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Review

Global Existence of Smooth Solutions to the Incompressible 2D Navier-Stokes-Landau-Lifshitz Equations with the Dzyaloshinskii-Moriya Interaction and V-Flow Term

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Abstract: The paper focuses on the incompressible Navier-Stokes equations coupled with the Landau-Lifshitz-Gilbert equations, incorporating the Dzyaloshinskii-Moriya (DM) interaction and a V-flow term, derived as models for magnetoviscoelastic materials. It is demonstrated that under assumption small initial data there exist the global smooth solutions for this coupled system posed on $\mathbb{T}^2$ or $\mathbb{R}^2$.

***Keywords*:** incompressible Navier-Stokes-Landau-Lifshitz equations, Dzyaloshinskii-Moriya interaction, global smooth solution, small initial data

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Abbreviation

Symbol	Description
Ω	Computational domain ($\mathbb{T}^2$ or $\mathbb{R}^2$)
u	Velocity field of the fluid
P	Pressure field
d	Magnetization vector field
V	External velocity field affecting magnetization
μ	Fluid viscosity coefficient
λ	Coupling coefficient between fluid and magnetization
α	Non-adiabatic torque strength
β	Gilbert damping coefficient
κ	DM interaction strength
f	External applied field
$\Delta_V d$	V-Laplacian: $\Delta d + V \cdot \nabla d$

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$\nabla \cdot u = 0$	Incompressibility condition
$\nabla d \odot \nabla d$	Dyadic product of magnetization gradients

1. Introduction

1.1 *Main models*

Consideration is given to a class of specialized partial differential equations in fluid dynamics: the incompressible Navier-Stokes-Landau-Lifshitz (NSLL) equations integrated with the Dzyaloshinskii-Moriya (DM) interaction and a V-flow term:

$$u_t + u \cdot \nabla u + \nabla P = \mu \Delta u - \lambda \nabla \cdot (\nabla d \odot \nabla d), \tag{1}$$

$$d_t + u \cdot \nabla d + \alpha d \times (\Delta d + V \cdot \nabla d + \nabla \times d) + \beta d \times (d \times (\Delta d + V \cdot \nabla d + \nabla \times d)) = d \times f, \tag{2}$$

$$\nabla \cdot u = 0. \tag{3}$$

Here, u and d denote the velocity and magnetization field vectors, respectively. The constants μ and λ are positive and represent the shear viscosity and the coupling between kinetic and potential energy. The Gilbert damping coefficient $\beta > 0$ is associated with the gyroscopic ratio, and the torque characterized by α is typically non-adiabatic, indicating its strength. The pressure P is an undetermined scalar function. The matrix $\nabla d \odot \nabla d$ has entries defined by $\partial_{x_i} d \cdot \partial_{x_j} d$. The vector field $V = V(x)$ is assumed smooth in domain Ω, and the operator

$$\Delta_V d := \Delta d + V \cdot \nabla d = \Delta d + \nabla_V d$$

is referred to as the V-Laplacian, defined concerning the metric in Ω. This coupled parabolic system integrates the incompressible Navier-Stokes equations with the Landau-Lifshitz-Gilbert equations, embedding the DM interaction in a strongly coupled framework.

We examine the Cauchy problem for equations (1)-(3) with initial conditions:

$$u(0, x) = u_0(x), \quad d(0, x) = d_0(x), \quad x \in \Omega$$

for the given initial data $(u_0, d_0) : \Omega \to \mathbb{R}^3 \times \mathbb{S}^2$ satisfying $\nabla \cdot u_0 = 0$, where two-dimensional (2D) domain $\Omega = \mathbb{T}^2$ or $\mathbb{R}^2$.

1.2 *Related models*

We now introduce models associated with equations (1)-(3). The micromagnetic energy functional [1, 2] is defined as:

$$\begin{aligned} \mathscr{E}(d, f) &:= \frac{l_{ex}^2}{2} \int_\Omega |\nabla d|^2 dx + \kappa \int_\Omega d \cdot \mathrm{curl} d\, dx - \frac{1}{2} \int_\Omega d \cdot \pi(d)\, dx - \int_\Omega d \cdot f\, dx \\ &:= \mathscr{E}_{ex}(d) + \mathscr{E}_{DM}(d) - \mathscr{E}_{low}(d) - \mathscr{E}_{app}(d, f). \end{aligned} \tag{4}$$

Where, $\mathscr{E}_{ex}$ represents the exchange energy with exchange length l_{ex}, and $\mathscr{E}_{DM}$ corresponds to the energy contribution from the Dzyaloshinskii-Moriya (DM) interaction, characterized by strength $\kappa \in \mathbb{R}$. The term $\mathscr{E}_{low}$ accounts for lower-order effects like anisotropy and stray field energy, while $\mathscr{E}_{app}$ represents the energy from an external time-dependent field $f \in C^1(\Omega \times [0, \infty); \mathbb{R}^3)$.

From this energy functional, the gauged Landau-Lifshitz-Gilbert system is derived:

$$\partial_t d + \alpha d \times (\Delta d + V \cdot \nabla d + \nabla \times d) + \beta d \times (d \times (\Delta d + V \cdot \nabla d + \nabla \times d)) = d \times f.$$

In a two-dimensional domain $\Omega \subset \mathbb{R}^2$, the system remains applicable, where $\nabla \times d$ is typically defined as $\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, 0\right) \times d$.

Neglecting the DM term and setting $V = 0$, $f = 0$, the system (1)-(3) simplifies to the incompressible Navier-Stokes-Landau-Lifshitz (NSLL) equations:

$$\begin{cases} u_t + u \cdot \nabla u - \mu \Delta u + \nabla P = -\lambda \nabla \cdot (\nabla d \odot \nabla d), \\ \nabla \cdot u = 0, \\ d_t + u \cdot \nabla d + \alpha d \times \Delta d + \beta d \times (d \times \Delta d) = 0. \end{cases} \tag{5}$$

Setting $\alpha = 0$, $V = 0$, and $f = 0$, the model relates to liquid crystal theory, particularly the simplified Ericksen-Leslie system describing nematic liquid crystal hydrodynamics:

$$\begin{cases} u_t + u \cdot \nabla u - \mu \Delta u + \nabla P = -\lambda \nabla \cdot (\nabla d \odot \nabla d), \\ \nabla \cdot u = 0, \\ d_t + u \cdot \nabla d = \beta (\Delta d + |\nabla d|^2 d). \end{cases} \tag{6}$$

Further reduction with $\beta = 0$ leads to the Navier-Stokes-Schrödinger (NSS) flow from a Riemannian manifold (M, g) into $\mathbb{S}^2$:

$$\begin{cases} u_t + u \cdot \nabla u - \mu \Delta u + \nabla P = -\lambda \nabla \cdot (\nabla d \odot \nabla d), \\ \nabla \cdot u = 0, \\ d_t + u \cdot \nabla d + \alpha (d \times \Delta d) = 0. \end{cases} \tag{7}$$

1.3 *Background*

Research on the Navier-Stokes equations has advanced significantly since Leray's foundational work [3]. Pioneering contributions by Ladyzhenskaya [4], Temam [5], and Lions [6] have deepened our understanding of these equations.

Nonetheless, the regularity of solutions in three dimensions remains unresolved, standing as one of the Clay Millennium Prize problems.

Parallel to this, the Landau-Lifshitz equation, introduced in 1935, has been extensively studied by both mathematicians and physicists. Visintin [7] first explored weak solutions incorporating magnetostrictive effects in 1985. Subsequently, Sulem et al. [8] demonstrated that the dissipation-free Landau-Lifshitz equation (or Schrödinger flow into $\mathbb{S}^2$) admits global weak solutions and small-data global solutions in suitable Sobolev spaces. These insights were further refined by Ding and Wang [9, 10], who examined Schrödinger flows from closed Riemannian manifolds to Kähler manifolds, establishing local existence and regularity.

In 1992, Alouges and Soyeur [11] identified non-uniqueness in weak solutions of the Landau-Lifshitz-Gilbert (LLG) equation under Neumann boundary conditions in a unit ball in $\mathbb{R}^3$. Guo and Hong [12] applied harmonic map techniques in 1993 to establish global existence and uniqueness of partially regular weak solutions for two-dimensional Landau-Lifshitz equations. This was complemented by Chen et al. [13], who proved that weak solutions with finite energy must be Chen-Struwe solutions, thereby ensuring uniqueness.

Further contributions include Wang's [14] 1998 work on weak solutions for the Schrödinger flow from Euclidean domains or closed Riemannian manifolds into $\mathbb{S}^2$. Carbou and Fabrie [15, 16] extended these results by proving the local and global existence of regular solutions for the Landau-Lifshitz equations in both two and three-dimensional bounded domains. Bejenaru et al. [17] later addressed global well-posedness in critical Sobolev spaces for dimensions $n \geq 2$.

Di Fratta and co-authors [1] established weak-strong uniqueness for the LLG equation in micromagnetics. Building on Wang's [14] approach, Chen and Wang [18] explored global weak solutions for Landau-Lifshitz flows with Neumann boundary conditions. Recently, Wang [19] confirmed the global existence of smooth solutions for the gauged LLG flow in periodic domains with small initial data.

Turning to the Navier-Stokes-Landau-Lifshitz (NSLL) equations, Fan et al. [20] initiated the study of system (1)-(3) in 2010, establishing regularity criteria for smooth solutions in Besov and multiplier spaces. Wang and Guo [21] later proved the existence of weak solutions in two-dimensional periodic domains. Wei et al. [22] achieved global small smooth solutions in $\mathbb{R}^3$ and analyzed their decay behavior, with further improvements by Duan and Zhao [23]. Zhai et al. [24] demonstrated the global existence of unique solutions in $\mathbb{R}^3$ Besov spaces, relaxing smallness conditions on the initial velocity field. Wang and Wang [25] extended global smooth solution results to both two- and three-dimensional settings, while Qiu and Wang [26, 27] established blow-up criteria for strong solutions.

