Contemporary Mathematics
https://ojs.wiserpub.com/index.php/CM/

Research Article

Global Weak Solutions for the 1D Pollutant Transport Model: The Case of $\varepsilon = 0$

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Received: 2 January 2025; **Revised:** 24 March 2025; **Accepted:** 21 April 2025

Abstract: This paper investigates the existence of global weak solutions for a one-dimensional pollutant transport model. The study builds upon the previous work. Our primary advancement involves demonstrating that the global weak solutions of a model studied in previous work remain valid when the coefficient associated with the regularizing term $\varepsilon \partial_x^2 h_2$ is eliminated ($\varepsilon = 0$). To achieve this, we have refined certain a priori estimates, ensuring they are independent of the parameter ε.

***Keywords*:** shallow water equations, bilayer models, approximate solutions, pollutant transport model

MSC: 35D05, 35Q30, 76N108

1. Introduction

This paper examines the existence of global weak solutions for a one-dimensional pollutant transport model. The model consists of the Saint-Venant equations, which represent the hydrodynamic component, coupled with a transport equation describing the morphodynamic behavior. This framework is particularly useful for simulating scenarios such as the evolution of a pollutant fluid within a water body. The Saint-Venant equations are expressed as follows:

$$\partial_t h_1 + \partial_x (h_1 u) = 0, \tag{1}$$

$$\partial_t (h_1 u) + \partial_x (h_1 u^2) + \frac{1}{2} g \partial_x h_1^2 - 4\nu_1 \partial_x (h_1 \partial_x u) + \frac{u}{\beta} - h_1 \partial_x (\sigma \partial_x^2 h_1 - V(h_1))$$

$$+ r_1 h_1 |u|^2 u + rgh_1 \partial_x h_2 + rgh_2 \partial_x (h_1 + h_2) = 0, \tag{2}$$

and the pollutant transport equation is given by:

DOI: https://doi.org/10.37256/cm.6420256337

$$\partial_t h_2 + \partial_x(h_2 u) - \partial_x\left((a{h_2}^2 + b{h_2}^3)\partial_x p_2\right) = 0, \tag{3}$$

with

$$\partial_x p_2 = \rho_2 g \partial_x(h_1 + h_2) \quad \text{and} \quad V(h_1) = \frac{1}{h_1^3} - \frac{\alpha}{h_1^4} \quad (\alpha > 0), \tag{4}$$

notice that here $(t, x) \in (0, T) \times [0, 1]$, defines the domain of interest.

The model describes a two-layer system of immiscible fluids. Here, h_1 and h_2 represent the heights of the water and pollutant layers, respectively, while u corresponds to the water velocity. The parameters ν_1 and p_2 correspond to the kinematic viscosity and pressure, respectively, and g denotes the gravitational constant. The coefficients σ, r_1, and β represent the interfacial tension coefficient, the quadratic friction coefficient, and the positive slip length parameter, respectively. The parameters a and b are associated with the interfacial friction and the viscosity coefficient of the pollutant, respectively. The constants α and ε are positive, while r represents the density ratio, defined as $r = \dfrac{\rho_2}{\rho_1}$, where ρ_1 and ρ_2 are the densities of water and the pollutant, respectively. The term $V(h_1)$ denotes the Van der Waals force, expressed as $V(h_1) = \dfrac{1}{h_1^3} - \dfrac{\alpha}{h_1^4}$ $(\alpha > 0)$; see [1–3].

In [2], the authors introduced the term $\varepsilon\partial_x^2 h_2$ into the transport equation (3) to establish the existence of global weak solutions for the system (1)-(3). This additional term, $\varepsilon\partial_x^2 h_2$, enables the derivation of the entropy inequality. The model analyzed in [2] is described as follows:

$$\partial_t h_1 + \partial_x(h_1 u) = 0, \tag{5}$$

$$\partial_t(h_1 u) + \partial_x(h_1 u^2) + \frac{1}{2} g \partial_x h_1^2 - 4\nu_1 \partial_x(h_1 \partial_x u) + \frac{u}{\beta} - h_1 \partial_x(\sigma \partial_x^2 h_1 - V(h_1))$$

$$+ r_1 h_1 |u|^2 u + rgh_1 \partial_x h_2 + rgh_2 \partial_x(h_1 + h_2) = 0, \tag{6}$$

$$\partial_t h_2 + \partial_x(h_2 u) - \varepsilon\partial_x^2 h_2 - \partial_x\left((a{h_2}^2 + b{h_2}^3)\partial_x p_2\right) = 0. \tag{7}$$

Our objective is to demonstrate that the existence result established in [2] can still be achieved without the inclusion of the regularizing term $\varepsilon\partial_x^2 h_2$.

We augment the system (1)-(3) by incorporating the initial conditions for our study

$$h_1(0, x) = h_{1_0}(x), \quad h_2(0, x) = h_{2_0}(x), \quad (h_1 u)(0, x) = \mathbf{m}_0(x) \ \text{ in } \ [0, 1]. \tag{8}$$

$$h_{1_0}, h_{2_0} \in L^2(0, 1), \quad \partial_x(h_{1_0}) \in L^2(0, 1),$$

$$\partial_x \mathbf{m}_0 \in L^1(0, 1), \quad \mathbf{m}_0 = 0 \quad \text{if} \quad h_{1_0} = 0,$$

$$\frac{|\mathbf{m}_0|^2}{h_{1_0}} \in L^1(0,\, 1), \quad \varphi(h_{1_0}) \in L^1(0,\, 1), \tag{9}$$

where $\varphi(h_1) = 4\nu_1 \log h_1$.

The energy inequality associated to the system (5)-(7) (see [2]) is:

$$\frac{d}{dt}\int_0^1 \left[\frac{1}{2}h_1|u|^2 + U(h_1) + \frac{1}{2}g(1-r)|h_1|^2 + \frac{1}{2}rg|h_1+h_2|^2 + \frac{1}{2}\sigma|\partial_x h_1|^2\right]$$

$$+4\nu_1\int_0^1 h_1|\partial_x u|^2 + \frac{1}{\beta}\int_0^1 |u|^2 + \frac{1}{2}gr\varepsilon\int_0^1 |\partial_x h_2|^2$$

$$+r_1\int_0^T\int_0^1 h_1|u|^4 + \rho_2 rg^2\int_0^1 h_2^2|\partial_x(h_1+h_2)|^2(a+bh_2) \leq \frac{1}{2}rg\varepsilon\int_0^1 |\partial_x h_1|^2 \tag{10}$$

where the potential function U is defined as the indefinite integral of V defined by

$$U(h_1) = -\frac{1}{2h_1^2} + \frac{\alpha}{3h_1^3}, \quad h_1 > 0.$$

The BD entropy inequality (see [2]) associated to the system (5)-(7) reads as

$$\frac{d}{dt}\int_0^1 \left[\frac{1}{2}h_1|u+\partial_x\varphi(h_1)|^2 - \frac{1}{\beta}\varphi(h_1) + \frac{1}{2}g(1-r)|h_1|^2 + \frac{1}{2}rg|h_1+h_2|^2 + \frac{1}{2}\sigma|\partial_x h_1|^2 + U(h_1)\right]$$

$$+\frac{1}{\beta}\int_0^1 |u|^2 + 4\nu_1\int_0^1 \left(g + gr\frac{h_2}{h_1} + V'(h_1)\right)|\partial_x h_1|^2 + rg\int_0^1 \left(\varepsilon + 4\nu_1\frac{h_2}{h_1}\right)\partial_x h_1\partial_x h_2 + 4\nu_1\sigma\int_0^1 |\partial_x^2 h_1|^2$$

$$+r_1\int_0^T\int_0^1 h_1|u|^4 + gr\varepsilon\int_0^1 |\partial_x h_2|^2 + rg^2\int_0^1 h_2^2(a+bh_2)\left(\partial_x(h_1+h_2)\right)^2 \leq \frac{1}{2}rg\varepsilon\int_0^1 |\partial_x h_1|^2 \tag{11}$$

where

$$\varphi(h_1) = 4\nu_1 \log h_1. \tag{12}$$

The BD entropy, initially introduced in [4], is a mathematical entropy that facilitates obtaining regularity on $\partial_x\sqrt{h_1}$.

We define $(h_1,\, h_2,\, u)$ as a weak solution of (1)-(3), with initial condition satisfying the entropy inequality (11) for all smooth test functions $\phi = \phi(t,\, x)$ such that $\phi(T,\, .) = 0$. Specifically, we have:

$$h_{01}\phi(0,.)-\int_0^T\int_0^1 h_1\partial_t\phi-\int_0^T\int_0^1 h_1u\partial_x\phi=0, \tag{13}$$

$$-h_{02}\phi(0,.)-\int_0^T\int_0^1 h_2\partial_t\phi-\int_0^T\int_0^1 h_2u\partial_x\phi+\int_0^T\int_0^1\left((a{h_2}^2+b{h_2}^3)\partial_x p_2\right)\partial_x\phi=0, \tag{14}$$

$$h_{01}u_{01}\phi(0,.)-\int_0^T\int_0^1 h_1u\partial_t\phi-\int_0^T\int_0^1 h_1u^2\partial_x\phi+4\nu_1\int_0^T\int_0^1 h_1\partial_x u\partial_x\phi$$

$$+\frac{1}{\beta}\int_0^T\int_0^1 u\phi+\int_0^T\int_0^1(\sigma\partial_x^2h_1-V(h_1))\phi\partial_x h_1+\int_0^T\int_0^1(\sigma\partial_x^2h_1-V(h_1))h_1\partial_x\phi$$

$$-\frac{1}{2}g\int_0^T\int_0^1 h_1^2\partial_x\phi-rg\int_0^T\int_0^1 h_2h_1\partial_x\phi+r_1\int_0^T\int_0^1 h_1|u|^2u\phi$$

$$-rg\int_0^T\int_0^1\phi h_2\partial_x h_1-rg\int_0^T\int_0^1(h_1+h_2)h_2\partial_x\phi-rg\int_0^T\int_0^1(h_1+h_2)\partial_x h_2\phi=0. \tag{15}$$

Numerous studies have addressed the existence of global weak solutions for pollutant transport models: see for instance [2, 5, 6]. The construction of global weak solutions is a crucial step in establishing their existence. For a shallow water model, Bresch and Desjardins demonstrated in [7] how to construct global weak solutions in the two-dimensional case. In [1, 8], the authors constructed approximate solutions to establish the existence of global weak solutions for one-dimensional lubrication models. Similarly, in [9], the authors developed sequences of suitably smooth approximate solutions for the one-dimensional pollutant transport model analyzed in [2]. For an alternative approach to constructing sequences of suitably smooth approximate solutions, see, for example [10].

In our analysis, our contribution is to show that when $\varepsilon=0$ (the coefficient of the regularising term $\varepsilon\partial_x^2h_2$ is zero), the existence result got in [2] is preserved.

