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Research Article

Sandwich Weighted Composition Operators Between Weighted Bloch Spaces

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Abstract: In this paper, we integrate two foundational operators: a weighted composition operator, denoted by $W_{\varphi,\ \psi}$ and two differentiation operators to construct a sandwich weighted composition operator, referred to as $SW_{\varphi,\ \psi}$. We conduct a comprehensive analysis of the norms and essential norms of this operator within the framework of weighted Bloch spaces.

***Keywords*:** linear operators, Banach spaces of analytic functions

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1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$, and let $dA(z)$ represent the normalized area measure on $\mathbb{D}$. The class $H(\mathbb{D})$ consists of all holomorphic functions defined on $\mathbb{D}$, while $S(\mathbb{D})$ denotes the family of all holomorphic self-maps of $\mathbb{D}$.

A strictly positive continuous function $\nu : \mathbb{D} \to \mathbb{R}_+$ is called a *weight*, and it is said to be *radial* if it satisfies the condition $\nu(z) = \nu(|z|)$ for all $z \in \mathbb{D}$. In this paper, we focus on the case where both ν and μ are radial, non-increasing weights that converge to zero as $|z| \to 1$.

The associated weighted space of analytic functions, denoted by H_ν^∞, is defined as

$$H_\nu^\infty = \left\{ f \in H(\mathbb{D}) : \|f\|_{H_\nu^\infty} := \sup_{z\in\mathbb{D}} \nu(z)|f(z)| < \infty \right\}.$$

This space forms a well-established Banach space, equipped with the norm $\|\cdot\|_{H_\nu^\infty}$.

Furthermore, the associated weight $\overline{\nu}$ is defined by

$$\overline{\nu}(z) := \left(\sup\left\{|f(z)| : f \in H_\nu^\infty,\ \|f\|_{H_\nu^\infty} \le 1\right\}\right)^{-1},\ z \in \mathbb{D}.$$

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We introduce the notation $\eta(z) = 1-|z|^2$, which serves as a fundamental component in the study of standard weights. Standard weights are represented by the expression $v_\alpha(z) = (\eta(z))^\alpha$, where $\alpha > 0$. It is obvious that $\overline{v}_\alpha = v_\alpha$. Furthermore, the weighted Bloch space $\mathscr{B}^\alpha = \mathscr{B}^\alpha(\mathbb{D})$ comprises analytic functions whose derivatives are in $H^\infty_{v_\alpha}$ The space $\mathscr{B}^\alpha$ is characterized as a Banach space equipped with the norm defined by $\|f\|_{\mathscr{B}^\alpha} = |f(0)| + \|f'\|_{H^\infty_{v_\alpha}}$. For more about these spaces we refer the readers to [1]. The investigation of products involving composition, multiplication, and differentiation operators has garnered significant scholarly attention within the fields of functional analysis and operator theory, particularly in the context of holomorphic function spaces. This paper establishes bounds for the essential norm of the sandwich weighted composition operator between Bloch-type spaces.

To enhance the clarity and readability of this work, we adopt the following conventions: we assume that α and β are positive real numbers, ψ is a function in $H(\mathbb{D})$, and φ belongs to the space $S(\mathbb{D})$.

The *sandwich weighted composition operator* $SW_{\varphi,\,\psi}$ is defined by

$$(SW_{\varphi,\,\psi}f)(z) = (DW_{\varphi,\,\psi}Df)(z) = \psi'(z)f'(\varphi(z)) + \psi(z)\varphi'(z)f''(\varphi(z)),\ f \in H(\mathbb{D}).$$

This operator effectively integrates the two foundational operators: a weighted composition operator, denoted as $W_{\varphi,\,\psi}$ with two differentiation operators and is obtained by sandwiching weighted composition between two differentiation operators. The composition operator C_φ has garnered extensive scholarly attention, with established boundedness on nearly all classical spaces of analytic functions, as evidenced by various studies, including those noted in sources [2, 3]. In contrast, the differentiation operator D typically demonstrates unbounded behavior within these spaces. Notably, the pioneering work of Hibschweiler and Portony [4] examined the initial product of these operators, assessing their boundedness and compactness in the context of Bergman and Hardy spaces through Carleson-type measures. Following this, numerous researchers have further investigated the products of these operators, as referenced in [3-37].

Our objective is to investigate the boundedness and compactness of the sandwich weighted composition operator within Bloch-type spaces, while also providing estimates for both the operator norm and the essential norm associated with its action in these settings.

The *essential norm* of a bounded linear operator $T : X \to Y$ between normed spaces, denoted by $\|T\|_{e,\,X\to Y}$, is defined by the expression

$$\|T\|_{e,\,X\to Y} = \inf\{\|T-K\| : K \text{ is a compact operator from } X \text{ to } Y\}.$$

As such, the essential norm serves as a critical tool for understanding the implications of operator theory in both mathematical and applied frameworks. Constants will be denoted by C, remaining positive and varying with each occurrence. Furthermore, we will use the notation $A \asymp B$ to imply that $B \lesssim A \lesssim B$, where the expression $A \lesssim B$ indicates the existence of a positive constant C such that $A \le CB$.

2. Methods

In this section, we present several auxiliary results that will serve as foundational components in the proof of the main results contained within this paper. These preliminary findings are essential for establishing the necessary framework and will facilitate a clearer understanding of the subsequent arguments and outcomes.

The following theorem serves as a foundational framework for our subsequent discussions and analyses. For the proof, we direct the readers to Theorem 2.1 presented in [21]. Additionally, further insights can be gleaned from the Theorem 2.4 in [11].

Lemma 1

(a) $W_{\varphi,\,\psi} : H^\infty_v \to H^\infty_\mu$ is bounded if and only if

$$\|W_{\varphi,\ \psi}\|_{H_\nu^\infty\to H_\mu^\infty} \asymp \sup_{n\geq 0}\frac{\|\psi\varphi^n\|_\mu}{\|z^n\|_\nu} \asymp \sup_{z\in\mathbb{D}}\frac{\mu(z)}{\overline{\nu}(\varphi(z))}|\psi(z)| < \infty;$$

(b) If $W_{\varphi,\ \psi}: H_\nu^\infty \to H_\mu^\infty$ is bounded, then

$$\|W_{\varphi,\ \psi}\|_{e,\ H_\nu^\infty\to H_w^\infty} \asymp \limsup_{n\to\infty}\frac{\|\psi\varphi^n\|_\mu}{\|z^n\|_\nu} \asymp \limsup_{|\varphi(z)|\to 1}\frac{\mu(z)}{\overline{\nu}(\varphi(z))}|\psi(z)|.$$

The next results establish limits of norm of certain sequences in H_{ν_α}, see Lemma 2.1 in [11]. This result highlights the asymptotic behavior of the norm of the function z^n within the context of $H_{\nu_\alpha}^\infty$ spaces.