Recent developments in NSLL systems have integrated the DM term and V-flow. Qiu et al. [28] investigated blow-up criteria for local smooth solutions in $\mathbb{R}^n$ ($n = 2, 3$), deriving Serrin-type conditions and extending results in Besov spaces. These findings clarify the dynamics of fluid and magnetic interactions in systems featuring the DM term and V-flow, building on foundational studies by Fan et al. [20] and Wei et al. [22].

Moreover, this research delves into the physical implications of the DM interaction, prevalent in materials with broken inversion symmetry like magnetic thin films, fostering topological structures such as skyrmions. The V-flow term, analogous to the Gilbert damping, encapsulates dissipative processes, enriching the theoretical landscape of NSLL systems and their applications in material science and engineering.

In the context of the Ericksen-Leslie system, Lin and Liu [29, 30] pioneered the analysis of an approximation model based on Ginzburg-Landau functionals, proving global existence in two and three dimensions and establishing partial regularity results. Lin et al. [31], along with Hong [32], confirmed the global existence of weak solutions in two dimensions, while Lin and Wang [33] extended these results to three dimensions in 2016.

Research on coupled fluid-magnetic interactions has gained significant attention in recent years. Various mathematical models describe the motion of viscoelastic and magnetoviscoelastic fluids. Artemov and Baranovskii [34] studied mixed boundary-value problems in viscoelastic media, providing analytical techniques for handling complex boundary conditions similar to those in magnetoviscoelastic models. Baranovskii [35] analyzed non-isothermal flows of second-grade fluids, deriving exact solutions for fluid motion between parallel plates, which enhances understanding of viscoelastic flow behavior in velocity-magnetization coupling. Benešová et al. [36] examined weak solutions to an evolutionary magnetoelasticity model, offering insights into its mathematical structure. Kalousek et al. [37] studied weak and strong solutions in magnetoviscoelasticity, providing key results on regularity and stability. Kalousek and

Schlömerkemper [38] further explored dissipative solutions, highlighting energy dissipation mechanisms. Schlömerkemper and Žabenský [39] investigated uniqueness in magneto-viscoelastic flows, crucial for well-posedness.

Building on these works, our research advances the study of viscoelastic and magnetoviscoelastic systems by incorporating Dzyaloshinskii-Moriya interaction and V-flow effects, yielding new insights into their global solution properties.

1.4 *Main results*

We now summarize the primary findings of this study, consolidating results posed on periodic and whole-space domains.

Theorem 1 Let Ω be either the 2D torus $\mathbb{T}^2$ or the Euclidean plane $\mathbb{R}^2$. Assume that $V \in L^2([0, T], L^\infty(\Omega))$, $f \in L^2([0, T], L^2(\Omega)) \cap L^4([0, T], L^4(\Omega))$, and $|d_0| = 1$. Suppose $\|u_0\|_{L^2(\Omega)}$, $\|\nabla d_0\|_{L^2(\Omega)}$, and $\|f\|_{L^2([0, T], L^2(\Omega))}$ are sufficiently small to ensure $\tilde{\kappa}_1 > 0$ and $\hat{\kappa}_1 > 0$, as defined in Lemma 5. The following conditions apply based on the domain:

• For $\Omega = \mathbb{T}^2$:

- For $m = 1$: $u_0 \in H^1(\Omega)$, $\nabla d_0 \in H^1(\Omega)$, $f \in L^2([0, T], H^1(\Omega))$, and $V \in L^4([0, T], W^{1,4}(\Omega))$;
- For $m = 2$: $u_0 \in H^2(\Omega)$, $\nabla d_0 \in H^2(\Omega)$, $f \in L^2([0, T], H^2(\Omega))$, and $V \in L^4([0, T], W^{2,4}(\Omega))$;
- For $m \geq 2$: $u_0 \in H^{m+1}(\Omega)$, $\nabla d_0 \in H^{m+1}(\Omega)$, $f \in L^2([0, T], H^m(\Omega))$, and $V \in L^\infty([0, T], H^m(\Omega))$.

• For $\Omega = \mathbb{R}^2$:

- For $m = 1$: $u_0 \in H^1(\Omega)$, $\nabla d_0 \in H^1_Q(\Omega)$, $f \in L^2([0, T], H^1(\Omega))$, and $V \in L^4([0, T], W^{1,4}(\Omega))$;
- For $m = 2$: $u_0 \in H^2(\Omega)$, $\nabla d_0 \in H^2_Q(\Omega)$, $f \in L^2([0, T], H^2(\Omega))$, and $V \in L^4([0, T], W^{2,4}(\Omega))$;
- For $m \geq 2$: $u_0 \in H^{m+1}(\Omega)$, $\nabla d_0 \in H^{m+1}_Q(\Omega)$, and $V \in L^\infty([0, T], H^m(\Omega))$.

Then, the incompressible Navier-Stokes-Landau-Lifshitz equations (1)-(3) admit a global smooth solution:

$$u \in L^\infty([0, T], H^m(\Omega)) \cap L^2([0, T], H^{m+1}(\Omega)), \quad m \geq 1,$$

$$d \in \begin{cases} L^\infty([0, T], H^{m+1}(\Omega)) \cap L^2([0, T], H^{m+2}(\Omega)), & \text{if } \Omega = \mathbb{T}^2, \quad m \geq 1, \\ L^\infty([0, T], H^{m+1}_Q(\Omega)) \cap L^2([0, T], H^{m+2}_Q(\Omega)), & \text{if } \Omega = \mathbb{R}^2, \quad m \geq 1. \end{cases}$$

Here, $H^k_Q := W^{k,2}_Q$ and

$$W^{k,p}_Q = \{f : \mathbb{R}^2 \to \mathbb{R}^3 \mid |f(x)| = 1, \text{ a.e.}, \ f - Q \in W^{k,p}\},$$

with the induced distance $d^{k,p}_Q(f, g) = \|f - g\|_{W^{k,p}_Q}$, and where Q is an arbitrary point on $\mathbb{S}^2$.

Remark 1 From equation (1), we can easily derive the equation for the pressure P as:

$$\Delta P = -\nabla \cdot (u \cdot \nabla u) - \lambda \nabla \cdot (\nabla \cdot (\nabla d \odot \nabla d)).$$

Therefore, we only need to estimate u and d in our analysis.

The NSLL equations describe the coupling of incompressible fluid motion with magnetization evolution. These equations naturally arise in the study of magnetorheological fluids, with applications in adaptive damping systems, tunable materials, and spintronic devices.

Unlike classical magnetohydrodynamics (MHD), the NSLL system incorporates the Landau-Lifshitz-Gilbert (LLG) equation, governing magnetization precession and damping. This study extends the NSLL equations by incorporating the Dzyaloshinskii-Moriya (DM) interaction and a V-flow term. The DM interaction, an antisymmetric exchange effect due to spin-orbit coupling, stabilizes chiral magnetic structures such as skyrmions. The V-Laplacian, defined as $\Delta_V d = \Delta d + V \cdot \nabla d$, modifies the conventional Laplacian term by incorporating an external vector field V, influencing anisotropic magnetization diffusion. This extension allows for richer physical dynamics, including field-driven alignment and directional anisotropic dissipation. Additionally, the inclusion of the V-Laplacian term in our model captures further anisotropic effects in the magnetic field dynamics. This term models the influence of an external vector field V, which can represent physical phenomena such as material inhomogeneities or external field variations.

Previous studies have primarily addressed NSLL systems with either simplified coupling terms or restrictive conditions. This study contributes in the following aspects:

1. Establishing the global existence of smooth solutions for NSLL systems incorporating both DM interaction and V-flow, extending prior works that largely neglect these effects.

2. Analyzing the system in a two-dimensional domain ($\mathbb{T}^2$ or $\mathbb{R}^2$), where global smooth solutions are achievable, contrasting with three-dimensional cases, where stronger nonlinear effects hinder uniform bounds on ∇d and u. A crucial challenge in extending these results to three dimensions lies in the breakdown of uniform *a priori* estimates and the limitations of Gronwall-type inequalities. The Gagliardo-Nirenberg (G-N) inequality behaves differently in three dimensions compared to two dimensions, making it impossible to derive the necessary uniform estimates. Moreover, the supercritical term $|\nabla d|^2 d$ and the subcritical term $u \times (\nabla \times u)$ cannot be treated using the same techniques, further complicating the analysis. Despite these obstacles, 3D studies are crucial for bulk magnetorheological fluids, plasma magnetohydrodynamics, and engineering simulations. Future research should explore asymptotic methods, computational analysis, and experimental validation to bridge the gap between 2D and 3D models.

3. Providing a physical interpretation of key model parameters, including the Gilbert damping coefficient (β) and the non-adiabatic torque strength (α), emphasizing their significance in applications such as spintronic devices, magnetic interfaces, and skyrmion dynamics. The parameter α represents the strength of the non-adiabatic torque, influencing the precessional motion of the magnetization. The Gilbert damping coefficient β is associated with energy dissipation, affecting the rate at which the system relaxes to equilibrium. Their physical significance is essential in understanding the behavior of magnetoviscoelastic materials under different conditions.

Using the methodologies from [10, 40], local existence of smooth solutions for the NSLL system (1)-(3) can be established, assuming that V, f, u_0, and d_0 possess sufficient regularity. However, extending these results to global solutions requires establishing a series of *a priori* estimates via the classical energy method, as discussed in [18]. In order to establish uniform energy estimates for the incompressible Navier-Stokes-Landau-Lifshitz (NSLL) equations, we make use of the classical Gronwall inequalities. These inequalities allow us to control the growth of integral and differential inequalities, which naturally arise in our energy estimates.