The system (1)-(3) is supplemented with the following boundary conditions:

$$u=0 \quad \text{at} \quad x=0,1 \tag{16}$$

$$\partial_x h_i=0, \quad i=1,2 \quad \text{at} \quad x=0,1 \tag{17}$$

The structure of our paper is as follows. In Section 2, we recall the energy equality associated with the system (1)-(3) and derive a version of the BD entropy identity. These energy and BD entropy identities establish the regularities required to prove our existence results. Additionally, we present an existence theorem for global weak solutions to (1)-(3). In Section 3, following the approaches of Bresch and Desjardins in [7] and Kitavtsev et al. in [1], we propose an approximating system that satisfies analogous energy and entropy identities. These identities are crucial for proving the existence of global weak solutions to our model, using the techniques developed in [1]. In Section 4, we establish the existence of global weak solutions for the case $\beta=\infty$ by taking the limit in (1)-(3).

2. A priori estimates

In this section, we recall the classical energy inequality and the BD entropy associated to our system (1)-(2). In addition, we give a priori estimates derived from energy and BD entropy inequalities.

Lemma 1 (Energy inequality) For classical solutions of the system (1)-(3), the following inequality holds

$$\frac{d}{dt}\int_0^1\left[\frac{1}{2}h_1|u_1|^2+U(h_1)+\frac{1}{2}g(1-r)|h_1|^2+\frac{1}{2}rg|h_1+h_2|^2+\frac{1}{2}\sigma|\partial_x h_1|^2\right]$$

$$+4\nu_1\int_0^1 h_1|\partial_x u_1|^2+\frac{1}{\beta}\int_0^1|u_1|^2+r_1\int_0^T\int_0^1 h_1|u_1|^4+\rho_2 rg^2\int_0^1 h_2^2|\partial_x(h_1+h_2)|^2(a+bh_2)=0 \tag{18}$$

where the potential function U is the indefinite integral of V defined by

$$U(h_1)=-\frac{1}{2h_1^2}+\frac{\alpha}{3h_1^3},\quad h_1>0.$$

Proof. See [2]. $\square$

Remark 1 The term $\int_0^1\int_0^T U(h_1)|\partial_x h_1|^2$ can be absorbed by $\int_0^1\int_0^T\left(\frac{\alpha}{6h_1^3}-\frac{2}{3\alpha^2}\right)|\partial_x h_1|^2$ thanks to the work done in [1].

Corollary 1 Let (h_1, h_2, u_1) be a solution of model (1)-(3). Then, thanks to Lemma 1 we have:

$$h_1\in L^\infty(0, T; L^2(0, 1)),$$

$$\partial_x h_1\in L^\infty(0, T; L^2(0, 1)),$$

$$(h_1+h_2)\in L^\infty(0, T; L^2(0, 1)),$$

$$\sqrt{h_1}\partial_x u_1\in L^2(0, T; L^2(0, 1)),$$

$$u_1\in L^2(0, T; L^2(0, 1)),$$

$$h_2\sqrt{a+bh_2}(\partial_x(h_1+h_2))\in L^2(0, T; L^2(0, 1)),$$

$$h_1^{-\frac{3}{2}}\in L^\infty(0, T; L^2(0, 1)),$$

$$\sqrt{h_1}|u_1|^2\in L^2(0, T; L^2(0, 1)).$$

Remark 2

• Since h_2 is in $L^\infty(0,\, T;\, L^2(0,\, 1))$ so $\sqrt{a+bh_2}$ is in $L^\infty(0,\, T;\, L^p(0,\, 1))$ for all finite p with $p \geq 4$ in order to ensure that h_2 is in $L^\infty(0,\, T;\, L^2(0,\, 1))$.

• Otherwise, as $\sqrt{a+bh_2}h_2\partial_x(h_1+h_2) \in L^2(0,\, T;\, L^2(0,\, 1))$ and $\sqrt{a+bh_2}$ is in $L^\infty(0,\, T;\, L^p(0,\, 1))$ so $h_2\partial_x(h_1+h_2)$ is in $L^\infty(0,\, T;\, L^p(0,\, 1))$, $p \geq 4$.

• Using the inequality

$$(a-b)^2 \geq \frac{1}{2}b^2 - a^2,$$

one obtains

$$\left[h_2(\partial_x(h_1-h_2))\right]^2 \geq h_2^2\frac{|\partial_x h_2|^2}{2} - h_2^2|\partial_x h_1|^2, \tag{19}$$

which gives:

$$h_2^2|\partial_x h_2|^2 \leq 2h_2^2|\partial_x(h_1+h_2)|^2 + 2h_2^2|\partial_x h_1|^2.$$

Since $h_2\partial_x(h_2+h_1) \in L^\infty(0,\, T;\, L^p(0,\, 1))$ and $h_2\partial_x h_1 \in L^\infty(0,\, T;\, L^p(0,\, 1))$, so, by considering (19), we have:

$$h_2\partial_x h_2 \in L^2(0,\, T;\, L^2(0,\, 1)).$$

Additionally, we will require certain extra regularities on h_1, which will be obtained using an additional BD entropy equality introduced in the following lemma.

Lemma 2 For smooth solutions $(h_1,\, h_2,\, u)$ of the system (1)-(3) that satisfy the classical energy equality stated in Lemma 1, the following mathematical BD entropy inequality holds:

$$\begin{aligned}
&\frac{d}{dt}\int_0^1 \left[\frac{1}{2}h_1|u_1+\partial_x\varphi(h_1)|^2 - \frac{1}{\beta}\varphi(h_1) + \frac{1}{2}g(1-r)|h_1|^2 + \frac{1}{2}rg|h_1+h_2|^2 + \frac{1}{2}\sigma|\partial_x h_1|^2 + U(h_1)\right] \\
&+\frac{1}{\beta}\int_0^1 |u_1|^2 + 4\nu_1\int_0^1 \left(g+gr\frac{h_2}{h_1}+V'(h_1)\right)|\partial_x h_1|^2 + 4\nu_1 rg\int_0^1 \frac{h_2}{h_1}\partial_x h_1\partial_x h_2 + 4\nu_1\sigma\int_0^1 |\partial_x^2 h_1|^2 \\
&+4\nu_1 r_1\int_0^T\int_0^1 |u_1|^2 u_1\partial_x h_1 + rg^2\int_0^1 h_2^2(a+bh_2)\Big(\partial_x(h_1+h_2)\Big)^2 = 0
\end{aligned} \tag{20}$$

where

$$\varphi(h_1) = 4\nu_1 \log h_1. \tag{21}$$

Proof. See [2, 9]. □

These results are necessary to establish the aforementioned lemma.

Proposition 1 If h_1 possesses the regularities outlined in Corollary 1, then there exist constants c_1 and c_2 such that $0 < c_1 < h_1 < c_2$.

Proof. See [1, 2, 9]. We use the bound on $\partial_x h_{1_\eta}$ to establish a Hölder inequality, giving an upper estimate of h_{1_η}. Then, by combining the estimates $L^\infty(0, T; L^2(0, 1))$ on $(h_{1_\eta})^{-3/2}$ and $\partial_x h_{1_\eta}$, we obtain a bound on $\|1/\sqrt{h_{1_\eta}}\|_{W^{1,1}(0,1)}$. Finally, the continuous injection $W^{1,1}(0, 1) \hookrightarrow L^\infty(0, 1)$ guarantees a positive lower bound for h_{1_η}. □

Remark 3 In Lemma 2, all the terms, except

$$\int_0^T \int_0^1 |u_1|^2 u_1 \partial_x h_1 \quad \text{and} \quad 4\nu_1 \int_0^T \int_0^1 \frac{h_2}{h_1} \partial_x h_1 \partial_x h_2$$

are controlled because they exhibit the appropriate sign. The approach for controlling the term $\int_0^T \int_0^1 |u_1|^2 u_1 \partial_x h_1$ is inspired by [5]. What remains is to manage the term $\int_0^T \int_0^1 4\nu_1 \frac{h_2}{h_1} \partial_x h_1 \partial_x h_2$.

Proposition 2 A constant C exists such that

$$\left\| 4\nu_1 \frac{h_2}{h_1} \partial_x h_1 \partial_x h_2 \right\|_{L^2(0, T; L^2(0, 1))} \leq C.$$

Proof. We have:

$$4\nu_1 \frac{h_2}{h_1} \partial_x h_1 \partial_x h_2 = \partial_x \varphi(h_1) h_2 \partial_x h_2,$$

we can write:

$$\int_0^T \int_0^1 \left| \frac{h_2}{h_1} \partial_x h_1 \partial_x h_2 \right| \leq \frac{1}{2} \int_0^T \int_0^1 |\partial_x \varphi(h_1)|^2 + \frac{1}{2} \int_0^T \int_0^1 |h_2 \partial_x h_2|^2.$$

Let us now look at the first term to the right of the above inequality because for the second one, $h_2 \partial_x h_2$ is in $L^2(0, T; L^2(0, 1))$. Since:

$$\int_0^T \int_0^1 |\partial_x \varphi(h_1)|^2 = \int_0^T \int_0^1 \frac{|\partial_x h_1|^2}{h_1^2} \leq \int_0^T \int_0^1 |\partial_x h_1|^2 \leq \mathbf{C},$$

then $\partial_x \varphi(h_1)$ is in $L^2(0, T; L^2(0, 1))$. Which completes the proof. □

Lemma 3 For classical solutions of the system (1)-(3) with h_1 as the first component, the following holds:

$$\frac{1}{4} \int_0^1 h_1 |\partial_x \varphi(h_1)|^2 \leq \frac{1}{2} \int_0^1 h_1 (u + \partial_x \varphi(h_1))^2 + 2E(h_1, h_2, u_1) + \frac{1}{3\alpha^2} \tag{22}$$

with

$$E(h_1, h_2, u_1) = \int_0^1 \left[\frac{1}{2}h_1|u_1|^2 + U(h_1) + \frac{1}{2}rg|h_1+h_2|^2 + \frac{1}{2}g(1-r)|h_1|^2 + \frac{1}{2}\sigma|\partial_x h_1|^2\right].$$

Proof. See [2] □

Corollary 2 Let (h_1, h_2, u_1) be a solution of model (1)-(3).

Then, thanks to Lemma 2 and Lemma 3 we have:

$$\sqrt{h_1} \in L^\infty(0, T; L^2(0, 1)),$$

$$\partial_x \sqrt{h_1} \in L^\infty(0, T; L^2(0, 1)),$$

$$\partial_x^2 h_1 \in L^2(0, T; L^2(0, 1)).$$

Remark 4 To demonstrate that $\partial_x h_{2_\varepsilon}$ belongs to $L^\infty(0, T; L^2(0, 1))$, we proceed by leveraging the system's Partial Differential Equations (PDEs), energy estimates, and Sobolev embeddings. Here is the detailed reasoning:

• Governing Equation for h_{2_ε}

The evolution of h_{2_ε} is governed by:

$$\partial_t h_{2_\varepsilon} + \partial_x(h_{2_\varepsilon} u_\varepsilon) - \partial_x\big((ah_{2_\varepsilon}^2 + bh_{2_\varepsilon}^3)\partial_x p_{2_\varepsilon}\big) = 0.$$

Key terms: 1. Advection term: $\partial_x(h_{2_\varepsilon} u_\varepsilon)$, 2. Diffusion term: $\partial_x\big((ah_{2_\varepsilon}^2 + bh_{2_\varepsilon}^3)\partial_x p_{2_\varepsilon}\big)$.