Lemma 2 For any positive α, the following condition holds:

$$(n+1)^\alpha\|z^n\|_{H_{\nu_\alpha}^\infty} \to (2\alpha/e)^\alpha \text{ as } n \text{ approaches infinity.}$$

The subsequent lemmas present the notable characterization of Bloch-type spaces $\mathscr{B}^\alpha$, as referenced in [1]. Furthermore, the growth behavior of functions and their derivatives within the context of $\mathscr{B}^\alpha$ spaces is pivotal to our investigation.

Lemma 3 For any integer $m\in\mathbb{N}$ and positive parameter α, the norm of the derivative $\|f'\|_{H_{\nu_\alpha}^\infty}$ is asymptotically equivalent to the supremum $\sup_{z\in\mathbb{D}}(\eta(z))^{\alpha+m-1}|f^{(m)}(z)| < \infty$. Specifically, for functions $f\in\mathscr{B}^\alpha$ and $z\in\mathbb{D}$, we have

$$|f^{(m)}(z)| \lesssim \frac{\|f\|_{\mathscr{B}^\alpha}}{(\eta(z))^{\alpha+m-1}}.$$

The subsequent criterion for compactness is derived from a straightforward modification of Proposition 3.11 as presented in [2].

Lemma 4 Suppose that $SW_{\varphi,\ \psi}: \mathscr{B}^\alpha \to \mathscr{B}^\beta$ is bounded. Then $SW_{\varphi,\ \psi}: \mathscr{B}^\alpha \to \mathscr{B}^\beta$ is compact if for any sequence $\{f_k\}$ that is bounded in $\mathscr{B}^\alpha$ and converges uniformly to zero on compact subsets of $\mathbb{D}$, the norm $\|SW_{\varphi,\ \psi}f_k\|_{\mathscr{B}^\beta}$ approaches zero as $k\to\infty$.

3. Results: the operator $SW_{\varphi,\ \psi}$: $\mathscr{B}^\alpha \to \mathscr{B}^\beta$

In this section, we thoroughly examine the boundedness and compactness of the sandwich weighted composition operator $SW_{\varphi,\ \psi}$ between weighted Bloch spaces. Furthermore, we provide detailed estimates for the operator norm of $SW_{\varphi,\ \psi}$ acting between these spaces.

Theorem 1 $SW_{\varphi,\ \psi}: \mathscr{B}^\alpha \to \mathscr{B}^\beta$ is bounded if and only if

$$\text{(a) } L_1 = \sup_{z\in\mathbb{D}}\frac{(\eta(z))^\beta}{\eta(\varphi(z))^\alpha}|\psi''(z)| \asymp \sup_{n\geq 0}(n+1)^\alpha\|\psi''\varphi^n\|_{H_{\nu_\beta}^\infty} < \infty;$$

$$\text{(b) } L_2 = \sup_{z\in\mathbb{D}}\frac{(\eta(z))^\beta}{\eta(\varphi(z))^{\alpha+1}}|2\psi'(z)\varphi'(z)+\psi(z)\varphi''(z)| \asymp \sup_{n\geq 0}(n+1)^\alpha\|(2\psi'\varphi'+\psi\varphi'')\varphi^n\|_{H_{\nu_\beta}^\infty} < \infty;$$

(c) $L_3 = \sup_{z\in\mathbb{D}} \dfrac{(\eta(z))^\beta}{\eta(\varphi(z))^{\alpha+2}}|\psi(z)(\varphi'(z))^2| \asymp \sup_{n\geq 0}(n+1)^{\alpha+2}\|(\psi(\varphi')^2)\varphi^n\|_{H^\infty_{v_\beta}} < \infty.$

Furthermore, if $SW_{\varphi,\,\psi} : \mathscr{B}^\alpha \to \mathscr{B}^\beta$ is bounded, then the operator norm satisfies the following estimate:

$$L_1 + L_2 + L_3 \lesssim \|SW_{\varphi,\,\psi}\|_{\mathscr{B}^\alpha \to \mathscr{B}^\beta} \lesssim \frac{|\psi'(0)|}{(\eta(\varphi(0))^\alpha} + \frac{|\psi(0)\varphi'(0)|}{(\eta(\varphi(0))^{\alpha+1}} + L_1 + L_2 + L_3. \tag{1}$$

Proof. Suppose that conditions (a), (b) and (c) are satisfied. Let f belong to $\mathscr{B}^\alpha$. Then we can apply Lemma 3, to establish an upper bound of $\|SW_{\varphi,\,\psi}\|_{\mathscr{B}^\alpha \to \mathscr{B}^\beta}$. Specifically, it can be shown that:

$$\begin{aligned}
&(\eta(z))^\beta|(SW_{\varphi,\,\psi}f)'(z)|\\
\leq&(\eta(z))^\beta|\psi''(z)||f'(\varphi(z))| + (\eta(z))^\beta|2\psi'(z)\varphi'(z)+\psi(z)\varphi''(z)||f''(\varphi(z))| + (\eta(z))^\beta|\psi(z)(\varphi'(z))^2||f'''(\varphi(z))|\\
\lesssim&\left(\frac{(\eta(z))^\beta}{(\eta(\varphi(z))^\alpha}|\psi''(z)| + \frac{(\eta(z))^\beta}{(\eta(\varphi(z))^{\alpha+1}}|2\psi'(z)\varphi'(z)+\psi(z)\varphi''(z)| + \frac{(\eta(z))^\beta}{(\eta(\varphi(z))^{\alpha+2}}|\psi(z)(\varphi'(z))^2|\right)\|f\|_{\mathscr{B}^\alpha},
\end{aligned}$$

and

$$\begin{aligned}
|(SW_{\varphi,\,\psi}f)(0)| &= |\psi'(0)f'(\varphi(0)) + \psi(0)\varphi'(0)f''(\varphi(0))|\\
&\lesssim \left(\frac{|\psi'(0)|}{(\eta(\varphi(0))^\alpha} + \frac{|\psi(0)\varphi'(0)|}{(\eta(\varphi(0))^{\alpha+1}}\right)\|f\|_{\mathscr{B}^\alpha},
\end{aligned}$$

which leads us to infer that

$$\|SW_{\varphi,\,\psi}f\|_{\mathscr{B}^\beta} \lesssim \left(\frac{|\psi'(0)|}{(\eta(\varphi(0))^\alpha} + \frac{|\psi(0)\varphi'(0)|}{(\eta(\varphi(0))^{\alpha+1}} + L_1 + L_2 + L_3\right)\|f\|_{\mathscr{B}^\alpha}.$$

