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Research Article

Interval Boundary Correction for Interval Linear Program

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Abstract: An interval linear programming problem is a bounded problem when all of its deterministic problems are bounded. However, if the best deterministic problem is unbounded or the worst deterministic problem is infeasible, a pre-processing step on tightening interval parameter boundaries should be employed to obtain an adjusted bounded interval linear program. With the help of the duality theorem, we propose an interval boundary correction algorithm designed to minimize the loss of interval information. The algorithm makes relatively small adjustments with respect to the interval width of each interval parameter boundary. Some numerical examples illustrating this correction algorithm are provided.

***Keywords*:** interval linear programming, infeasible systems, interval boundary correction, duality theorem

MSC: 90C70, 90C31, 90C46

1. Introduction

In an interval linear programming problem, interval parameters could appear at three positions: the interval objective coefficient, the interval coefficient matrix, and the interval right-hand side. These interval parameters represent data that arise from an uncertain information. An optimal solution to an interval linear program, as found in the literature, can typically be represented in two distinct forms: either as an interval vector solution or as a real vector solution. The choice between these representations often depends on the preferences and aims of the decision maker or the specific application context. When a range of possible outcomes is preferred, the methods reported in [1–7] offer various optimal interval vector solutions, capturing different deterministic cases within the interval linear program. In contrast, examples of optimal real vector solutions can be found in [8–10], which explore optimistic solutions for interval linear programs under different semantics. The classification of all semantics, including combinations of tolerance, control, left-localized, and right-localized, is thoroughly documented in [11–20].

When attempting to obtain an optimal interval vector solution to the problem, the corresponding interval linear program should be bounded, meaning that any deterministic linear program generated by selecting any value from each interval parameter must yield an optimal solution. Nevertheless, several methods in the literature directly assume the boundedness of each deterministic problem without addressing this aspect explicitly. Tong [21] introduced the Best and Worst Case (BWC) method in 1994, which obtains an optimal interval solution for an interval linear program by using the optimal solutions of the deterministic best and worst scenarios. Another notable method is the Two-Step Method (TSM)

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proposed by Guo et al. in 1995 [22]. This method employs the concept of gray integer programming to formulate two deterministic problems instead of best and worst models. Numerous other methods have been developed to improve the optimal interval solution creating their own two deterministic problems with specific constraints. Prominent examples include IILP [1], ITSM [6], MILP [7], IMILP [1], SOM-2 [4], ISOM-2 [5], ThSM-I [3], ThSM-II [3], and RTSM [2]. The most recent review of these methods can be found in [5].

All methods mentioned above provide an optimal interval vector solution only when the two deterministic problems are bounded, while the boundedness of each deterministic problem of an interval linear program: min $[\underline{c},\, \overline{c}]^{\mathrm{T}}x$ subject to $[\underline{A},\, \overline{A}]x \geq [\underline{b},\, \overline{b}]$, $x \geq \vec{0}$ depends on the worst deterministic problem: $\min_{x\in\Omega_1}\, \overline{c}^{\mathrm{T}}x$ and the best deterministic problem: $\min_{x\in\Omega_2}\, \underline{c}^{\mathrm{T}}x$, where $\Omega_1 = \left\{x \mid \underline{A}x \geq \overline{b},\, x \geq 0\right\}$ and $\Omega_2 = \left\{x \mid \overline{A}x \geq \underline{b},\, x \geq 0\right\}$. The set Ω_2 is a super set of Ω_1, since $\overline{A}x \geq \underline{A}x \geq \overline{b} \geq \underline{b}$. Therefore, we will obtain the best and worst case problems by using Ω_2 and Ω_1 as their feasible regions, respectively. However, these sets Ω_1 and Ω_2 could be empty or unbounded, potentially leading to infeasible or unbounded linear programs. Consequently, without interval boundary correction to ensure the boundedness of these two deterministic problems, existing methods may fail to provide a real value optimal vector of an interval linear program. Hence, a pre-processing step is essential to adjust the interval data and ensure the boundedness of the interval linear program.

A linear program is unbounded then its dual problem will be infeasible. Therefore, our pre-processing method needs to be able to prevent infeasibility in the infeasible worst deterministic problem and infeasibility in the dual of the unbounded best deterministic problem, in order to obtain the boundedness of the interval linear program. According to the comprehensive review by Chinneck [23], there are three primary approaches to detect and/or repair the infeasibility of a linear programming problem. The first approach is to identify an Irreducible Infeasible Subset (IIS). Which is the smallest subset of constraints that causes the problem to be infeasible. If any constraint is removed from this set, the problem becomes feasible. Practical methods for identifying IISs in linear programs were first developed by [24]. The second approach is to find a Maximum Feasible Subset (MAX FS). This approach focuses on finding the largest feasible subset of constraints, minimizing the number of constraints that need to be removed to restore feasibility. Research on this topic includes Amaldi et al. [25] and Chinneck [26]. It is worth noting that IIS and MAX FS are not necessarily complements of each other. A detailed review of methods for identifying IISs and MAX FS is available in [23]. The last approach is to minimize the adjustments of constraints. This approach involves determining the smallest adjustments required to make the constraints feasible. The 'smallest adjustment' could be defined using various norms, such as the l_1 norm: $\|D\|_{l_1} = \sum_{ij}\left|d_{ij}\right|$, the l_∞ norm: $\|D\|_{l_\infty} = \max_{ij}\left|d_{ij}\right|$, the matrix-induced infinity norm: $\|D\|_\infty = \max_i \sum_j \left|d_{ij}\right|$, and the Frobenius norm: $\|D\|_F^2 = \sum_{i=1}^m \sum_{j=1}^n \left|d_{ij}\right|^2$. The optimal adjustment of constraint coefficient matrix and the right-hand side under these norms requires solving a finite number of linear programming problems [27].

Under the restriction that the corrected or adjusted interval parameters must remain a subset of the original intervals, and with the aim of preserving as much of the original interval information as possible, these three approaches for addressing infeasibility are not applicable, as they may concern only a few infeasible constraints while leaving the rest unchanged or provide the new intervals that extend beyond the original bounds. Instead, we propose distributing the necessary corrections across all constraints to transform the original infeasible linear programming problem into a feasible one.

The outline of the paper is as follows. This work builds on the foundational background discussed in Section 2. In Section 3, we introduce a linear programming model that incorporates a penalty for infeasibility or elastic terms into an infeasible linear program. We prove the boundedness of the problem using the linear duality theorem and identify a threshold penalty cost, μ_T, which guarantees that the linear programming problem yields the same optimal solution for any penalty cost μ, where $\mu \geq \mu_T > 0$. The boundary correction algorithm for infeasible worst problem is provided in Section 4. The modification of a linear program with penalty on infeasibility or elastic terms is presented by adding a few more constraints to ensure that the new boundary parameters are in the original ones and the adjustments are proportional to the interval widths. The method can also be adapted for the unbounded best deterministic problem, where the dual problem becomes infeasible. We mention the boundary correction for an unbounded best problem in Section 5. In Section 6, we provide small numerical examples to illustrate the approach, demonstrating the adjusted intervals and showing boundedness for both best-case and worst-case scenarios. The paper concludes with a summary of our findings in the final section.

2. Preliminaries

The background knowledge and involved definitions used in this paper are defined in this section. Denote that $\mathbb{R}$ and $\mathbb{IR}$ are the sets of real and interval real number, respectively.