• Energy Estimate Framework

To show that $\partial_x h_{2_\varepsilon} \in L^\infty(0, T; L^2(0, 1))$, we compute the energy estimates derived from the governing equation.

• Testing with $-\partial_x^2 h_{2_\varepsilon}$

Testing the equation with $-\partial_x^2 h_{2_\varepsilon}$:

$$\int_0^1 \partial_t h_{2_\varepsilon}(-\partial_x^2 h_{2_\varepsilon})dx + \int_0^1 \partial_x(h_{2_\varepsilon} u_\varepsilon)(-\partial_x^2 h_{2_\varepsilon})dx - \int_0^1 \partial_x\big((ah_{2_\varepsilon}^2 + bh_{2_\varepsilon}^3)\partial_x p_{2_\varepsilon}\big)(-\partial_x^2 h_{2_\varepsilon})dx = 0.$$

• Step 1: Time Derivative Term Integration by parts gives:

$$\int_0^1 \partial_t h_{2,\varepsilon}(-\partial_x^2 h_{2,\varepsilon})dx = -\int_0^1 \partial_t h_{2,\varepsilon}\partial_x^2 h_{2,\varepsilon}dx$$

$$= -\int_0^1 \partial_x(\partial_t h_{2,\varepsilon}\partial_x h_{2,\varepsilon})dx + \int_0^1 \partial_x\partial_t h_{2,\varepsilon}\partial_x h_{2,\varepsilon}dx$$

$$= \int_0^1 \partial_x\partial_t h_{2,\varepsilon}\partial_x h_{2,\varepsilon}dx = \frac{1}{2}\frac{d}{dt}\int_0^1 |\partial_x h_{2,\varepsilon}|^2 dx.$$

• Step 2: Advection Term for the advection term:

$$\int_0^1 \partial_x(h_{2_\varepsilon}u_\varepsilon)(-\partial_x^2 h_{2_\varepsilon})dx = \int_0^1 h_{2_\varepsilon}u_\varepsilon(\partial_x^3 h_{2_\varepsilon})dx.$$

This term requires regularity assumptions for u_ε and h_{2_ε}. Typically, $u_\varepsilon \in L^\infty(0, T; H^1(0, 1))$, ensuring boundedness.
• Step 3: Diffusion Term for the diffusion term:

$$-\int_0^1 \partial_x\big((ah_{2_\varepsilon}^2 + bh_{2_\varepsilon}^3)\partial_x p_{2_\varepsilon}\big)(-\partial_x^2 h_{2_\varepsilon})dx.$$

Integration by parts gives:

$$\int_0^1 (ah_{2_\varepsilon}^2 + bh_{2_\varepsilon}^3)(\partial_x p_{2_\varepsilon})(\partial_x^3 h_{2_\varepsilon})dx.$$

The nonlinear terms $ah_{2_\varepsilon}^2$ and $bh_{2_\varepsilon}^3$ are controlled by energy bounds on h_{2_ε} and $\partial_x h_{2_\varepsilon}$.
• Energy Balance
Combining the terms, we obtain:

$$\frac{d}{dt}\int_0^1 |\partial_x h_{2_\varepsilon}|^2 dx + C\int_0^1 |\partial_x^2 h_{2_\varepsilon}|^2 dx \leq \int_0^1 F(h_{2_\varepsilon}, u_\varepsilon, \partial_x p_{2_\varepsilon})dx,$$

where F contains terms involving h_{2_ε}, u_ε, and $\partial_x p_{2_\varepsilon}$.
• The energy inequality implies:

$$\partial_x h_{2_\varepsilon} \in L^\infty(0, T; L^2(0, 1)).$$

This holds under regularity assumptions on h_{2_ε} and u_ε, supported by the dissipative structure of the system.
Remark 5
1. We have the additional regularities thanks to Corollary 1:
(a) $\partial_t h_1 \in L^\infty(0, T; W^{-1, 2}(0, 1))$.
(b) h_1 is bounded in $L^2(0, T; H^1(0, 1))$.
(c) $h_1 u_1$ is bounded in $L^3(0, T; L^3(0, 1)) \cap L^\infty(0, T; L^2(0, 1)) \cap L^2(0, T; W^{1, 1}(0, 1))$.
2. h_1 is bounded in $L^\infty(0, T; L^\infty(0, 1))$ and h_2 is bounded in $L^\infty(0, T; L^2(0, 1))$.
3. $\sqrt{h_1}$ is bounded in $L^2(0, T; H^1(0, 1))$.
Theorem 1 A global weak solution exists for the system (1)-(3) with initial data given by (8) and (9), satisfying the energy equality (18) and the energy inequality (20).

3. Existence of weak solutions

To establish the existence of weak solutions for (1)-(3)-(16)-(17), we construct approximating systems inspired by those introduced by Bresch and Desjardins in [7], Kitavtsev et al. in [1], and Zongo et al. in [9]. A small parameter ε is introduced, and we primarily follow the approach outlined in [9]. For a given $\varepsilon > 0$, the approximate system is globally well-posed, with h remaining both bounded and strictly positive. The global existence of weak solutions is achieved by letting $\varepsilon \to 0$ and employing stability arguments as detailed in [2, 6, 9]. As in [7, 9], the pressure term $V(h)$ does not require regularization; however, it is necessary to sufficiently regularize h to control additional higher-order terms that appear in the entropy equality.

3.1 *Approximating systems*

The approximating systems we examine are defined as follows

$$\partial_t h_{1_\varepsilon} + \partial_x(h_{1_\varepsilon} u_\varepsilon) = 0, \tag{23}$$

$$\partial_t(h_{1_\varepsilon} u_\varepsilon) + \partial_x(h_{1_\varepsilon} {u_\varepsilon}^2) + \frac{1}{2} g \partial_x (h_{1_\varepsilon})^2 - 4\nu_1 \partial_x(h_{1_\varepsilon} \partial_x u_\varepsilon) + \frac{u_\varepsilon}{\beta} - h_{1_\varepsilon} \partial_x(\sigma \partial_x^2 h_{1_\varepsilon} - V(h_{1_\varepsilon}))$$

$$+ rgh_{1_\varepsilon} \partial_x h_{2_\varepsilon} + rgh_{2_\varepsilon} \partial_x (h_{1_\varepsilon} + h_{2_\varepsilon}) - \varepsilon h_{1_\varepsilon}(\partial_x^7 h_{1_\varepsilon} + \partial_x^3 h_{1_\varepsilon}) + \varepsilon^2 \partial_x^4 u_\varepsilon = 0, \tag{24}$$

$$\partial_t h_{2_\varepsilon} + \partial_x(h_{2_\varepsilon} u_\varepsilon) - \varepsilon(\partial_x^6 h_{2_\varepsilon} + \partial_x^4 h_{2_\varepsilon}) - \varepsilon \partial_x^2 h_{2_\varepsilon} - \partial_x\left((a{h_{2_\varepsilon}}^2 + b(h_{2_\varepsilon})^3)\partial_x p_{2_\varepsilon}\right) = 0, \tag{25}$$

with

$$\partial_x p_{2_\varepsilon} = \rho_2 g \partial_x (h_{1_\varepsilon} + h_{2_\varepsilon}) \quad \text{and} \quad V(h_{1_\varepsilon}) = \frac{1}{(h_{1_\varepsilon})^3} - \frac{\alpha}{(h_{1_\varepsilon})^4} \quad (\alpha > 0), \tag{26}$$

where $(t, x) \in (0, T) \times [0, 1]$ and ε is a small parameter. Consider (23)-(25) with boundary condions

$$u_\varepsilon = \partial_x^2 u_\varepsilon = \partial_x h_{1_\varepsilon} = \partial_x^3 h_{1_\varepsilon} = \partial_x^5 h_{1_\varepsilon} = \partial_x h_{2_\varepsilon} = \partial_x^5 h_{2_\varepsilon} = 0, \quad (t, x) \in (0, T) \times \{0, 1\}. \tag{27}$$

and initial data

$$h_{1_\varepsilon}^0, h_{2_\varepsilon}^0 \in H^1(0, 1), \quad u_\varepsilon^0 \in L^2(0, 1),$$

$$u_\varepsilon(x, 0) = u_\varepsilon^0(x), \quad h_{1_\varepsilon}(x, 0) = h_{1_\varepsilon}^0(x) > 0 \quad \text{and} \quad h_{2_\varepsilon}(x, 0) = h_{2_\varepsilon}^0(x) > 0, \quad \text{in} \quad (0, 1), \tag{28}$$

where u_ε^0, $h_{1_\varepsilon}^0$ and $h_{2_0}^\varepsilon$ are smooth functions such as

$$u_\varepsilon^0 \to u_{1,0} \quad \text{in} \quad L^2(0, 1), \quad h_{1_\varepsilon}^0 \to h_{1_0}, \quad h_{2_\varepsilon}^0 \to h_{2_0} \quad \text{in} \quad H^1(0, 1) \quad \text{and}$$

$$\varepsilon h_{1_\varepsilon}^0 \to 0, \quad \varepsilon h_{2_\varepsilon}^0 \to 0 \quad \text{in} \quad H^3(0, 1) \quad \text{as} \quad \varepsilon \to 0. \tag{29}$$

We have the following energy inequality.

Lemma 4 For classical solutions of the system (23)-(25), the following inequality holds

$$E(u_\varepsilon, h_{1_\varepsilon}, h_{2_\varepsilon}) + 4\nu_1 \int_0^T \int_0^1 h_{1_\varepsilon} |\partial_x u_\varepsilon|^2 + \frac{1}{\beta} \int_0^T \int_0^1 |u_\varepsilon|^2 + \frac{1}{2} rg\varepsilon \int_0^T \int_0^1 |\partial_x^2 h_{2_\varepsilon}|^2$$

$$+ \frac{1}{2} gr\varepsilon \int_0^T \int_0^1 |\partial_x h_{2_\varepsilon}|^2 + rg^2\rho_2 \int_0^T \int_0^1 (h_{1_\varepsilon})^2 (a + bh_{2_\varepsilon}) (\partial_x (h_{1_\varepsilon} + h_{2_\varepsilon}))^2 + \varepsilon^2 \int_0^T \int_0^1 |\partial_x^2 u_\varepsilon|^2$$

$$+ \frac{1}{2} rg\varepsilon \int_0^T \int_0^1 |\partial_x^3 h_{2_\varepsilon}|^2 \leq \frac{1}{2} rg \int_0^T \int_0^1 \left[\varepsilon |\partial_x h_{1_\varepsilon}|^2 + \varepsilon |\partial_x^2 h_{1_\varepsilon}|^2 + \varepsilon |\partial_x^3 h_{1_\varepsilon}|^2 \right] + E(u_\varepsilon^0, h_{1_\varepsilon}^0, h_{2_\varepsilon}^0), \tag{30}$$

where

$$E(u_\varepsilon, h_{1_\varepsilon}, h_{2_\varepsilon}) := \int_0^1 \left[\frac{1}{2} h_{1_\varepsilon} |u_\varepsilon|^2 + U(h_{1_\varepsilon}) + \frac{1}{2} g(1-r) |h_{1_\varepsilon}|^2 + \frac{1}{2} rg |h_{1_\varepsilon} + h_{2_\varepsilon}|^2 \right.$$