Thus, we arrive at the conclusion that

$$\|SW_{\varphi,\,\psi}\|_{\mathscr{B}^\alpha \to \mathscr{B}^\beta} \lesssim \frac{|\psi'(0)|}{(\eta(\varphi(0))^\alpha} + \frac{|\psi(0)\varphi'(0)|}{(\eta(\varphi(0))^{\alpha+1}} + L_1 + L_2 + L_3. \tag{2}$$

Conversely, suppose that $SW_{\varphi,\,\psi} : \mathscr{B}^\alpha \to \mathscr{B}^\beta$ is bounded. Then considering $f(z) = z,\ f(z) = z^2/2!$ and $f(z) = z^3/3!$, respectively in $\mathscr{B}^\alpha$, using the fact that $|\varphi(z)| < 1$, and the resulting inequalities, we have that

$$\sup_{z\in\mathbb{D}}(\eta(z))^{\beta}|\psi''(z)| \lesssim \|SW_{\varphi,\ \psi}\|_{\mathscr{B}^{\alpha}\to\mathscr{B}^{\beta}}, \tag{3}$$

$$\sup_{z\in\mathbb{D}}(\eta(z))^{\beta}|2\psi'(z)\varphi'(z)+\psi(z)\varphi''(z)| \lesssim \|SW_{\varphi,\ \psi}\|_{\mathscr{B}^{\alpha}\to\mathscr{B}^{\beta}}, \tag{4}$$

and

$$\sup_{z\in\mathbb{D}}(\eta(z))^{\beta}|\psi(z)(\varphi'(z))^2| \lesssim \|SW_{\varphi,\ \psi}\|_{\mathscr{B}^{\alpha}\to\mathscr{B}^{\beta}}. \tag{5}$$

Consider the function

$$f_{\varphi(\rho)}(z) = A_1(\alpha)\frac{\eta(\varphi(\rho))}{(1-\overline{\varphi(\rho)}z)^{\alpha}} + A_2(\alpha)\frac{(\eta(\varphi(\rho)))^2}{(1-\overline{\varphi(\rho)}z)^{\alpha+1}}$$

$$+A_3(\alpha)\frac{(\eta(\varphi(\rho)))^3}{(1-\overline{\varphi(\rho)}z)^{\alpha+2}} + A_4(\alpha)\frac{(\eta(\varphi(\rho)))^4}{(1-\overline{\varphi(\rho)}z)^{\alpha+3}},$$

where $A_1(\alpha) = -\alpha$, $A_2(\alpha) = -6$, $A_3(\alpha) = 3\alpha+12$ and $A_4(\alpha) = -(2\alpha+6)$. Then

$$f'_{\varphi(\rho)}(z) = \Bigg(B_1(\alpha)\frac{\eta(\varphi(\rho))}{(1-\overline{\varphi(\rho)}z)^{\alpha+1}} + B_2(\alpha)\frac{(\eta(\varphi(\rho)))^2}{(1-\overline{\varphi(\rho)}z)^{\alpha+2}}$$

$$+B_3(\alpha)\frac{(\eta(\varphi(\rho)))^3}{(1-\overline{\varphi(\rho)}z)^{\alpha+3}} + B_4(\alpha)\frac{(\eta(\varphi(\rho)))^4}{(1-\overline{\varphi(\rho)}z)^{\alpha+4}}\Bigg)\overline{\varphi(\rho)}.$$

where $B_1(\alpha) = \alpha A_1(\alpha)$, $B_2(\alpha) = (\alpha+1)A_2(\alpha)$, $B_3(\alpha) = (\alpha+2)A_3(\alpha)$ and $B_4(\alpha) = (\alpha+3)A_4(\alpha)$. Since $\eta(\varphi(\rho))/|1-\overline{\varphi(\rho)}z| \lesssim 1$, so we have that

$$|f'_{\varphi(\rho)}(z)| \lesssim \frac{\eta(\varphi(\rho))}{(1-\overline{\varphi(\rho)}z)^{\alpha+1}}.$$

Thus $f_{\varphi(\rho)} \in \mathscr{B}^{\alpha}$ and moreover $\|f_{\varphi(\rho)}\|_{\mathscr{B}^{\alpha}} \lesssim 1$. Also

$$f''_{\varphi(\rho)}(z) = \Bigg(C_1(\alpha)\frac{\eta(\varphi(\rho))}{(1-\overline{\varphi(\rho)}z)^{\alpha+2}} + C_2(\alpha)\frac{(\eta(\varphi(\rho)))^2}{(1-\overline{\varphi(\rho)}z)^{\alpha+3}}$$

$$+C_3(\alpha)\frac{(\eta(\varphi(\rho)))^3}{(1-\overline{\varphi(\rho)}z)^{\alpha+4}} + C_4(\alpha)\frac{(\eta(\varphi(\rho)))^4}{(1-\overline{\varphi(\rho)}z)^{\alpha+5}}\Bigg)(\overline{\varphi(\rho)})^2,$$

where $C_1(\alpha) = (\alpha+1)B_1(\alpha)$, $C_2(\alpha) = (\alpha+2)B_2(\alpha)$, $C_3(\alpha) = (\alpha+3)B_3(\alpha)$ and $C_4(\alpha) = (\alpha+4)B_4(\alpha)$, and

$$f'''_{\varphi(\rho)}(z) = \left(D_1(\alpha)\frac{\eta(\varphi(\rho))}{(1-\overline{\varphi(\rho)}z)^{\alpha+3}} + D_2(\alpha)\frac{(\eta(\varphi(\rho)))^2}{(1-\overline{\varphi(\rho)}z)^{\alpha+4}} \right.$$

$$\left. + D_3(\alpha)\frac{(\eta(\varphi(\rho)))^3}{(1-\overline{\varphi(\rho)}z)^{\alpha+5}} + D_4(\alpha)\frac{(\eta(\varphi(\rho)))^4}{(1-\overline{\varphi(\rho)}z)^{\alpha+6}} \right)(\overline{\varphi(\rho)})^3,$$