Let $\mathbf{A} \in \mathbb{IR}^{m\times n}$, $\boldsymbol{b} \in \mathbb{IR}^{m}$, and $\boldsymbol{c} \in \mathbb{IR}^{n}$, where m and n be the given positive integers. An interval linear problem with an interval vector decision variable $x \in \mathbb{R}^n$ is defined as

$$\min\ \boldsymbol{c}^{\mathrm{T}}x$$

$$\text{subject to}: \boldsymbol{A}x \geq \boldsymbol{b} \tag{1}$$

$$x \geq \vec{0}_n,$$

where $\boldsymbol{c} = [\underline{c},\ \overline{c}] = \{c \in \mathbb{R}^n : \underline{c} \leq c \leq \overline{c}\}$, $\boldsymbol{A} = [\underline{A},\ \overline{A}] = \{A \in \mathbb{R}^{m\times n} : \underline{A} \leq A \leq \overline{A}\}$, and $\boldsymbol{b} = [\underline{b},\ \overline{b}] = \{b \in \mathbb{R}^m : \underline{b} \leq b \leq \overline{b}\}$. Each $(i,\ j)^{\text{th}}$ component of an interval matrix $\boldsymbol{A}$ can be written as follows, for all $i \in \{1,\ \ldots,\ m\}$ and $j \in \{1,\ \ldots,\ n\}$.

$$\boldsymbol{A} = [\underline{A},\ \overline{A}] = \begin{bmatrix} [\underline{a}_{11},\ \overline{a}_{11}] & [\underline{a}_{12},\ \overline{a}_{12}] & \cdots & [\underline{a}_{1n},\ \overline{a}_{1n}] \\ [\underline{a}_{21},\ \overline{a}_{21}] & [\underline{a}_{22},\ \overline{a}_{22}] & \cdots & [\underline{a}_{2n},\ \overline{a}_{2n}] \\ \vdots & \vdots & \ddots & \vdots \\ [\underline{a}_{m1},\ \overline{a}_{m1}] & [\underline{a}_{m2},\ \overline{a}_{m2}] & \cdots & [\underline{a}_{mn},\ \overline{a}_{mn}] \end{bmatrix},$$

where

$$\underline{A} = \begin{bmatrix} \underline{a}_{11} & \underline{a}_{12} & \cdots & \underline{a}_{1n} \\ \underline{a}_{21} & \underline{a}_{22} & \cdots & \underline{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \underline{a}_{m1} & \underline{a}_{m2} & \cdots & \underline{a}_{mn} \end{bmatrix} \text{ and } \overline{A} = \begin{bmatrix} \overline{a}_{11} & \overline{a}_{12} & \cdots & \overline{a}_{1n} \\ \overline{a}_{21} & \overline{a}_{22} & \cdots & \overline{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a}_{m1} & \overline{a}_{m2} & \cdots & \overline{a}_{mn} \end{bmatrix}.$$

The interval vectors $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{x}$ can be written in the same manner. The relationship between the left-hand side $\boldsymbol{Ax}$ and the right-hand side $\boldsymbol{b}$ can be analyzed component-wise. Additionally, $\boldsymbol{A}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ can be represented in terms of the midpoint and half-width of each interval component, as follows:

$$\boldsymbol{A} = [\boldsymbol{A}_c - \Delta,\ \boldsymbol{A}_c + \Delta]$$

$$\boldsymbol{b} = [\boldsymbol{b}_c - \delta,\ \boldsymbol{b}_c + \delta]$$

$$\boldsymbol{c} = [\boldsymbol{c}_c - \lambda,\ \boldsymbol{c}_c + \lambda],$$

where $\boldsymbol{A}_c,\ \boldsymbol{b}_c,\ \boldsymbol{c}_c$ represent the midpoint matrix/vectors and $\Delta,\ \delta,\ \lambda$ denote the half-width matrices/vectors of $\boldsymbol{A}$, $\boldsymbol{b}$ and $\boldsymbol{c}$, respectively.

An interval linear program (1) is not well-defined due to the presence of interval-valued data. It indicates that the coefficient matrix is not explicitly known. Instead, each element of the matrix is constrained to lie within a specified interval. Specifically, selecting any $a_{ij} \in [\underline{a}_{ij},\ \overline{a}_{ij}]$, $b_i \in [\underline{b}_i,\ \overline{b}_i]$, and $c_j \in [\underline{c}_j,\ \overline{c}_j]$, results in the deterministic linear program (2). This is referred to as the characteristic problem in [5] and is well-defined as follows:

$$\min \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to: } \sum_{j=1}^{n} a_{ij} x_j \geq b_i,\ i = 1,\ 2,\ \ldots,\ m, \tag{2}$$

$$x_j \geq 0,\ j = 1,\ 2,\ \ldots,\ n.$$

For any deterministic linear program

$$P:\ \min\ c^{\mathrm{T}} x \text{ subject to } Ax \geq b,\ x \geq \vec{0}_n,$$

its corresponding dual problem is defined by

$$D:\ \max\ b^{\mathrm{T}} y \text{ subject to } y^{\mathrm{T}} A \leq c^{\mathrm{T}},\ y \geq \vec{0}_m.$$

The dual of the dual problem reverts to the original primal deterministic problem. The established relationship between a primal problem and its dual is summarized in the following table.

Table 1. Relationship between primal and dual linear programming problems

		Dual		
		Optimal	Infeasible	Unbounded
	Optimal	Yes		
Primal	Infeasible		Yes	Yes
	Unbounded		Yes	

If a problem is bounded, its dual must also be bounded. Conversely, if a problem is unbounded, its dual will be infeasible. However, if a problem is infeasible, its dual may either be infeasible or unbounded, requiring further investigation to ascertain its precise nature.

3. Threshold penalty for infeasible linear programming problem

For any $x \geq 0$ the following inequalities hold:

$$\underline{c}^{\mathrm{T}}x \leq c^{\mathrm{T}}x \leq \overline{c}^{\mathrm{T}}x$$

and

$$\overline{A}x \geq Ax \geq \underline{A}x \geq \overline{b} \geq b \geq \underline{b},$$

where $A \in \left[\underline{A}, \overline{A}\right]$, $b \in \left[\underline{b}, \overline{b}\right]$ and $c \in [\underline{c}, \overline{c}]$. Define Ω_1 and Ω_2 as the smallest and largest feasible regions among all feasible regions of the deterministic problems of the interval linear program (1), respectively:

$$\Omega_1 = \left\{x \in \mathbb{R}^n | \underline{A}x \geq \overline{b},\ x \geq \vec{0}_n\right\},$$

$$\Omega_2 = \left\{x \in \mathbb{R}^n | \overline{A}x \geq \underline{b},\ x \geq \vec{0}_n\right\}.$$

Thus, the worst deterministic linear program for the interval problem (1) is

$$\min_{x \in \Omega_1} \overline{c}^{\mathrm{T}}x,$$

while the best deterministic linear program is:

$$\min_{x \in \Omega_2} \underline{c}^{\mathrm{T}}x.$$

It is not hard to see that the optimality of every deterministic problem depends on the analysis of both the worst and the best deterministic problems. Table 2 outlines that these two models can be categorized into many types depending on the type (infeasible, bounded, or unbounded) of the best deterministic problem.

Table 2. All possible types (infeasible, bounded, or unbounded) of the worst and best deterministic problems of an interval linear program

The best problem	The worst problem
infeasible	infeasible
bounded	infeasible bounded
unbounded	infeasible bounded unbounded

If $\Omega_2 = \emptyset$, Ω_1 must also be empty. Therefore, the feasibility of the best deterministic problem is important. An infeasible best model suggests that the given interval linear program may be inherently flawed. Conversely, if the best model is feasible, it could still be unbounded. Additionally, if the worst model is infeasible while $\Omega_2 \neq \emptyset$, parameter adjustments are necessary to achieve a bounded interval linear program. In situations where both the best and worst problems are unbounded, it becomes possible to identify and address the problematic original interval parameters.