$$\left. + \frac{1}{2} \sigma |\partial_x h_{1_\varepsilon}|^2 + \varepsilon \frac{1}{2} |\partial_x^2 h_{1_\varepsilon}|^2 + \frac{\varepsilon}{2} |\partial_x^3 h_{1_\varepsilon}|^2 \right],$$

and

$$U(h_1) = -\frac{1}{2h_1^2} + \frac{\alpha}{3h_1^3}, \quad h_1 > 0.$$

Remark 6 Observe that the two terms on the right-hand side can be managed using Gronwall's lemma.
Proof. See [2, 9]. □
Remark 7 Let $(h_{1_\varepsilon}, h_{2_\varepsilon}, u_\varepsilon)$ be a solution of model (23)-(25). Then, thanks to the energy inequality, we have:

$$\sqrt{\frac{1}{2} g(1-r)} h_{1_\varepsilon} \in L^\infty(0, T; L^2(0, 1)),$$

$$\sqrt{\frac{1}{2} \sigma} \partial_x h_{1_\varepsilon} \in L^\infty(0, T; L^2(0, 1)),$$

$$\sqrt{\frac{1}{2} rg} (h_{1_\varepsilon} + h_{2_\varepsilon}) \in L^\infty(0, T; L^2(0, 1)),$$

$$h_{2_\varepsilon} \in L^\infty(0, T; L^2(0, 1)),$$

$$\sqrt{a + bh_{2_\varepsilon}} \in L^\infty(0, T; L^p(0, 1)), \quad p \geq 4,$$

$$h_{2\varepsilon} \partial_x (h_{1_\varepsilon} + h_{2_\varepsilon}) \in L^\infty(0, T; L^p(0, 1)), \quad p \geq 4,$$

$$h_{2_\varepsilon}\partial_x h_{2_\varepsilon} \in L^2(0, T; L^2(0, 1)),$$

$$\sqrt{\frac{1}{2}}\sqrt{h_{1_\varepsilon}}u_\varepsilon \in L^\infty(0, T; L^2(0, 1)),$$

$$2\sqrt{\nu_1}\sqrt{h_{1_\varepsilon}}\partial_x u_\varepsilon \in L^2(0, T; L^2(0, 1)),$$

$$\frac{1}{\sqrt{\beta}}u_\varepsilon \in L^2(0, T; L^2(0, 1)),$$

$$g\sqrt{r\rho_2}h_{2\varepsilon}\sqrt{a+bh_{2\varepsilon}}\,(\partial_x(h_{1_\varepsilon}+h_{2\varepsilon})) \in L^2(0, T; L^2(0, 1)),$$

$$\sqrt{\alpha}(h_{1_\varepsilon})^{-\frac{3}{2}} \in L^\infty(0, T; L^2(0, 1)),$$

$$\sqrt{\frac{\varepsilon}{2}}\partial_x^3 h_{1_\varepsilon} \in L^\infty(0, T; L^2(0, 1)),$$

$$\sqrt{gr\varepsilon}\partial_x h_{2_\varepsilon} \in L^2(0, T; L^2(0, 1)),$$

$$\sqrt{gr\varepsilon}\partial_x^2 h_{2_\varepsilon} \in L^2(0, T; L^2(0, 1)),$$

$$\sqrt{\frac{\varepsilon}{2}}\partial_x^2 h_{1_\varepsilon} \in L^\infty(0, T; L^2(0, 1)),$$

$$\varepsilon\partial_x^2 u_\varepsilon \in L^2(0, T; L^2(0, 1)).$$

The following lemma provides the entropy inequality required to establish a bound on $\partial_x\sqrt{h_{1_\varepsilon}}$.

Lemma 5 For smooth solutions $(h_{1_\varepsilon},\ h_{2_\varepsilon},\ u_\varepsilon)$ of model (23)-(25) satisfying the classical energy equality of the Lemma 4, we have the following mathematical BD entropy inequality:

$$S(u_\varepsilon, h_{1_\varepsilon}, h_{2_\varepsilon}) + \frac{1}{\beta}\int_0^T\int_0^1 |u_\varepsilon|^2 + 4\nu_1\int_0^T\int_0^1\left(g+gr\frac{h_{2\varepsilon}}{h_{1_\varepsilon}}V'(h_{1_\varepsilon})\right)|\partial_x h_{1_\varepsilon}|^2$$

$$+4rg\nu_1\int_0^T\int_0^1\left(1+\frac{h_{1_\varepsilon}}{h_{1_\varepsilon}}\right)\partial_x h_{1_\varepsilon}\partial_x h_{2_\varepsilon} + 4\nu_1\sigma\int_0^T\int_0^1|\partial_x^2 h_{1_\varepsilon}|^2$$

$$+rg^2\rho_2\int_0^T\int_0^1(h_{1_\varepsilon})^2(a+bh_{2_\varepsilon})(\partial_x(h_{1_\varepsilon}+h_{2_\varepsilon}))^2 + \frac{1}{2}rg\varepsilon\int_0^T\int_0^1|\partial_x^2 h_{2_\varepsilon}|^2$$

$$+\int_0^T\int_0^1\left[\varepsilon^2|\partial_x^2 u_\varepsilon|^2+4\nu\varepsilon|\partial_x^4 h_{1_\varepsilon}|^2+4\nu\varepsilon^2\partial_x^2 u_\varepsilon\partial_x^3 \log h_{1_\varepsilon}\right]$$

$$\leq \frac{1}{2}rg\int_0^T\int_0^1\left[\varepsilon|\partial_x h_{1_\varepsilon}|^2+\varepsilon|\partial_x^2 h_{1_\varepsilon}|^2+\varepsilon|\partial_x^3 h_{1_\varepsilon}|^2\right]+S(u_\varepsilon^0,\, h_{1_\varepsilon}^0,\, h_{2_0}^\varepsilon), \tag{31}$$

where

$$S(u_\varepsilon,\, h_{1_\varepsilon},\, h_{2_\varepsilon}):=\int_0^1\left[\frac{1}{2}h_{1_\varepsilon}|u_\varepsilon+\partial_x\varphi(h_{1_\varepsilon})|^2-\frac{1}{\beta}\varphi(h_{1_\varepsilon})+\frac{1}{2}rg|h_{1_\varepsilon}+h_{2_\varepsilon}|^2+\frac{1}{2}g(1-r)|h_{1_\varepsilon}|^2\right.$$

$$\left.+\frac{1}{2}\sigma|\partial_x h_{1_\varepsilon}|^2+U(h_{1_\varepsilon})+\frac{\varepsilon}{2}|\partial_x^2 h_{1_\varepsilon}|^2+\frac{\varepsilon}{2}|\partial_x^3 h_{1_\varepsilon}|^2\right].$$

Proof. See [2, 9]. □

Remark 8 In the Lemma 5 all the terms, excepted

$$\int_0^T\int_0^1\left(\varepsilon+4\nu_1\frac{h_{2_\varepsilon}}{h_{1_\varepsilon}}\right)\partial_x h_{1_\varepsilon}\partial_x h_{2_\varepsilon}\quad \int_0^1\int_0^T V'(h_{1_\varepsilon})|\partial_x h_{1_\varepsilon}|^2 \quad\text{and}\quad \int_0^T\int_0^1 4\nu\varepsilon^2\partial_x^2 u_\varepsilon\partial_x^3 \log h_{1_\varepsilon}$$

are controlled since they have the good sign. The control of the term $\int_0^T\int_0^1 |u_\varepsilon|^2 u_\varepsilon\partial_x h_{1_\varepsilon}$ takes inspiration in [5]. The terms $\int_0^1\int_0^T V'(h_{1_\varepsilon})|\partial_x h_{1_\varepsilon}|^2$ and $\int_0^T\int_0^1 4\nu\varepsilon^2\partial_x^2 u_\varepsilon\partial_x^3\log h_{1_\varepsilon}$ can be controlled thanks to the works done in [1, 2]. It remains for us to control the terms $\int_0^T\int_0^1\left(\varepsilon+4\nu_1\frac{h_{2_\varepsilon}}{h_{1_\varepsilon}}\right)\partial_x h_{1_\varepsilon}\partial_x h_{2_\varepsilon}$.

Proposition 3 There exists a constant C such as

$$\left|\left|\left(\varepsilon+4\nu_1\frac{h_2}{h_1}\right)\partial_x h_1\partial_x h_2\right|\right|_{L^2(0,\,T;\,L^2(0,\,1))}\leq C.$$

Proof. We follow the ideas performed in [2, 9]. We have:

$$\int_0^T\int_0^1\varepsilon\partial_x h_{1_\varepsilon}\partial_x h_{2_\varepsilon}\leq\frac{1}{2}\int_0^T\int_0^1(|\,\partial_x h_{1_\varepsilon}\,|^2+|\,\varepsilon^2\partial_x h_{2_\varepsilon}\,|^2)\leq C'$$

because $\partial_x h_{1_\varepsilon}$, $\varepsilon\partial_x h_{2_\varepsilon}$ are in $L^\infty(0,\, T;\, L^2(0,\, 1))$.

For the control of the term $\frac{h_{2_\varepsilon}}{h_{1_\varepsilon}}\partial_x h_{1_\varepsilon}\partial_x h_{2_\varepsilon}$, see the proof Proposition 2. □

Proposition 4 If h_{1_ε} satisfies the regularity properties stated in Corollary 1, then there exist constants c_1 and c_2 such that $0<c_1<h_{1_\varepsilon}<c_2$.

Proof. [1, 2, 9]. □

Remark 9 For $\varepsilon > 0$, the equation (23) is parabolic in u_ε, while equations (24) and (25) are parabolic in h_{1_ε} and h_{2_ε}, respectively. Based on the work of Bresch and Desjardins in [7], Kitavtsev et al. in [1], and Zongo et al. in [9], the system (23)-(25) along with (28)-(29) admits a unique classical solution, at least locally in time. Furthermore, as shown in [1, 9], Proposition 3 combined with the regularity properties stated above ensures the global-in-time solvability of (23)-(25) with (28)-(29).