where $D_1(\alpha) = (\alpha+2)C_1(\alpha)$, $D_2(\alpha) = (\alpha+3)C_2(\alpha)$, $D_3(\alpha) = (\alpha+4)C_3(\alpha)$ and $D_4(\alpha) = (\alpha+5)C_4(\alpha)$. Therefore, we see that

$$f_{\varphi(\rho)}(\varphi(\rho)) = 0,\ f'_{\varphi(\rho)}(\varphi(\rho)) = 0,\ f''_{\varphi(\rho)}(\varphi(\rho)) = 0, \tag{6}$$

and

$$f'''_{\varphi(\rho)}(\varphi(\rho)) = E(\alpha)\frac{(\overline{\varphi(\rho)})^3}{(\eta(\varphi(\rho)))^{\alpha+2}}, \tag{7}$$

where $E(\alpha) = -(18\alpha^2 + 84\alpha + 108)$. Using (6) and (7), we obtain that

$$\begin{aligned}
\|SW_{\varphi,\ \psi}\|_{\mathscr{B}^\alpha \to \mathscr{B}^\beta} \gtrsim & \|SW_{\varphi,\ \psi} f_{\varphi(\rho)}\|_{\mathscr{B}^\beta} \\
\geq & (\eta(\rho))^\beta |\psi''(\rho)||f'(\varphi(\rho))| \\
& + (\eta(\rho))^\beta |2\psi'(\rho)\varphi'(\rho) + \psi(\rho)\varphi''(\rho)||f''(\varphi(\rho))| \\
& + (1-|\rho|^2)^\beta |\psi(\rho)(\varphi'(\rho))^2||f'''(\varphi(\rho))| \\
\gtrsim & \frac{(\eta(\rho))^\beta |\psi(\rho)(\varphi'(\rho))^2|}{(\eta(\varphi(\rho)))^{\alpha+2}} |\varphi(\rho)|^3.
\end{aligned} \tag{8}$$

Thus for fixed $\delta \in (0,\ 1)$, from (8) we obtain that

$$\sup_{|\varphi(\rho)|>\delta} \frac{(\eta(\rho))^\beta |\psi(\rho)(\varphi'(\rho))^2|}{(\eta(\varphi(\rho)))^{\alpha+2}} \lesssim \|SW_{\varphi,\ \psi}\|_{\mathscr{B}^\alpha \to \mathscr{B}^\beta}. \tag{9}$$

By employing equation (5), we can ascertain that

$$\sup_{|\varphi(\rho)|\le\delta}\frac{(\eta(\rho))^{\beta}|\psi(\rho)(\varphi'(\rho))^2|}{(\eta(\varphi(\rho)))^{\alpha+2}}\lesssim\frac{1}{(\eta(\delta))^{\alpha+2}}\|SW_{\varphi,\ \psi}\|_{\mathscr{B}^{\alpha}\to\mathscr{B}^{\beta}}. \tag{10}$$

Hence from (9) and (10), we have that

$$L_3\lesssim\|SW_{\varphi,\ \psi}\|_{\mathscr{B}^{\alpha}\to\mathscr{B}^{\beta}}. \tag{11}$$

Again consider the function

$$g_{\varphi(\rho)}(z)=F_1(\alpha)\frac{\eta(\varphi(\rho))}{(1-\overline{\varphi(\rho)}z)^{\alpha}}+F_2(\alpha)\frac{(\eta(\varphi(\rho)))^2}{(1-\overline{\varphi(\rho)}z)^{\alpha+1}}+F_3(\alpha)\frac{(\eta(\varphi(\rho)))^3}{(1-\overline{\varphi(\rho)}z)^{\alpha+2}}+F_4(\alpha)\frac{(\eta(\varphi(\rho)))^4}{(1-\overline{\varphi(\rho)}z)^{\alpha+3}},$$

where $F_1(\alpha)=-(6\alpha+18)$, $F_2(\alpha)=(18\alpha+48)$, $F_3(\alpha)=-(18\alpha+42)$ and $F_4(\alpha)=(6\alpha+12)$. Proceeding as above we can easily show that $g_{\varphi(\rho)}\in\mathscr{B}^{\alpha}$ and $\|g_{\varphi(\rho)}\|_{\mathscr{B}^{\alpha}}\lesssim 1$. Moreover,

$$g_{\varphi(\rho)}(\varphi(\rho))=0,\ g'_{\varphi(\rho)}(\varphi(\rho))=0,\ f'''_{\varphi(\rho)}(\varphi(\rho))=0,\ \text{and}$$

$$g''_{\varphi(\rho)}(\varphi(\rho))=\frac{-12(\overline{\varphi(\rho)})^2}{(\eta(\varphi(\rho)))^{\alpha+1}}.$$

Using these facts, we obtain that

$$\|SW_{\varphi,\ \psi}\|_{\mathscr{B}^{\alpha}\to\mathscr{B}^{\beta}}\gtrsim\|SW_{\varphi,\ \psi}g_{\varphi(\rho)}\|_{\mathscr{B}^{\beta}}\gtrsim\frac{(\eta(\rho))^{\beta}|2\psi'(\rho)\varphi'(\rho)+\psi(\rho)\varphi''(\rho)|}{(\eta(\varphi(\rho)))^{\alpha+1}}|\varphi(\rho)|^2.$$

Thus for fixed $\delta\in(0,\ 1)$, we have that

$$\sup_{|\varphi(\rho)|>\delta}\frac{(\eta(\rho))^{\beta}|2\psi'(\rho)\varphi'(\rho)+\psi(\rho)\varphi''(\rho)|}{(\eta(\varphi(\rho)))^{\alpha+1}}\lesssim\|SW_{\varphi,\ \psi}\|_{\mathscr{B}^{\alpha}\to\mathscr{B}^{\beta}}. \tag{12}$$

By using (4), we have that

$$\sup_{|\varphi(\rho)|\le\delta}\frac{(\eta(\rho))^{\beta}|2\psi'(\rho)\varphi'(\rho)+\psi(\rho)\varphi''(\rho)|}{(\eta(\varphi(\rho)))^{\alpha+1}}\lesssim\frac{1}{(\eta(\delta))^{\alpha+1}}\|SW_{\varphi,\ \psi}\|_{\mathscr{B}^{\alpha}\to\mathscr{B}^{\beta}}. \tag{13}$$

Combining (12) and (13), we have that

$$L_2\lesssim\|SW_{\varphi,\ \psi}\|_{\mathscr{B}^{\alpha}\to\mathscr{B}^{\beta}}. \tag{14}$$