The organization of the paper is as follows. In Section 2, we present some lemmas that will be used in the subsequent sections. In Section 3, we provide basic energy estimates for the solutions of equations (1)-(3). In Section 4, we prove the existence of global smooth solutions to the incompressible Navier-Stokes-Landau-Lifshitz equations (1)-(3) in two dimensions, under assumption small initial data. Finally, Section 5 concludes the paper with a summary of key findings and potential directions for future research.

2. Preliminaries

In this section, we introduce some lemmas which should be used in the following sections.

Lemma 1 (Gronwall inequality (integral form)) (i) Let $\xi(t)$ be a nonnegative, summable function on $[0, T]$ which satisfies for a.e. t the integral inequality

$$\xi(t) \leq C_1 \int_0^T \xi(s)ds + C_2$$

for constants $C_1, C_2 > 0$. Then

$$\xi \leq C_2(1 + C_1 t e^{C_1 t})$$

for a.e. $0 \leq t \leq T$.

(ii) In particular, if

$$\xi(t) \leq C_1 \int_0^t \xi(s)\mathrm{d}s$$

for a.e. $0 \leq t \leq T$, then

$$\xi(t) = 0.$$

Lemma 2 (Gronwall inequality (differential form)) (i) Let $\eta(\cdot)$ be a nonnegative, absolutely continuous function on $[0, T]$, which satisfies for a.e. t the differential inequality

$$\eta'(t) \leq \phi(t)\eta(t) + \psi(t),$$

where $\phi(t)$ and $\psi(t)$ are nonnegative, summable functions on $[0, T]$. Then

$$\eta(t) \leq \exp\left\{\int_0^t \phi(s)ds\right\}\left[\eta(0) + \int_0^t \psi(s)ds\right]$$

for all $0 \leq t \leq T$.

(ii) In particular, if

$$\eta' \leq \phi\eta, \ \text{ on } \ [0, T], \text{ and } \ \eta(0) = 0,$$

then

$$\eta \equiv 0 \ \text{ on } \ [0, T].$$

Applying the integral form of Gronwall's inequality to the energy functional, we obtain the uniform bound on $|\nabla d|_{H^m}(m = 0, 1, 2, 3, \cdots)$, which plays a crucial role in proving the global existence of smooth solutions.

Lemma 3 (Gagliardo-Nirenberg inequality) Let Ω be $\mathbb{R}^n$ or a bounded Lipshitz domain in $\mathbb{R}^n$ with $\partial\Omega$, and let u be any function in $W^{m,\,r}(\Omega)\cap W^{k,\,q}(\Omega)$, $1\le r,\,q\le+\infty$. For any integer j, $0\le k<j<m$, and for any number α in the interval $0<\theta\le 1$, set

$$\frac{1}{p}-\frac{j}{n}=\theta\left(\frac{1}{r}-\frac{m}{n}\right)+(1-\theta)\left(\frac{1}{q}-\frac{k}{n}\right).$$

Then

$$\|\nabla^j u\|_{L^p(\Omega)}\le C_G(\|u\|_{W^{m,\,r}(\Omega)})^{\theta}(\|u\|_{W^{k,\,q}(\Omega)})^{1-\theta}.$$

Here the positive constant C_G depends on $j,\,p,\,m,\,r,\,k,\,q,\,n,\,\theta$.

For the sake of brevity denoting $D^\tau=\prod_{i=1}^n\partial_{x_i}^{\tau_i}$, $|\tau|=\sum_{i=1}^n\tau_i$, where τ_i are nonnegative integers. Let D^k denote any kind of D^τ where $|\tau|=k$. We denote $\bar\beta=\max\{\beta,\,\mu\}$. This abbreviation will be used throughout this paper. $\tilde C(\varepsilon)$ is a constant which appears in the following Young inequality with ε:

$$ab\le\varepsilon a^p+\tilde C(\varepsilon)b^q,\ \left(a,\,b>0,\,\varepsilon>0,\,1<p,\,q<+\infty,\,\frac{1}{p}+\frac{1}{q}=1\right),\ \tilde C(\varepsilon)=(\varepsilon p)^{-q/p}q^{-1}.$$

3. A priori estimates

In this section, a priori estimates in arbitrary dimension are derived. The spaces worked in this section are $\Omega=\mathbb{T}^2$ or $\mathbb{R}^2$.

Lemma 4 If $f\in L^2([0,\,T],\,L^2(\Omega))$ and $V\in L^2([0,\,T],\,L^\infty(\Omega))$, we have

$$\sup_{0\le t\le T}\|d\|^2_{L^2(\Omega)}\le\|d_0\|^2_{L^2(\Omega)},\tag{8}$$

$$\sup_{0\le t\le T}\left(\|u\|^2_{L^2(\Omega)}+\|\nabla d\|^2_{L^2(\Omega)}\right)+\mu\int_0^T 2\|\nabla u\|^2_{L^2(\Omega)}\mathrm{d}t+\frac{\beta\lambda}{2}\int_0^T\|d\times\Delta d\|^2_{L^2(\Omega)}\mathrm{d}t\le\tilde K.\tag{9}$$

Here

$$\tilde K=\left(1+\frac{1}{\lambda}\right)\exp\left\{\frac{2\lambda(\beta^2+\alpha^2)}{\beta}\int_0^T(\|V\|^2_{L^\infty(\Omega)}+1)\mathrm{d}t\right\}\times\left\{\|u_0\|^2_{L^2(\Omega)}+\lambda\|\nabla d_0\|^2_{L^2(\Omega)}+\frac{2\lambda^2}{\beta}\int_0^T\|f\|^2_{L^2(\Omega)}\mathrm{d}t\right\}.$$

Proof. We begin to multiply the equation (2) by d, and integrate by parts respect to x over Ω, noticing the following facts,

$$\langle d_t,\,d\rangle=\partial_t\frac{1}{2}\|d\|^2_{L^2(\Omega)},\quad\langle u\cdot\nabla d,\,d\rangle=\int_\Omega u\cdot\nabla(\frac{1}{2}|d|^2)\mathrm{d}x=-\int_\Omega\nabla\cdot u\frac{1}{2}|d|^2\mathrm{d}x=0,$$

$$\langle \alpha d \times \Delta d,\, d \rangle = 0, \quad \langle \beta d \times (d \times \Delta d),\, d \rangle = 0.$$

The equation (8) can be obtained easily. We only emphasize that the equation (9) is available. Similar to the beginning, we multiply the equation (1) by u, and integrate by parts, then get

$$\langle u_t,\, u \rangle = \frac{1}{2}\partial_t \|u\|^2_{L^2(\Omega)}, \quad \langle u \cdot \nabla u,\, u \rangle = \int_\Omega u \cdot \nabla(\frac{1}{2}|u|^2)\mathrm{d}x = -\int_\Omega \nabla \cdot u \frac{1}{2}|u|^2 \mathrm{d}x = 0,$$

$$\langle \nabla P,\, u \rangle = -\langle P,\, \nabla \cdot u \rangle = 0, \quad \langle \mu \Delta u,\, u \rangle = -\mu \|\nabla u\|^2_{L^2(\Omega)},$$

$$-\langle \lambda \nabla \cdot (\nabla d \odot \nabla d),\, u \rangle = -\lambda \int_\Omega \sum_{i,\, j,\, k=1}^{n} \partial_i(\partial_i d_j \partial_k d_j) u_k \mathrm{d}x = \lambda \int_\Omega \sum_{i,\, j,\, k=1}^{2} \partial_i d_j \partial_k d_j \partial_i u_k \mathrm{d}x.$$

It gives as follows to multiply the equation (2) by $-\lambda \Delta d$ and integrate by parts:

$$-\lambda \langle d_t,\, \Delta d \rangle = \partial_t \frac{1}{2}\lambda \|\nabla d\|^2_{L^2(\Omega)},$$

$$-\lambda \langle u \cdot \nabla d,\, \Delta d \rangle = -\lambda \int_\Omega \sum_{i,\, j,\, k=1}^{n} u_i \partial_i d_j \partial_k^2 d_j dx$$

$$= \lambda \int_\Omega \sum_{i,\, j,\, k=1}^{n} (\partial_k u_i \partial_i d_j \partial_k d_j + u_i \partial_i \partial_k d_j \partial_k d_j)\mathrm{d}x = \lambda \int_\Omega \sum_{i,\, j,\, k=1}^{2} \partial_k u_i \partial_i d_j \partial_k d_j \mathrm{d}x,$$

$$-\lambda \langle \alpha d \times \Delta d,\, \Delta d \rangle = 0, \quad -\langle \beta \lambda d \times (d \times \Delta d),\, \Delta d \rangle = \beta \lambda \|d \times \Delta d\|^2_{L^2(\Omega)},$$

$$|-\lambda \alpha \langle d \times (V \cdot \nabla d + \nabla \times d),\, \Delta d \rangle| = |\lambda \alpha \langle V \cdot \nabla d + \nabla \times d,\, d \times \Delta d \rangle|$$

$$\leq \frac{\beta}{4}\|d \times \Delta d\|^2_{L^2(\Omega)} + \frac{\lambda^2 \alpha^2}{\beta}\|\nabla d\|^2_{L^2(\Omega)}(\|V\|^2_{L^\infty(\Omega)} + 1),$$

$$|-\lambda \beta \langle d \times (d \times (V \cdot \nabla d + \nabla \times d)),\, \Delta d \rangle| = |\lambda \beta \langle d \times (V \cdot \nabla d + \nabla \times d),\, d \times \Delta d \rangle|$$

$$\leq \frac{\beta}{4}\|d \times \Delta d\|^2_{L^2(\Omega)} + \beta \lambda^2 \|\nabla d\|_{L^2(\Omega)}(\|V\|^2_{L^\infty(\Omega)} + 1),$$

$$|-\lambda \langle d \times f,\, \Delta d \rangle| = |\lambda \langle f,\, d \times \Delta d \rangle| \leq \frac{\beta}{4}\|d \times \Delta d\|^2_{L^2(\Omega)} + \frac{\lambda^2}{\beta}\|f\|^2_{L^2(\Omega)}.$$