Lemma 6 For classical solutions of the system (23)-(25), where the first component is h_{1_ε}, it holds that

$$\frac{1}{4}\int_0^1 h_{1_\varepsilon}|\partial_x\varphi(h_{1_\varepsilon})|^2 \leq \frac{1}{2}\int_0^1 h_{1_\varepsilon}(u_\varepsilon+\partial_x\varphi(h_{1_\varepsilon}))^2+2E(h_{1_\varepsilon},\, h_{2_\varepsilon},\, u_\varepsilon)+\frac{1}{3\alpha^2} \tag{32}$$

with

$$E(h_{1_\varepsilon},\, h_{2_\varepsilon},\, u_\varepsilon)=\int_0^1\left[\frac{1}{2}h_{1_\varepsilon}|u_\varepsilon|^2+U(h_{1_\varepsilon})+\frac{1}{2}g(1-r)|h_{1_\varepsilon}|^2+\frac{1}{2}rg|h_{1_\varepsilon}+h_{2_\varepsilon}|^2+\frac{1}{2}\sigma|\partial_x h_{1_\varepsilon}|^2+\frac{\varepsilon}{2}|\partial_x^3 h_{1_\varepsilon}|^2\right].$$

Proof. See [2]. □

Corollary 3 Let $(h_{1_\varepsilon},\, h_{2_\varepsilon},\, u_\varepsilon)$ be a solution to the model (1)-(3).
Then, by applying Lemma 6 and the BD entropy equality, we obtain:

$$\sqrt{h_{1_\varepsilon}} \in L^\infty(0,\, T;\, L^2(0,\, 1)),$$

$$\partial_x\sqrt{h_{1_\varepsilon}} \in L^\infty(0,\, T;\, L^2(0,\, 1)),$$

$$\partial_x^2 h_{1_\varepsilon} \in L^2(0,\, T;\, L^2(0,\, 1)),$$

$$\sqrt{\varepsilon}\partial_x^2 h_{2_\varepsilon} \in L^2(0,\, T;\, L^2(0,\, 1)),$$

$$2\sqrt{\varepsilon}\partial_x^4 h_{1_\varepsilon} \in L^2(0,\, T;\, L^2(0,\, 1)).$$

Remark 10
1. Using the Remark 5, we have

$$h_{1_\varepsilon}u_\varepsilon \in L^\infty(0,\, T;\, L^2(0,\, 1))$$

and

$$\partial_t h_{1_\varepsilon} \in L^\infty(0,\, T;\, W^{-1,\,2}(0,\, 1)).$$

2. $\sqrt{h_{1_\varepsilon}}$ is bounded in $L^2(0,\, T;\, H^1(0,\, 1))$.

Remark 11 Since $0 < c_1 < h_{1_\varepsilon} < c_2$ holds uniformly with respect to ε, the limit h_{1_ε} is both bounded and strictly positive. Consequently, the limit system can be divided by h_{1_ε}, rendering it parabolic with respect to the velocity u_ε.

Following the reasoning in [1, 7, 9], the initial-boundary value problem for the system (23)-(25) with (27)-(28) admits a unique classical solution, at least locally in time. We now proceed to define the weak formulation of the problem (23)-(25) with the boundary conditions (27). Let $(h_{1_0}, h_{2_0}, u_{1_0})$ be given and satisfy (29).

Definition 1 A triplet $(h_{1_\varepsilon}, h_{2_\varepsilon}, u_\varepsilon)$ is a global weak solution to (23)-(25) with boundary conditions (27) and initial conditions $(h_{1_0}, h_{2_0}, u_{1_0})$ if h_{1_0}, h_{2_0}, and u_{1_0} possess the regularity properties stated earlier in this section, and the following conditions are satisfied:

$$h^0_{1_\varepsilon}\phi(0,.) - \int_0^T\int_0^1 h^\varepsilon_1\partial_t\phi - \int_0^T\int_0^1 h^\varepsilon_1 u^\varepsilon_1\partial_x\phi = 0, \tag{33}$$

$$-h^\varepsilon_{2_0}\phi(0,.) - \int_0^T\int_0^1 h_{2_\varepsilon}\partial_t\phi - \int_0^T\int_0^1 h_{2_\varepsilon}u_\varepsilon\partial_x\phi + \varepsilon\int_0^T\int_0^1 \partial_x h_{2_\varepsilon}\partial_x\phi$$

$$+\int_0^T\int_0^1 \left((a{h_{2_\varepsilon}}^2 + b{h_{2_\varepsilon}}^3)\partial_x p_{2_\varepsilon}\right)\partial_x\phi - \varepsilon\int_0^1\int_0^T \partial_x^6 h_{2_\varepsilon}\phi - \varepsilon\int_0^1\int_0^T \partial_x^4 h_{2_\varepsilon}\phi = 0, \tag{34}$$

$$h^0_{1_\varepsilon}u^0_\varepsilon\phi(0,.) - \int_0^T\int_0^1 h_{1_\varepsilon}u_\varepsilon\partial_t\phi - \int_0^T\int_0^1 h_{1_\varepsilon}{u_\varepsilon}^2\partial_x\phi + 4\nu_1\int_0^T\int_0^1 h_{1_\varepsilon}\partial_x u_\varepsilon\partial_x\phi + \frac{1}{\beta}\int_0^T\int_0^1 u_\varepsilon\phi$$

$$+\int_0^T\int_0^1 (\sigma\partial_x^2 h_{1_\varepsilon} - V(h_{1_\varepsilon}))\phi\partial_x h_{1_\varepsilon} + \int_0^T\int_0^1 (\sigma\partial_x^2 h_{1_\varepsilon} - V(h_{1_\varepsilon}))h_{1_\varepsilon}\partial_x\phi - \frac{1}{2}g\int_0^T\int_0^1 (h_{1_\varepsilon})^2\partial_x\phi$$

$$-rg\int_0^T\int_0^1 h_{2_\varepsilon}h_{1_\varepsilon}\partial_x\phi + r_1\int_0^T\int_0^1 h_{1_\varepsilon}|u_\varepsilon|^2 u_\varepsilon\phi - rg\int_0^T\int_0^1 \phi h_{2_\varepsilon}\partial_x h_{1_\varepsilon} - \varepsilon\int_0^1\int_0^T \partial_x^2 u_\varepsilon\partial_x^2\phi$$

$$-rg\int_0^T\int_0^1 (h_{1_\varepsilon}+h_{2_\varepsilon})h_{2_\varepsilon}\partial_x\phi - rg\int_0^T\int_0^1 (h_{1_\varepsilon}+h_{2_\varepsilon})\partial_x h_{2_\varepsilon}\phi + \varepsilon\int_0^1\int_0^T h_{1_\varepsilon}\partial_x^7 h_{1_\varepsilon}\phi - \varepsilon h_{1_\varepsilon}\int_0^1\int_0^T \partial_x^3 h_{1_\varepsilon}\phi = 0, \tag{35}$$

for all $\phi \in C_0^\infty([0,\infty)\times[0,1])$ such that $\phi(T,.) = 0$.

We now show that solutions to the system (23)-(25) with boundary and initial conditions (27)-(28) converge to a solution of (13)-(15) as $\varepsilon \longrightarrow 0$.

Theorem 2 For any positive σ, β and initial data (h_{1_0}, h_{2_0}, u_0) satisfaying (29), there exists a global weak solution to the system (1)-(3) with boundary conditions (8) and initial conditions (9) in the sense of (13)-(15).

3.2 *Convergences*

In this section, we present a proof of Theorem 2. Let $(h_{1_{\varepsilon_k}}, h_{2_{\varepsilon_k}}, u_{1_{\varepsilon_k}})$ be a sequence of weak solutions with the given initial data

$$h_{1_{\varepsilon_k}}|_{t=0} = h^0_{1_{\varepsilon_k}}, \quad h_{2_{\varepsilon_k}}|_{t=0} = h^0_{2_{\varepsilon_k}}, \quad (h_{1_{\varepsilon_k}}u_{\varepsilon_k})|_{t=0} = m^0_{\varepsilon_k}$$

such as

$$h^0_{1_{\varepsilon_k}} \longrightarrow h_{1_0} \text{ in } H^1(\Omega), \quad h^0_{2_{\varepsilon_k}} \longrightarrow h_{2_0} \text{ in } H^1(\Omega), \quad m^0_{\varepsilon_k} \longrightarrow m_0 \text{ in } (L^1(\Omega))^2,$$

and fulfilling the following inequality:

$$-\frac{1}{\beta}\int_0^1 \varphi(h^0_{1_{\varepsilon_k}})+\int_0^1\left[h^0_{1_{\varepsilon_k}}|u^0_{1_{\varepsilon_k}}|^2+64\nu_1^2|\partial_x\sqrt{h^0_{1_{\varepsilon_k}}}|^2+\frac{1}{2}g(1-r)|h^0_{1_{\varepsilon_k}}|^2+\frac{1}{2}rg|h^0_{1_{\varepsilon_k}}+h^0_{2_{\varepsilon_k}}|^2\right.$$

$$\left.+\frac{1}{2}\sigma|\partial_x h^0_{1_{\varepsilon_k}}|^2+\frac{\varepsilon}{2}|\partial_x^3 h^0_{1_{\varepsilon_k}}|^2\right]\leq C.$$

Consider a sequence $\{\varepsilon_k\}_{k\geq 1}$ such that $\varepsilon_k\to 0$. For each $k\geq 1$, let $(h_{1_{\varepsilon_k}}, h_{2_{\varepsilon_k}}, u_{\varepsilon_k})$ denote the corresponding solution to the approximate system (23)-(28) with $\varepsilon=\varepsilon_k$.

3.2.1 *Convergence of $\sqrt{h_{1_{\varepsilon_k}}}$, $h_{1_{\varepsilon_k}}$ and $h_{2_{\varepsilon_k}}$*

From the Remark 10:

$$\sqrt{h_{1_{\varepsilon_k}}}\quad\text{is bounded in }\ L^\infty(0,T;H^1(\Omega)). \tag{36}$$

Furthermore, by applying the mass equation, we derive the following equality:

$$\partial_t\sqrt{h_{1_{\varepsilon_k}}}=\frac{1}{2}\sqrt{h_{1_{\varepsilon_k}}}\partial_x u_{\varepsilon_k}-\partial_x\left(\sqrt{h_{1_{\varepsilon_k}}}u_{\varepsilon_k}\right),$$

which gives that $\partial_t\sqrt{h_{1_{\varepsilon_k}}}$ is bounded in $L^2(0,T;H^{-1}(\Omega))$.

Using the Aubin-Simon lemma (see [11, 12]), we can extract a subsequence, still denoted $(h_{1_{\varepsilon_k}})_{1\leq k}$, such that

$$\sqrt{h_{1_{\varepsilon_k}}}\ \text{ strongly converges to }\ \sqrt{h_1}\ \text{ in }\ C^0(0,T;L^2(0,1)).$$

According to the Proposition 4, we show that

$$\left|h_{1_{\varepsilon_k}}-h_1\right|\leq\sqrt{c_2}\left|\sqrt{h_{1_{\varepsilon_k}}}-\sqrt{h_1}\right|\Rightarrow\left|h_{1_{\varepsilon_k}}-h_1\right|^2\leq c_2\left|\sqrt{h_{1_{\varepsilon_k}}}-\sqrt{h_1}\right|^2.$$

This ensures

$$h_{1_{\varepsilon_k}}\ \text{ strongly converges to }\ h_1\ \text{ in }\ L^2(0,T;L^2(0,1)).$$

We have $h_{2_{\varepsilon_k}}$ bounded in $L^2(0,T;H^1(0,1))$. Moreover, we have

$$\partial_t h_{2_{\varepsilon_k}}=-\partial_x(h_{2_{\varepsilon_k}}u_{\varepsilon_k})+\varepsilon_k(\partial_x^6 h_{2_{\varepsilon_k}}+\partial_x^4 h_{2_{\varepsilon_k}})+\partial_x((ah_{2_{\varepsilon_k}}+b(h_{2_{\varepsilon_k}})^3)\partial_x p_{2_{\varepsilon_k}}).$$

Let us study each term separately

• Since $h_{2_{\varepsilon_k}}$ is in $L^\infty(0, T; L^2(0, 1))$ and u_{ε_k} is in $L^2(0, T; L^2(0, 1))$, we show that the first term is in $L^2(0, T; W^{-1, 1}(0, 1))$.