Finally, consider the function

$$h_{\varphi(\rho)}(z) = G_1(\alpha)\frac{\eta(\varphi(\rho))}{(1-\overline{\varphi(\rho)}z)^{\alpha}} + G_2(\alpha)\frac{(\eta(\varphi(\rho)))^2}{(1-\overline{\varphi(\rho)}z)^{\alpha+1}}$$

$$+ G_3(\alpha)\frac{(\eta(\varphi(\rho)))^3}{(1-\overline{\varphi(\rho)}z)^{\alpha+2}} + G_4(\alpha)\frac{(\eta(\varphi(\rho)))^4}{(1-\overline{\varphi(\rho)}z)^{\alpha+3}},$$

where $G_1(\alpha) = -(\alpha+2)(\alpha+3)/(\alpha+1)$, $G_2(\alpha) = (\alpha+3)(3\alpha+4)/(\alpha+1)$, $G_3(\alpha) = -(3\alpha+8)$ and $G_4(\alpha) = \alpha+2$. Proceeding as above we can easily show that $h_{\varphi(\rho)} \in \mathscr{B}^{\alpha}$ and $\|h_{\varphi(\rho)}\|_{\mathscr{B}^{\alpha}} \lesssim 1$. Moreover,

$$h_{\varphi(\rho)}(\varphi(\rho)) = 0,\ h''_{\varphi(\rho)}(\varphi(\rho)) = 0,\ h'''_{\varphi(\rho)}(\varphi(\rho)) = 0, \text{ and}$$

$$h'_{\varphi(\rho)}(\varphi(\rho)) = \frac{2}{\alpha+1}\frac{\varphi'(\rho)}{(1-|\varphi(\rho)|^2)^{\alpha}}.$$

Using these facts, we obtain that

$$\|SW_{\varphi,\,\psi}\|_{\mathscr{B}^{\alpha}\to\mathscr{B}^{\beta}} \gtrsim \|SW_{\varphi,\,\psi}h_{\varphi(\rho)}\|_{\mathscr{B}^{\beta}} \gtrsim \frac{(\eta(\rho))^{\beta}|\psi''(\rho)|}{(\eta(\varphi(\rho)))^{\alpha}}|\varphi(\rho)|.$$

Thus we have that

$$\sup_{|\varphi(\rho)|>\delta}\frac{(\eta(\rho))^{\beta}|\psi''(\rho)|}{(\eta(\varphi(\rho)))^{\alpha}} \lesssim \|SW_{\varphi,\,\psi}\|_{\mathscr{B}^{\alpha}\to\mathscr{B}^{\beta}}. \tag{15}$$

By using (3), we obtain that

$$\sup_{|\varphi(\rho)|\le\delta}\frac{(\eta(\rho))^{\beta}|\psi''(\rho)|}{(\eta(\varphi(\rho)))^{\alpha}} \lesssim \frac{1}{(1-|\delta|^2)^{\alpha}}\|SW_{\varphi,\,\psi}\|_{\mathscr{B}^{\alpha}\to\mathscr{B}^{\beta}}. \tag{16}$$

Combining (15) and (16), we have that

$$L_1 \lesssim \|SW_{\varphi,\,\psi}\|_{\mathscr{B}^{\alpha}\to\mathscr{B}^{\beta}}. \tag{17}$$

Combining (11), (14) and (17), we see that

$$L_1 + L_2 + L_3 \lesssim \|SW_{\varphi,\,\psi}\|_{\mathscr{B}^{\alpha}\to\mathscr{B}^{\beta}}. \tag{18}$$

Therefore from (2) and (18), we see that (1) holds.

By employing Lemma 1 with $\nu(z) = (\eta(z))^\alpha$ and $\mu(z) = (\eta(z))^\beta$, along with Lemma 2 and the subsequent inequalities,

$$L_1 \asymp \sup_{n\geq 0} \frac{(n+1)^\alpha \|\psi''\varphi^n\|_{H^\infty_{\nu_\beta}}}{(n+1)^\alpha \|z^n\|_{H^\infty_{\nu_\alpha}}} \asymp \sup_{n\geq 0}(n+1)^{\alpha+1}\|\psi''\varphi^n\|_{H^\infty_{\nu_\beta}} < \infty,$$

$$L_2 \asymp \sup_{n\geq 0} \frac{(n+1)^{\alpha+1} \|(2\psi'\varphi' + \psi\varphi'')\varphi^n\|_{H^\infty_{\nu_\beta}}}{(n+1)^{\alpha+1} \|z^n\|_{H^\infty_{\nu_{\alpha+1}}}} \asymp \sup_{n\geq 0}(n+1)^{\alpha+1}\|(2\psi'\varphi' + \psi\varphi'')\varphi^n\|_{H^\infty_{\nu_\beta}} < \infty, \text{ and}$$

$$L_3 \asymp \sup_{n\geq 0} \frac{(n+1)^{\alpha+2} \|(\psi(\varphi')^2)\varphi^n\|_{H^\infty_{\nu_\beta}}}{(n+1)^{\alpha+2} \|z^n\|_{H^\infty_{\nu_{\alpha+2}}}} \asymp \sup_{n\geq 0}(n+1)^{\alpha+2}\|(\psi(\varphi')^2)\varphi^n\|_{H^\infty_{\nu_\beta}} < \infty,$$

it is established that the conditions labeled as (a), (b), and (c) are satisfied, thereby completing the proof.

Theorem 2 Suppose that $SW_{\varphi,\,\psi} : \mathscr{B}^\alpha \to \mathscr{B}^\beta$ is bounded. Then the following conditions are equivalent:

(a) $SW_{\varphi,\,\psi} : \mathscr{B}^\alpha \to \mathscr{B}^\beta$ is compact;

(b) The following conditions hold:

$$\lim_{|\varphi(z)|\to 1} \frac{(\eta(z))^\beta |\psi''(z)|}{(\eta(\varphi(z)))^\alpha} = 0.$$

$$\lim_{|\varphi(z)|\to 1} \frac{(\eta(z))^\beta}{(\eta(\varphi(z)))^{\alpha+1}} |2\psi'(z)\varphi'(z) + \psi(z)\varphi''(z)| = 0.$$

$$\lim_{|\varphi(z)|\to 1} \frac{(\eta(z))^\beta}{(\eta(\varphi(z)))^{\alpha+2}} |\psi(z)(\varphi'(z))^2| = 0.$$

