sin\left(\frac{1}{2}(x-Vt)\sqrt{-\lambda^2+4\mu}\right)-w_2\cos\left(\frac{1}{2}(x-Vt)\sqrt{-\lambda^2+4\mu}\right)\right)}{w_2\sin\left(\frac{1}{2}(x-Vt)\sqrt{-\lambda^2+4\mu}\right)+w_1\cos\left(\frac{1}{2}(x-Vt)\sqrt{-\lambda^2+4\mu}\right)}, \tag{21}$$

with w_1, w_2 as arbitrary parameters.

Case-II

$$c=-\frac{30(V^2-k^2)}{A_2{}^2(\lambda^4-8\lambda^2\mu+16\mu^2)},$$

$$A_0=\mu A_2,\ \ A_1=\lambda A_2,$$

$$a_1=-\frac{4V^2\lambda^2a_2-16V^2\mu a_2+5V^2-5k^2}{4(\lambda^2-4\mu)},$$

$$b_2=\frac{-4\lambda^4b_1+32\lambda^2\mu b_1-64\mu^2b_1+V^2-k^2}{4V^2(\lambda^4-8\lambda^2\mu+16\mu^2)}, \tag{22}$$

equipped with V, k, A_2, a_2, b_1 as free parameters.

As a result, with the help of (22) and using (18) along with $\tau = -Vt + x$, $F(\tau) = q(x, t)$, we have obtained following solutions structures for equation (3):

For the instance $\lambda^2 - 4\mu > 0$

$$q(x, t) = -\frac{1}{4}\frac{A_2\left(\lambda^2 - 4\mu\right)\left({w_1}^2 - {w_2}^2\right)}{\left(w_2 \sinh\left(\frac{1}{2}(x - Vt)\sqrt{\lambda^2 - 4\mu}\right) + w_1 \cosh\left(\frac{1}{2}(x - Vt)\sqrt{\lambda^2 - 4\mu}\right)\right)^2}. \tag{23}$$

In (23), for $w_1 = 0$, $w_2 \neq 0$, one recovers singular solitary waves while for $w_2 = 0$, $w_1 \neq 0$, one obtains solitary waves.

For the instance $\lambda^2 - 4\mu < 0$

$$q(x, t) = \frac{1}{4}\frac{A_2\left({w_1}^2 + {w_2}^2\right)\left(\lambda^2 - 4\mu\right)}{\left(w_2 \sin\left(\frac{1}{2}(x - Vt)\sqrt{-\lambda^2 + 4\mu}\right) + w_1 \cos\left(\frac{1}{2}(x - Vt)\sqrt{-\lambda^2 + 4\mu}\right)\right)^2}, \tag{24}$$

with w_1, w_2 as arbitrary parameters.

2.3 *n* = 5/2

Here, we consider $n = \frac{5}{2}$ for (3) and then reduced ODE is rewritten as follows:

$$(V^2 b_2 + b_1)F'''' + (a_2 V^2 + a_1)F'' + cF(\tau)^5 + (-k^2 + V^2)F = 0. \tag{25}$$

Homogeneous balance among highest order linear and most nonlinear derivative containing terms in equation (25) gives $m = 1$ and hence generate following solution structure:

$$F(\tau) = A_0 + A_1\left(\frac{G'(\tau)}{G(\tau)}\right), \tag{26}$$

with A_i, $i = 0$, 1 are constants, to be determined in the mean process of calculations and $G(\tau)$ following the auxiliary equation (8). Using (26) in equation (25), along with (8), we have furnished following case to exploring variety of solutions of equation (3):

$$c = -\frac{16(V^2 - k^2)}{(\lambda^4 - 8\lambda^2\mu + 16\mu^2){A_1}^4},$$

$$A_0 = \frac{1}{2}A_1\lambda, \quad a_1 = -\frac{3V^2\lambda^2 a_2 - 12V^2\mu a_2 - 10V^2 + 10k^2}{3(\lambda^2 - 4\mu)},$$

$$b_1 = -\frac{3V^2\lambda^4 b_2 - 24V^2\lambda^2\mu b_2 + 48V^2\mu^2 b_2 - 2V^2 + 2k^2}{3(\lambda^4 - 8\lambda^2\mu + 16\mu^2)}, \tag{27}$$

equipped with A_1, V, a_2, b_2, k as free parameters.

As a result, with the help of (27) and using (26) along with $\tau = -Vt + x$, $F(\tau) = q(x, t)$, we have obtained following solutions structures for equation (3):

For the instance $\lambda^2 - 4\mu > 0$

$$q(x, t) = \frac{1}{2}\frac{A_1\sqrt{\lambda^2-4\mu}\left(w_1\sinh\left(\frac{1}{2}(-Vt+x)\sqrt{\lambda^2-4\mu}\right) - w_2\cosh\left(\frac{1}{2}(-Vt+x)\sqrt{\lambda^2-4\mu}\right)\right)}{w_2\sinh\left(\frac{1}{2}(-Vt+x)\sqrt{\lambda^2-4\mu}\right) - w_1\cosh\left(\frac{1}{2}(-Vt+x)\sqrt{\lambda^2-4\mu}\right)}. \tag{28}$$

In (28), for $w_1 = 0$, $w_2 \neq 0$, one recovers singular solitary waves while for $w_2 = 0$, $w_1 \neq 0$, one obtains shock waves.

For the instance $\lambda^2 - 4\mu < 0$

$$q(x, t) = -\frac{1}{2}\frac{A_1\sqrt{-\lambda^2+4\mu}\left(w_1\sin\left(\frac{1}{2}(x-Vt)\sqrt{-\lambda^2+4\mu}\right) - w_2\cos\left(\frac{1}{2}(x-Vt)\sqrt{-\lambda^2+4\mu}\right)\right)}{w_2\sin\left(\frac{1}{2}(x-Vt)\sqrt{-\lambda^2+4\mu}\right) + w_1\cos\left(\frac{1}{2}(x-Vt)\sqrt{-\lambda^2+4\mu}\right)}, \tag{29}$$

with w_1, w_2 as arbitrary parameters.

3. Perturbed 6BE

This section is devoted to explore some different types of solutions for 6BE Model with Perturbation terms. It is remarkable to address that perturbed 6BE is addressed very first time in current study as follow:

$$q_{tt} - k^2 q_{xx} + c\left(q^{2n}\right)_{xx} + a_1 q_{xxxx} + a_2 q_{xxtt} + b_1 q_{xxxxxx} + b_2 q_{xxxxtt}$$

$$= \theta q_x q_{xx} + \delta q^m q_x + \Lambda q q_{xxx} + \nu q q_x q_{xx} + \xi q_x q_{xxx} + \psi q_{xxxxx} + \kappa q q_{xxxxx}, \tag{30}$$

with $q = q(x, t)$.

Utilizing $\tau = -Vt + x$, along with $q(x, t) = F(\tau)$, we have recovered the following ordinary differential equation from the equation (30):

$$(V^2 b_2 + b_1)F'''''' + (-\kappa F(\tau) - \psi)F''''' + (a_2 V^2 + a_1)F'''' + (-\Lambda F(\tau) - \xi F')F''' - \delta F(\tau)^m F'$$

$$+\left((-\theta - F(\tau)\nu)F' + V^2 - k^2 + 2cF(\tau)^{2n-1}n\right)F'' + c(4F(\tau)^{2n-2}n^2 - 2F(\tau)^{2n-2}n)(F')^2 = 0. \tag{31}$$

Now considering $n = \frac{3}{2}$, $m = 1$, the equation (31), turns out subsequently as

$$(V^2b_2+b_1)F'''''' + (-\kappa F(\tau)-\psi)F''''' + (a_2V^2+a_1)F'''' + (-\Lambda F(\tau)-\xi F')F''' - \delta F(\tau)F'$$

$$\left((-\theta - F(\tau)\nu)F' + V^2 - k^2 + 3cF(\tau)^2\right)F'' + c(6F(\tau))(F')^2 = 0. \tag{32}$$

Next equation (32) has been examined with the help of extended $\left(\frac{G'}{G}\right)$-expansion method for recovering the new analytic solutions as in subsequent portions of this paper: Speculating the homogeneous balance among highest order derivative term and highly non linear terms in equation (32) leads to the following solution structure for the equation (32) is assumed:

$$F(\tau) = A_0 + A_1\left(\frac{G'(\tau)}{G(\tau)}\right) + A_2\left(\frac{G(\tau)}{G'(\tau)}\right), \tag{33}$$

with $G(\tau)$ following the auxiliary equation

$$G'' + \lambda G' + \mu G = 0, \tag{34}$$

It is worthy to notice that A_0, A_1, A_2 are parameters, to be evaluated in the mean process of calculations. Utilizing (33) into (32), along with (34), we furnish the subsequent parameter values for extracting the solutions to equation (32) as per mentioned details:

Category-I

$$\Lambda = \frac{Z_1}{9\kappa\,(\lambda^2-4\mu)\,(3V^2\lambda\,b_2-\kappa A_0+3\lambda\,b_1)\,(V^2b_2+b_1)},$$

$$\delta = -\frac{Z_2}{9\kappa\,(3V^2\lambda\,b_2-\kappa A_0+3\lambda\,b_1)\,(V^2b_2+b_1)},$$

$$\psi = \frac{Z_3}{3\,(V^2b_2+b_1)\,(\lambda^2-4\mu)^2\,\kappa^2},$$

$$\theta = -\frac{Z_4}{3\kappa\,(\lambda^2-4\mu)\,(3V^2\lambda\,b_2-\kappa A_0+3\lambda\,b_1)\,(V^2b_2+b_1)},$$

$$\xi = -\frac{2Z_5}{(9V^2\lambda^2b_2-4V^2\mu\,b_2+\lambda^2b_1-4\mu\,b_1)\,(\lambda^2-4\mu)\,\kappa\,(V^2b_2+b_1)},$$

$$A_1 = 0,\ A_2 = \frac{6\left(V^2b_2+b_1\right)\mu}{\kappa}, \tag{35}$$

along with

$$
\begin{aligned}
Z_1 = & 9V^6\lambda^4\nu b_2{}^3 + 486V^6c\lambda^3 b_2{}^3 - 72V^6\lambda^2\mu\nu b_2{}^3 - 30V^4\kappa^2\lambda^5 b_2{}^2 - 324V^6c\lambda\mu b_2{}^3 + 144V^6\mu^2\nu b_2{}^3 \\
& + 240V^4\kappa^2\lambda^3\mu b_2{}^2 + 27V^4\lambda^4\nu b_1 b_2{}^2 - 432V^4c\kappa\lambda^2 A_0 b_2{}^2 + 1{,}458V^4c\lambda^3 b_1 b_2{}^2 + 15V^4\kappa^2\lambda^3 a_2 b_2 \\
& - 480V^4\kappa^2\lambda\mu^2 b_2{}^2 - 216V^4\lambda^2\mu\nu b_1 b_2{}^2 + 10V^2\kappa^3\lambda^4 A_0 b_2 - 60V^2\kappa^2\lambda^5 b_1 b_2 + 108V^4c\kappa\mu A_0 b_2{}^2 \\
& - 972V^4c\lambda\mu b_1 b_2{}^2 - 60V^4\kappa^2\lambda\mu a_2 b_2 + 432V^4\mu^2\nu b_1 b_2{}^2 - 80V^2\kappa^3\lambda^2\mu A_0 b_2 + 480V^2\kappa^2\lambda^3\mu b_1 b_2 \\
& + 27V^2\lambda^4\nu b_1{}^2 b_2 + 135V^2c\kappa^2\lambda A_0{}^2 b_2 - 864V^2c\kappa\lambda^2 A_0 b_1 b_2 + 1{,}458V^2c\lambda^3 b_1{}^2 b_2 - 5V^2\kappa^3\lambda^2 A_0 a_2 \\
& + 160V^2\kappa^3\mu^2 A_0 b_2 + 15V^2\kappa^2\lambda^3 a_1 b_2 + 15V^2\kappa^2\lambda^3 a_2 b_1 - 960V^2\kappa^2\lambda\mu^2 b_1 b_2 - 216V^2\lambda^2\mu\nu b_1{}^2 b_2 \\
& + 10\kappa^3\lambda^4 A_0 b_1 - 30\kappa^2\lambda^5 b_1{}^2 + 15V^4\kappa^2\lambda b_2 + 216V^2c\kappa\mu A_0 b_1 b_2 - 972V^2c\lambda\mu b_1{}^2 b_2 - 15V^2k^2\kappa^2\lambda b_2 \\
& + 20V^2\kappa^3\mu A_0 a_2 - 60V^2\kappa^2\lambda\mu a_1 b_2 - 60V^2\kappa^2\lambda\mu a_2 b_1 + 432V^2\mu^2\nu b_1{}^2 b_2 - 80\kappa^3\lambda^2\mu A_0 b_1 \\
& + 240\kappa^2\lambda^3\mu b_1{}^2 + 9\lambda^4\nu b_1{}^3 - 15c\kappa^3 A_0{}^3 + 135c\kappa^2\lambda A_0{}^2 b_1 - 432c\kappa\lambda^2 A_0 b_1{}^2 + 486c\lambda^3 b_1{}^3 - 5\kappa^3\lambda^2 A_0 a_1 \\
& + 160\kappa^3\mu^2 A_0 b_1 + 15\kappa^2\lambda^3 a_1 b_1 - 480\kappa^2\lambda\mu^2 b_1{}^2 - 72\lambda^2\mu\nu b_1{}^3 - 5V^2\kappa^3 A_0 + 15V^2\kappa^2\lambda b_1 \\
& + 108c\kappa\mu A_0 b_1{}^2 - 324c\lambda\mu b_1{}^3 + 5k^2\kappa^3 A_0 - 15k^2\kappa^2\lambda b_1 + 20\kappa^3\mu A_0 a_1 - 60\kappa^2\lambda\mu a_1 b_1 + 144\mu^2\nu b_1{}^3,
\end{aligned}
$$

$$
\begin{aligned}
Z_2 = & 9V^6\lambda^4\nu b_2{}^3 - 72V^6\lambda^2\mu\nu b_2{}^3 - 12V^4\kappa^2\lambda^5 b_2{}^2 + 648V^6c\lambda\mu b_2{}^3 + 144V^6\mu^2\nu b_2{}^3 + 96V^4\kappa^2\lambda^3\mu b_2{}^2 \\
& + 27V^4\lambda^4\nu b_1 b_2{}^2 - 108V^4c\kappa\lambda^2 A_0 b_2{}^2 + 6V^4\kappa^2\lambda^3 a_2 b_2 - 192V^4\kappa^2\lambda\mu^2 b_2{}^2 - 216V^4\lambda^2\mu\nu b_1 b_2{}^2 \\
& + 4V^2\kappa^3\lambda^4 A_0 b_2 - 24V^2\kappa^2\lambda^5 b_1 b_2 - 216V^4c\kappa\mu A_0 b_2{}^2 + 1{,}944V^4c\lambda\mu b_1 b_2{}^2 - 24V^4\kappa^2\lambda\mu a_2 b_2 \\
& + 432V^4\mu^2\nu b_1 b_2{}^2 - 32V^2\kappa^3\lambda^2\mu A_0 b_2 + 192V^2\kappa^2\lambda^3\mu b_1 b_2 + 27V^2\lambda^4\nu b_1{}^2 b_2 + 54V^2c\kappa^2\lambda A_0{}^2 b_2 \\
& - 216V^2c\kappa\lambda^2 A_0 b_1 b_2 - 2V^2\kappa^3\lambda^2 A_0 a_2 + 64V^2\kappa^3\mu^2 A_0 b_2 + 6V^2\kappa^2\lambda^3 a_1 b_2 + 6V^2\kappa^2\lambda^3 a_2 b_1 \\
& - 384V^2\kappa^2\lambda\mu^2 b_1 b_2 - 216V^2\lambda^2\mu\nu b_1{}^2 b_2 + 4\kappa^3\lambda^4 A_0 b_1 - 12\kappa^2\lambda^5 b_1{}^2 + 6V^4\kappa^2\lambda b_2 \\
& - 432V^2c\kappa\mu A_0 b_1 b_2 + 1{,}944V^2c\lambda\mu b_1{}^2 b_2 - 6V^2k^2\kappa^2\lambda b_2 + 8V^2\kappa^3\mu A_0 a_2 - 24V^2\kappa^2\lambda\mu a_1 b_2
\end{aligned}
$$

$$-24V^2\kappa^2\lambda\,\mu\,a_2b_1+432V^2\mu^2\nu\,b_1{}^2b_2-32\,\kappa^3\lambda^2\mu A_0b_1+96\,\kappa^2\lambda^3\mu\,b_1{}^2+9\,\lambda^4\nu\,b_1{}^3-6\,c\kappa^3A_0{}^3$$

$$+54\,c\kappa^2\lambda\,A_0{}^2b_1-108\,c\kappa\,\lambda^2A_0b_1{}^2-2\,\kappa^3\lambda^2A_0a_1+64\,\kappa^3\mu^2A_0b_1+6\,\kappa^2\lambda^3a_1b_1-192\,\kappa^2\lambda\,\mu^2b_1{}^2$$

$$-72\,\lambda^2\mu\,\nu\,b_1{}^3-2V^2\kappa^3A_0+6V^2\kappa^2\lambda\,b_1-216\,c\kappa\,\mu\,A_0b_1{}^2+648\,c\lambda\,\mu\,b_1{}^3+2k^2\kappa^3A_0-6k^2\kappa^2\lambda\,b_1$$

$$+8\,\kappa^3\mu\,A_0a_1-24\,\kappa^2\lambda\,\mu\,a_1b_1+144\,\mu^2\nu\,b_1{}^3,$$