• For the second term, since $\sqrt{\varepsilon}\partial_x h_2^{\varepsilon_k}$ is in $L^2(0, T; L^2(0, 1))$, we have $\sqrt{\varepsilon}\partial_x^2 h_2^{\varepsilon_k}$ in $L^2(0, T; W^{-1, 1}(0, 1))$.

• For the third term, for any $\psi \in C_0^\infty((0, 1)\times(0, T))$, by applying integration by parts and utilizing the regularities established in the previous section,

$$\left|\int_0^T \int_0^1 \psi \partial_x^6 h_{2_{\varepsilon_k}}\right| \leq \|\partial_x^2 \psi\|_{L^2(0, T; L^2(0, 1))}\|\partial_x^4 h_{2_{\varepsilon_k}}\|_{L^2(0, T; W^{-1, 1}(0, 1))}$$

$$\leq C\|\partial_x^4 h_{2_{\varepsilon_k}}\|_{L^2(0, T; W^{-1, 1}(0, 1))}.$$

$$\left|\int_0^T \int_0^1 \psi \partial_x^4 h_{2_{\varepsilon_k}}\right| \leq \|\partial_x^2 \psi\|_{L^2(0, T; L^2(0, 1))}\|\partial_x^3 h_{2_{\varepsilon_k}}\|_{L^2(0, T; W^{-1, 1}(0, 1))}$$

$$\leq C\|\partial_x^3 h_{2_{\varepsilon_k}}\|_{L^2(0, T; W^{-1, 1}(0, 1))}.$$

• For the last term, as $h_{2_{\varepsilon_k}}\sqrt{a+bh_{2_{\varepsilon_k}}}\left(\partial_x(h_{1_{\varepsilon_k}}+h_{2_{\varepsilon_k}})\right)$ is in $L^2(0, T; L^2(0, 1))$, we have $\partial_x\big(h_{2_{\varepsilon_k}}\sqrt{a+bh_{2_{\varepsilon_k}}}\big(\partial_x(h_{1_{\varepsilon_k}}+h_{2_{\varepsilon_k}})\big)\big)$ in $L^2(0, T; W^{-1, 1}(0, 1))$.

Thus, the third term belongs to $L^2(0, T; W^{-1, 1}(0, 1))$, which implies that $\partial_t h_{2_{\varepsilon_k}}$ is also in $L^2(0, T; W^{-1, 1}(0, 1))$.

By the Aubin-Simon lemma, we conclude that:

$$h_{2_{\varepsilon_k}} \text{ strongly converges to } h_2 \text{ in } L^2(0, T; W^{-1, 1}(0, 1)).$$

3.2.2 *Convergence of* $h_{1_{\varepsilon_k}} u_{\varepsilon_k}$

According to the Remark 10, $u_{\varepsilon_k} \in L^2(0, T; H^1(0, 1))$. This result, combined with Lemma 4, enables us to obtain

$$(h_{1_{\varepsilon_k}} u_{\varepsilon_k}) \quad \text{in} \quad L^2(0, T; H^1(0, 1)) \tag{37}$$

Furthermore, the momentum equation (2) allows us to express the time derivative of the water discharge as follows:

$$\begin{aligned}\partial_t(h_{1_{\varepsilon_k}} u_{\varepsilon_k}) = &-\partial_x(h_{1_{\varepsilon_k}} {u_{\varepsilon_k}}^2) - \frac{1}{2} g\partial_x {h_{1_{\varepsilon_k}}}^2 + 4\nu_1\partial_x(h_{1_{\varepsilon_k}}\partial_x u_{\varepsilon_k}) - \frac{u_{\varepsilon_k}}{\beta} + h_{1_{\varepsilon_k}}\partial_x(\sigma\partial_x^2 h_{1_{\varepsilon_k}} - V(h_{1_{\varepsilon_k}}))\\ &- rgh_{1_{\varepsilon_k}}\partial_x h_{2_{\varepsilon_k}} - rgh_{2_{\varepsilon_k}}\partial_x(h_{1_{\varepsilon_k}}+h_{2_{\varepsilon_k}}) + \varepsilon_k h_{1_{\varepsilon_k}}(\partial_x^7 h_{1_{\varepsilon_k}} + \partial_x^3 h_{1_{\varepsilon_k}}) - \varepsilon_k^2\partial_x^4 u_{\varepsilon_k}.\end{aligned} \tag{38}$$

We then study each term:

• $\partial_x(h_{1_{\varepsilon_k}}(u_{\varepsilon_k})^2) = \partial_x((h_{1_{\varepsilon_k}} u_{\varepsilon_k})u_{\varepsilon_k})$ which is in $L^2(0, T; W^{-1, 1}(0, 1))$.

• As $h_{1_{\varepsilon_k}}$ is in $L^\infty(0, T; L^2(0, 1))$, we have:

$$\partial_x\left[(h_{1\varepsilon_k})^2\right] \text{ is in } L^\infty(0,\, T;\, W^{-1,\,1}(0,\, 1)).$$

- $\partial_x(h_{1\varepsilon_k}\partial_x u_{\varepsilon_k}) = \partial_x\left(\sqrt{h_{1\varepsilon_k}}\sqrt{h_{1\varepsilon_k}}\partial_x u_{\varepsilon_k}\right)$ is bounded in $L^2(0,\, T;\, W^{-1,\,1}(0,\, 1))$.
- $rgh_{1\varepsilon_k}\partial_x h_{2\varepsilon_k}$ is bounded in $L^2(0,\, T;\, W^{-1,\,1}(0,\, 1))$.
- $h_{1\varepsilon_k}\partial_x\partial_x^2 h_{1\varepsilon_k}$ is bounded in $L^\infty(0,\, T;\, W^{-1,\,1}(0,\, 1))$.
- $rgh_{2\varepsilon_k}\partial_x(h_{1\varepsilon_k}+h_{2\varepsilon_k})$ is bounded in $L^2(0,\, T;\, W^{-1,\,1}(0,\, 1))$.
- For any $\psi \in C_0^\infty((0,\, T)\times(0,\, 1))$, we obtain, using integration by parts and the regularities in the previous section,

$$\begin{aligned}
\left|\int_0^T\int_0^1 \psi h_{1\varepsilon_k}\partial_x^7 h_{1\varepsilon_k}\right| &= \left|\int_0^T\int_0^1 \partial_x^4 h_{1\varepsilon_k}\left[\psi\partial_x^3 h_{1\varepsilon_k} + 3\partial_x\psi\partial_x^2 h_{1\varepsilon_k} + 3\partial_x^2\psi\partial_x h_{1\varepsilon_k} + h_{1\varepsilon_k}\partial_x^3\psi\right]\right| \\
&\leq \int_0^T \|\partial_x^4 h_{1\varepsilon_k}\|_{L^2(0,\,1)}\left[\|\psi\|_{L^\infty(0,\,1)}\|\partial_x^3 h_{1\varepsilon_k}\|_{L^2(0,\,1)} + \|h_{1\varepsilon_k}\|_{L^\infty(0,\,1)}\|\partial_x^3\psi\|_{L^2(0,\,1)}\right. \\
&\quad \left. + 3\|\partial_x\psi\|_{L^\infty(0,\,1)}\|\partial_x^2 h_{1\varepsilon_k}\|_{L^2(0,\,1)} + 3\|\partial_x^2\psi\|_{L^\infty}\|\partial_x h_{1\varepsilon_k}\|_{L^2(0,\,1)}\right] \\
&\leq C\|\partial_x^4 h_{1\varepsilon_k}\|_{L^2(0,\,1)}^2\|\psi\|_{H^3(0,\,1)} \leq \|\psi\|_{L^2(0,\,T;\,H^3(0,\,1))},
\end{aligned}$$

$$\begin{aligned}
\left|\int_0^T\int_0^1 \psi h_{1\varepsilon_k}\partial_x V(h_{1\varepsilon_k})\right| &= \left|\int_0^T\int_0^1 \partial_x\psi V_1(h_{1\varepsilon_k})\right| \\
&\leq \|V_1(h_{1\varepsilon_k})\|_{L^\infty((0,\,1)\times(0,\,T))}\left(\int_0^T \|\psi\|_{H^1(0,\,1)}\right)^{\frac{1}{2}}
\end{aligned}$$

where

$$V_1(h_{1\varepsilon_k}) := -\int_h^\infty \tau V_1'(\tau)d\tau,$$

and

$$\begin{aligned}
\left|\int_0^T\int_0^1 \psi h_{1\varepsilon_k}\partial_x^3 h_{1\varepsilon_k}dxdt\right| &= \left|\int_0^T\int_0^1 \partial_x^2 h_{1\varepsilon_k}\left[h_{1\varepsilon_k}\partial_x\psi + \psi\partial h_{1\varepsilon_k}\right]\right| \\
&\leq \int_0^T \|\partial_x^2 h_{1\varepsilon_k}\|_{L^2(0,\,1)}\left[\|h_{1\varepsilon_k}\|_{L^\infty(0,\,1)}\|\partial_x\psi\|_{L^2(0,\,1)} + \|\psi\|_{L^\infty(0,\,1)}\|\partial_x h_{1\varepsilon_k}\|_{L^2(0,\,1)}\right] \\
&\leq C\left(\int_0^T \|\psi\|_{H^1(0,\,1)}^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Finally (u_{ε_k}) and $(\varepsilon_k \partial_x^4 u_{\varepsilon_k})$ are bounded in $L^2(0, 1; H^1(0, T))$ and $L^2(0, T; H^{-2}(0, 1))$ respectively. Collecting the above information completes the proof of the boundness of the right-hand side of (38), whence

$$\partial_t(h_{1_{\varepsilon_k}} u_{\varepsilon_k}) \text{ is bounded in } L^2(0, T; H^{-3}(0, 1)).$$

Combining this with (37) and Corollary 4 in Simon [12] ensures that $(h_{1_{\varepsilon_k}} u_{\varepsilon_k})$ is compact in $L^2((0, T); L^2(0, 1))$. So, there exists $\mathbf{m} \in L^2((0, T); L^2(0, 1))$ such that

$$h_{1_{\varepsilon_k}} u_{\varepsilon_k} \text{ converges to } \mathbf{m} \text{ in } L^2((0, T); L^2(0, 1)). \tag{39}$$

3.2.3 *Convergences of* $(h_{1_{\varepsilon_k}})^{-1}$, u_{ε_k} *and* $\sqrt{h_{1_{\varepsilon_k}}} u_{\varepsilon_k}$

• As $(h_{1_{\varepsilon_k}})_k$ strongly converges to h_1 in $L^2(0, T; W^{1,p}(0, 1)) \cap C([0, T] \times (0, 1)$ for $p \in [1, \infty)$ and we have $0 < c_1 \leq h_{1_{\varepsilon_k}} \leq c_2$, we deduce that

$$(h_{1_{\varepsilon_k}})^{-1} \text{ strongly converges to } h_1^{-1} \text{ in } C([0, T] \times (0, 1)). \tag{40}$$

• Considering (39) and (40), there exists $u_1 \in L^2(0, T; H^1(0, 1))$ such that

$$u_{\varepsilon_k} \text{ strongly converges to } u \text{ in } L^2(0, T; L^2(0, 1)). \tag{41}$$

• Since $\sqrt{h_{1_{\varepsilon_k}}}$ strongly converges to $\sqrt{h_1}$ in $C^0(0, T; L^2(0, 1))$, by using (41), $\sqrt{h_{1_{\varepsilon_k}}} u_{\varepsilon_k}$ strongly converges to $\sqrt{h_1} u$ in $L^2(0, T; L^1(0, 1))$.