Various methods (see [1–7]) have been proposed to address an interval solution $\boldsymbol{x} = [\underline{x},\ \overline{x}]$ to an interval linear programming problem (1), where x^* in $[\underline{x},\ \overline{x}]$ denotes an optimal solution to a deterministic problem:

$$\min\ c^{\mathrm{T}}x \ \text{ subject to } \ Ax \geq b,\ x \geq 0,$$

for some $A \in \left[\underline{A},\ \overline{A}\right]$, $b \in \left[\underline{b},\ \overline{b}\right]$ and $c \in [\underline{c},\ \overline{c}]$. These methods assume that every deterministic problem within the given interval linear program is optimal. However, there are no reports about a pre-processor for checking boundedness of a given interval linear program. Therefore, the objective of this paper is to transform an infeasible worst deterministic linear program and an unbounded best deterministic linear program into the bounded problems. By refining the interval parameter boundaries, the adjusted interval linear program ensures optimality across all scenarios. It is important to note here that this paper does not focus on identifying an Irreducible Infeasible Subsystem (IIS).

Before addressing both the infeasible worst and unbounded best problems of (1), let us first outline how to transform an infeasible linear program into a feasible one using penalties on infeasibility. Murty et al. [28] proposed measuring the best adjustment of an infeasible linear program to a feasible one by minimizing the total penalty for variable penalties. This approach is equivalent to a method introduced by Chinneck and Dravnieks [24], which they referred to as a version of problem P_μ in Theorem 3 without the $c^T x$ term in the objective function. This method, known as elastic programming, uses elastic variables, denoted by s. However, there has been no rigid proof of the boundedness of the adjusted feasible problem. Therefore, we address and prove it in Theorem 3.

Theorem 1 (Transformation of an infeasible problem to a bounded problem)

Let A be an $m \times n$ matrix and let b and c be an $m \times 1$ and $n \times 1$ vectors, respectively. Suppose the set $\{x \in \mathbb{R}^n \mid Ax \geq b,\ x \geq \vec{0}_n\}$ is empty. Then, the set $\Omega = \{(x,\ s) \in \mathbb{R}^{m+n} \mid Ax + s \geq b,\ s \geq \vec{0}_m,\ x \geq \vec{0}_n\}$ is nonempty and the following problem P_μ is bounded:

$$\text{Problem } P_\mu:\ \min c^{\mathrm{T}}x + \mu \sum_{i=1}^{m} s_i$$

$$\text{subject to}: Ax + s \geq b$$

$$s \geq \vec{0}_m$$

$$x \geq \vec{0}_n,$$

for any given the penalty $\mu > 0$.

Proof. The fact that the set $\left\{x \in \mathbb{R}^n \mid Ax \geq b,\ x \geq \vec{0}_n\right\}$ is empty implies that for any $x \geq \vec{0}_n$, there exists its corresponding nonempty subset $M_x \subset M = \{1,\ 2\ldots,\ m\}$ such that

$$(Ax)_i < b_i,\ \forall i \in M_x,$$

and

$$(Ax)_j \geq b_j,\ \forall j \in M \backslash M_x.$$

Therefore, we can define $s_i = b_i - (Ax)_i$, for all $i \in M_x$ and $s_i = 0$, for all $i \in M \backslash M_x$. This choice of s_i's ensures that the set $\Omega = \left\{(x,\ s) \in \mathbb{R}^{m+n} \mid Ax + s \geq b,\ s \geq \vec{0}_m,\ x \geq \vec{0}_n\right\}$ is nonempty.

Furthermore, s_i is allowed to be any value that greater or equal to $\max\{b_i - (Ax)_i,\ 0\}$, which results in the unboundedness of the following problem

$$\max_{(x,\ s) \in \Omega} c^{\mathrm{T}} x + \mu \sum_{i=1}^{m} s_i, \tag{3}$$

since we can keep increasing s_i, where $i \in M_x$. By the duality theorem, the dual problem of (3) must be infeasible. The feasible region of the dual problem of (3) is:

$$\Omega_{D_1} = \left\{y \mid y^{\mathrm{T}}(A\ \ I) \leq [-c^{\mathrm{T}} - \mu^{\mathrm{T}}],\ y \geq \vec{0}_m\right\} = \emptyset.$$

If Problem P_μ is also unbounded, then the feasible region of the dual problem of P_μ would be infeasible, and thus:

$$\Omega_{D_2} = \left\{y \mid y^{\mathrm{T}}(A\ \ I) \leq [c^{\mathrm{T}} \mu^{\mathrm{T}}],\ y \geq \vec{0}_m\right\} = \emptyset.$$

The fact that both Ω_{D_1} and Ω_{D_2} are empty implies that $\vec{0}_m \notin \Omega_{D_1}$ and $\vec{0}_m \notin \Omega_{D_2}$. In other words, we have

$$\vec{0}_{m+n}^{\mathrm{T}} > [-c^{\mathrm{T}} - \mu^{\mathrm{T}}]$$

and

$$\vec{0}_{m+n}^{\mathrm{T}} > [c^{\mathrm{T}} \mu^{\mathrm{T}}],$$

which lead to a contradiction. Therefore, Problem P_μ must be bounded. □

The following theorem aims to find a threshold penalty value μ_T, so that the optimal solution to Problem P_μ, remains the same for any $\mu \geq \mu_T$.

Theorem 2 (Threshold penalty value μ_T)

Let A be an $m \times n$ matrix and let b and c be an $m \times 1$ and $n \times 1$ vectors, respectively. Suppose that the set $\left\{x \in \mathbb{R}^n \mid Ax \geq b,\ x \geq \vec{0}_n\right\}$ is empty. Then, there exists a nonnegative value $\mu_T > 0$ such that for any $\mu \geq \mu_T$, Problems P_μ and P_{μ_T} yield the same optimal solution, where Problem P_μ is defined as:

$$\text{Problem } P_\mu: \ \min c^{\mathrm{T}} x + \mu \sum_{i=1}^{m} s_i$$

$$\text{subject to}: Ax + s \geq b$$

$$s \geq \vec{0}_m$$

$$x \geq \vec{0}_n.$$

Proof. By Theorem 1, Problem P_μ is bounded, for any $\mu > 0$. Consequently, its dual problem is also bounded. The dual of Problem P_μ is given by:

$$\text{Problem } D_\mu: \ \max b^{\mathrm{T}} y$$

$$\text{subject to}: y^{\mathrm{T}}[A \ \ I] \leq [c^{\mathrm{T}} \ \ \overbrace{\mu \ \mu \ \ldots \ \mu}^{\text{m terms}}] \tag{4}$$

$$y \geq \vec{0}_m$$

Define T as

$$T = \begin{bmatrix} (A^{\mathrm{T}})_{n\times m} & \multirow{2}{*}{I_{m+n}} \\ I_m & \end{bmatrix}$$

where $C = [c^{\mathrm{T}} \ \ \overbrace{\mu \ \mu \ \ldots \ \mu}^{\text{m terms}}]$. Thus, $T \begin{bmatrix} y \\ w \end{bmatrix} = C^{\mathrm{T}}$, where $w \in \mathbb{R}^{m+n}$ is a slack variable of Constraint (4).