$$Z_3=3V^4\kappa^2\lambda^5b_2{}^2+324V^6c\lambda\,\mu\,b_2{}^3-24V^4\kappa^2\lambda^3\mu\,b_2{}^2-54V^4c\kappa\,\lambda^2A_0b_2{}^2+3V^4\kappa^2\lambda^3a_2b_2+48V^4\kappa^2\lambda\,\mu^2b_2{}^2$$

$$-V^2\kappa^3\lambda^4A_0b_2+6V^2\kappa^2\lambda^5b_1b_2-108V^4c\kappa\,\mu\,A_0b_2{}^2+972V^4c\lambda\,\mu\,b_1b_2{}^2-12V^4\kappa^2\lambda\,\mu\,a_2b_2$$

$$+8V^2\kappa^3\lambda^2\mu\,A_0b_2-48V^2\kappa^2\lambda^3\mu\,b_1b_2+27V^2c\kappa^2\lambda\,A_0{}^2b_2-108V^2c\kappa\,\lambda^2A_0b_1b_2-V^2\kappa^3\lambda^2A_0a_2$$

$$-16V^2\kappa^3\mu^2A_0b_2+3V^2\kappa^2\lambda^3a_1b_2+3V^2\kappa^2\lambda^3a_2b_1+96V^2\kappa^2\lambda\,\mu^2b_1b_2-\kappa^3\lambda^4A_0b_1+3\,\kappa^2\lambda^5b_1{}^2$$

$$+3V^4\kappa^2\lambda\,b_2-216V^2c\kappa\,\mu\,A_0b_1b_2+972V^2c\lambda\,\mu\,b_1{}^2b_2-3V^2k^2\kappa^2\lambda\,b_2+4V^2\kappa^3\mu\,A_0a_2$$

$$-12V^2\kappa^2\lambda\,\mu\,a_1b_2-12V^2\kappa^2\lambda\,\mu\,a_2b_1+8\,\kappa^3\lambda^2\mu\,A_0b_1-24\,\kappa^2\lambda^3\mu\,b_1{}^2-3\,c\kappa^3A_0{}^3+27\,c\kappa^2\lambda\,A_0{}^2b_1$$

$$-54\,c\kappa\,\lambda^2A_0b_1{}^2-\kappa^3\lambda^2A_0a_1-16\,\kappa^3\mu^2A_0b_1+3\,\kappa^2\lambda^3a_1b_1+48\,\kappa^2\lambda\,\mu^2b_1{}^2-V^2\kappa^3A_0+3V^2\kappa^2\lambda\,b_1$$

$$-108\,c\kappa\,\mu\,A_0b_1{}^2+324\,c\lambda\,\mu\,b_1{}^3+k^2\kappa^3A_0-3k^2\kappa^2\lambda\,b_1+4\,\kappa^3\mu\,A_0a_1-12\,\kappa^2\lambda\,\mu\,a_1b_1,$$

$$Z_4=-18V^6\lambda^4\nu\,b_2{}^3+162V^6c\lambda^3b_2{}^3+36V^6\lambda^2\mu\,\nu\,b_2{}^3+15V^4\kappa^2\lambda^5b_2{}^2+972V^6c\lambda\,\mu\,b_2{}^3+144V^6\mu^2\nu\,b_2{}^3$$

$$-120V^4\kappa^2\lambda^3\mu\,b_2{}^2+18V^4\kappa\,\lambda^3\nu\,A_0b_2{}^2-54V^4\lambda^4\nu\,b_1b_2{}^2-324V^4c\kappa\,\lambda^2A_0b_2{}^2+486V^4c\lambda^3b_1b_2{}^2$$

$$-3V^4\kappa^2\lambda^3a_2b_2+240V^4\kappa^2\lambda\,\mu^2b_2{}^2-72V^4\kappa\,\lambda\,\mu\,\nu\,A_0b_2{}^2+108V^4\lambda^2\mu\,\nu\,b_1b_2{}^2-5V^2\kappa^3\lambda^4A_0b_2$$

$$+30V^2\kappa^2\lambda^5b_1b_2-324V^4c\kappa\,\mu\,A_0b_2{}^2+2{,}916V^4c\lambda\,\mu\,b_1b_2{}^2+12V^4\kappa^2\lambda\,\mu\,a_2b_2+432V^4\mu^2\nu\,b_1b_2{}^2$$

$$+40V^2\kappa^3\lambda^2\mu\,A_0b_2-240V^2\kappa^2\lambda^3\mu\,b_1b_2-3V^2\kappa^2\lambda^2\nu\,A_0{}^2b_2+36V^2\kappa\,\lambda^3\nu\,A_0b_1b_2-54V^2\lambda^4\nu\,b_1{}^2b_2$$

$$+135V^2c\kappa^2\lambda\,A_0{}^2b_2-648V^2c\kappa\,\lambda^2A_0b_1b_2+486V^2c\lambda^3b_1{}^2b_2+V^2\kappa^3\lambda^2A_0a_2-80V^2\kappa^3\mu^2A_0b_2$$

$$
\begin{aligned}
&-3V^2\kappa^2\lambda^3 a_1 b_2 - 3V^2\kappa^2\lambda^3 a_2 b_1 + 480V^2\kappa^2\lambda\,\mu^2 b_1 b_2 + 12V^2\kappa^2\mu\,\nu A_0{}^2 b_2 - 144V^2\kappa\lambda\,\mu\,\nu A_0 b_1 b_2 \\
&+108V^2\lambda^2\mu\,\nu\, b_1{}^2 b_2 - 5\kappa^3\lambda^4 A_0 b_1 + 15\kappa^2\lambda^5 b_1{}^2 + 15V^4\kappa^2\lambda\, b_2 - 648V^2 c\kappa\mu A_0 b_1 b_2 \\
&+2{,}916V^2 c\lambda\,\mu\, b_1{}^2 b_2 - 15V^2k^2\kappa^2\lambda\, b_2 - 4V^2\kappa^3\mu A_0 a_2 + 12V^2\kappa^2\lambda\,\mu\, a_1 b_2 + 12V^2\kappa^2\lambda\,\mu\, a_2 b_1 \\
&+432V^2\mu^2\nu\, b_1{}^2 b_2 + 40\kappa^3\lambda^2\mu A_0 b_1 - 120\kappa^2\lambda^3\mu\, b_1{}^2 - 3\kappa^2\lambda^2\nu A_0{}^2 b_1 + 18\kappa\lambda^3\nu A_0 b_1{}^2 - 18\lambda^4\nu\, b_1{}^3 \\
&-15c\kappa^3 A_0{}^3 + 135c\kappa^2\lambda A_0{}^2 b_1 - 324c\kappa\lambda^2 A_0 b_1{}^2 + 162c\lambda^3 b_1{}^3 + \kappa^3\lambda^2 A_0 a_1 - 80\kappa^3\mu^2 A_0 b_1 \\
&-3\kappa^2\lambda^3 a_1 b_1 + 240\kappa^2\lambda\,\mu^2 b_1{}^2 + 12\kappa^2\mu\,\nu A_0{}^2 b_1 - 72\kappa\lambda\,\mu\,\nu A_0 b_1{}^2 + 36\lambda^2\mu\,\nu\, b_1{}^3 - 5V^2\kappa^3 A_0 \\
&+15V^2\kappa^2\lambda\, b_1 - 324c\kappa\mu A_0 b_1{}^2 + 972c\lambda\,\mu\, b_1{}^3 + 5k^2\kappa^3 A_0 - 15k^2\kappa^2\lambda\, b_1 - 4\kappa^3\mu A_0 a_1 + 12\kappa^2\lambda\,\mu\, a_1 b_1 \\
&+144\mu^2\nu\, b_1{}^3,
\end{aligned}
$$