Combining the above equations yields

$$\partial_t \frac{1}{2}\|u\|^2_{L^2(\Omega)} + \partial_t \frac{1}{2}\lambda\|\nabla d\|^2_{L^2(\Omega)} + \mu\|\nabla u\|^2_{L^2(\Omega)} + \frac{\beta}{4}\lambda\|d\times\Delta d\|^2_{L^2(\Omega)}$$

$$\leq \frac{\lambda^2(\beta^2+\alpha^2)}{\beta}(\|V\|^2_{L^\infty(\Omega)}+1)\|\nabla d\|^2_{L^2(\Omega)} + \frac{\lambda^2}{\beta}\|f\|^2_{L^2(\Omega)}.$$

The application of Gronwall's inequality guarantees that the energy functionals $\|u(t)\|_{L^2}$ and $\|\nabla d(t)\|_{L^2}$ remain uniformly bounded for all $t \in [0, T]$. Suitable *a prior* bounds on solutions are essential for proving global smooth solutions, as they prevent the energy from blowing up in finite time. The exponential factor in the bound reflects the influence of the velocity field V, the damping coefficient β, and the external force f, indicating their impact on system stability. Then, using the Gronwall inequality leads to

$$\sup_{0\leq t\leq T}\|u\|^2_{L^2(\Omega)} + \lambda\sup_{0\leq t\leq T}\|\nabla d\|^2_{L^2(\Omega)} + 2\mu\int_0^T\|\nabla u\|^2_{L^2(\Omega)}\mathrm{d}t + \frac{\beta\lambda}{2}\int_0^T\|d\times\Delta d\|^2_{L^2(\Omega)}\mathrm{d}t$$

$$\leq \exp\left\{\frac{2\lambda(\beta^2+\alpha^2)}{\beta}\int_0^T(\|V\|^2_{L^\infty(\Omega)}+1)\mathrm{d}t\right\}\times\left\{\|u_0\|^2_{L^2(\Omega)} + \lambda\|\nabla d_0\|^2_{L^2(\Omega)} + \frac{2\lambda^2}{\beta}\int_0^T\|f\|^2_{L^2(\Omega)}\mathrm{d}t\right\}.$$

It can be shown

$$\sup_{0\leq t\leq T}\|u\|^2_{L^2(\Omega)} + \sup_{0\leq t\leq T}\|\nabla d\|^2_{L^2(\Omega)} + 2\mu\int_0^T\|\nabla u\|^2_{L^2(\Omega)}\mathrm{d}t + \frac{\beta\lambda}{2}\int_0^T\|d\times\Delta d\|^2_{L^2(\Omega)}\mathrm{d}t \leq \tilde{K},$$

where

$$\tilde{K} = \left(1+\frac{1}{\lambda}\right)\exp\left\{\frac{2\lambda(\beta^2+\alpha^2)}{\beta}\int_0^T(\|V\|^2_{L^\infty(\Omega)}+1)\mathrm{d}t\right\}\times\left\{\|u_0\|^2_{L^2(\Omega)} + \lambda\|\nabla d_0\|^2_{L^2(\Omega)} + \frac{2\lambda^2}{\beta}\int_0^T\|f\|^2_{L^2(\Omega)}\mathrm{d}t\right\}.$$

□

Next, for simplicity, since $|d| = 1$, write the equations (1)-(3) in the form:

$$u_t + u\cdot\nabla u + \nabla P = \mu\Delta u - \lambda\nabla\cdot(\nabla d\odot\nabla d), \tag{10}$$

$$d_t + u\cdot\nabla d + \alpha d\times(\Delta d + V\cdot\nabla d + \nabla\times d)$$

$$= \beta\Delta d + \beta|\nabla d|^2 d - \beta d\times(d\times(V\cdot\nabla d + \nabla\times d)) + d\times f, \tag{11}$$

$$\nabla\cdot u = 0. \tag{12}$$

Indeed, it can be proved clearly.

4. Proof of Theorem 1

In this section, the global regularity of the equations (1)-(3) (or (10)-(11)) with initial data under some conditions posed on $\Omega = \mathbb{T}^2$ or $\mathbb{R}^2$ shall be proved.

Firstly, the estimates about $\sup_{0\le t\le T}\|u\|^2_{H^1(\Omega)}$ and $\sup_{0\le t\le T}\|Dd\|^2_{H^1(\Omega)}$ of equations (10)-(11) are stated as the follows.

Lemma 5 Assume that the smooth local solutions u and d are of the system (10)-(11) with $\nabla\cdot u = 0$. Let $u_0 \in H^1(\Omega)$, $\nabla d_0 \in H^1(\Omega)$, $V \in L^2([0,\, T],\, L^\infty(\Omega)) \cap L^4([0,\, T],\, W^{1,4}(\Omega))$, $f \in L^4([0,\, T],\, L^4(\Omega)) \cap L^2([0,\, T],\, H^1(\Omega))$. If $\|u_0\|_{L^2(\Omega)}$, $\|\nabla d_0\|_{L^2(\Omega)}$, and $\|f\|_{L^2([0,\, T],\, L^2(\Omega))}$ satisfy the conditions:

$$\tilde{\kappa}_1 = \frac{\beta}{2} - (\beta C_G^3 \tilde{K} + 2\beta C_G^2 \tilde{K}^{\frac{1}{2}}) - 3|\alpha| C_G^2 \tilde{K}^{\frac{1}{2}} > 0,$$

$$\hat{\kappa}_1 = \mu - \tilde{C}\left(\frac{\beta}{20C_G^2}\right)\tilde{K}^{\frac{4}{3}} - \tilde{C}\left(\frac{\beta}{20|\lambda| C_G^2}\right)\tilde{K}^{\frac{4}{3}} - C_G^2 \tilde{K}^{\frac{1}{2}} > 0,$$

where

$$\tilde{K} = \left(1+\frac{1}{\lambda}\right)\exp\left\{\frac{2\lambda(\beta^2+\alpha^2)}{\beta}\int_0^T (\|V\|^2_{L^\infty(\Omega)}+1)\mathrm{d}t\right\} \times \left\{\|u_0\|^2_{L^2(\Omega)} + \lambda\|\nabla d_0\|^2_{L^2(\Omega)} + \frac{2\lambda^2}{\beta}\int_0^T \|f\|^2_{L^2(\Omega)}\mathrm{d}t\right\}.$$

For any $T > 0$ such that

$$\sup_{0\le t\le T}\|u\|^2_{H^1(\Omega)} + \sup_{0\le t\le T}\|Dd\|^2_{H^1(\Omega)} + 2\hat{\kappa}_1\int_0^T \|u\|^2_{H^2(\Omega)}\mathrm{d}t + 2\tilde{\kappa}_1\int_0^T \|Dd\|^2_{H^2(\Omega)}\mathrm{d}t \le \tilde{K}_1.$$

Here $\tilde{K}_1$ is defined in the proof, and $\bar{\beta} = \max\{\beta,\, \mu\}$.

Proof. We differentiate the equation (11) with D^2, take inner product with $D^2 d$ and integrate over Ω, we have the helpful equality:

$$\begin{aligned}
&\langle D^2\partial_t d,\, D^2 d\rangle + \langle D^2(u\cdot\nabla d),\, D^2 d\rangle + \alpha\langle D^2(d\times\Delta d),\, D^2 d\rangle \\
&= \beta\langle D^2\Delta d,\, D^2 d\rangle + \beta\langle D^2(|\nabla d|^2 d),\, D^2 d\rangle - \langle \alpha D^2(d\times(V\cdot\nabla d+\nabla d)),\, D^2 d\rangle \\
&\quad - \langle \beta D^2(d\times(d\times(V\cdot\nabla d+\nabla\times d))),\, D^2 d\rangle + \langle D^2(d\times f),\, D^2 d\rangle.
\end{aligned}$$

Notice the Gagliardo-Nirenberg inequalities,

$$\|D^2 d\|_{L^2(\Omega)} \le C_G\|Dd\|^{\frac{1}{2}}_{H^2(\Omega)}\|Dd\|^{\frac{1}{2}}_{L^2(\Omega)},$$

$$\|Dd\|_{L^4(\Omega)} \le C_G \|Dd\|_{H^2(\Omega)}^{\frac{1}{4}} \|Dd\|_{L^2(\Omega)}^{\frac{3}{4}},$$

$$\|D^2 d\|_{L^4(\Omega)} \le C_G \|Dd\|_{H^2(\Omega)}^{\frac{3}{4}} \|Dd\|_{L^2(\Omega)}^{\frac{1}{4}},$$

$$\|Dd\|_{L^6(\Omega)} \le C_G \|Dd\|_{H^2(\Omega)}^{\frac{1}{3}} \|Dd\|_{L^2(\Omega)}^{\frac{2}{3}},$$

$$\|Du\|_{L^4(\Omega)} \le C_G \|u\|_{H^2(\Omega)}^{\frac{3}{4}} \|u\|_{L^2(\Omega)}^{\frac{1}{4}},$$

$$\|u\|_{L^4(\Omega)} \le C_G \|u\|_{H^2(\Omega)}^{\frac{1}{4}} \|u\|_{L^2(\Omega)}^{\frac{3}{4}},$$

we can estimate the nonlinear terms as following:

$$\begin{aligned}
& |\langle D^2(u\cdot\nabla d),\, D^2 d\rangle| \\
&= |\langle D^2 u\cdot\nabla d + 2Du\cdot\nabla Dd + u\cdot\nabla D^2 d,\, D^2 d\rangle| \\
&= |\langle Du\cdot\nabla Dd,\, D^2 d\rangle - \langle Du\cdot\nabla d,\, D^3 d\rangle| \\
&= |-\langle D\nabla\cdot u Dd,\, D^2 d\rangle - \langle Du\cdot\nabla D^2 d,\, Dd\rangle - \langle Du\cdot\nabla d,\, D^3 d\rangle| \\
&= |-\langle Du\cdot\nabla D^2 d,\, Dd\rangle - \langle Du\cdot\nabla d,\, D^3 d\rangle| \\
&\le 2\|Du\|_{L^4(\Omega)} \|Dd\|_{L^4(\Omega)} \|D^3 d\|_{L^2(\Omega)} \\
&\le 2C_G^2 \|Dd\|_{H^2(\Omega)}^{\frac{5}{4}} \|Dd\|_{L^2(\Omega)}^{\frac{3}{4}} \|u\|_{H^2(\Omega)}^{\frac{3}{4}} \|u\|_{L^2(\Omega)}^{\frac{1}{4}} \\
&\le \frac{\beta}{10}\|Dd\|_{H^2(\Omega)}^2 + \tilde{C}\left(\frac{\beta}{20C_G^2}\right) \|Dd\|_{L^2(\Omega)}^2 \|u\|_{L^2(\Omega)}^{\frac{2}{3}} \|u\|_{H^2(\Omega)}^2 \\
&\le \frac{\beta}{10}\|Dd\|_{H^2(\Omega)}^2 + \tilde{C}\left(\frac{\beta}{20C_G^2}\right) \tilde{K}^{\frac{4}{3}} \|u\|_{H^2(\Omega)}^2,
\end{aligned}$$

$$
\begin{aligned}
&\quad |\beta\langle D^2(|\nabla d|^2 d),\, D^2 d\rangle| \\
&= |-\beta\langle D(|\nabla d|^2 d),\, D^3 d\rangle| \\
&= |\beta\langle |\nabla d|^2 Dd + 2d\cdot\nabla d\nabla Dd,\, D^3 d\rangle| \\
&\le \beta\left(\|Dd\|_{L^6(\Omega)}^3 + 2\|d\|_{L^\infty(\Omega)}\|Dd\|_{L^4(\Omega)}\|D^2 d\|_{L^4(\Omega)}\right)\|D^3 d\|_{L^2(\Omega)} \\
&\le \beta\left(C_G^3\|Dd\|_{L^2(\Omega)}^2 + 2C_G^2\|Dd\|_{L^2(\Omega)}\right)\|Dd\|_{H^2(\Omega)}^2 \\
&\le \beta(C_G^3\tilde{K} + 2C_G^2\tilde{K}^{\frac{1}{2}})\|Dd\|_{H^2(\Omega)}^2, \\
&\quad |\alpha\langle D^2(d\times\Delta d),\, D^2 d\rangle| \\
&= |-\alpha\langle D^2(d\times\nabla d),\, \nabla D^2 d\rangle| \\
&= |\alpha\langle D^2 d\times\nabla d + 2Dd\times D\nabla d + d\times\nabla D^2 d,\, \nabla D^2 d\rangle| \\
&\le |\alpha|\left(\|D^2 d\|_{L^4(\Omega)}\|Dd\|_{L^4(\Omega)} + 2\|Dd\|_{L^4(\Omega)}\|D^2 d\|_{L^4(\Omega)}\right)\|D^3 d\|_{L^2(\Omega)} \\
&\le 3|\alpha|C_G^2\|Dd\|_{H^2(\Omega)}^2\|Dd\|_{L^2(\Omega)} \le 3|\alpha|C_G^2\tilde{K}^{\frac{1}{2}}\|Dd\|_{H^2(\Omega)}^2, \\
&\quad |\alpha\langle D^2(d\times(V\cdot\nabla d + \nabla\times d)),\, D^2 d\rangle| \\
&= |-\alpha\langle D(d\times(V\cdot\nabla d + \nabla\times d)),\, D^3 d\rangle| \\
&= |\alpha\langle Dd\times(V\cdot\nabla d) + d\times(DV\cdot\nabla d + V\cdot\nabla Dd) + d\times(\nabla\times Dd),\, D^3 d\rangle| \\
&\le |\alpha|\Big(\|Dd\|_{L^4(\Omega)}^2(\|V\|_{L^\infty(\Omega)} + 1) + \|d\|_{L^\infty(\Omega)}\|DV\|_{L^4(\Omega)}\|Dd\|_{L^4(\Omega)} \\
&\quad + \|d\|_{L^\infty(\Omega)}(1 + \|V\|_{L^\infty(\Omega)})\|D^2 d\|_{L^2(\Omega)}\Big)\|D^3 d\|_{L^2(\Omega)} \\
&\le |\alpha|C_G^2\|Dd\|_{H^2(\Omega)}^{\frac{3}{2}}\|Dd\|_{L^2(\Omega)}^{\frac{3}{2}}(\|V\|_{L^\infty(\Omega)} + 1) + |\alpha|C_G\|DV\|_{L^4(\Omega)}\|Dd\|_{H^2(\Omega)}^{\frac{5}{4}}\|Dd\|_{L^2(\Omega)}^{\frac{3}{4}}
\end{aligned}
$$

$$+|\alpha|C_G\|Dd\|_{H^2(\Omega)}^{\frac{3}{2}}(1+\|V\|_{L^\infty(\Omega)})\|Dd\|_{L^2(\Omega)}^{\frac{1}{2}}$$

$$\leq \frac{\beta}{10}\|Dd\|_{H^2(\Omega)}^2+\tilde{C}\left(\frac{\beta}{50C_G^2|\alpha|}\right)(\|V\|_{L^\infty(\Omega)}^4+1)\|Dd\|_{L^2(\Omega)}^6$$

$$+\tilde{C}\left(\frac{\beta}{50C_G|\alpha|}\right)\|DV\|_{L^4(\Omega)}^{\frac{8}{3}}\|Dd\|_{L^2(\Omega)}^2+\tilde{C}\left(\frac{\beta}{50C_G|\alpha|}\right)\|Dd\|_{L^2(\Omega)}^2(1+\|V\|_{L^\infty(\Omega)}^4)$$

$$\leq \frac{\beta}{10}\|Dd\|_{H^2(\Omega)}^2+\tilde{C}\left(\frac{\beta}{50C_G^2|\alpha|}\right)(\|V\|_{L^\infty(\Omega)}^4+1)\tilde{K}^3$$

$$+\left(\tilde{C}\left(\frac{\beta}{50C_G|\alpha|}\right)\|DV\|_{L^4(\Omega)}^{\frac{8}{3}}+\tilde{C}\left(\frac{\beta}{50C_G|\alpha|}\right)(1+\|V\|_{L^\infty(\Omega)}^4)\right)\tilde{K},$$

$$\left|\beta\left\langle D^2\Big(d\times\big(d\times(V\cdot\nabla d+\nabla\times d)\big)\Big),D^2d\right\rangle\right|$$

$$=\left|-\beta\left\langle D\Big(d\times\big(d\times(V\cdot\nabla d+\nabla\times d)\big)\Big),D^3d\right\rangle\right|$$

$$=\beta|\left\langle Dd\times\big(d\times(V\cdot\nabla d+\nabla\times d)\big),D^3d\right\rangle+\left\langle d\times Dd\times(V\cdot\nabla d+\nabla\times d),D^3d\right\rangle$$

$$+\left\langle d\times d\times(DV\cdot\nabla d+V\cdot\nabla Dd+\nabla\times Dd),D^3d\right\rangle|$$

$$\leq \beta\Big(\|d\|_{L^\infty(\Omega)}2\|Dd\|_{L^4(\Omega)}^2(\|V\|_{L^\infty(\Omega)}+1)+\|d\|_{L^\infty(\Omega)}^2\|DV\|_{L^4(\Omega)}\|Dd\|_{L^4(\Omega)}$$

$$+\|d\|_{L^\infty(\Omega)}^2(\|V\|_{L^\infty(\Omega)}+1)\|D^2d\|_{L^2(\Omega)}\Big)\|D^3d\|_{L^2(\Omega)}$$

$$\leq 2\beta C_G^2\|Dd\|_{H^2(\Omega)}^{\frac{3}{2}}\|Dd\|_{L^2(\Omega)}^{\frac{3}{2}}(\|V\|_{L^\infty(\Omega)}+1)+\beta C_G\|DV\|_{L^4(\Omega)}\|Dd\|_{H^2(\Omega)}^{\frac{5}{4}}\|Dd\|_{L^2(\Omega)}^{\frac{3}{4}}$$

$$+\beta C_G\|Dd\|_{H^2(\Omega)}^{\frac{3}{2}}\|Dd\|_{L^2(\Omega)}^{\frac{1}{2}}(\|V\|_{L^\infty(\Omega)}+1)$$

$$\leq \frac{\beta}{10}\|Dd\|_{H^2(\Omega)}^2+\tilde{C}\left(\frac{1}{100C_G^2}\right)(1+\|V\|_{L^\infty(\Omega)}^4)\|Dd\|_{L^2(\Omega)}^6+\tilde{C}\left(\frac{1}{50C_G}\right)\|DV\|_{L^4(\Omega)}^{\frac{8}{3}}\|Dd\|_{L^2(\Omega)}^2$$

$$+\tilde{C}\left(\frac{1}{50C_G}\right)(1+\|V\|_{L^\infty(\Omega)}^4)\|Dd\|_{L^2(\Omega)}^2$$

$$\leq \frac{\beta}{10}\|Dd\|^2_{H^2(\Omega)} + \tilde{C}\left(\frac{1}{100C_G^2}\right)(1+\|V\|^4_{L^\infty(\Omega)})\tilde{K}^3 + \left(\tilde{C}\left(\frac{1}{50C_G}\right)\|DV\|^{\frac{8}{3}}_{L^4(\Omega)}\right.$$