Let Ψ be the set of all invertible $(m+n) \times (m+n)$ sub-matrices of T. For any $B \in \Psi$, let β_{ij} denote the $(i, \ j)$-th element of B^{-1}. We then have:

$$y_i = \sum_{j=1}^{n} \beta_{ij} c_j + \mu \sum_{j=n+1}^{m+n} \beta_{ij}, \ \forall i = 1, \ 2, \ \ldots, \ m$$

$$w_k = \sum_{j=1}^{n} \beta_{kj} c_j + \mu \sum_{j=n+1}^{m+n} \beta_{kj}, \ \forall k = m+1, \ m+2, \ \ldots, \ 2m+n.$$

Define $\alpha_i = \sum_{j=1}^{n} \beta_{ij} c_j$ and $\gamma_i = \sum_{j=n+1}^{m+n} \beta_{ij}$, for each $i = 1, \ 2, \ \ldots, \ 2m+n$. For optimality, we require $y \geq \vec{0}_m$ and $w \geq \vec{0}_{m+n}$, which translates to

$$\alpha_i + \gamma_i \mu \geq 0, \ \forall i = 1, \ 2, \ \ldots, \ 2m+n. \tag{5}$$

The terms α_i and γ_i corresponding to an optimal basis matrix of Problem D_ν must satisfy (5).

Case analysis:

Case (i): If α_i and $\gamma_i \geq 0$.

In this case (5) is always satisfied for any $\mu \geq 0$.

Case (ii): If $\alpha_i < 0$ and $\gamma_i > 0$.

Here, (5) is satisfied only if $\mu \geq -\dfrac{\alpha_i}{\gamma_i} = \left|\dfrac{\alpha_i}{\gamma_i}\right|$.

Case (iii): If α_i and $\gamma_i < 0$.

In this scenario, (5) is satisfied only if $\mu \leq -\dfrac{\alpha_i}{\gamma_i} < 0$. However, since μ must be nonnegative, α_i and γ_i cannot both be less than 0, as this would contradict the optimality of Problem D_μ.

Case (iv): If $\alpha_i > 0$ and $\gamma_i < 0$.

Here, (5) is satisfied only if $\mu \leq -\dfrac{\alpha_i}{\gamma_i} = \left|\dfrac{\alpha_i}{\gamma_i}\right|$. For $\mu \geq \left|\dfrac{\alpha_i}{\gamma_i}\right|$, the corresponding matrix B would not yield optimality for Problem D_μ.

If Case (iv) appears for every matrix $B \in \Psi$, Problem D_μ would not have optimality when

$$\mu > \mathbb{M} = \max\left\{\left|\frac{\alpha_i}{\gamma_i}\right| \mid \alpha_i = \sum_{j=1}^{n} \beta_{ij} c_j, \ \gamma_i = \sum_{j=n+1}^{m+n} \beta_{ij}, \ \text{for } \beta_{ij} \text{ being the } (i, \ j)\text{-th element of } B^{-1}, \ \forall B \in \Psi\right\}$$

So, Case (iv) will not happen when we consider the optimality of D_μ, for $\mu > \mathbb{M}$. By defining

$$\mu_T = 1 + \mathbb{M},$$

Thus, an optimal basis matrix for Problem D_{μ_T} will also serve Problem D_μ when $\mu \geq \mu_T$. Therefore, Problems P_μ and P_{μ_T} have the same optimal solution when $\mu \geq \mu_T$. □

4. Boundary correction for infeasible worst problem

Given that the best deterministic model is feasible, this section addresses the scenario where the worst determinsitic model

$$\min_{x \in \Omega_1} \overline{c}^{\mathrm{T}} x,$$

is infeasible; i.e., the set $\Omega_1 = \left\{x \in \mathbb{R}^n | \underline{A} x \geq \overline{b}, \ x \geq \vec{0}_n\right\}$ is empty.

Since we are working under the scope of interval linear program, the standard approaches for correcting infeasibility [23–27] are unable to spread the elastic terms in Problem P_μ to achieve a new lower bound of interval coefficient matrix with the objective to balance the proportional changes across all elements of $\underline{A}$ and $\overline{b}$. In order to do that we modify Problem P_μ to become Problem I, which aims to balance or even up the elastic variables of all constraints as much as possible by adding Constraint (8) and the penalizing the uneven elastic term between any two constraints. Constraint (9) in

Problem I guarantees that the corrected lower bounds will not exceed the upper bounds of the original interval parameters. Corollary 1 shows that Problem I is bounded.

Corollary 1 Let $\Omega_1 = \emptyset$ and $\Omega_2 \neq \emptyset$. Then Problem I is bounded, for any given $\mu_1, \mu_2 > 0$ and $p = \binom{m}{2}$.

$$\text{Problem } I: \ \min \ \bar{c}^{\mathrm{T}} x + \mu_1 \sum_{i=1}^{m} s_i + \mu_2 \sum_{k=1}^{p} (\theta_k + \omega_k) \tag{6}$$

$$\text{subject to} : (\underline{A}x)_i + s_i \geq \bar{b}_i, \ \forall i = 1, \ \ldots, \ m \tag{7}$$

$$(2\Delta x)_i + (2\delta)_i \geq s_i, \ \forall i = 1, \ \ldots, \ m \tag{8}$$

$$s_{i_k} - s_{j_k} + \theta_k - \omega_k = 0, \ \forall k = 1, \ \ldots, \ p, \tag{9}$$

$$i_k < j_k \text{ where } i_k, \ j_k \in \{1, \ \ldots, \ m\}$$

$$\vec{\theta}, \ \vec{\omega} \geq \vec{0}_p, \ \vec{s} \geq \vec{0}_m, \ \vec{x} \geq \vec{0}_n.$$

Proof. Let $\mu_1, \ \mu_2 > 0$. Consider Problem I_0 which is Problem I without the Constraints (8)-(9).

$$\text{Problem } I_0 : \min \ \bar{c}^{\mathrm{T}} x + \mu_1 \sum_{i=1}^{m} s_i + \mu_2 \sum_{k=1}^{p} (\theta_k + \omega_k) \tag{10}$$

$$\text{subject to} : (\underline{A}x)_i + s_i \geq \bar{b}_i, \ \forall i = 1, \ \ldots, \ m \tag{11}$$

$$\vec{\theta}, \ \vec{\omega} \geq \vec{0}_p, \ \vec{s} \geq \vec{0}_m, \ \vec{x} \geq \vec{0}_n.$$

Problem I_0 is bounded by Theorem 1, as we can treat $\vec{\theta}$ and $\vec{\omega}$ as $\vec{0}_p$. To obtain the minimum objective value, let Ω_{I_0} and Ω_I be the feasible regions of Problems I_0 and I, respectively. We have $\Omega_I \subseteq \Omega_{I_0}$. Since $\Omega_2 \neq \emptyset$, there is $x \geq \vec{0}_n$ satisfying $\bar{A}x \geq \underline{b}$. Given that $\bar{A} = \underline{A} + 2\Delta$ and $\bar{b} = \underline{b} + 2\delta$, we have

$$\bar{A}x = \underline{A}x + 2\Delta x \geq \underline{b} = \bar{b} - 2\delta.$$

Thus, for each i, there exists $s_i = (2\Delta x)_i + 2\delta_i \geq 0$ such that $(\underline{A}x)_i + s_i \geq \bar{b}_i$. For any two terms s_{i_k} and s_{j_k}, where $i_k < j_k$ and $k = 1, \ 2, \ \ldots, \ p,$ we can define

$$\theta_k = \begin{cases} \left| s_{i_k} - s_{j_k} \right|, & \text{if } s_{i_k} - s_{j_k} < 0, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\omega_k = \begin{cases} s_{i_k} - s_{j_k}, & \text{if } s_{i_k} - s_{j_k} \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, Ω_I is nonempty. Hence, Problem I is bounded. $\square$

Remarks on Problem I:

1. The terms θ_k and ω_k may result in some s_i^* being greater than $(2\Delta x^*)_i + 2\delta_i$, when $(x^*,\ s^*)$ is an optimal solution for Problem I without including Constraint (8). To adjust boundaries $\underline{A}$ and $\overline{b}$ so that $\Omega_{1_{\text{new}}} \neq \emptyset$, we have to increase the lower bound $\underline{A}$ by no more than 2Δ and decrease the upper bound $\overline{b}$ by no more than 2δ. Thus Constraint (8) must be incorporated into Problem I when we assume that perturbations are restricted to within the original interval parameters.