$$
\begin{aligned}
Z_5 = {}& 9V^6\lambda^4\nu\, b_2{}^3 - 72V^6\lambda^2\mu\,\nu\, b_2{}^3 - 30V^4\kappa^2\lambda^5 b_2{}^2 + 1{,}620V^6 c\lambda\,\mu\, b_2{}^3 + 144V^6\mu^2\nu\, b_2{}^3 + 240V^4\kappa^2\lambda^3\mu\, b_2{}^2 \\
&+27V^4\lambda^4\nu\, b_1 b_2{}^2 - 270V^4 c\kappa\lambda^2 A_0 b_2{}^2 + 15V^4\kappa^2\lambda^3 a_2 b_2 - 480V^4\kappa^2\lambda\,\mu^2 b_2{}^2 - 216V^4\lambda^2\mu\,\nu\, b_1 b_2{}^2 \\
&+10V^2\kappa^3\lambda^4 A_0 b_2 - 60V^2\kappa^2\lambda^5 b_1 b_2 - 540V^4 c\kappa\mu A_0 b_2{}^2 + 4{,}860V^4 c\lambda\,\mu\, b_1 b_2{}^2 - 60V^4\kappa^2\lambda\,\mu\, a_2 b_2 \\
&+432V^4\mu^2\nu\, b_1 b_2{}^2 - 80V^2\kappa^3\lambda^2\mu A_0 b_2 + 480V^2\kappa^2\lambda^3\mu\, b_1 b_2 + 27V^2\lambda^4\nu\, b_1{}^2 b_2 + 135V^2 c\kappa^2\lambda A_0{}^2 b_2 \\
&-540V^2 c\kappa\lambda^2 A_0 b_1 b_2 - 5V^2\kappa^3\lambda^2 A_0 a_2 + 160V^2\kappa^3\mu^2 A_0 b_2 + 15V^2\kappa^2\lambda^3 a_1 b_2 + 15V^2\kappa^2\lambda^3 a_2 b_1 \\
&-960V^2\kappa^2\lambda\,\mu^2 b_1 b_2 - 216V^2\lambda^2\mu\,\nu\, b_1{}^2 b_2 + 10\kappa^3\lambda^4 A_0 b_1 - 30\kappa^2\lambda^5 b_1{}^2 + 15V^4\kappa^2\lambda\, b_2 \\
&-1{,}080V^2 c\kappa\mu A_0 b_1 b_2 + 4{,}860V^2 c\lambda\,\mu\, b_1{}^2 b_2 - 15V^2k^2\kappa^2\lambda\, b_2 + 20V^2\kappa^3\mu A_0 a_2 - 60V^2\kappa^2\lambda\,\mu\, a_1 b_2 \\
&-60V^2\kappa^2\lambda\,\mu\, a_2 b_1 + 432V^2\mu^2\nu\, b_1{}^2 b_2 - 80\kappa^3\lambda^2\mu A_0 b_1 + 240\kappa^2\lambda^3\mu\, b_1{}^2 + 9\lambda^4\nu\, b_1{}^3 - 15c\kappa^3 A_0{}^3 \\
&+135c\kappa^2\lambda A_0{}^2 b_1 - 270c\kappa\lambda^2 A_0 b_1{}^2 - 5\kappa^3\lambda^2 A_0 a_1 + 160\kappa^3\mu^2 A_0 b_1 + 15\kappa^2\lambda^3 a_1 b_1 - 480\kappa^2\lambda\,\mu^2 b_1{}^2 \\
&-72\lambda^2\mu\,\nu\, b_1{}^3 - 5V^2\kappa^3 A_0 + 15V^2\kappa^2\lambda\, b_1 - 540c\kappa\mu A_0 b_1{}^2 + 1{,}620c\lambda\,\mu\, b_1{}^3 + 5k^2\kappa^3 A_0 - 15k^2\kappa^2\lambda\, b_1 \\
&+20\kappa^3\mu A_0 a_1 - 60\kappa^2\lambda\,\mu\, a_1 b_1 + 144\mu^2\nu\, b_1{}^3,
\end{aligned}
$$

its worth to mention that all the found parameters are equipped with V, c, A_0, b_1, b_2, a_1, k, κ, λ, μ, ν as free parameters.

Using solution of equation (34) along with parameter values (35), we may write the subsequent solution for equation (30), by reverting back to original variables x, t:

Regrading the instance $\lambda^2 - 4\mu > 0$, we have extracted following solution structure for equation (30):

$$q(x, t) = \left(\frac{2w_2 \sinh\left(\frac{(x-Vt)\sqrt{\lambda^2-4\mu}}{2}\right) + 2w_1 \cosh\left(\frac{(x-Vt)(\sqrt{\lambda^2-4\mu})}{2}\right)}{\sqrt{\lambda^2-4\mu}\left(w_1 \sinh\left(\frac{(x-Vt)\sqrt{\lambda^2-4\mu}}{2}\right) + w_2 \cosh\left(\frac{(x-Vt)(\sqrt{\lambda^2-4\mu})}{2}\right)\right)} - \frac{\lambda}{2} \right) A_2 + A_0. \quad (36)$$

In (36), for $w_1 = 0$, $w_2 \neq 0$, one recovers shock waves while for $w_2 = 0$, $w_1 \neq 0$, one obtains singular solitary waves.

Regrading the instance $\lambda^2 - 4\mu < 0$, we have extracted following solution structure for equation (30):

$$q(x, t) = \left(\frac{2w_2 \sin\left(\frac{(x-Vt)\sqrt{-\lambda^2+4\mu}}{2}\right) + 2w_1 \cos\left(\frac{(x-Vt)(\sqrt{-\lambda^2+4\mu})}{2}\right)}{\sqrt{-\lambda^2+4\mu}\left(w_1 \sin\left(\frac{(x-Vt)\sqrt{-\lambda^2+4\mu}}{2}\right) + w_2 \cos\left(\frac{(x-Vt)(\sqrt{-\lambda^2+4\mu})}{2}\right)\right)} - \frac{\lambda}{2} \right) A_2 + A_0. \quad (37)$$

Category-II

$$\Lambda = \frac{Z_6}{9\,(3V^2\lambda\, b_2 + \kappa A_0 + 3\,\lambda\, b_1)\,(\lambda^2 - 4\,\mu)\,(V^2 b_2 + b_1)\,\kappa},$$

$$\delta = -\frac{Z_7}{9\kappa\,(3V^2\lambda\, b_2 + \kappa A_0 + 3\,\lambda\, b_1)\,(V^2 b_2 + b_1)},$$

$$\psi = -\frac{Z_8}{3\kappa^2\,(\lambda^2 - 4\,\mu)\,(V^2\lambda^2 b_2 - 4V^2\mu\, b_2 + \lambda^2 b_1 - 4\,\mu\, b_1)},$$

$$\theta = -\frac{Z_9}{3\kappa\,(3V^2\lambda\, b_2 + \kappa A_0 + 3\,\lambda\, b_1)\,(\lambda^2 - 4\,\mu)\,(V^2 b_2 + b_1)},$$

$$\xi = \frac{2Z_{10}}{9\left(V^4\lambda^2 {b_2}^2 - 4V^4\mu\, {b_2}^2 + 2V^2\lambda^2 b_1 b_2 - 8V^2\mu\, b_1 b_2 + \lambda^2 {b_1}^2 - 4\,\mu\, {b_1}^2\right)\kappa\,(\lambda^2 - 4\,\mu)},$$