$$\left. + \tilde{C}\left(\frac{1}{50C_G}\right)(1+\|V\|^4_{L^\infty(\Omega)})\right)\tilde{K},$$

$$\left|\left\langle D^2(d\times f), D^2 d\right\rangle\right|$$

$$= \left|-\left\langle D(d\times f), D^3 d\right\rangle\right|$$

$$\leq \|Dd\|_{L^4(\Omega)}\|f\|_{L^4(\Omega)}\|D^3 d\|_{L^2(\Omega)} + \|d\|_{L^\infty(\Omega)}\|Df\|_{L^2(\Omega)}\|D^3 d\|_{L^2(\Omega)}$$

$$\leq C_G\|Dd\|^{\frac{5}{4}}_{H^2(\Omega)}\|Dd\|^{\frac{3}{4}}_{L^2(\Omega)}\|f\|_{L^4(\Omega)} + \|d\|_{L^\infty(\Omega)}\|Df\|_{L^2(\Omega)}\|D^3 d\|_{L^2(\Omega)}$$

$$\leq \frac{\beta}{10}\|Dd\|^2_{H^2(\Omega)} + \tilde{C}\left(\frac{\beta}{20C_G}\right)\|Dd\|^2_{L^2(\Omega)}\|f\|^{\frac{8}{3}}_{L^4(\Omega)} + \frac{5}{\beta}\|f\|^2_{H^1(\Omega)}$$

$$\leq \frac{\beta}{10}\|Dd\|^2_{H^2(\Omega)} + \tilde{C}\left(\frac{\beta}{20C_G}\right)\|f\|^{\frac{8}{3}}_{L^4(\Omega)}\tilde{K} + \frac{5}{\beta}\|f\|^2_{H^1(\Omega)}.$$

It can be obtained the following estimate to differentiate the equation (10) with D, multiply by Du and integrate it with respect to x over Ω,

$$\langle Du_t, Du\rangle + \langle D(u\cdot\nabla u), Du\rangle + \langle D\nabla P, Du\rangle = \langle \mu D\Delta u, Du\rangle - \langle \lambda D\nabla\cdot(\nabla d\odot\nabla d), Du\rangle.$$

Each term in the above equation shall be estimated. Indeed, calculating yields

$$|\langle D(u\cdot\nabla u), Du\rangle|$$

$$= |\langle Du\cdot\nabla u + u\cdot\nabla Du, Du\rangle|$$

$$= |\langle Du\cdot\nabla u, Du\rangle|$$

$$= |-\langle D(\nabla\cdot u)u, Du\rangle - \langle Du\cdot\nabla Du, u\rangle|$$

$$\leq \|u\|_{L^4(\Omega)}\|Du\|_{L^4(\Omega)}\|D^2 u\|_{L^2(\Omega)} \leq C_G^2\|u\|^2_{H^2(\Omega)}\|u\|_{L^2(\Omega)} \leq C_G^2\tilde{K}^{\frac{1}{2}}\|u\|^2_{H^2(\Omega)},$$

$$
\begin{aligned}
&|\lambda\langle D\cdot\nabla(\nabla d\odot\nabla d),\, Du\rangle|\\
&= \left|\lambda\langle \sum_{i,\,j,\,k=1}^{2} D(\partial_i(\partial_i d_k\partial_j d_k)),\, Du_j\rangle\right|\\
&= \left|\lambda \sum_{i,\,j,\,k=1}^{2}\Big(\langle D\partial_i^2 d_k\partial_j d_k+\partial_i^2 d_k\partial_j Dd_k+\partial_i Dd_k\partial_i\partial_j d_k+\partial_i d_k\partial_i\partial_j Dd_k,\, Du_j\rangle\Big)\right|\\
&= \left|\lambda \sum_{i,\,j,\,k=1}^{2}\Big(D\partial_i^2 d_k\partial_j d_k-\partial_j\partial_i^2 d_k Dd_k,\, Du_j\rangle\Big)\right|\\
&= |\lambda\langle Du\cdot\nabla d,\, D\Delta d\rangle-\lambda\langle Du\cdot\nabla\Delta d,\, Dd\rangle|\\
&\leq 2|\lambda|\|Du\|_{L^4(\Omega)}\|Dd\|_{L^4(\Omega)}\|D^3 d\|_{L^2(\Omega)}\\
&\leq 2|\lambda|C_G^2\|Dd\|_{H^2(\Omega)}^{\frac{5}{4}}\|Dd\|_{L^2(\Omega)}^{\frac{3}{4}}\|u\|_{H^2(\Omega)}^{\frac{3}{4}}\|u\|_{L^2(\Omega)}^{\frac{1}{4}}\\
&\leq \frac{\beta}{10}\|Dd\|_{H^2(\Omega)}^2+\tilde{C}\left(\frac{\beta}{20|\lambda|C_G^2}\right)\|Dd\|_{L^2(\Omega)}^2\|u\|_{L^2(\Omega)}^{\frac{2}{3}}\|u\|_{H^2(\Omega)}^2\\
&\leq \frac{\beta}{10}\|Dd\|_{H^2(\Omega)}^2+\tilde{C}\left(\frac{\beta}{20|\lambda|C_G^2}\right)\tilde{K}^{\frac{4}{3}}\|u\|_{H^2(\Omega)}^2.
\end{aligned}
$$

Combining the above estimates, one obtains

$$
\begin{aligned}
&\partial_t \frac{1}{2}\|D^2 d\|^2_{L^2(\Omega)} + \partial_t \frac{1}{2}\|Du\|^2_{L^2(\Omega)}\\
&+\left(\frac{\beta}{2} - (\beta C_G^3 \tilde{K} + 2\beta C_G^2 \tilde{K}^{\frac{1}{2}}) - 3|\alpha| C_G^2 \tilde{K}^{\frac{1}{2}}\right)\|Dd\|^2_{H^2(\Omega)}\\
&+\left(\mu - \tilde{C}\left(\frac{\beta}{20C_G^2}\right)\tilde{K}^{\frac{4}{3}} - \tilde{C}\left(\frac{\beta}{20|\lambda|C_G^2}\right)\tilde{K}^{\frac{4}{3}} - C_G^2 \tilde{K}^{\frac{1}{2}}\right)\|u\|^2_{H^2(\Omega)}\\
\leq\ &\beta\|Dd\|^2_{H^1(\Omega)} + \mu\|u\|^2_{H^1(\Omega)} + \left(\tilde{C}\left(\frac{\beta}{50C_G^2|\alpha|}\right) + \tilde{C}\left(\frac{1}{100C_G^2}\right)\right)(1 + \|V\|^4_{L^\infty})\tilde{K}^3\\
&+\left(\tilde{C}\left(\frac{\beta}{50C_G|\alpha|}\right) + \tilde{C}\left(\frac{1}{50C_G}\right)\right)\|DV\|^{\frac{8}{3}}_{L^4(\Omega)}\tilde{K} + \left(\tilde{C}\left(\frac{\beta}{50C_G|\alpha|}\right) + \tilde{C}\left(\frac{1}{50C_G}\right)\right)(1 + \|V\|^4_{L^\infty(\Omega)})\tilde{K}\\
&+\tilde{C}\left(\frac{\beta}{20C_G}\right)\|f\|^{\frac{8}{3}}_{L^4(\Omega)}\tilde{K} + \frac{5}{\beta}\|f\|^2_{H^1(\Omega)}.
\end{aligned}
$$

Assuming that the initial data u_0 and d_0 meet with the conditions:

$$
\tilde{\kappa}_1 = \frac{\beta}{2} - (\beta C_G^3 \tilde{K} + 2\beta C_G^2 \tilde{K}^{\frac{1}{2}}) - 3|\alpha| C_G^2 \tilde{K}^{\frac{1}{2}} > 0,
$$

$$
\hat{\kappa}_1 = \mu - \tilde{C}(\frac{\beta}{20C_G^2})\tilde{K}^{\frac{4}{3}} - \tilde{C}(\frac{\beta}{20|\lambda|C_G^2})\tilde{K}^{\frac{4}{3}} - C_G^2 \tilde{K}^{\frac{1}{2}} > 0.
$$

Due to the integral form of Gronwall's inequality, the quantities $\sup_{0\leq t\leq T}\|Dd\|^2_{H^1(\Omega)}$ and $\sup_{0\leq t\leq T}\|u\|^2_{H^1(\Omega)}$ remain uniformly bounded. This result is crucial for establishing the global existence of smooth solutions, as it prevents energy accumulation over time. The term $\tilde{K}_1$ in our bound depends on the initial conditions (u_0, d_0) and external influences such as V and f. The exponential dependence on time in the bound reflects how the system dynamics are influenced by the nonlinearity and coupling effects. If the initial data or external forces remain small, this bound guarantees that the system remains well-behaved for all $t \in [0, T]$. Therefore, using the Gronwall inequality, it leads to

$$
\sup_{0\leq t\leq T}\|Dd\|^2_{H^1(\Omega)} + \sup_{0\leq t\leq T}\|u\|^2_{H^1(\Omega)} + 2\tilde{\kappa}_1\int_0^T \|Dd\|^2_{H^2(\Omega)}\mathrm{d}t + 2\hat{\kappa}_1\int_0^T \|u\|^2_{H^2(\Omega)}\mathrm{d}t \leq \tilde{K}_1,
$$

where

$$\tilde{K}_1 = \left(1+2\bar{\beta}T\exp(2\bar{\beta}T)\right)\Bigg(\|Dd_0\|^2_{H^1(\Omega)} + \|u_0\|^2_{H^1(\Omega)} + \tilde{K} + 2\int_0^T \Bigg(\Bigg(\tilde{C}\left(\frac{\beta}{50C_G^2|\alpha|}\right)$$

$$+\tilde{C}\left(\frac{1}{100C_G^2}\right)\Bigg)(1+\|V\|^4_{L^\infty})\tilde{K}^3 + \left(\tilde{C}\left(\frac{\beta}{50C_G|\alpha|}\right)+\tilde{C}\left(\frac{1}{50C_G}\right)\right)\|DV\|^{\frac{8}{3}}_{L^4(\Omega)}\tilde{K}$$

$$+\left(\tilde{C}\left(\frac{\beta}{50C_G|\alpha|}\right)+\tilde{C}\left(\frac{1}{50C_G}\right)\right)(1+\|V\|^4_{L^\infty(\Omega)})\tilde{K}+\tilde{C}\left(\frac{\beta}{20C_G}\right)\|f\|^{\frac{8}{3}}_{L^4(\Omega)}\tilde{K}+\frac{5}{\beta}\|f\|^2_{H^1(\Omega)}\Bigg)\mathrm{d}t\Bigg).$$

Here $\bar{\beta} = \max\{\beta, \mu\}$. Hence this lemma is proved completely. □

Next, the estimates of $\sup_{0\le t\le T}\|u\|_{H^2(\Omega)}$ and $\sup_{0\le t\le T}\|Dd\|_{H^2(\Omega)}$ of (10) and (11) shall be provided by the lemma.