2. To minimize the term $\sum_{k=1}^{\binom{m}{2}} (\theta_k + \omega_k)$ in the objective function, the optimal solution $(\theta_k^*,\ \omega_k^*)$ can be viewed as $\left(\left|s_{i_k}^* - s_{j_k}^*\right|,\ 0\right)$ if $s_{i_k}^* - s_{j_k}^* < 0$ and $(\theta_k^*,\ \omega_k^*) = (0,\ s_{i_k}^* - s_{j_k}^*)$ when $s_{i_k}^* - s_{j_k}^* > 0$.

3. The term $\sum_{k=1}^{\binom{m}{2}} (\theta_k + \omega_k)$ is instrumental in redistributing the elastic terms more evenly across all constraints.

Corollary 2 Let $\Omega_1 = \emptyset$ and $\Omega_2 \neq \emptyset$. For any $\mu > 0$, let Problem I_μ be Problem I with $\mu = \mu_1 = \mu_2 > 0$. Then, there exists $\mu_T > 0$ such that Problem I_μ obtains the same optimal solution as Problem I_{μ_T}, for every $\mu \geq \mu_T$.

Proof. The proof follows the proof of Theorem 2 in the similar fashion. $\square$

The statement of Corollary 2 is not true, when $\mu_1 \neq \mu_2$. The terms μ_1 and μ_2 must have a certain relationship to be able to obtain the same optimal solution.

Let $\left\{ y \in \mathbb{R}^{2m+2\binom{m}{2}} \;\middle|\; T_I y \leq [c^{\mathrm{T}}\ \overbrace{\mu_1\ \mu_1\ \ldots\ \mu_1}^{\text{m terms}}\ \overbrace{\mu_2\ \mu_2\ \ldots\ \mu_2}^{\binom{m}{2}}]^{\mathrm{T}},\ y \geq \vec{0} \right\}$ be the feasible region for the dual problem D_I of Problem I, where T_I is the corresponding coefficient matrix of Problem D_I generated from the coefficient matrix of Problem I. Let Ψ_I be the set of all largest size invertible sub-matrices of T_I, and $B_I \in \Psi$ where β_{ij} be the $(i,\ j)$ element of B_I^{-1}. Therefore,

$$y_i = \sum_{j=1}^{n} \beta_{ij} c_j + \mu_1 \sum_{j=n+1}^{m+n} \beta_{ij} + \mu_2 \sum_{j=m+n+1}^{m+n+\binom{m}{2}} \beta_{ij},\ \ \forall i = 1,\ 2,\ \ldots,\ 2m + 2\binom{m}{2}$$

Define the constants $\alpha_i = \sum_{j=1}^{n} \beta_{ij} c_j$, $\gamma_i = \sum_{j=n+1}^{m+n} \beta_{ij}$, and $\eta_i = \sum_{j=m+n+1}^{m+n+\binom{m}{2}} \beta_{ij}$, for each $i = 1,\ 2,\ \ldots,\ 2m + 2\binom{m}{2}$. The optimality requires

$$y_i = \alpha_i + \gamma_i \mu_1 + \eta_i \mu_2 \geq 0,\ \forall i = 1,\ 2,\ \ldots,\ 2m + 2\binom{m}{2}. \tag{12}$$

For any given $\mu_1 > 0$ and $\mu_2 > 0$, there is the corresponding optimal basis matrix, which generate the terms α_i, γ_i, and η_i. To maintain the same optimal basis while changing μ_1, and μ_2, the penality terms must satisfy the relationship (12).

Interval parameters with different radius should not use the same amount of adjustment. We introduce an interval boundary correction algorithm that adjust each interval parameter relatively comparing with its width. The penalty terms in the objective function focus on to both balance and minimize the elastic terms s_i's for each i^{th} constraint. We could also add a restriction $\theta_k+\omega_k \leq K$ (optional), where K is a given non-negative number, to ensure that s_i's will not differ from each other more than K units and still get a reasonably small value of s_i's. However, this restriction may lead infeasibility to Problem I if K is too small.

4.1 *Boundary correction method for infeasible worst model*

Let x^* be an optimal solution to Problem I. The lower bound coefficient $\underline{A}$ and the upper bound right-hand-side $\overline{b}_i$ have to be adjusted to the new boundaries $\underline{A}_{\text{new}}$ and $\overline{b}_{i\text{new}}$, where $\underline{A} \leq \underline{A}_{\text{new}} \leq \overline{A}$ and $\underline{b}_i \leq \overline{b}_{i\text{new}} \leq \overline{b}_i$. Let h_{ij_1}, h_{ij_2} and k_i be the extra positive amount adding/subtracting to the boundaries $\underline{a}_{ij_1}$, $\underline{a}_{ij_2}$ and $\overline{b}_i$ for each i^{th} constraint, respectively. By adjusting the boundaries relatively to their width through each i^{th} constraint, we have to preserve the ratio

$$H_i = \frac{h_{ij_1}}{\Delta_{ij_1}} = \frac{h_{ij_2}}{\Delta_{ij_2}} = \frac{k_i}{\delta_i} \geq 0, \tag{13}$$

given that $\Delta_{ij} > 0$ and $\delta_i > 0$.

In the case of $\Delta_{ij} = 0$ and $\delta_i = 0$, we have the corresponding constant coefficient a_{ij} and the constant right-hand-side b_i, which have no boundaries to adjust. The extra amount adding to the boundaries will fulfill the elastic amount s_i^*, therefore

$$h_{i1}x_1^* + h_{i2}x_2^* + \ldots + h_{in}x_n^* + k_i \geq s_i^*,\ \forall i = 1,\ 2,\ \ldots,\ m.$$

This inequality can be written in term of the ratio H_i as follows:

$$H_i(\Delta_{i1}x_1^* + \Delta_{i2}x_2^* + \ldots + \Delta_{in}x_n^* + \delta_i) \geq s_i^*,\ \forall i = 1,\ 2,\ \ldots,\ m.$$

Hence, we will use the ratio H_i to generate the new boundaries, where

$$H_i = \frac{s_i^*}{(\Delta_{i1}x_1^* + \Delta_{i2}x_2^* + \ldots + \Delta_{in}x_n^* + \delta_i)},\ \forall i = 1,\ 2,\ \ldots,\ m. \tag{14}$$

Using Constraint (8) to preserve the other boundaries, therefore

$$2\Delta_{i1}x_1^* + 2\Delta_{i2}x_2^* + \ldots + 2\Delta_{in}x_n^* + 2\delta_i \geq H_i(\Delta_{i1}x_1^* + \Delta_{i2}x_2^* + \ldots + \Delta_{in}x_n^* + \delta_i),$$

which means that H_i is naturally limited by 2. We can now use

$$h_{ij} = \Delta_{ij}H_i,\ \forall j = 1,\ 2,\ \ldots,\ n \text{ and } k_i = \delta_i H_i$$

to generate $\underline{A}_{\text{new}}$ and $\overline{b}_{i\text{new}}$. This is the pre-process step of adjusting infeasible worst model to become a bounded one. Note that $H_i \neq 0$ if and only if $\Delta_{ij}x_j^* \neq 0$, for some i or $\delta_i \neq 0$. If $\Delta_{i1}x_1^* + \Delta_{i2}x_2^* + \ldots + \Delta_{in}x_n^* + \delta_i = 0$, it implies that $s_i = 0$ and no we will not do further adjustment for all intervals in the i^{th} constraint. Moreover, the proportion between h_{ij} and Δ_{ij}

could depend on the decision maker and the importance of that particular interval information. The inequality (13) can be modified to be

$$H_i = \frac{l_{ij_1} h_{ij_1}}{\Delta_{ij_1}} = \frac{l_{ij_2} h_{ij_2}}{\Delta_{ij_2}} = \frac{l_i k_i}{\delta_i} \geq 0, \tag{15}$$

where l_{ij_1}, l_{ij_2}, l_i are positive constants given by the decision maker. Hence, H_i in (14) will be changed accordingly.