$$A_1 = -\frac{6(V^2 b_2 + b_1)}{\kappa},\ A_2 = 0, \quad (38)$$

along with

$$
\begin{aligned}
Z_6 = & -9V^6\lambda^4\nu b_2{}^3 + 486V^6c\lambda^3 b_2{}^3 + 72V^6\lambda^2\mu\nu b_2{}^3 - 30V^4\kappa^2\lambda^5 b_2{}^2 - 324V^6c\lambda\mu b_2{}^3 \\
& - 144V^6\mu^2\nu b_2{}^3 + 240V^4\kappa^2\lambda^3\mu b_2{}^2 - 27V^4\lambda^4\nu b_1 b_2{}^2 + 432V^4c\kappa\lambda^2 A_0 b_2{}^2 + 1{,}458V^4c\lambda^3 b_1 b_2{}^2 \\
& + 15V^4\kappa^2\lambda^3 a_2 b_2 - 480V^4\kappa^2\lambda\mu^2 b_2{}^2 + 216V^4\lambda^2\mu\nu b_1 b_2{}^2 - 10V^2\kappa^3\lambda^4 A_0 b_2 \\
& - 60V^2\kappa^2\lambda^5 b_1 b_2 - 108V^4c\kappa\mu A_0 b_2{}^2 - 972V^4c\lambda\mu b_1 b_2{}^2 - 60V^4\kappa^2\lambda\mu a_2 b_2 \\
& - 432V^4\mu^2\nu b_1 b_2{}^2 + 80V^2\kappa^3\lambda^2\mu A_0 b_2 + 480V^2\kappa^2\lambda^3\mu b_1 b_2 - 27V^2\lambda^4\nu b_1{}^2 b_2 \\
& + 135V^2c\kappa^2\lambda A_0{}^2 b_2 + 864V^2c\kappa\lambda^2 A_0 b_1 b_2 + 1{,}458V^2c\lambda^3 b_1{}^2 b_2 + 5V^2\kappa^3\lambda^2 A_0 a_2 \\
& - 160V^2\kappa^3\mu^2 A_0 b_2 + 15V^2\kappa^2\lambda^3 a_1 b_2 + 15V^2\kappa^2\lambda^3 a_2 b_1 - 960V^2\kappa^2\lambda\mu^2 b_1 b_2 \\
& + 216V^2\lambda^2\mu\nu b_1{}^2 b_2 - 10\kappa^3\lambda^4 A_0 b_1 - 30\kappa^2\lambda^5 b_1{}^2 + 15V^4\kappa^2\lambda b_2 - 216V^2c\kappa\mu A_0 b_1 b_2 \\
& - 972V^2c\lambda\mu b_1{}^2 b_2 - 15V^2k^2\kappa^2\lambda b_2 - 20V^2\kappa^3\mu A_0 a_2 - 60V^2\kappa^2\lambda\mu a_1 b_2 - 60V^2\kappa^2\lambda\mu a_2 b_1 \\
& - 432V^2\mu^2\nu b_1{}^2 b_2 + 80\kappa^3\lambda^2\mu A_0 b_1 + 240\kappa^2\lambda^3\mu b_1{}^2 - 9\lambda^4\nu b_1{}^3 + 15c\kappa^3 A_0{}^3 + 135c\kappa^2\lambda A_0{}^2 b_1 \\
& + 432c\kappa\lambda^2 A_0 b_1{}^2 + 486c\lambda^3 b_1{}^3 + 5\kappa^3\lambda^2 A_0 a_1 - 160\kappa^3\mu^2 A_0 b_1 + 15\kappa^2\lambda^3 a_1 b_1 - 480\kappa^2\lambda\mu^2 b_1{}^2 \\
& + 72\lambda^2\mu\nu b_1{}^3 + 5V^2\kappa^3 A_0 + 15V^2\kappa^2\lambda b_1 - 108c\kappa\mu A_0 b_1{}^2 - 324c\lambda\mu b_1{}^3 - 5k^2\kappa^3 A_0 \\
& - 15k^2\kappa^2\lambda b_1 - 20\kappa^3\mu A_0 a_1 - 60\kappa^2\lambda\mu a_1 b_1 - 144\mu^2\nu b_1{}^3, \\
Z_7 = & -9V^6\lambda^4\nu b_2{}^3 + 72V^6\lambda^2\mu\nu b_2{}^3 - 12V^4\kappa^2\lambda^5 b_2{}^2 + 648V^6c\lambda\mu b_2{}^3 - 144V^6\mu^2\nu b_2{}^3 \\
& + 96V^4\kappa^2\lambda^3\mu b_2{}^2 - 27V^4\lambda^4\nu b_1 b_2{}^2 + 108V^4c\kappa\lambda^2 A_0 b_2{}^2 + 6V^4\kappa^2\lambda^3 a_2 b_2 - 192V^4\kappa^2\lambda\mu^2 b_2{}^2 \\
& + 216V^4\lambda^2\mu\nu b_1 b_2{}^2 - 4V^2\kappa^3\lambda^4 A_0 b_2 - 24V^2\kappa^2\lambda^5 b_1 b_2 + 216V^4c\kappa\mu A_0 b_2{}^2 + 1{,}944V^4c\lambda\mu b_1 b_2{}^2 \\
& - 24V^4\kappa^2\lambda\mu a_2 b_2 - 432V^4\mu^2\nu b_1 b_2{}^2 + 32V^2\kappa^3\lambda^2\mu A_0 b_2 + 192V^2\kappa^2\lambda^3\mu b_1 b_2 - 27V^2\lambda^4\nu b_1{}^2 b_2 \\
& + 54V^2c\kappa^2\lambda A_0{}^2 b_2 + 216V^2c\kappa\lambda^2 A_0 b_1 b_2 + 2V^2\kappa^3\lambda^2 A_0 a_2 - 64V^2\kappa^3\mu^2 A_0 b_2 + 6V^2\kappa^2\lambda^3 a_1 b_2
\end{aligned}
$$

$$+6V^2\kappa^2\lambda^3 a_2 b_1 - 384V^2\kappa^2\lambda\,\mu^2 b_1 b_2 + 216V^2\lambda^2\mu\,\nu\, b_1{}^2 b_2 - 4\kappa^3\lambda^4 A_0 b_1 - 12\kappa^2\lambda^5 b_1{}^2$$

$$+6V^4\kappa^2\lambda\, b_2 + 432V^2 c\kappa\,\mu A_0 b_1 b_2 + 1{,}944V^2 c\lambda\,\mu\, b_1{}^2 b_2 - 6V^2k^2\kappa^2\lambda\, b_2 - 8V^2\kappa^3\mu A_0 a_2$$

$$-24V^2\kappa^2\lambda\,\mu\, a_1 b_2 - 24V^2\kappa^2\lambda\,\mu\, a_2 b_1 - 432V^2\mu^2\nu\, b_1{}^2 b_2 + 32\kappa^3\lambda^2\mu A_0 b_1 + 96\kappa^2\lambda^3\mu\, b_1{}^2$$

$$-9\lambda^4\nu\, b_1{}^3 + 6c\kappa^3 A_0{}^3 + 54c\kappa^2\lambda A_0{}^2 b_1 + 108c\kappa\lambda^2 A_0 b_1{}^2 + 2\kappa^3\lambda^2 A_0 a_1 - 64\kappa^3\mu^2 A_0 b_1$$

$$+6\kappa^2\lambda^3 a_1 b_1 - 192\kappa^2\lambda\,\mu^2 b_1{}^2 + 72\lambda^2\mu\,\nu\, b_1{}^3 + 2V^2\kappa^3 A_0 + 6V^2\kappa^2\lambda\, b_1 + 216c\kappa\,\mu A_0 b_1{}^2$$

$$+648c\lambda\,\mu\, b_1{}^3 - 2k^2\kappa^3 A_0 - 6k^2\kappa^2\lambda\, b_1 - 8\kappa^3\mu A_0 a_1 - 24\kappa^2\lambda\,\mu\, a_1 b_1 - 144\mu^2\nu\, b_1{}^3,$$

$$Z_8 = 3V^4\kappa^2\lambda^5 b_2{}^2 + 324V^6 c\lambda\,\mu\, b_2{}^3 - 24V^4\kappa^2\lambda^3\mu\, b_2{}^2 + 54V^4 c\kappa\lambda^2 A_0 b_2{}^2 + 3V^4\kappa^2\lambda^3 a_2 b_2$$

$$+48V^4\kappa^2\lambda\,\mu^2 b_2{}^2 + V^2\kappa^3\lambda^4 A_0 b_2 + 6V^2\kappa^2\lambda^5 b_1 b_2 + 108V^4 c\kappa\,\mu A_0 b_2{}^2$$

$$+972V^4 c\lambda\,\mu\, b_1 b_2{}^2 - 12V^4\kappa^2\lambda\,\mu\, a_2 b_2 - 8V^2\kappa^3\lambda^2\mu A_0 b_2 - 48V^2\kappa^2\lambda^3\mu\, b_1 b_2$$

$$+27V^2 c\kappa^2\lambda A_0{}^2 b_2 + 108V^2 c\kappa\lambda^2 A_0 b_1 b_2 + V^2\kappa^3\lambda^2 A_0 a_2 + 16V^2\kappa^3\mu^2 A_0 b_2$$

$$+3V^2\kappa^2\lambda^3 a_1 b_2 + 3V^2\kappa^2\lambda^3 a_2 b_1 + 96V^2\kappa^2\lambda\,\mu^2 b_1 b_2 + \kappa^3\lambda^4 A_0 b_1 + 3\kappa^2\lambda^5 b_1{}^2 + 3V^4\kappa^2\lambda\, b_2$$

$$+216V^2 c\kappa\,\mu A_0 b_1 b_2 + 972V^2 c\lambda\,\mu\, b_1{}^2 b_2 - 3V^2k^2\kappa^2\lambda\, b_2 - 4V^2\kappa^3\mu A_0 a_2 - 12V^2\kappa^2\lambda\,\mu\, a_1 b_2$$

$$-12V^2\kappa^2\lambda\,\mu\, a_2 b_1 - 8\kappa^3\lambda^2\mu A_0 b_1 - 24\kappa^2\lambda^3\mu\, b_1{}^2 + 3c\kappa^3 A_0{}^3 + 27c\kappa^2\lambda A_0{}^2 b_1$$

$$+54c\kappa\lambda^2 A_0 b_1{}^2 + \kappa^3\lambda^2 A_0 a_1 + 16\kappa^3\mu^2 A_0 b_1 + 3\kappa^2\lambda^3 a_1 b_1 + 48\kappa^2\lambda\,\mu^2 b_1{}^2 + V^2\kappa^3 A_0$$

$$+3V^2\kappa^2\lambda\, b_1 + 108c\kappa\,\mu A_0 b_1{}^2 + 324c\lambda\,\mu\, b_1{}^3 - k^2\kappa^3 A_0 - 3k^2\kappa^2\lambda\, b_1 - 4\kappa^3\mu A_0 a_1 - 12\kappa^2\lambda\,\mu\, a_1 b_1,$$

$$Z_9 = 18V^6\lambda^4\nu\, b_2{}^3 + 162V^6 c\lambda^3 b_2{}^3 - 36V^6\lambda^2\mu\,\nu\, b_2{}^3 + 15V^4\kappa^2\lambda^5 b_2{}^2 + 972V^6 c\lambda\,\mu\, b_2{}^3$$

$$-144V^6\mu^2\nu\, b_2{}^3 - 120V^4\kappa^2\lambda^3\mu\, b_2{}^2 + 18V^4\kappa\lambda^3\nu A_0 b_2{}^2 + 54V^4\lambda^4\nu\, b_1 b_2{}^2 + 324V^4 c\kappa\lambda^2 A_0 b_2{}^2$$