(c) φ and ψ satisfy the following conditions:

$$\lim_{n\to\infty}(n+1)^\alpha \|\psi''\varphi^n\|_{H^\infty_{\nu_\beta}} = 0.$$

$$\lim_{n\to\infty}(n+1)^{\alpha+1} \|(2\psi'\varphi' + \psi\varphi'')\varphi^n\|_{H^\infty_{\nu_\beta}} = 0.$$

$$\lim_{n\to\infty}(n+1)^{\alpha+2} \|(\psi_2(\varphi')^2)\varphi^n\|_{H^\infty_{\nu_\beta}} = 0.$$

Proof. By taking $\nu(z) = (\eta(z))^\alpha$ and $\mu(z) = (\eta(z))^\beta$, using Lemma 1 and Lemma 2, we can easily show that $(b) \Leftrightarrow (c)$. $(b) \Rightarrow (a)$: Suppose that conditions in (b) hold. Let $\{f_m\}$ be a bounded sequence within the space $\mathscr{B}^\alpha$, constrained by the condition $\sup_m \|f_m\|_{\mathscr{B}^\alpha} \leq 1$ and converging to zero uniformly on compact subsets of $\mathbb{D}$. By conditions in (b), it follows that for any specified $\varepsilon > 0$, there is $\delta > 0$ such that

$$\max\left\{\frac{(\eta(z))^{\beta}|\psi''(z)|}{(\eta(\varphi(z)))^{\alpha}},\ \frac{(\eta(z))^{\beta}}{(\eta(\varphi(z)))^{\alpha+1}}|2\psi'(z)\varphi'(z)+\psi(z)\varphi''(z)|,\ \frac{(\eta(z))^{\beta}}{(\eta(\varphi(z)))^{\alpha+2}}|\psi(z)(\varphi'(z))^{2}|\right\}<\varepsilon, \tag{19}$$

whenever $\delta<|\varphi(z)|<1$ for some $\delta\in(0,\ 1)$. Let $K=\{z\in\mathbb{D}:|z|\leq\delta\}$. It is evident that K constitutes a compact subset of the open unit disk $\mathbb{D}$. Consequently, we have

$$\begin{aligned}
\|SW_{\varphi,\ \psi}f_m\|_{\mathscr{B}^{\beta}} =&|\psi'(0)f_m'(\varphi(0))+\psi(0)\varphi'(0)f_m''(\varphi(0))|+\sup_{\rho\in\mathbb{D}}(\eta(z))^{\beta}|(SW_{\varphi,\ \psi})'f_m(\rho)|\\
\leq&|\psi'(0)||f_m'(\varphi(0))|+|\psi(0)\varphi'(0)||f_m''(\varphi(0))|\\
&+\left(\sup_{\rho\in\mathbb{D}:\varphi(\rho)\in K}+\sup_{\rho\in\mathbb{D}:\delta<|\varphi(\rho)|<1}\right)(\eta(z))^{\beta}|\psi''(\rho)||f_m'(\varphi(\rho))|\\
&+\left(\sup_{\rho\in\mathbb{D}:\varphi(\rho)\in K}+\sup_{\rho\in\mathbb{D}:\delta<|\varphi(\rho)|<1}\right)(\eta(z))^{\beta}|2\psi'(\rho)\varphi'(\rho)+\psi(\rho)\varphi''(\rho)||f_m''(\varphi(\rho))|\\
&+\left(\sup_{\rho\in\mathbb{D}:\varphi(\rho)\in K}+\sup_{\rho\in\mathbb{D}:\delta<|\varphi(\rho)|<1}\right)(\eta(z))^{\beta}|\psi(\rho)(\varphi'(\rho))^{2}||f_m'''(\varphi(\rho))|\\
\lesssim&|\psi'(0)||f_m'(\varphi(0))|+|\psi(0)\varphi'(0)||f_m''(\varphi(0))|+Q_1\sup_{z\in K}|f_m'(z)|+Q_2\sup_{z\in K}|f_m''(z)|+Q_3\sup_{z\in K}|f_m(z)|\\
&+\left(\frac{(\eta(z))^{\beta}|\psi''(z)|}{(\eta(\varphi(z)))^{\alpha}}+\frac{(\eta(z))^{\beta}}{(\eta(\varphi(z)))^{\alpha+1}}|2\psi'(z)\varphi'(z)+\psi(z)\varphi''(z)|\right.\\
&\left.+\frac{(\eta(z))^{\beta}}{(\eta(\varphi(z)))^{\alpha+2}}|\psi(z)(\varphi'(z))^{2}|\right)\|f_m\|_{\mathscr{B}^{\alpha}},
\end{aligned} \tag{20}$$

where we have used the fact that $Q_1=\sup_{z\in\mathbb{D}}(\eta(z))^{\beta}|\psi''(z)|$, $Q_2=\sup_{z\in\mathbb{D}}(\eta(z))^{\beta}|2\psi'(z)\varphi'(z)+\psi(z)\varphi''(z)|$, and $Q_3=\sup_{z\in\mathbb{D}}(\eta(z))^{\beta}|\psi(z)(\varphi'(z))^{2}|$ are finite. Utilizing (19) alongside the conditions $|f_m'(\varphi(0))|<\varepsilon$, $|f_m''(\varphi(0))|<\varepsilon$, $\sup_{z\in K}|f_m'(z)|<\varepsilon$, $\sup_{z\in K}|f_m''(z)|<\varepsilon$, and $\sup_{z\in K}|f_m'''(z)|<\varepsilon$ for all $m\geq N_0$ in (20), we conclude that $\|SW_{\varphi,\ \psi}f_m\|_{\mathscr{B}^{\beta}}<\varepsilon$ for $m\geq N_0$. Since $\varepsilon>0$ is arbitrary, it follows that $\|SW_{\varphi,\ \psi}f_m\|_{\mathscr{B}^{\beta}}\to 0$ as $m\to\infty$. Consequently, by invoking Lemma 4, we assert that the operator $SW_{\varphi,\ \psi}:\mathscr{B}^{\alpha}\to\mathscr{B}^{\beta}$ is compact.

$(a)\Rightarrow(b)$: If $SW_{\varphi,\ \psi}:\mathscr{B}^{\alpha}\to\mathscr{B}^{\beta}$ is compact, we consider a sequence $\{\rho_m\}_{m\in\mathbb{N}}$ in $\mathbb{D}$ such that $|\varphi(\rho_m)|\to 1$ as $m\to\infty$. If such a sequence does not exist, then conditions in (b) are trivially satisfied.