Lemma 6 Assume that u and d are the smooth local solution of the system (10)-(12) with $\nabla\cdot u = 0$. Set $u_0 \in H^2(\Omega)$, $\nabla d_0 \in H^2(\Omega)$, $V \in L^4([0, T]; L^\infty(\Omega)) \cap L^4([0, T], W^{2,4}(\Omega))$, $f \in L^2([0, T], H^2(\Omega)) \cap L^4([0, T]; L^4(\Omega))$. Then there holds that

$$\sup_{0\le t\le T}\|u\|^2_{H^2(\Omega)} + \sup_{0\le t\le T}\|Dd\|^2_{H^2(\Omega)} + \beta\int_0^T \|Dd\|^2_{H^3(\Omega)}\mathrm{d}t + \mu\int_0^T \|u\|^2_{H^3(\Omega)}\mathrm{d}t \le \tilde{K}_2,$$

here $\tilde{K}_2$ is defined in the proof.

Proof. Differentiating the equation (11) by D^3, taking inner product with D^3d and integrating over Ω, it is inferred that

$$\langle D^3\partial_t d, D^3d\rangle + \lambda\langle D^3(u\cdot\nabla d), D^3d\rangle + \alpha\langle D^3(d\times\Delta d), D^3d\rangle + \alpha\langle D^3(d\times(V\cdot\nabla d+\nabla\times d)), D^3d\rangle$$

$$= \beta\langle D^3\Delta d, D^3d\rangle + \beta\langle D^3(|\nabla d|^2 d), D^3d\rangle - \beta\langle D^3(d\times(d\times(V\cdot\nabla d+\nabla\times d))), D^3d\rangle + \langle D^3(d\times f), D^3d\rangle.$$

According to the Gagliardo-Nirenberg inequalities, we see

$$\|Dd\|_{L^8(\Omega)} \le C_G\|Dd\|^{\frac{1}{4}}_{H^3(\Omega)}\|Dd\|^{\frac{3}{4}}_{L^2(\Omega)},$$

$$\|D^2d\|_{L^4(\Omega)} \le C_G\|Dd\|^{\frac{1}{4}}_{H^3(\Omega)}\|Dd\|^{\frac{3}{4}}_{H^1(\Omega)},$$

$$\|D^3d\|_{L^2(\Omega)} \le C_G\|Dd\|^{\frac{1}{2}}_{H^3(\Omega)}\|Dd\|^{\frac{1}{2}}_{H^1(\Omega)}$$

$$\|Dd\|_{L^4(\Omega)} \le C_G\|Dd\|^{\frac{1}{2}}_{H^1(\Omega)}\|Dd\|^{\frac{1}{2}}_{L^2(\Omega)},$$

$$\|D^3d\|_{L^4(\Omega)} \le C_G\|Dd\|^{\frac{3}{4}}_{H^3(\Omega)}\|Dd\|^{\frac{1}{4}}_{H^1(\Omega)},$$

$$\|D^2u\|_{L^4(\Omega)} \leq C_G\|u\|_{H^3(\Omega)}^{\frac{3}{4}}\|u\|_{H^1(\Omega)}^{\frac{1}{4}},$$

$$\|Du\|_{L^4(\Omega)} \leq C_G\|u\|_{H^3(\Omega)}^{\frac{1}{4}}\|u\|_{H^1(\Omega)}^{\frac{3}{4}},$$

then we derive

$$\begin{aligned}
&|\langle D^3(u\cdot\nabla d), D^3 d\rangle| \\
=& |\langle D^3u\cdot\nabla d + 3D^2u\cdot\nabla Dd + 3Du\cdot\nabla D^2 d + u\cdot\nabla D^3 d, D^3 d\rangle| \\
=& |-\langle D^2u\cdot\nabla Dd, D^3 d\rangle - \langle D^2u\cdot\nabla d, D^4 d\rangle \\
&-3\langle D^2\nabla\cdot uDd, D^3 d\rangle - 3\langle D^2u\cdot\nabla D^3 d, Dd\rangle \\
&-3\langle D\nabla\cdot uD^2 d, D^2 d\rangle - 3\langle Du\cdot\nabla D^3 d, D^2 d\rangle| \\
=& |\langle D^2u\cdot\nabla D^3 d, Dd\rangle + \langle D^2\nabla\cdot uDd, D^3 d\rangle - \langle D^2u\cdot\nabla d, D^4 d\rangle \\
&-3\langle D^2u\cdot\nabla D^3 d, Dd\rangle - 3\langle Du\cdot\nabla D^3 d, D^2 d\rangle| \\
=& |-\langle D^2u\cdot\nabla d, D^4 d\rangle + \langle D^2u\cdot\nabla D^3 d, Dd\rangle \\
&-3\langle D^2u\cdot\nabla D^3 d, Dd\rangle - 3\langle Du\cdot\nabla D^3 d, D^2 d\rangle| \\
=& |\langle D^2u\cdot\nabla d, D^4 d\rangle - 2\langle D^2u\cdot\nabla D^3 d, Dd\rangle - 3\langle Du\cdot\nabla D^3 d, D^2 d\rangle| \\
\leq& 3\|D^2u\|_{L^4(\Omega)}\|Dd\|_{L^4(\Omega)}\|D^4 d\|_{L^2(\Omega)} + 3\|Du\|_{L^4(\Omega)}\|D^2 d\|_{L^4(\Omega)}\|D^4 d\|_{L^2(\Omega)} \\
\leq& 3C_G^2\|u\|_{H^3(\Omega)}^{\frac{3}{4}}\|u\|_{H^1(\Omega)}^{\frac{1}{4}}\|Dd\|_{H^3(\Omega)}\|Dd\|_{H^1(\Omega)}^{\frac{1}{2}}\|Dd\|_{L^2(\Omega)}^{\frac{1}{2}} \\
&+3C_G^2\|u\|_{H^3(\Omega)}^{\frac{1}{4}}\|u\|_{H^1(\Omega)}^{\frac{3}{4}}\|Dd\|_{H^3(\Omega)}^{\frac{5}{4}}\|Dd\|_{H^1(\Omega)}^{\frac{3}{4}} \\
\leq& \frac{\beta}{14}\|Dd\|_{H^3(\Omega)}^2 + \tilde{C}\left(\frac{\beta}{84C_G^2}\right)\|u\|_{H^3(\Omega)}^{\frac{3}{2}}\|u\|_{H^1(\Omega)}^{\frac{1}{2}}\|Dd\|_{H^1(\Omega)}\|Dd\|_{L^2(\Omega)}
\end{aligned}$$

$$+\tilde{C}\left(\frac{\beta}{84C_G^2}\right)\|u\|_{H^3(\Omega)}^{\frac{2}{3}}\|u\|_{H^1(\Omega)}^2\|Dd\|_{H^1(\Omega)}^2$$

$$\leq \frac{\beta}{14}\|Dd\|_{H^3(\Omega)}^2+\frac{\mu}{6}\|u\|_{H^3(\Omega)}^2+\tilde{C}\left(\frac{\mu}{12\tilde{C}\left(\frac{\beta}{84C_G^2}\right)}\right)\|Dd\|_{H^1(\Omega)}^4\|Dd\|_{L^2(\Omega)}^4\|u\|_{H^1(\Omega)}^2$$

$$+\tilde{C}\left(\frac{\mu}{12\tilde{C}\left(\frac{\beta}{84C_G^2}\right)}\right)\|Dd\|_{H^1(\Omega)}^3\|u\|_{H^1(\Omega)}^3$$

$$\leq \frac{\beta}{14}\|Dd\|_{H^3(\Omega)}^2+\frac{\mu}{6}\|u\|_{H^3(\Omega)}^2+\tilde{C}\left(\frac{\mu}{12\tilde{C}\left(\frac{\beta}{84C_G^2}\right)}\right)\tilde{K}_1^3\tilde{K}^2+\tilde{C}\left(\frac{\mu}{12\tilde{C}\left(\frac{\beta}{84C_G^2}\right)}\right)\tilde{K}_1^3$$

$$|\beta\langle D^3(|\nabla d|^2 d),\, D^3 d\rangle|$$

$$=|-\beta\langle D^2(|\nabla d|^2 d),\, D^4 d\rangle|$$

$$=|\beta\langle|\nabla d|^2D^2d+4Dd\nabla Dd\cdot\nabla d+2|D\nabla d|^2d+2\nabla dD^2\nabla d\cdot d,\, D^4d\rangle|$$

$$\leq \beta\Big(5\|Dd\|_{L^8(\Omega)}^2\|D^2d\|_{L^4(\Omega)}+2\|D^2d\|_{L^4(\Omega)}^2\|d\|_{L^\infty(\Omega)}$$