$$+486V^4 c\lambda^3 b_1 b_2{}^2 - 3V^4\kappa^2\lambda^3 a_2 b_2 + 240V^4\kappa^2\lambda\,\mu^2 b_2{}^2 - 72V^4\kappa\lambda\,\mu\,\nu A_0 b_2{}^2$$

$$-108V^4\lambda^2\mu\,\nu\, b_1 b_2{}^2 + 5V^2\kappa^3\lambda^4 A_0 b_2 + 30V^2\kappa^2\lambda^5 b_1 b_2 + 324V^4 c\kappa\,\mu A_0 b_2{}^2 + 2{,}916V^4 c\lambda\,\mu\, b_1 b_2{}^2$$

$$+12V^4\kappa^2\lambda\,\mu\,a_2b_2-432V^4\mu^2\nu\,b_1{b_2}^2-40V^2\kappa^3\lambda^2\mu\,A_0b_2-240V^2\kappa^2\lambda^3\mu\,b_1b_2$$

$$+3V^2\kappa^2\lambda^2\nu\,{A_0}^2b_2+36V^2\kappa\,\lambda^3\nu\,A_0b_1b_2+54V^2\lambda^4\nu\,{b_1}^2b_2$$

$$+135V^2c\kappa^2\lambda\,{A_0}^2b_2+648V^2c\kappa\,\lambda^2A_0b_1b_2+486V^2c\lambda^3{b_1}^2b_2-V^2\kappa^3\lambda^2A_0a_2$$

$$+80V^2\kappa^3\mu^2A_0b_2-3V^2\kappa^2\lambda^3a_1b_2-3V^2\kappa^2\lambda^3a_2b_1+480V^2\kappa^2\lambda\,\mu^2b_1b_2$$

$$-12V^2\kappa^2\mu\,\nu\,{A_0}^2b_2-144V^2\kappa\,\lambda\,\mu\,\nu\,A_0b_1b_2-108V^2\lambda^2\mu\,\nu\,{b_1}^2b_2+5\kappa^3\lambda^4A_0b_1$$

$$+15\kappa^2\lambda^5{b_1}^2+15V^4\kappa^2\lambda\,b_2+648V^2c\kappa\,\mu\,A_0b_1b_2+2{,}916V^2c\lambda\,\mu\,{b_1}^2b_2-15V^2k^2\kappa^2\lambda\,b_2$$

$$+4V^2\kappa^3\mu\,A_0a_2+12V^2\kappa^2\lambda\,\mu\,a_1b_2+12V^2\kappa^2\lambda\,\mu\,a_2b_1-432V^2\mu^2\nu\,{b_1}^2b_2$$

$$-40\kappa^3\lambda^2\mu\,A_0b_1-120\kappa^2\lambda^3\mu\,{b_1}^2+3\kappa^2\lambda^2\nu\,{A_0}^2b_1+18\kappa\,\lambda^3\nu\,A_0{b_1}^2$$

$$+18\lambda^4\nu\,{b_1}^3+15c\kappa^3{A_0}^3+135c\kappa^2\lambda\,{A_0}^2b_1+324c\kappa\,\lambda^2A_0{b_1}^2+162c\lambda^3{b_1}^3-\kappa^3\lambda^2A_0a_1$$

$$+80\kappa^3\mu^2A_0b_1-3\kappa^2\lambda^3a_1b_1+240\kappa^2\lambda\,\mu^2{b_1}^2-12\kappa^2\mu\,\nu\,{A_0}^2b_1-72\kappa\,\lambda\,\mu\,\nu\,A_0{b_1}^2$$

$$-36\lambda^2\mu\,\nu\,{b_1}^3+5V^2\kappa^3A_0+15V^2\kappa^2\lambda\,b_1+324c\kappa\,\mu\,A_0{b_1}^2+972c\lambda\,\mu\,{b_1}^3$$

$$-5k^2\kappa^3A_0-15k^2\kappa^2\lambda\,b_1+4\kappa^3\mu\,A_0a_1+12\kappa^2\lambda\,\mu\,a_1b_1-144\mu^2\nu\,{b_1}^3,$$

$$Z_{10}=-9V^6\lambda^4\nu\,{b_2}^3+72V^6\lambda^2\mu\,\nu\,{b_2}^3-30V^4\kappa^2\lambda^5{b_2}^2$$

$$+1{,}620V^6c\lambda\,\mu\,{b_2}^3-144V^6\mu^2\nu\,{b_2}^3+240V^4\kappa^2\lambda^3\mu\,{b_2}^2-27V^4\lambda^4\nu\,b_1{b_2}^2$$

$$+270V^4c\kappa\,\lambda^2A_0{b_2}^2+15V^4\kappa^2\lambda^3a_2b_2-480V^4\kappa^2\lambda\,\mu^2{b_2}^2+216V^4\lambda^2\mu\,\nu\,b_1{b_2}^2$$

$$-10V^2\kappa^3\lambda^4A_0b_2-60V^2\kappa^2\lambda^5b_1b_2+540V^4c\kappa\,\mu\,A_0{b_2}^2+4{,}860V^4c\lambda\,\mu\,b_1{b_2}^2$$

$$-60V^4\kappa^2\lambda\,\mu\,a_2b_2-432V^4\mu^2\nu\,b_1{b_2}^2+80V^2\kappa^3\lambda^2\mu\,A_0b_2+480V^2\kappa^2\lambda^3\mu\,b_1b_2$$

$$-27V^2\lambda^4\nu\,{b_1}^2b_2+135V^2c\kappa^2\lambda\,{A_0}^2b_2+540V^2c\kappa\,\lambda^2A_0b_1b_2+5V^2\kappa^3\lambda^2A_0a_2$$

$$-160V^2\kappa^3\mu^2A_0b_2+15V^2\kappa^2\lambda^3a_1b_2+15V^2\kappa^2\lambda^3a_2b_1-960V^2\kappa^2\lambda\,\mu^2b_1b_2$$

$$+216V^2\lambda^2\mu\,\nu\,{b_1}^2b_2-10\,\kappa^3\lambda^4A_0b_1-30\,\kappa^2\lambda^5{b_1}^2$$

$$+15V^4\kappa^2\lambda\,b_2+1{,}080V^2c\kappa\,\mu\,A_0b_1b_2+4{,}860V^2c\lambda\,\mu\,{b_1}^2b_2-15V^2k^2\kappa^2\lambda\,b_2$$

$$-20V^2\kappa^3\mu\,A_0a_2-60V^2\kappa^2\lambda\,\mu\,a_1b_2-60V^2\kappa^2\lambda\,\mu\,a_2b_1-432V^2\mu^2\nu\,{b_1}^2b_2$$

$$+80\,\kappa^3\lambda^2\mu\,A_0b_1+240\,\kappa^2\lambda^3\mu\,{b_1}^2-9\,\lambda^4\nu\,{b_1}^3+15\,c\kappa^3{A_0}^3+135\,c\kappa^2\lambda\,{A_0}^2b_1$$

$$+270\,c\kappa\,\lambda^2A_0{b_1}^2+5\,\kappa^3\lambda^2A_0a_1-160\,\kappa^3\mu^2A_0b_1+15\,\kappa^2\lambda^3a_1b_1$$

$$-480\,\kappa^2\lambda\,\mu^2{b_1}^2+72\,\lambda^2\mu\,\nu\,{b_1}^3+5V^2\kappa^3A_0+15V^2\kappa^2\lambda\,b_1+540\,c\kappa\,\mu\,A_0{b_1}^2$$

$$+1{,}620\,c\lambda\,\mu\,{b_1}^3-5\,k^2\kappa^3A_0-15\,k^2\kappa^2\lambda\,b_1-20\,\kappa^3\mu\,A_0a_1-60\,\kappa^2\lambda\,\mu\,a_1b_1-144\,\mu^2\nu\,{b_1}^3,$$

its worth to mention that all the obtained parameter values are laced with V, c, A_0, b_1, b_2, a_1, k, and λ, κ, μ, ν as free parameters.

Using solution of equation (34) along with parameter values (33) and (38), we may write the subsequent solution for equation (30), by reverting back to original variables x, t:

Regrading the instance $\lambda^2-4\mu>0$, we have extracted following solution structure for equation (30):

$$q(x,\,t)=\left(\frac{\sqrt{\lambda^2-4\mu}\left(w_1\sinh\left(\frac{(x-Vt)\sqrt{\lambda^2-4\mu}}{2}\right)+w_2\cosh\left(\frac{(x-Vt)(\sqrt{\lambda^2-4\mu})}{2}\right)\right)}{2w_2\sinh\left(\frac{(x-Vt)\sqrt{\lambda^2-4\mu}}{2}\right)+2w_1\cosh\left(\frac{(x-Vt)(\sqrt{\lambda^2-4\mu})}{2}\right)}-\frac{\lambda}{2}\right)A_1+A_0. \quad (39)$$

In (39), for $w_1=0$, $w_2\neq 0$, one recovers singular solitary waves while for $w_2=0$, $w_1\neq 0$, one obtains shock waves.