Let $f_{\varphi(\rho_m)}$, $g_{\varphi(\rho_m)}$ and $h_{\varphi(\rho_m)}$ be defined as in Theorem 1. According to the aforementioned theorem, these functions lie within the space $\mathscr{B}^{\alpha}$ and it holds that $\sup_m\|K_m\|_{\mathscr{B}^{\alpha}}\leq 1$, where K_m represents $f_{\varphi(\rho_m)}$, $g_{\varphi(\rho_m)}$ or $h_{\varphi(\rho_m)}$. Furthermore, K_m converges uniformly to zero on compact subsets of $\mathbb{D}$. Following the arguments in Theorem 1, it can be established that

$$\|SW_{\varphi,\ \psi}h_{\varphi(\rho_m)}\|_{\mathscr{B}^\beta} \gtrsim \frac{(\eta(\rho_m))^\beta|\psi''(\rho_m)|}{(\eta(\varphi(\rho_m)))^\alpha}|\varphi(\rho_m)|$$

$$\|SW_{\varphi,\ \psi}g_{\varphi(\rho_m)}\|_{\mathscr{B}^\beta} \gtrsim \frac{(\eta(\rho_m))^\beta}{(\eta(\varphi(\rho_m)))^{\alpha+1}}|2\psi'(\rho_m)\varphi'(\rho_m)+\psi(\rho_m)\varphi''(\rho_m)||\varphi(\rho_m)|^2, \text{ and}$$

$$\|SW_{\varphi,\ \psi}f_{\varphi(\rho_m)}\|_{\mathscr{B}^\beta} \gtrsim \frac{(\eta(\rho_m))^\beta}{(\eta(\varphi(\rho_m)))^{\alpha+2}}|\psi(\rho_m)(\varphi'(\rho_m))^2||\varphi(\rho_m)|^3.$$

Since the operator $SW_{\varphi,\ \psi}$ maps $\mathscr{B}^\alpha$ to $\mathscr{B}^\beta$ and is compact, we conclude that $\|SW_{\varphi,\ \psi}f_m\|_{\mathscr{B}^\beta}$ $\|SW_{\varphi,\ \psi}g_m\|_{\mathscr{B}^\beta}$ and $\|SW_{\varphi,\ \psi}f_m\|_{\mathscr{B}^\beta}$ converge to zero as $m\to\infty$. Therefore, the conditions in part (b) are addressed, thereby completing the proof. □

4. Essential norm of $SW_{\varphi,\psi}$: $\mathscr{B}^\alpha \to \mathscr{B}^\beta$

In this section, we will estimate the essential norm of the operator $SW_{\varphi,\ \psi}:\mathscr{B}^\alpha\to\mathscr{B}^\beta$. This estimation is articulated in terms of the functions $\psi_1,\ \psi_2,\ \varphi$ and their derivatives.

Theorem 3 Let $SW_{\varphi,\ \psi}:\mathscr{B}^\alpha\to\mathscr{B}^\beta$ is bounded. Then

$$\begin{aligned}\|SW_{\varphi,\ \psi}\|_{e,\ \mathscr{B}^\alpha\to\mathscr{B}^\beta} \asymp \max\Bigg\{ &\limsup_{|\varphi(z)|\to 1}\frac{(\eta(z))^\beta|\psi''(z)|}{(\eta(\varphi(z)))^\alpha},\\ &\limsup_{|\varphi(z)|\to 1}\frac{(\eta(z))^\beta|2\psi'(z)\varphi'(z)+\psi(z)\varphi''(z)|}{(\eta(\varphi(z)))^{\alpha+1}},\\ &\limsup_{|\varphi(z)|\to 1}\frac{(\eta(z))^\beta|\psi_2(z)(\varphi'(z))^2|}{(\eta(\varphi(z)))^{\alpha+2}}\Bigg\}\\ \asymp \max\Bigg\{ &\limsup_{n\to\infty}(n+1)^\alpha\|\psi''\varphi^n\|_{H^\infty_{v_\beta}},\\ &\limsup_{n\to\infty}(n+1)^{\alpha+1}\|(2\psi'\varphi'+\psi\varphi'')\varphi^n\|_{H^\infty_{v_\beta}},\\ &\limsup_{n\to\infty}(n+1)^{\alpha+2}\|(\psi(\varphi')^2)\varphi^n\|_{H^\infty_{v_\beta}}\Bigg\}.\end{aligned}$$

Proof. Using the identity

$$(SW_{\varphi,\ \psi}f)' = W_{\varphi,\ \psi''}f' + W_{\varphi,\ 2\psi'\varphi'+\psi\varphi''}f'' + W_{\varphi,\ \psi(\varphi')^2}f''',$$

we have that

$$\|SW_{\varphi,\ \psi}\|_{e,\ \mathscr{B}^{\alpha}\to\mathscr{B}^{\beta}} \lesssim \|W_{\varphi,\ \psi''}\|_{e,\ H^{\infty}_{v_{\alpha}}\to\mathscr{B}^{\beta}} + \|W_{\varphi,\ 2\psi'\varphi'+\psi\varphi''}\|_{e,\ H^{\infty}_{v_{\alpha+1}}\to\mathscr{B}^{\beta}}$$

$$+ \|W_{\varphi,\ \psi(\varphi')^2}\|_{e,\ H^{\infty}_{v_{\alpha+2}}\to\mathscr{B}^{\beta}}.$$

By Lemma 1 and Lemma 2, it follows that

$$\|W_{\varphi,\ \psi''}\|_{e,\ H^{\infty}_{v_{\alpha}}\to H^{\infty}_{v_{\beta}}} \asymp \limsup_{|\varphi(z)|\to 1} \frac{(\eta(z))^{\beta}|\psi_1''(z)|}{(\eta(\varphi(z)))^{\alpha}}$$

$$\asymp \limsup_{n\to\infty} \frac{\|\psi_1''\varphi^n\|_{H^{\infty}_{v_{\beta}}}}{\|z^n\|_{H^{\infty}_{v_{\alpha}}}}$$

$$\asymp \limsup_{n\to\infty}(n+1)^{\alpha}\|\psi_1''\varphi^n\|_{H^{\infty}_{v_{\beta}}},$$

$$\|W_{\varphi,\ 2\psi'\varphi'+\psi\varphi''}\|_{e,\ H^{\infty}_{v_{\alpha+1}}\to H^{\infty}_{v_{\beta}}} \asymp \limsup_{|\varphi(z)|\to 1} \frac{(\eta(z))^{\beta}|2\psi'(z)\varphi'(z)+\psi(z)\varphi''(z)|}{(\eta(\varphi(z)))^{\alpha+1}}$$