Boundary correction algorithm for infeasible worst model

Step 0: For $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, input Δ_{ij}, δ_i, $\underline{a}_{ij}$, $\overline{b}_i$.

Step 1: Get the optimal solution $(x^*, s^*, \theta^*, \omega^*)$ to Problem I.

Step 2: For $i = 1, 2, \ldots, m$,

$H_0 \leftarrow \Delta_{i1} x_1^* + \Delta_{i2} x_2^* + \ldots + \Delta_{in} x_n^* + \delta_i$

If $H_0 = 0$, then $\underline{a}_{ij_{\text{new}}} \leftarrow \underline{a}_{ij}$, $j = 1, 2, \ldots, n$ and $\overline{b}_{i_{\text{new}}} \leftarrow \overline{b}_i$.

Otherwise, $H_i \leftarrow \dfrac{s_i}{H_0}$.

Step 3: $h_{ij} \leftarrow \Delta_{ij} H_i$ and $k_i \leftarrow \delta_i H_i$.

Step 4: $\underline{a}_{ij_{\text{new}}} \leftarrow \underline{a}_{ij} + h_{ij}$ and $\overline{b}_{i_{\text{new}}} \leftarrow \overline{b}_i - k_i$.

5. Boundary correction for unbounded best problem

When the best deterministic model is unbounded, we may adjust the boundaries $\overline{A}$ and $\underline{c}$ so that the problem becomes bounded. The adjustment process is done through the infeasible dual problem of the unbounded best model using the similar idea from Theorem 2 as follows.

Theorem 3 Let $\Omega_1 \neq \emptyset$. Suppose that the best deterministic problem

$$\min \underline{c}^{\mathrm{T}} x \text{ subject to} : \overline{A} x \geq \underline{b},\ x \geq \vec{0}_n,$$

is unbounded. Therefore, Dual Problem : $\max \underline{b}^{\mathrm{T}} y$ subject to : $y^{\mathrm{T}} \overline{A} \leq \underline{c}^{\mathrm{T}}$, $y \geq \vec{0}_m$ is an infeasible problem and Problem II is bounded, for any given μ_1 , $\mu_2 > 0$.

$$\text{Problem } II: \ \max \underline{b}^{\mathrm{T}} y - \mu_1 \sum_{j=1}^{n} s_j - \mu_2 \sum_{k=1}^{\binom{n}{2}} (\theta_k + \omega_k)$$

$$\text{subject to} : (y^{\mathrm{T}} \overline{A})_j - s_j \leq (\underline{c})_j,\ \forall j = 1, \ldots, n$$

$$(2\Delta y)_j + (2\lambda)_j \geq s_j,\ \forall j = 1, \ldots, n$$

$$s_{i_k} - s_{j_k} + \theta_k - \omega_k = 0,\ \forall k = 1, \ldots, q,$$

$$i_k < j_k \text{ where } i_k,\ j_k \in \{1, \ldots, n\}$$

$$\vec{\theta},\ \vec{\omega} \geq \vec{0}_q,\ \vec{s} \geq \vec{0}_n,\ \vec{y} \geq \vec{0}_m,$$

where $q = \binom{n}{2}$.

Proof. The proof can be done in the similar fashion as the one in Corollary 1. □

Corollary 3 Let $\Omega_1 \neq \emptyset$ and Ω_2 be an unbounded set. For any $\mu > 0$, let Problem II_μ be Problem II with $\mu = \mu_1 = \mu_2 > 0$. Then, There exists $\mu_T > 0$ such that Problem II_μ obtains the same optimal solution as Problem II_{μ_T}, for every $\mu \geq \mu_T$.

Proof. The proof follows the proof of Theorem 2 in the similar fashion. □

We can now extend the approach used for boundary correction in Subsection 4.1 to address the elastic terms s_k's in Problem II. By modifying the boundaries of the interval coefficients in the infeasible dual problem

$$\max \underline{b}^{\mathrm{T}} y \text{ subject to} : y^{\mathrm{T}} \overline{A} \leq \underline{c}^{\mathrm{T}},\ y \geq \vec{0}_m,$$

we obtain an adjusted dual problem that is bounded. Consequently, the originally unbounded problem

$$\min \underline{c}^{\mathrm{T}} x \text{ subject to} : \overline{A} x \geq \underline{b},\ x \geq \vec{0}_n$$

will also become bounded upon applying these adjusted intervals.

Boundary correction algorithm for unbounded best model

Step 0: For $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, input Δ_{ij}, λ_j, $\overline{a}_{ij}$, $\underline{b}_i$.

Step 1: Get the optimal solution $(y^*, s^*, \theta^*, \omega^*)$ to Problem II.

Step 2: For $j = 1, 2, \ldots, n$,

$K_0 \leftarrow \Delta_{1j} y_1^* + \Delta_{2j} y_2^* + \ldots + \Delta_{mj} y_m^* + \lambda_j$

If $K_0 = 0$, then $\overline{a}_{ij_{\text{new}}} \leftarrow \overline{a}_{ij}$, $i = 1, 2, \ldots, m$ and $\underline{c}_{j_{\text{new}}} \leftarrow \underline{c}_j$.

Otherwise, $K_j \leftarrow \dfrac{s_j}{K_0}$.

Step 3: $h_{ij} \leftarrow \Delta_{ij} K_j$ and $k_j \leftarrow \delta_j K_j$.

Step 4: $\overline{a}_{ij_{\text{new}}} \leftarrow \overline{a}_{ij} - h_{ij}$ and $\underline{c}_{j_{\text{new}}} \leftarrow \underline{c}_j + k_i$.

Problems I and II may initially appear to be substantial in scale. However, incorporating this preprocessing step becomes essential when a modeler assigns equal significance to each interval and/or constraint. It may be unnecessary to compare the infeasibility of every individual constraint, as the relative importance of each interval and constraint can guide the prioritization of which infeasibility measures to compare.

6. Numerical examples and application on diet problem

We provide two numerical examples to illustrate each step of the algorithms. Moreover, an application on a diet problem is presented to show the usefulness of our methods.