Regrading the instance $\lambda^2-4\mu<0$, we have extracted following solution structure for equation (30):

$$q(x,\,t)=\left(\frac{\sqrt{-\lambda^2+4\mu}\left(w_1\sin\left(\frac{(x-Vt)\sqrt{-\lambda^2+4\mu}}{2}\right)+w_2\cos\left(\frac{(x-Vt)(\sqrt{-\lambda^2+4\mu})}{2}\right)\right)}{2w_2\sin\left(\frac{(x-Vt)\sqrt{-\lambda^2+4\mu}}{2}\right)+2w_1\cos\left(\frac{(x-Vt)(\sqrt{-\lambda^2+4\mu})}{2}\right)}-\frac{\lambda}{2}\right)A_1+A_0. \quad (40)$$

4. Extended tanh approach

Furthermore, the solutions to Eq. (30) are furnished through the extended Tanh approach. This technique proposed the following solution structure for ODE (32), and then finally leading to formation of soliton solutions to governing model (30):

$$F(\tau) = A_0 + A_1 \tanh(p\tau) + \frac{A_2}{\tanh(p\tau)}. \tag{41}$$

Substituting Eq. (41) into Eq. (32), we have produced a family of polynomials containing $\tanh(p\tau)$ and $\frac{1}{\tanh(p\tau)}$. When we set the coefficients of these polynomials to zero, the following result is obtained:

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$$\Lambda = -\frac{L_1}{144\, p^2 A_0 \kappa^2 (V^2 b_2 + b_1)},$$

$$\delta = \frac{2L_2}{9(V^2 b_2 + b_1)\,\kappa^2 A_0},$$

$$\psi = \frac{L_3}{768\, \kappa\, p^4 (V^2 b_2 + b_1)},$$

$$\theta = \frac{L_4}{48 p^2 A_0 \kappa^2 (V^2 b_2 + b_1)},$$

$$\xi = -\frac{L_5}{\left(1{,}152\, V^4 {b_2}^2 + 2{,}304\, V^2 b_1 b_2 + 1{,}152\, {b_1}^2\right) \kappa\, p^4}$$

$$A_1 = -\frac{6p\left(V^2 b_2 + b_1\right)}{\kappa}, \quad A_2 = -\frac{6p\left(V^2 b_2 + b_1\right)}{\kappa}, \tag{42}$$

along with

$$\begin{aligned} L_1 = {} & 2{,}304\, V^6 \nu\, p^4 {b_2}^3 + 6{,}912\, V^4 \nu\, p^4 b_1 {b_2}^2 - 432\, V^4 c\kappa\, p^2 A_0 {b_2}^2 \\ & + 2{,}560\, V^2 \kappa^3 p^4 A_0 b_2 + 6{,}912\, V^2 \nu\, p^4 {b_1}^2 b_2 \\ & - 864\, V^2 c\kappa\, p^2 A_0 b_1 b_2 - 80\, V^2 \kappa^3 p^2 A_0 a_2 + 2{,}560\, \kappa^3 p^4 A_0 b_1 \\ & + 2{,}304\, \nu\, p^4 {b_1}^3 - 15\, c\kappa^3 {A_0}^3 - 432\, c\kappa\, p^2 A_0 {b_1}^2 \\ & - 80\, \kappa^3 p^2 A_0 a_1 - 5\, V^2 \kappa^3 A_0 + 5\, k^2 \kappa^3 A_0, \end{aligned}$$

$$
\begin{aligned}
L_2 = {} & 1{,}152V^6\nu p^4{b_2}^3 + 3{,}456V^4\nu p^4 b_1{b_2}^2 \\
& + 432V^4c\kappa p^2A_0{b_2}^2 + 512V^2\kappa^3p^4A_0b_2 + 3{,}456V^2\nu p^4{b_1}^2b_2 \\
& + 864V^2c\kappa p^2A_0b_1b_2 - 16V^2\kappa^3p^2A_0a_2 + 512\kappa^3p^4A_0b_1 \\
& + 1{,}152\nu p^4{b_1}^3 - 3c\kappa^3{A_0}^3 + 432c\kappa p^2A_0{b_1}^2 \\
& - 16\kappa^3p^2A_0a_1 - V^2\kappa^3A_0 + k^2\kappa^3A_0, \\
L_3 = {} & A_0\left(432V^4cp^2{b_2}^2 - 256V^2\kappa^2p^4b_2 + 864V^2cp^2b_1b_2\right. \\
& - 16V^2\kappa^2p^2a_2 - 256\kappa^2p^4b_1 - 3c\kappa^2{A_0}^2 + 432cp^2{b_1}^2 \\
& \left. + \left(-16\kappa^2p^2a_1 - V^2\kappa^2 + k^2\kappa^2\right)A_0, \right.\\
L_4 = {} & 2{,}304V^6\nu p^4{b_2}^3 + 6{,}912V^4\nu p^4b_1{b_2}^2 \\
& + 1{,}296V^4c\kappa p^2A_0{b_2}^2 - 1{,}280V^2\kappa^3p^4A_0b_2 - 48V^2\kappa^2\nu p^2{A_0}^2b_2 \\
& + 6{,}912V^2\nu p^4{b_1}^2b_2 + 2{,}592V^2c\kappa p^2A_0b_1b_2 \\
& + 16V^2\kappa^3p^2A_0a_2 - 1{,}280\kappa^3p^4A_0b_1 - 48\kappa^2\nu p^2{A_0}^2b_1 \\
& + 2{,}304\nu p^4{b_1}^3 - 15c\kappa^3{A_0}^3 + 1{,}296c\kappa p^2A_0{b_1}^2 \\
& + 16\kappa^3p^2A_0a_1 - 5V^2\kappa^3A_0 + 5k^2\kappa^3A_0, \\
L_5 = {} & 2{,}304V^6\nu p^4{b_2}^3 + 6{,}912V^4\nu p^4b_1{b_2}^2 + 2{,}160V^4c\kappa p^2A_0{b_2}^2 \\
& + 2{,}560V^2\kappa^3p^4A_0b_2 + 6{,}912V^2\nu p^4{b_1}^2b_2 \\
& + 4{,}320V^2c\kappa p^2A_0b_1b_2 - 80V^2\kappa^3p^2A_0a_2 \\
& + 2{,}560\kappa^3p^4A_0b_1 + 2{,}304\nu p^4{b_1}^3 - 15c\kappa^3{A_0}^3 \\
& + 2{,}160c\kappa p^2A_0{b_1}^2 - 80\kappa^3p^2A_0a_1 - 5V^2\kappa^3A_0 + 5k^2\kappa^3A_0.
\end{aligned} \tag{43}
$$

Employing the aforementioned results (42), (43) along with eq. (41) for Eq. (32), the shock wave solution is revealed for equation (30) as follows:

$$q(x,\,t)=A_0-\frac{6p\left(V^2b_2+b_1\right)\tanh\left(p\left(-Vt+x\right)\right)}{\kappa}-\frac{6p\left(V^2b_2+b_1\right)}{\kappa\tanh\left(p\left(-Vt+x\right)\right)},\tag{44}$$

equipped with V, c, A_0, b_1, b_2, a_1, a_2, κ, p, ν as free parameters.

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$$\Lambda=-\frac{L_6}{36p^2A_0\kappa^2\left(V^2b_2+b_1\right)},$$

$$\delta=\frac{2L_7}{9\left(V^2b_2+b_1\right)\kappa^2A_0},$$

$$\psi=\frac{L_8}{48\kappa p^4\left(V^2b_2+b_1\right)},$$

$$\theta=\frac{L_9}{p^2A_0 12\kappa^2\left(V^2b_2+b_1\right)},$$

$$\xi=-\frac{L_{10}}{72\,\kappa p^4\left(V^2b_2+b_1\right)^2},$$

$$A_1=0,\quad A_2=-\frac{6p\left(V^2b_2+b_1\right)}{\kappa},\tag{45}$$

$$\begin{aligned}L_6={}&144V^6\nu p^4{b_2}^3+432V^4\nu p^4b_1{b_2}^2-108V^4c\kappa p^2A_0{b_2}^2\\&+160V^2\kappa^3p^4A_0b_2+432V^2\nu p^4{b_1}^2b_2\\&-216V^2c\kappa p^2A_0b_1b_2-20V^2\kappa^3p^2A_0a_2\\&+160\,\kappa^3p^4A_0b_1+144\,\nu p^4{b_1}^3-15\,c\kappa^3{A_0}^3\\&-108\,c\kappa p^2A_0{b_1}^2-20\,\kappa^3p^2A_0a_1-5V^2\kappa^3A_0+5k^2\kappa^3A_0,\end{aligned}$$