$$\asymp \limsup_{n\to\infty} \frac{\|(2\psi'\varphi'+\psi\varphi'')\varphi^n\|_{H^{\infty}_{v_{\beta}}}}{\|z^n\|_{H^{\infty}_{v_{\alpha}}}}$$

$$\asymp \limsup_{n\to\infty}(n+1)^{\alpha}\|(2\psi'\varphi'+\psi\varphi'')\varphi^n\|_{H^{\infty}_{v_{\beta}}}, \text{ and}$$

$$\|W_{\varphi,\ \psi(\varphi')^2}\|_{e,\ H^{\infty}_{v_{\alpha+2}}\to H^{\infty}_{v_{\beta}}} \asymp \limsup_{|\varphi(z)|\to 1} \frac{(\eta(z))^{\beta}|\psi_2(z)(\varphi'(z))^2|}{(\eta(\varphi(z))))^{\alpha+2}}$$

$$\asymp \limsup_{n\to\infty} \frac{\|(\psi_2(\varphi')^2)\varphi^n\|_{H^{\infty}_{v_{\beta}}}}{\|z^n\|_{H^{\infty}_{v_{\alpha+2}}}}$$

$$\asymp \limsup_{n\to\infty}(n+1)^{\alpha+2}\|(\psi_2(\varphi')^2)\varphi^n\|_{H^{\infty}_{v_{\beta}}},$$

and so

$$\|SW_{\varphi,\ \psi}\|_{e,\ \mathscr{B}^\alpha\to\mathscr{B}^\beta} \lesssim \max\Bigg\{\limsup_{n\to\infty}(n+1)^{\alpha-1}\|\psi''\varphi^n\|_{H^\infty_{v_\beta}},$$

$$\limsup_{n\to\infty}(n+1)^{\alpha}\|(2\psi'\varphi'+\psi\varphi'')\varphi^n\|_{H^\infty_{v_\beta}},$$

$$\limsup_{n\to\infty}(n+1)^{\alpha+2}\|((\psi_2(\varphi')^2))\varphi^n\|_{H^\infty_{v_\beta}}\Bigg\}.$$

This effectively establishes the upper bound. Assume that $\{z_k\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_k)|\to 1$ as $k\to\infty$. We define the functions $f_{\varphi(z_m)}$, $g_{\varphi(z_m)}$ and $h_{\varphi(z_m)}$ as per Theorem 2. Then as in Theorem 2, these functions converge uniformly to zero on compact subsets of $\mathbb{D}$. Fixing $s\in(0,\ 1)$, we consider $\varphi_s(z)=s\varphi(z)$, and by Theorem 2 $SW_{\varphi_s,\ \psi}:\mathscr{B}^\alpha\to\mathscr{B}^\beta$ is compact. Consequently, by Lemma 4 we have that $\|SW_{\varphi_s,\ \psi}f_{\varphi(z_m)}\|_{\mathscr{B}^\beta}\to 0$, $\|SW_{\varphi_s,\ \psi}g_{\varphi(z_m)}\|_{\mathscr{B}^\beta}\to 0$ and $\|SW_{\varphi_s,\ \psi}h_{\varphi(z_m)}\|_{\mathscr{B}^\beta}\to 0$ as $m\to\infty$. Thus

$$\begin{aligned}\|SW_{\varphi,\ \psi}-SW_{\varphi_s,\ \psi}\|_{\mathscr{B}^\alpha\to\mathscr{B}^\beta} &\gtrsim \limsup_{m\to\infty}\|SW_{\varphi,\ \psi}f_m\|_{\mathscr{B}^\beta}\\ &\gtrsim \limsup_{m\to\infty}\frac{(\eta(\rho_m))^\beta}{(\eta(\varphi(\rho_m)))^{\alpha+2}}|\psi(z)(\varphi'(z))^2||\varphi(\rho_m)|^2\\ &\asymp \limsup_{n\to\infty}(n+1)^{\alpha+2}\|(\psi(\varphi')^2)\varphi^n\|_{H^\infty_{v_\beta}},\end{aligned}$$

$$\begin{aligned}\|SW_{\varphi,\ \psi}-SW_{\varphi_s,\ \psi}\|_{\mathscr{B}^\alpha\to\mathscr{B}^\beta} &\gtrsim \limsup_{m\to\infty}\|SW_{\varphi,\ \psi}g_m\|_{\mathscr{B}^\beta}\\ &\gtrsim \limsup_{m\to\infty}\frac{(\eta(\rho_m))^\beta}{(\eta(\varphi(\rho_m)))^{\alpha+1}}|2\psi'(z)\varphi'(z)+\psi(z)\varphi''(z)||\varphi(\rho_m)|^2\\ &\asymp \limsup_{n\to\infty}(n+1)^{\alpha+1}\|(2\psi'\varphi'+\psi\varphi'')\varphi^n\|_{H^\infty_{v_\beta}}, \text{ and}\end{aligned}$$

$$\begin{aligned}\|SW_{\varphi,\ \psi}-SW_{\varphi_s,\ \psi}\|_{\mathscr{B}^\alpha\to\mathscr{B}^\beta} &\gtrsim \limsup_{m\to\infty}\|SW_{\varphi,\ \psi}h_m\|_{\mathscr{B}^\beta}\\ &\gtrsim \limsup_{m\to\infty}\frac{(\eta(\rho_m))^\beta|\psi''(z_m)|}{(\eta(\varphi(z_m)))^{\alpha}}|\varphi(z_m)|\\ &\asymp \limsup_{n\to\infty}(n+1)^{\alpha}\|\psi''\varphi^n\|_{H^\infty_{v_\beta}}.\end{aligned}$$

From the aforementioned three inequalities, we derive the lower estimate, thereby concluding the proof. □

5. Conclusions

In the realm of functional analysis, the exploration of operators and their properties is paramount for advancing the understanding of various function spaces. This paper aims to derive explicit bounds for the essential norm of the sandwich weighted composition operator $SW_{\varphi,\ \psi}$, particularly in the context of Bloch type spaces. The Bloch spaces, which consist of functions that exhibit a certain degree of smoothness, provide a natural setting to study the properties of such operators. Establishing bounds for this norm and essential norm is instrumental, as it not only elucidates the operator's characteristics but also contributes to the broader discourse on composition operators and their interactions with differential operations. Furthermore, the results obtained may serve as a foundation for future inquiries into more complex scenarios, including higher-dimensional function spaces and non-linear operator dynamics.

Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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