Example 1 This problem illustrates the case when $\Omega_1 = \emptyset$ and Ω_2 is a bounded set.

$$
\begin{aligned}
&\min\ [-3,\ -2]\, x_1 + [2,\ 4]\, x_2 \\
&\text{subject to}: [1,\ 4]\, x_1 + [-5,\ -3]\, x_2 \geq [-5,\ -1], \\
&\qquad [-4,\ -3]\, x_1 + [0.5,\ 1]\, x_2 \geq [-4,\ 1], \\
&\qquad x_1,\ x_2 \geq 0.
\end{aligned} \tag{16}
$$

Then, $\Delta = \begin{bmatrix} 1.5 & 1 \\ 0.5 & 0.25 \end{bmatrix}$ and $\delta = [2\ \ 2.5]^{\mathrm{T}}$. We found that

$$\Omega_1 = \{(x_1,\ x_2) \mid x_1 - 5x_2 \geq -1,\ -4x_1 + 0.5x_2 \geq 1,\ x_1,\ x_2 \geq 0\}$$

is an empty set and

$$\Omega_2 \{(x_1,\ x_2) \mid 4x_1 - 3x_2 \geq -5,\ -3x_1 + x_2 \geq -4,\ x_1,\ x_2 \geq 0\}$$

is a bounded set. By Corollary 1, We obtain the following problem:

$$
\begin{aligned}
&\min\ -2x_1 + 4x_2 + 100s_1 + 100s_2 + 100\theta_1 + 100\omega_1 \\
&\text{subject to}: x_1 - 5x_2 + s_1 \geq -1, \\
&\qquad -4x_1 + 0.5x_2 + s_2 \geq 1, \\
&\qquad 3x_1 + 2x_2 + 4 \geq s_1, \\
&\qquad x_1 + 0.5x_2 + 5 \geq s_2, \\
&\qquad s_1 - s_2 - \theta_1 + \omega_1 = 0, \\
&\qquad \theta_1,\ \omega_1 \geq 0 \\
&\qquad s_1,\ s_2 \geq 0 \\
&\qquad x_1,\ x_2 \geq 0,
\end{aligned} \tag{17}
$$

where $\mu_1 = 100$ and $\mu_2 = 100$. By solving (17), we obtain $x^* = [0\ 0.364]^{\mathrm{T}}$, $s^* = [0.818\ 0.818]^{\mathrm{T}}$, $(\theta_1^* \omega_1^*) = \vec{0}$ as an optimal solution. To adjust boundaries for the infeasible worst model $\min\limits_{(x_1,\, x_2) \in \Omega_1} -2x_1 + 4x_2$, we have to determine the ratio H_i, for all $i = 1,\ 2$ and extra amount adding to the boundaries $\underline{a}_{ij_1}$, $\underline{a}_{ij_2}$ and $\overline{b}_i$ for each constraint $i = 1,\ 2$. Since $H_1 = \dfrac{0.818}{0.364 + 2} = 0.346$ and $H_2 = \dfrac{0.818}{(0.25 \times 0.364) + 2.5} = 0.316$, the extra amounts are $h_{11} = 0.519$, $h_{12} = 0.346$, $k_1 = 0.692$, $h_{21} = 0.158$, $h_{22} = 0.079$, and $k_2 = 0.79$. After using h_{ij} and k_i to generate $\underline{A}_{\text{new}}$ and $\overline{b}_{i_{\text{new}}}$, the new bounded worst model can be written in the following:

$$
\begin{aligned}
&\min\ -2x_1 + 4x_2 \\
&\text{subject to}: 1.519x_1 - 4.654x_2 \geq -1.692, \\
&\qquad -3.842x_1 + 0.579x_2 \geq 0.21, \\
&\qquad x_1,\ x_2 \geq 0.
\end{aligned}
$$

Hence, the preprocessing technique provides the adjusted interval linear program (18) that all its deterministic problems are optimal.

$$
\begin{aligned}
&\min\ [-3,\ -2]\, x_1 + [2,\ 4]\, x_2 \\
&\text{subject to}: [1.519,\ 4]\, x_1 + [-4.654,\ -3]\, x_2 \geq [-5,\ -1.692], \\
&\qquad [-3.842,\ -3]\, x_1 + [0.579,\ 1]\, x_2 \geq [-4,\ 0.21], \\
&\qquad x_1,\ x_2 \geq 0.
\end{aligned} \tag{18}
$$

Example 2 This problem illustrates the case when $\Omega_1 = \emptyset$ and Ω_2 is an unbounded set.

$$
\begin{aligned}
&\min\ [-3.5,\ -2]\, x_1 + [1.8,\ 4]\, x_2 \\
&\text{subject to}: [-4,\ -3]\, x_1 + [1,\ 2]\, x_2 \geq [-6,\ -1], \\
&\qquad [2,\ 4]\, x_1 + [-8,\ -2]\, x_2 \geq [-4,\ 3], \\
&\qquad x_1,\ x_2 \geq 0.
\end{aligned} \tag{19}
$$

We have $\Delta = \begin{bmatrix} 0.5 & 0.5 \\ 1 & 3 \end{bmatrix}$ and $\delta = [2.5\ 3.5]^{\mathrm{T}}$. Then,

$$\Omega_1 = \{-2x_1 + x_2 \geq -1,\ 2x_1 - 8x_2 \geq 3,\ x_1,\ x_2 \geq 0\} = \emptyset$$

and

$$\Omega_2 = \{-3x_1 + 2x_2 \geq -6,\ 4x_1 - 2x_2 \geq -4,\ x_1,\ x_2 \geq 0\}$$

is an unbounded set. To adjust boundaries for the infeasible worst model, we can achieve the solution in the same manner as Example 1, the new bounded worst model can be written in the following:

$$\begin{aligned}
&\min\ -2x_1 + 4x_2 \\
&\text{subject to}: -3.70588x_1 + 1.294123x_2 \geq -2.47062, \\
&\qquad 2.400008x_1 - 6.79998x_2 \geq 1.599973, \\
&\qquad x_1,\ x_2 \geq 0.
\end{aligned}$$

According to Theorem 3, we obtain Model (20).

$$\begin{aligned}
&\max\ -6y_1 - 4y_2 - 100s_1 - 100s_2 - 100\theta_1 - 100\omega_1 \\
&\text{subject to}: -3y_1 + 4y_2 - s_1 \leq -3.5, \\
&\qquad 2y_1 - 2y_2 - s_2 \leq 1.8, \\
&\qquad y_1 + 2y_2 + 1.5 \geq s_1, \\
&\qquad y_1 + 6y_2 + 2.2 \geq s_2, \\
&\qquad s_1 - s_2 - \theta_1 + \omega_1 = 0, \\
&\qquad \theta_1,\ \omega_1 \geq 0 \\
&\qquad s_1,\ s_2 \geq 0 \\
&\qquad y_1,\ y_2 \geq 0,
\end{aligned} \tag{20}$$

where $\mu_1 = 100$ and $\mu_2 = 100$. By solving (20), ($y^* = [1.06\ 0]^{\mathrm{T}}$, $s^* = [0.32\ 0.32]^{\mathrm{T}}$, $(\theta_1^*,\ \omega_1^*) = \vec{0}$) is an optimal solution. To adjust boundaries for the infeasible dual best problem, we have to determine the ratio H_i, for all $i = 1,\ 2$ and extra amount adding to the boundaries $\overline{a}_{ij_1}$, $\overline{a}_{ij_2}$ and $\underline{b}_i$ for each constraint $i = 1,\ 2$. Since $H_1 = 0.25$ and $H_2 = 0.196$, the extra amounts are $h_{11} = 0.125,\ h_{12} = 0.25,\ k_1 = 0.1875,\ h_{21} = 0.098,\ h_{22} = 0.588$, and $k_2 = 0.2156$. After using h_{ij} and k_i to generate $\overline{A}_{\text{new}}$ and $\underline{b}_{i_{\text{new}}}$, the new dual best problem can be generated in the following:

$$
\begin{aligned}
&\max\ -6y_1 - 4y_2 \\
&\text{subject to} : -3.125y_1 + 3.75y_2 \leq -3.3125, \\
&\qquad 1.902y_1 - 2588y_2 \leq 2.0156, \\
&\qquad y_1,\ y_2 \geq 0.
\end{aligned}
$$

The corresponding new best problem can be rewritten as follows.

$$
\begin{aligned}
&\min -3.3125x_1 + 2.0156x_2 \\
&\text{subject to} : -3.125x_1 + 1.902x_2 \geq -6, \\
&\qquad 3.75x_1 - 2.588x_2 \geq -4, \\
&\qquad x_1,\ x_2 \geq 0.
\end{aligned}
$$

Hence, the preprocessing technique provides the adjusted interval linear program (21) that all its deterministic problems are optimal.

$$
\begin{aligned}
&\min\ [-3.3125,\ -2]\, x_1 + [2.0156,\ 4]\, x_2 \\
&\text{subject to} : [-3.70588,\ -3.125]\, x_1 + 1.294123,\ 1.902]\, x_2 \geq [-6,\ -2.47062], \\
&\qquad [2.400008,\ 3.75]\, x_1 + [-6.79998,\ -2.588]\, x_2 \geq [-4,\ 1.599973], \\
&\qquad x_1,\ x_2 \geq 0.
\end{aligned}
\tag{21}
$$

Example 3 Diet problem.