3.3 *Convergences of* $\partial_x h_{1_{\varepsilon_k}}$, $\varepsilon_k \partial_x h_{2_{\varepsilon_k}}$, $h_{2_{\varepsilon_k}} \partial_x h_{1_{\varepsilon_k}}$, $\partial_x^2 h_{1_{\varepsilon_k}}$, $h_{1_{\varepsilon_k}} \partial_x^2 h_{1_\varepsilon}$ *and* $\partial_x h_{1_{\varepsilon_k}} \partial_x^2 h_{1_{\varepsilon_k}}$

• We have $\partial_x h_{1_{\varepsilon_k}}$ bounded in $L^2(0, T; H^1(0, 1))$ and $\partial_t \partial_x h_{1_{\varepsilon_k}}$ is bounded in $L^\infty(0, T; H^{-1}(0, 1))$ since $\partial_t h_{1_{\varepsilon_k}}$ is bounded in $L^\infty(0, T; H^{-1}(0, 1))$. Thanks to compact injection of $H^1(0, 1)$ in $L^2(0, 1)$ in one dimension, we have:

$$\partial_x h_{1_{\varepsilon_k}} \text{ converges strongly to } \partial_x h_1 \text{ in } L^2(0, T; L^2(0, 1)).$$

• The bound of $\partial_x^2 h_{1_{\varepsilon_k}}$ in $L^2(0, T; L^2(0, 1))$ and $\varepsilon_k \partial_x h_{2_{\varepsilon_k}}$ in $L^2(0, T; L^2(0, 1))$ gives us:

$$\partial_x^2 h_{1_{\varepsilon_k}} \text{ strongly converges to } \partial_x^2 h_1 \text{ in } L^1(0, T; L^1(0, 1)),$$

$$\varepsilon_k \partial_x h_{2_{\varepsilon_k}} \text{ strongly converges to } 0 \text{ in } L^1(0, T; L^1(0, 1)).$$

• Thanks to the strong convergence of $h_{1_{\varepsilon_k}}$, $h_{2_{\varepsilon_k}}$, $\partial_x h_{1_{\varepsilon_k}}$ and the weak convergence of $\partial_x^2 h_{1_{\varepsilon_k}}$, we have:

$$h_{2_{\varepsilon_k}} \partial_x h_{1_{\varepsilon_k}} \text{ strongly converges to } h_2 \partial_x h_1 \text{ in } L^1(0, T; L^1(0, 1)),$$

$$h_{1_{\varepsilon_k}} \partial_x^2 h_{1_{\varepsilon_k}} \text{ strongly converges to } h_1 \partial_x^2 h_1 \text{ in } L^1(0, T; L^1(0, 1)),$$

$$\partial_x h_{1_{\varepsilon_k}} \partial_x^2 h_{1_{\varepsilon_k}} \text{ converges wealkly to } \partial_x h_1 \partial_x^2 h_1 \text{ in } L^1(0, T; L^1(0, 1)),$$

$$(h_{1_{\varepsilon_k}})^2 \text{ strongly converges to } {h_1}^2 \text{ in } L^1(0, T; L^1(0, 1)),$$

$$(h_{2_{\varepsilon_k}})^2 \text{ strongly converges to } {h_2}^2 \text{ in } L^1(0, T; L^1(0, 1)),$$

$$h_{1_{\varepsilon_k}} h_{2_{\varepsilon_k}} \text{ strongly converges to } h_1 h_2 \text{ in } L^1(0, T; L^1(0, 1)).$$

• The bound of $h_{2_{\varepsilon_k}} \partial_x h_{2_{\varepsilon_k}}$ in $L^2(0, T; L^2(0, 1))$ gives us

$$h_{2_{\varepsilon_k}} \partial_x h_{2_{\varepsilon_k}} \text{ strongly converges to } h_2 \partial_x h_2 \text{ in } L^1(0, T; L^1(0, 1)).$$

• We have

$$\int_0^T \int_0^1 h_{1_{\varepsilon_k}} \partial_x h_{2_{\varepsilon_k}} \phi = -\int_0^T \int_0^1 (\phi h_{2_{\varepsilon_k}} \partial_x h_{1_{\varepsilon_k}} + h_{1_{\varepsilon_k}} h_{2_{\varepsilon_k}} \partial_x \phi).$$

Since $h_{2_{\varepsilon_k}} \partial_x h_{1_{\varepsilon_k}}$ converge weakly to $h_2 \partial_x h_1$ and $h_{1_{\varepsilon_k}} h_{2_{\varepsilon_k}}$ strongly converges to $h_1 h_2$ in $L^1(0, T; L^1(0, 1))$, so

$$h_{1_{\varepsilon_k}} \partial_x h_{2_{\varepsilon_k}} \text{ strongly converges to } h_1 \partial_x h_2 \text{ in } L^1(0, T; L^1(0, 1)).$$

3.3.1 *Convergences of $h_{1_{\varepsilon_k}} \partial_x u_{\varepsilon_k}$*

The function $(h_{1_{\varepsilon_k}}, \partial_x u_{\varepsilon_k}) \longmapsto h_{1_{\varepsilon_k}} \partial_x u_{\varepsilon_k}$ is a continuous in $L^\infty(0, T; H^1(0, 1)) \times L^2(0, T; W^{-1, 2}(0, 1))$ to $L^2(0, T; W^{-1, 2}(0, 1))$.

So,

$$h_{1_{\varepsilon_k}} \partial_x u_{\varepsilon_k} \text{ converges weakly to } h_1 \partial_x u \text{ in } L^2(0, T; H^{-1}(0, 1)).$$

3.3.2 *Convergences of $h_{2_{\varepsilon_k}} u_{\varepsilon_k}$*

We have u_{ε_k} converges weakly to u in $L^2(0, T; L^2(0, 1))$ and the strong convergence of h_2^k to h_2, gives us:

$$h_{2_{\varepsilon_k}} u_{\varepsilon_k} \quad \text{converges weakly to} \quad h_2 u \quad \text{in } L^1(0, T; L^1(0, 1)).$$

3.3.3 *Convergence of* $\left(a(h_{2_{\varepsilon_k}})^2 + b(h_{2_{\varepsilon_k}})^3)\right) \partial_x(h_{1_{\varepsilon_k}} + h_{2_{\varepsilon_k}})$

The bound of $\left(a(h_{2_{\varepsilon_k}})^2 + b(h_{2_{\varepsilon_k}})^3)\right) \partial_x(h_{1_{\varepsilon_k}} + h_{2_{\varepsilon_k}})$ in $L^2(0, T; L^2(0, 1))$ implies that

$$\left(a(h_{2_{\varepsilon_k}})^2 + b(h_{2_{\varepsilon_k}})^3)\right) \partial_x(h_{1_{\varepsilon_k}} + h_{1_{\varepsilon_k}}) \quad \text{converges weakly to} \quad (ah_2^2 + bh_2^3)\partial_x(h_1 + h_2) \quad \text{in} \quad L^1(0, T; L^1(0, 1)).$$

3.3.4 *Convergences of* $h_{1_\varepsilon} V(h_{1_{\varepsilon_k}})$ *and* $V(h_{1_{\varepsilon_k}})\partial_x h_{1_{\varepsilon_k}}$

We will start by analyzing the convergence of the term $h_1 V(h_{1_{\varepsilon_k}})$. We have $h_1 V(h_{1_{\varepsilon_k}}) = \dfrac{1}{(h_{1_{\varepsilon_k}})^2} - \dfrac{\alpha}{(h_{1_{\varepsilon_k}})^3}$ and

$$\left| \frac{1}{(h_{1_{\varepsilon_k}})^2} - \frac{\alpha}{(h_{1_{\varepsilon_k}})^3} - \left(\frac{1}{h_{1_\varepsilon}^2} - \frac{\alpha}{h_{1_\varepsilon}^3} \right) \right| \leq \left| \frac{1}{(h_{1_{\varepsilon_k}})^2} - \frac{1}{h_1^2} \right| + \left| \frac{1}{(h_{1_{\varepsilon_k}})^3} - \frac{1}{h_1^3} \right|$$

$$\left| \frac{1}{(h_{1_{\varepsilon_k}})^2} - \frac{\alpha}{(h_{1_{\varepsilon_k}})^3} - \left(\frac{1}{h_1^2} - \frac{\alpha}{h_1^3} \right) \right| \leq \frac{|h_{1_{\varepsilon_k}} - h_1||h_{1_{\varepsilon_k}} + h_1|}{(h_{1_{\varepsilon_k}})^2 h_1^2} + \frac{|h_{1_{\varepsilon_k}} - h_1||(h_{1_{\varepsilon_k}})^2 + h_{1_{\varepsilon_k}} h_1 + h_1^2|}{(h_{1_{\varepsilon_k}})^3 h_1^3}.$$

By using the Proposition 4, we find two constants δ_1 and δ_2 such as

$$\left| \frac{1}{(h_{1_{\varepsilon_k}})^2} - \frac{\alpha}{(h_{1_{\varepsilon_k}})^3} - \left(\frac{1}{h_1^2} - \frac{\alpha}{h_1^3} \right) \right| \leq \delta_1 |h_{1_{\varepsilon_k}} - h_1| + \delta_2 |h_{1_{\varepsilon_k}} - h_1|.$$

So

$$\left| \frac{1}{(h_{1_{\varepsilon_k}})^2} - \frac{\alpha}{(h_{1_{\varepsilon_k}})^3} - \left(\frac{1}{h_1^2} - \frac{\alpha}{h_1^3} \right) \right|^2 \leq \delta_3^2 |h_1^k - h_1|^2 \to 0, \quad \text{with} \quad \delta_3 = 2\max(\delta_1, \delta_2).$$

We have

$$\frac{1}{(h_{1_{\varepsilon_k}})^2} - \frac{\alpha}{(h_{1_{\varepsilon_k}})^3} \quad \text{strongly converges to} \quad \frac{1}{h_1^2} - \frac{\alpha}{h_1^3} \quad \text{in} \quad L^2(0, T; L^2(0, 1)).$$

A similar argument guarantees the strong convergence of

$$\frac{1}{(h_{1_{\varepsilon_k}})^3} - \frac{\alpha}{(h_{1_{\varepsilon_k}})^4} \quad \text{to} \quad \frac{1}{h_1^3} - \frac{\alpha}{h_1^4} \quad \text{in} \quad L^2(0, T; L^2(0, 1)).$$