$$
\begin{aligned}
L_7 = {} & 72V^6\nu p^4 {b_2}^3 + 216V^4\nu p^4 b_1 {b_2}^2 + 108V^4 c\kappa p^2 A_0 {b_2}^2 \\
& + 32V^2\kappa^3 p^4 A_0 b_2 + 216V^2\nu p^4 {b_1}^2 b_2 + 216V^2 c\kappa p^2 A_0 b_1 b_2 \\
& - 4V^2\kappa^3 p^2 A_0 a_2 + 32\kappa^3 p^4 A_0 b_1 + 72\nu p^4 {b_1}^3 - 3c\kappa^3 {A_0}^3 \\
& + 108c\kappa p^2 A_0 {b_1}^2 - 4\kappa^3 p^2 A_0 a_1 - V^2\kappa^3 A_0 + k^2\kappa^3 A_0, \\
L_8 = {} & A_0\big(108V^4 cp^2 {b_2}^2 - 16V^2\kappa^2 p^4 b_2 + 216V^2 cp^2 b_1 b_2 \\
& - 4V^2\kappa^2 p^2 a_2 - 16\kappa^2 p^4 b_1 - 3c\kappa^2 {A_0}^2\big) \\
& + \big(108cp^2 {b_1}^2 - 4\kappa^2 p^2 a_1 - V^2\kappa^2 + k^2\kappa^2\big)A_0, \\
L_9 = {} & 144V^6\nu p^4 {b_2}^3 + 432V^4\nu p^4 b_1 {b_2}^2 + 324V^4 c\kappa p^2 A_0 {b_2}^2 \\
& - 80V^2\kappa^3 p^4 A_0 b_2 - 12V^2\kappa^2\nu p^2 {A_0}^2 b_2 + 432V^2\nu p^4 {b_1}^2 b_2 \\
& + 648V^2 c\kappa p^2 A_0 b_1 b_2 + 4V^2\kappa^3 p^2 A_0 a_2 - 80\kappa^3 p^4 A_0 b_1 \\
& - 12\kappa^2\nu p^2 {A_0}^2 b_1 + 144\nu p^4 {b_1}^3 - 15c\kappa^3 {A_0}^3 \\
& + 324c\kappa p^2 A_0 {b_1}^2 + 4\kappa^3 p^2 A_0 a_1 - 5V^2\kappa^3 A_0 + 5k^2\kappa^3 A_0, \\
L_{10} = {} & 144V^6\nu p^4 {b_2}^3 + 432V^4\nu p^4 b_1 {b_2}^2 + 540V^4 c\kappa p^2 A_0 {b_2}^2 \\
& + 160V^2\kappa^3 p^4 A_0 b_2 + 432V^2\nu p^4 {b_1}^2 b_2 \\
& + 1{,}080V^2 c\kappa p^2 A_0 b_1 b_2 - 20V^2\kappa^3 p^2 A_0 a_2 + 160\kappa^3 p^4 A_0 b_1 \\
& + 144\nu p^4 {b_1}^3 - 15c\kappa^3 {A_0}^3 + 540c\kappa p^2 A_0 {b_1}^2 \\
& - 20\kappa^3 p^2 A_0 a_1 - 5V^2\kappa^3 A_0 + 5k^2\kappa^3 A_0.
\end{aligned} \tag{46}
$$

Employing the aforementioned results (45), (46) along with eq. (41) for equation (32), the following soliton solution is revealed for equation (30) as follows:

$$q(x,t)=A_0-\frac{6p\left(V^2b_2+b_1\right)}{\kappa\tanh\left(p\left(-Vt+x\right)\right)}, \tag{47}$$

equipped with V, c, A_0, b_1, b_2, a_1, a_2, κ, p, ν as free parameters.

Cluster-III

$$\Lambda=-\frac{L_{11}}{36p^2A_0\kappa^2\left(V^2b_2+b_1\right)},$$

$$\delta=\frac{2L_{12}}{9\left(V^2b_2+b_1\right)\kappa^2A_0},$$

$$\psi=\frac{L_{13}}{48\kappa p^4\left(V^2b_2+b_1\right)},$$

$$\theta=\frac{L_{14}}{12p^2A_0\kappa^2\left(V^2b_2+b_1\right)},$$

$$\xi=-\frac{L_{15}}{\left(72V^4{b_2}^2+144V^2b_1b_2+72{b_1}^2\right)\kappa p^4},$$

$$A_1=-\frac{6p\left(V^2b_2+b_1\right)}{\kappa},\quad A_2=0, \tag{48}$$

$$\begin{aligned}L_{11}=&\,144V^6\nu p^4{b_2}^3+432V^4\nu p^4b_1{b_2}^2-108V^4c\kappa p^2A_0{b_2}^2\\&+160V^2\kappa^3p^4A_0b_2+432V^2\nu p^4{b_1}^2b_2-216V^2c\kappa p^2A_0b_1b_2\\&-20V^2\kappa^3p^2A_0a_2+160\kappa^3p^4A_0b_1+144\nu p^4{b_1}^3-15c\kappa^3{A_0}^3\\&-108c\kappa p^2A_0{b_1}^2-20\kappa^3p^2A_0a_1-5V^2\kappa^3A_0+5k^2\kappa^3A_0,\end{aligned}$$

$$L_{12} = 72V^6\nu p^4{b_2}^3 + 216V^4\nu p^4b_1{b_2}^2 + 108V^4c\kappa p^2A_0{b_2}^2$$

$$+ 32V^2\kappa^3p^4A_0b_2 + 216V^2\nu p^4{b_1}^2b_2 + 216V^2c\kappa p^2A_0b_1b_2$$

$$- 4V^2\kappa^3p^2A_0a_2 + 32\kappa^3p^4A_0b_1 + 72\nu p^4{b_1}^3 - 3c\kappa^3{A_0}^3$$

$$+ 108c\kappa p^2A_0{b_1}^2 - 4\kappa^3p^2A_0a_1 - V^2\kappa^3A_0 + k^2\kappa^3A_0,$$

$$L_{13} = A_0\big(108V^4cp^2{b_2}^2 - 16V^2\kappa^2p^4b_2 + 216V^2cp^2b_1b_2$$

$$- 4V^2\kappa^2p^2a_2 - 16\kappa^2p^4b_1 - 3c\kappa^2{A_0}^2\big)$$

$$+ A_0\left(108cp^2{b_1}^2 - 4\kappa^2p^2a_1 - V^2\kappa^2 + k^2\kappa^2\right),$$

$$L_{14} = 144V^6\nu p^4{b_2}^3 + 432V^4\nu p^4b_1{b_2}^2 + 324V^4c\kappa p^2A_0{b_2}^2$$

$$- 80V^2\kappa^3p^4A_0b_2 - 12V^2\kappa^2\nu p^2{A_0}^2b_2 + 432V^2\nu p^4{b_1}^2b_2$$

$$+ 648V^2c\kappa p^2A_0b_1b_2 + 4V^2\kappa^3p^2A_0a_2 - 80\kappa^3p^4A_0b_1$$

$$- 12\kappa^2\nu p^2{A_0}^2b_1 + 144\nu p^4{b_1}^3 - 15c\kappa^3{A_0}^3$$

$$+ 324c\kappa p^2A_0{b_1}^2 + 4\kappa^3p^2A_0a_1 - 5V^2\kappa^3A_0 + 5k^2\kappa^3A_0,$$

$$L_{15} = 144V^6\nu p^4{b_2}^3 + 432V^4\nu p^4b_1{b_2}^2 + 540V^4c\kappa p^2A_0{b_2}^2$$

$$+ 160V^2\kappa^3p^4A_0b_2 + 432V^2\nu p^4{b_1}^2b_2 + 1{,}080V^2c\kappa p^2A_0b_1b_2$$

$$- 20V^2\kappa^3p^2A_0a_2 + 160\kappa^3p^4A_0b_1 + 144\nu p^4{b_1}^3 - 15c\kappa^3{A_0}^3$$

$$+ 540c\kappa p^2A_0{b_1}^2 - 20\kappa^3p^2A_0a_1 - 5V^2\kappa^3A_0 + 5k^2\kappa^3A_0. \tag{49}$$

Employing the aforementioned results (48), (49) along with eq. (41), shock wave solution is revealed for equation (30):

$$q(x, t) = A_0 - \frac{6p\left(V^2b_2 + b_1\right)\tanh\left(p\left(-Vt + x\right)\right)}{\kappa}, \tag{50}$$

equipped with V, c, A_0, b_1, b_2, a_1, a_2, κ, p, ν as free parameters.

5. Conclusions

The current paper is loaded up with a plethora of results from 6BE. The unperturbed version oif the model was addressed by G'/G-expansion approach that revealed shock waves and solitary wave as well as singular solitary wave solutions to the model. Subsequently the perturbed version of the model is considered where the same approach revealed shock waves as well as solitary waves. Additionally the extended tanh approach yielded the shock wave solutions. The results came with parametric restrictions that must hold for such a range of solutions to exist. Thee results are just a starting point to various future avenues to walk on. Later the model will be considered for two-layered fluid flow with surface tension that would give additional set of results. The results of those research activities, that are under way, will be made available with time.

Conflict of interest

The authors claim that there is no conflict of interest.

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