Consider a scenario where a farmer develops two distinct chicken feed formulas: I and II. The nutrient profiles for these formulas and the typical nutrition concentrations (kilograms per 100 kilograms of chicken food) for in-production laying hens are detailed in Table 3, also presented in [29] and [9]. Since the manufacturing cost (dollars) per unit of Formula I is 5 to 6 times greater than that of Formula II, the farmer intends to blend these formulas so that the quantity of

Formula II is at least 5 to 6 times the quantity of Formula I, resulting in a 100 kg ready-to-feed meal. Consequently, this scenario gives rise to the following interval linear programming problem (22) focused on minimizing costs.

$$
\begin{aligned}
&\min\ [5,\ 6]\, x_1 + x_2 \\
&\text{subject to}: [0.43,\ 0.5]\, x_1 + [0.085,\ 0.11]\, x_2 \geq [16,\ 18], \\
&\qquad [0.005,\ 0.007]\, x_1 + [0.003,\ 0.004]\, x_2 \geq [0.35,\ 0.45], \\
&\qquad [0.008,\ 0.0105]\, x_1 + [0.006,\ 0.008]\, x_2 \geq [0.75,\ 0.85], \\
&\qquad [0.048,\ 0.058]\, x_1 + [0.018,\ 0.032]\, x_2 \geq [3.5,\ 4.5], \\
&\qquad [0.007,\ 0.008]\, x_1 + [0.003,\ 0.0033]\, x_2 \geq [0.35,\ 0.5], \\
&\qquad 0.04\, x_1 + 0.04\, x_2 \geq 4, \\
&\qquad [0.025,\ 0.04]\, x_1 + [0.035,\ 0.04]\, x_2 \geq [3,\ 4], \\
&\qquad x_1 + x_2 = 100, \\
&\qquad x_2 \geq [5,\ 6]\, x_1, \\
&\qquad x_1,\ x_2 \geq 0.
\end{aligned} \tag{22}
$$

Table 3. Typical nutrition concentrations for in-production laying hens (kilograms per 100 kilograms of chicken food) and available nutrients in chicken formulas I and II (kilograms per 100 kilograms of chicken food)

	Protein	Methionine	Lysine	Calcium	Phosphorous	Fat	Fiber
Formula I	[43, 50]	[0.50, 0.70]	[0.80, 1.05]	[4.80, 5.80]	[0.70, 0.80]	4.0	[2.50, 4.00]
Formula II	[8.50, 11]	[0.30, 0.40]	[0.60, 0.80]	[1.80, 3.20]	[0.30, 0.33]	4.0	[3.50, 4.00]
Typical Nutrition Concentrations for production-laying hen	[1,600, 1,800]	[35, 45]	[75, 85]	[350, 450]	[35, 50]	400	[300, 400]

By analyzing the worst and best models of Problem (22), we determined that best model has a bounded feasible set, whereas worst model is infeasible. According to Corollary 1 and the preprocessing techinque in Subsection 4.1, the adjusted worst problem (23) is bounded can be formulated as follows.

$$
\begin{aligned}
&\min\ 6x_1 + x_2 \\
&\text{subject to : } 0.4921x_1 + 0.1072x_2 \geq 16.227, \\
&\qquad 0.007x_1 + 0.004x_2 \geq 0.35, \\
&\qquad 0.0098x_1 + 0.0074x_2 \geq 0.77, \\
&\qquad 0.0577x_1 + 0.0316x_2 \geq 3.5308, \\
&\qquad 0.008x_1 + 0.0033x_2 \geq 0.35, \\
&\qquad 0.04x_1 + 0.04x_2 \geq 4, \\
&\qquad 0.0309x_1 + 0.037x_2 \geq 3.6086, \\
&\qquad -5.9845x_1 + x_2 \geq 0, \\
&\qquad x_1 + x_2 = 100, \\
&\qquad x_1,\ x_2 \geq 0.
\end{aligned} \tag{23}
$$

Table 4. Typical nutrition concentrations for in-production laying hens (kilograms per 100 kilograms of chicken food) and available nutrients in chicken formulas I and II (kilograms per 100 kilograms of chicken food)

	Protein	Methionine	Lysine	Calcium	Phosphorous	Fat	Fiber
Formula I	[43, 50]	[0.50, 0.70]	[0.80, 1.05]	[4.80, 5.80]	[0.70, 0.80]	4.0	[2.50, 4.00]
Adjusted Formula I	[49.21, 50]	[0.70, 0.70]	[0.98, 1.05]	[5.77, 5.80]	[0.80, 0.80]	4.0	[3.09, 4.00]
Formula II	[8.50, 11]	[0.30, 0.40]	[0.60, 0.80]	[1.80, 3.20]	[0.30, 0.33]	4.0	[3.50, 4.00]
Adjusted Formula II	[10.72, 11]	[0.40, 0.40]	[0.74, 0.80]	[3.16, 3.20]	[0.33, 0.33]	4.0	[3.7, 4.00]
Typical Nutrition Concentrations for production-laying hen	[1,600, 1,800]	[35, 45]	[75, 85]	[350, 450]	[35, 50]	400	[300, 400]
Adjusted Typical Nutrition Concentrations for production-laying hen	[1,600, 1,622.7]	[35, 35]	[75, 77]	[350, 353.1]	[35, 35]	400	[300, 360.9]

To guarantee that Problem (23) attains optimality across all scenarios, the farmer should implement minor adjustments to the formulas as detailed in Table 4. These revisions will yield a manufacturing cost range of 151.28 to 171.53 dollars per 100 kg of mixed chicken feed. The modeler may opt to maintain the specified nutrient concentration intervals, as these standards are mandatory for the chicken feed. In such cases, the boundaries for nutrient adjustments in Formulas I and II may diverge more significantly from those provided in Table 3, if feasible solutions still exist. Alternatively, we might reconfigure Constraint (8) to accommodate new adjusted nutrient intervals that may either overlap with or differ from the original ones.

7. Conclusions

An interval linear program possesses an optimal vector solution if all associated deterministic problems yield optimal outcomes. This paper address the concern on the issues of unbounded best and infeasible worst deterministic problems of an interval linear programming problem. The boundary correction method has been introduced to refine an inadequately defined interval linear program by tightening the interval parameters within their original boundaries, ensuring the boundedness of both the best and worst scenarios. It has been proved that a threshold penalty price exists, ensuring that the elastic terms of each constraint remains the same when the penalty exceeds this threshold.

Adjusting the boundaries of all interval parameters can be computationally expensive. However, in practical applications, the decision-maker may decide to adjust only a select few intervals that are less costly to modify. Future research aims to enhance computational efficiency by automatically assigning adjustments to the smallest possible interval parameters while remaining within their original boundaries.

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Conflict of interest

The authors have no competing interests to declare that are relevant to the content of this article.

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