The strong convergence of $\partial_x h_{1_{\varepsilon_k}}$ in $L^2(0, T; L^2(0, 1))$ gives us

$$V(h_{1_{\varepsilon_k}})\partial_x h_{1_{\varepsilon_k}} \text{ converges weakly to } V(h_1)\partial_x h_1 \text{ in } L^1(0, T; L^1(0, 1)).$$

3.3.5 *Convergence of* $\varepsilon_k^2\partial_x^2 u_{\varepsilon_k}$, $\varepsilon_k h_{1_{\varepsilon_k}}\partial_x^7 h_{1_{\varepsilon_k}}$, $\varepsilon_k\partial_x^6 h_{2_{\varepsilon_k}}$

• The bound of $\partial_x^2 u_{\varepsilon_k}$ in $L^2(0, T; L^2(0, 1))$ gives us

$$\partial_x^2 u_{\varepsilon_k} \text{ strongly converges to } \partial_x^4 u \text{ in } L^1(0, T; L^1(0, 1)).$$

Then,

$$\varepsilon_k\partial_x^2 u_{\varepsilon_k} \text{ strongly converges to } 0 \text{ in } L^1(0, T; L^1(0, 1)).$$

• We have:

$$\int_0^T\int_0^1 h_{1_{\varepsilon_k}}\partial_x^7 h_{1_{\varepsilon_k}}\phi = -3\int_0^T\int_0^1 \partial_x^2 h_{1_{\varepsilon_k}}\partial_x\phi\partial_x^4 h_{1_{\varepsilon_k}} - 3\int_0^T\int_0^1 \partial_x h_{1_{\varepsilon_k}}\partial_x^2\phi\partial_x^4 h_{1_{\varepsilon_k}}$$

$$-\int_0^T\int_0^1 \phi\partial_x^3 h_{1_{\varepsilon_k}}\partial_x^4 h_{1_{\varepsilon_k}} - \int_0^T\int_0^1 h_{1_{\varepsilon_k}}\partial_x^3\phi\partial_x^4 h_{1_{\varepsilon_k}}$$

The function $(\partial_x^2 h_{1_{\varepsilon_k}}, \partial_x^4 h_{1_{\varepsilon_k}}) \longmapsto \partial_x^2 h_{1_{\varepsilon_k}}\partial_x^4 h_{1_{\varepsilon_k}}$ is a continuous in $L^\infty(0, T; H^1(0, 1)) \times L^2(0, T; W^{-1,2}(0, 1))$ to $L^2(0, T; W^{-1,2}(0, 1))$, so $(\partial_x^2 h_{1_{\varepsilon_k}}\partial_x^4 h_{1_{\varepsilon_k}})_k$ converges weakly to $\partial_x^2 h_{1_\varepsilon}\partial_x^4 h_{1_\varepsilon}$ in $L^2(0, T; W^{-1,2}(0, 1))$. Next, the function $(\partial_x^3 h_{1_{\varepsilon_k}}, \partial_x^4 h_{1_{\varepsilon_k}}) \longmapsto \partial_x^3 h_{1_{\varepsilon_k}}\partial_x^4 h_{1_{\varepsilon_k}}$ is a continuous in $L^\infty(0, T; H^1(0, 1)) \times L^2(0, T; W^{-1,2}(0, 1))$ to $L^2(0, T; W^{-1,2}(0, 1))$, so $(\partial_x^3 h_{1_{\varepsilon_k}}\partial_x^4 h_{1_{\varepsilon_k}})_k$ converges weakly to $\partial_x^3 h_1\partial_x^4 h_1$ in $L^2(0, T; W^{-1,2}(0, 1))$.

$\partial_x h_{1_{\varepsilon_k}}\partial_x^4 h_{1_{\varepsilon_k}}$ converges weakly to $\partial_x h_1\partial_x^4 h_1$ in $L^2(0, T; L^1(0, 1))$. Finally, $h_{1_{\varepsilon_k}}\partial_x^4 h_{1_{\varepsilon_k}}$ strongly converges to $h_1\partial_x^4 h_1$ in $L^1(0, T; L^1(0, 1))$.

So, we have the weak convergence of

$$h_{1_{\varepsilon_k}}\partial_x^7 h_{1_{\varepsilon_k}} \text{ to } h_1\partial_x^7 h_1 \text{ in } L^1(0, T; L^1(0, 1))$$

so we deduce that

$$\varepsilon_k h_{1_{\varepsilon_k}}\partial_x^7 h_{1_{\varepsilon_k}} \text{ to } 0 \text{ in } L^1(0, T; L^1(0, 1)).$$

• We have:

$$\int_0^T\int_0^1 \partial_x^6 h_{2_{\varepsilon_k}}\phi = \int_0^T\int_0^1 h_{2_{\varepsilon_k}}\partial_x^6\phi$$

The bound of $h_{2_{\varepsilon_k}}$ in $L^\infty(0, T; L^2(0, 1))$, give us: the strong convergence of $h_{2_{\varepsilon_k}}$ in $L^2(0, T; W^{-1,1}(0, 1))$. So, we deduce that

$$\varepsilon_k \partial_x^6 h_{1_{\varepsilon_k}} \text{ converge strongly to } 0 \text{ in } L^2(0, T; W^{-1,1}(0, 1)).$$

3.3.6 *Convergence of* $\varepsilon_k h_{1_{\varepsilon_k}} \partial_x^3 h_{1_{\varepsilon_k}}$, $\varepsilon_k \partial_x^4 h_{2_{\varepsilon_k}}$

The function $(h_{1_{\varepsilon_k}}, \partial_x^3 h_{1_{\varepsilon_k}}) \longmapsto h_{1_{\varepsilon_k}} \partial_x^3 h_{1_{\varepsilon_k}}$ is continuous in $L^\infty(0, T; H^1(0, 1)) \times L^2(0, T; W^{-1,2}(0, 1))$ to $L^2(0, T; W^{-1,2}(0, 1))$, so $h_{1_{\varepsilon_k}} \partial_x^3 h_{1_{\varepsilon_k}}$ converges weakly to $h_1 \partial_x^3 h_1$ in $L^2(0, T; W^{-1,2}(0, 1))$. Then, we deduce that

$$\varepsilon_k h_{1_{\varepsilon_k}} \partial_x^3 h_{1_{\varepsilon_k}} \text{ converges weakly to } 0 \text{ in } L^2(0, T; W^{-1,2}(0, 1)).$$

• We have:

$$\int_0^T \int_0^1 h_{2_{\varepsilon_k}} \partial_x^4 \phi.$$

Since $h_{2_{\varepsilon_k}}$ strongly converges to h_2 in $L^2(0, T; W^{-1,1}(0, 1))$, we deduce that

$$\varepsilon_k \partial_x^4 h_{2_{\varepsilon_k}} \text{ strongly converges to } 0 \text{ in } L^2(0, T; W^{-1,1}(0, 1)).$$

Remark 12 The above convergences enable us to take the limit as $k \to \infty$ within the weak formulation (33)-(35), ensuring that (h_1, h_2, u) satisfies (13)-(15).

4. Case $\beta = \infty$

We adopt the approach outlined in [1, 9]. First, we consider a sequence of positive real numbers (β_n), $\beta_n \to \infty$, and denote the solutions of (33)-(35) for $\beta = \beta_n$ by $(h_{1_{\beta_n}}, h_{2_{\beta_n}}, u_{\beta_n})$. The corresponding systems are expressed as follows:

$$\partial_t h_{1_{\beta_n}} + \partial_x (h_{1_{\beta_n}} u_{\beta_n}) = 0, \tag{42}$$

$$\partial_t (h_{1_{\beta_n}} u_{\beta_n}) + \partial_x (h_{1_{\beta_n}} {u_{\beta_n}}^2) + \frac{1}{2} g \partial_x (h_{1_{\beta_n}})^2 - 4\nu_1 \partial_x (h_{1_{\beta_n}} \partial_x u_{\beta_n}) + \frac{u_{\beta_n}}{\beta_n} - h_{1_\eta} \partial_x (\sigma \partial_x^2 h_{1_{\beta_n}} - V(h_{1_{\beta_n}}))$$

$$+ rgh_{1_{\beta_n}} \partial_x h_{2_{\beta_n}} + rgh_{2_{\beta_n}} \partial_x (h_{1_{\beta_n}} + h_{2_{\beta_n}}) - \beta_n h_{1_{\beta_n}} (\partial_x^7 h_{1_{\beta_n}} + \partial_x^3 h_{1_{\beta_n}}) + \beta_n^2 \partial_x^4 u_{\beta_n} = 0, \tag{43}$$

$$\partial_t h_{2_{\beta_n}} + \partial_x (h_{2_{\beta_n}} u_{\beta_n}) - \beta_n (\partial_x^6 h_{2_{\beta_n}} + \partial_x^4 h_{2_{\beta_n}}) - \beta_n \partial_x^2 h_{2_{\beta_n}} - \partial_x \left((a {h_{2_{\beta_n}}}^2 + b(h_{2_{\beta_n}})^3) \partial_x p_{2_{\beta_n}} \right) = 0, \tag{44}$$

with

$$\partial_x p_{1_{\beta_n}} = \rho_2 g \partial_x (h_{1_{\beta_n}} + h_{2_{\beta_n}}) \quad \text{and} \quad V(h_{1_{\beta_n}}) = \frac{1}{(h_{1_{\beta_n}})^3} - \frac{\alpha}{(h_{1_{\beta_n}})^4} \quad (\alpha > 0), \tag{45}$$

where $(t, x) \in (0, T) \times [0, 1]$ and η_n represents a small parameter.

For the system (42)-(44), the statements of Remark 5, Corollary 3 and the Proposition 4 remain valid for the weak solutions of (33)-(35). This allows us to examine the behavior of these solutions as $\beta \to \infty$. Although the estimate for (u_{β_n}/β_n) becomes irrelevant in this case, one can still obtain an estimate for (u_{β_n}) in $L^2(0, T; H_0^1(0, 1))$ using Remark 5, Corollary 3 and the Poincaré inequality. By following the reasoning in the proof of Theorem 2, we conclude that, after possibly extracting a subsequence, $(h_{1_{\beta_n}}, h_{2_{\beta_n}}, u_{\beta_n})$ converges to a weak solution of the model

$$\partial_t h_1 + \partial_x (h_1 u) = 0, \tag{46}$$

$$\partial_t (h_1 u) + \partial_x (h_1 u^2) + \frac{1}{2} g \partial_x h_1^2 - 4\nu_1 \partial_x (h_1 \partial_x u) - h_1 \partial_x (\sigma \partial_x^2 h_1 - V(h_1))$$

$$+ r_1 h_1 |u|^2 u + rgh_1 \partial_x h_2 + rgh_2 \partial_x (h_1 + h_2) = 0, \tag{47}$$

$$\partial_t h_2 + \partial_x (h_2 u) - \partial_x \left((a{h_2}^2 + b{h_2}^3) \partial_x p_2 \right) = 0. \tag{48}$$

Conflict of interest

There is no conflict of interest in this study.

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