$$+2\|d\|_{L^\infty(\Omega)}\|Dd\|_{L^4(\Omega)}\|D^3d\|_{L^4(\Omega)}\Big)\|D^4d\|_{L^2(\Omega)}$$

$$\leq 5\beta C_G^3\|Dd\|_{H^3(\Omega)}^{\frac{7}{4}}\|Dd\|_{L^2(\Omega)}^{\frac{3}{2}}\|Dd\|_{H^1(\Omega)}^{\frac{3}{4}}+2\beta C_G^2\|Dd\|_{H^3(\Omega)}^{\frac{3}{2}}\|Dd\|_{H^1(\Omega)}^{\frac{3}{2}}$$

$$+2\beta C_G^2\|Dd\|_{H^3(\Omega)}^{\frac{7}{4}}\|Dd\|_{H^1(\Omega)}^{\frac{3}{4}}\|Dd\|_{L^2(\Omega)}^{\frac{1}{2}}$$

$$\leq \frac{\beta}{14}\|Dd\|_{H^3(\Omega)}^2+\tilde{C}\left(\frac{1}{210C_G^3}\right)\|Dd\|_{L^2(\Omega)}^{12}\|Dd\|_{H^1(\Omega)}^6$$

$$+\tilde{C}\left(\frac{1}{84C_G^2}\right)\|Dd\|_{H^1(\Omega)}^6+\tilde{C}\left(\frac{1}{84C_G^2}\right)\|Dd\|_{H^1(\Omega)}^6\|Dd\|_{L^2(\Omega)}^4$$

$$\leq \frac{\beta}{14}\|Dd\|_{H^3(\Omega)}^2+\tilde{C}\left(\frac{1}{210C_G^3}\right)\tilde{K}^6\tilde{K}_1^3+\tilde{C}\left(\frac{1}{84C_G^2}\right)\tilde{K}_1^3+\tilde{C}\left(\frac{1}{84C_G^2}\right)\tilde{K}_1^3\tilde{K}^2,$$

$$
\begin{aligned}
&|\alpha\langle D^3(d\times\Delta d),\, D^3 d\rangle|\\
&=|-\alpha\langle D^3(d\times\nabla d),\, \nabla D^3 d\rangle|\\
&=|-\alpha\langle D^3 d\times\nabla d+3D^2 d\times\nabla Dd+3Dd\times\nabla D^2 d+d\times\nabla D^3 d,\, \nabla D^3 d\rangle|\\
&\le|\alpha|\Big(4\|D^3 d\|_{L^4(\Omega)}\|Dd\|_{L^4(\Omega)}+3\|D^2 d\|_{L^4(\Omega)}^2\Big)\|D^4 d\|_{L^2(\Omega)}\\
&\le 4C_G^2|\alpha|\|Dd\|_{H^3(\Omega)}^{\frac{7}{4}}\|Dd\|_{H^1(\Omega)}^{\frac{3}{4}}\|Dd\|_{L^2(\Omega)}^{\frac{1}{2}}+3|\alpha|C_G^2\|Dd\|_{H^3(\Omega)}^{\frac{3}{2}}\|Dd\|_{H^1(\Omega)}^{\frac{3}{2}}\\
&\le\frac{\beta}{14}\|Dd\|_{H^3(\Omega)}^2+\tilde{C}\left(\frac{\beta}{112C_G^2|\alpha|}\right)\|Dd\|_{H^1(\Omega)}^6\|Dd\|_{L^2(\Omega)}^4+\tilde{C}\left(\frac{\beta}{84C_G^2|\alpha|}\right)\|Dd\|_{H^1(\Omega)}^6\\
&\le\frac{\beta}{14}\|Dd\|_{H^3(\Omega)}^2+\tilde{C}\left(\frac{\beta}{112C_G^2|\alpha|}\right)\tilde{K}_1^3\tilde{K}^2+\tilde{C}\left(\frac{\beta}{84C_G^2|\alpha|}\right)\tilde{K}_1^3,
\end{aligned}
$$

$$
\begin{aligned}
&\left|\alpha\left\langle D^3\Big(d\times(V\cdot\nabla d+\nabla\times d)\Big),\, D^3 d\right\rangle\right|\\
&=\left|-\alpha\left\langle D^2\Big(d\times(V\cdot\nabla d+\nabla\times d)\Big),\, D^4 d\right\rangle\right|\\
&=\Big|-\alpha\langle D^2 d\times(V\cdot\nabla d+\nabla\times d)+2Dd\times(DV\cdot\nabla d+V\cdot\nabla Dd+\nabla\times(Dd)),\, D^4 d\rangle\\
&\quad-\alpha\langle d\times(D^2 V\cdot\nabla d+2DV\cdot\nabla Dd+V\cdot\nabla D^2 d+\nabla\times(D^2 d)),\, D^4 d\rangle\Big|\\
&\le|\alpha|\Big(3\|D^2 d\|_{L^4(\Omega)}\|Dd\|_{L^4(\Omega)}(1+\|V\|_{L^\infty(\Omega)})+2\|Dd\|_{L^8(\Omega)}^2\|DV\|_{L^4(\Omega)}\\
&\quad+\|d\|_{L^\infty(\Omega)}\|D^2 V\|_{L^4(\Omega)}\|Dd\|_{L^4(\Omega)}+2\|d\|_{L^\infty(\Omega)}\|DV\|_{L^4(\Omega)}\|D^2 d\|_{L^4(\Omega)}\\
&\quad+(1+\|V\|_{L^\infty(\Omega)})\|d\|_{L^\infty(\Omega)}\|D^3 d\|_{L^2(\Omega)}\Big)\|D^4 d\|_{L^2(\Omega)}\\
&\le|\alpha|\Big(3C_G^2\|Dd\|_{H^3(\Omega)}^{\frac{5}{4}}\|Dd\|_{H^1(\Omega)}^{\frac{5}{4}}\|Dd\|_{L^2(\Omega)}^{\frac{1}{2}}(1+\|V\|_{L^\infty(\Omega)})+2C_G^2\|Dd\|_{H^3(\Omega)}^{\frac{3}{2}}\|Dd\|_{L^2(\Omega)}^{\frac{3}{2}}\|DV\|_{L^4(\Omega)}\\
&\quad+C_G\|D^2 V\|_{L^4(\Omega)}\|Dd\|_{H^3(\Omega)}\|Dd\|_{H^1(\Omega)}^{\frac{1}{2}}\|Du\|_{L^2(\Omega)}^{\frac{1}{2}}+2C_G\|DV\|_{L^4(\Omega)}\|Dd\|_{H^3(\Omega)}^{\frac{5}{4}}\|Dd\|_{H^1(\Omega)}^{\frac{3}{4}}
\end{aligned}
$$

$$+C_G\|Dd\|^{\frac{3}{2}}_{H^3(\Omega)}\|Dd\|^{\frac{1}{2}}_{H^1(\Omega)}(1+\|V\|_{L^\infty(\Omega)})\Big)$$

$$\leq \frac{\beta}{14}\|Dd\|^2_{H^3(\Omega)}+\tilde{C}\left(\frac{\beta}{294|\alpha|C_G^2}\right)\|Dd\|^{\frac{10}{3}}_{H^1(\Omega)}\|Dd\|^{\frac{4}{3}}_{L^2(\Omega)}\left(1+\|V\|^{\frac{8}{3}}_{L^\infty(\Omega)}\right)$$

$$+\tilde{C}\left(\frac{\beta}{196|\alpha|C_G^2}\right)\|Dd\|^6_{H^1(\Omega)}\|DV\|^4_{L^4(\Omega)}+\tilde{C}\left(\frac{\beta}{98|\alpha|C_G}\right)\|D^2V\|^2_{L^2(\Omega)}\|Dd\|_{H^1(\Omega)}\|Du\|_{L^2(\Omega)}$$

$$+\tilde{C}\left(\frac{\beta}{196|\alpha|C_G}\right)\|DV\|^{\frac{8}{3}}_{L^4(\Omega)}\|Dd\|^2_{H^1(\Omega)}+\tilde{C}\left(\frac{\beta}{98|\alpha|C_G}\right)\|Dd\|^2_{H^1(\Omega)}\left(1+\|V\|^4_{L^\infty(\Omega)}\right)$$

$$\leq \frac{\beta}{14}\|Dd\|^2_{H^3(\Omega)}+\tilde{C}\left(\frac{\beta}{294|\alpha|C_G^2}\right)\tilde{K}_1^{\frac{5}{3}}\tilde{K}^{\frac{2}{3}}\left(1+\|V\|^{\frac{8}{3}}_{L^\infty(\Omega)}\right)$$

$$+\tilde{C}\left(\frac{\beta}{196|\alpha|C_G^2}\right)\tilde{K}_1^3\|DV\|^4_{L^4(\Omega)}+\tilde{C}\left(\frac{\beta}{98|\alpha|C_G}\right)\|D^2V\|^2_{L^2(\Omega)}\tilde{K}_1^{\frac{1}{2}}\tilde{K}^{\frac{1}{2}}$$

$$+\tilde{C}\left(\frac{\beta}{196|\alpha|C_G}\right)\|DV\|^{\frac{8}{3}}_{L^4(\Omega)}\tilde{K}_1+\tilde{C}\left(\frac{\beta}{98|\alpha|C_G}\right)\tilde{K}_1\left(1+\|V\|^4_{L^\infty(\Omega)}\right),$$

$$\left|\beta\left\langle D^3\Big(d\times\big(d\times(V\cdot\nabla d+\nabla\times d)\big)\Big),D^3d\right\rangle\right|$$

$$=\left|-\beta\left\langle D^2\Big(d\times\big(d\times(V\cdot\nabla d+\nabla\times d)\big)\Big),D^4d\right\rangle\right|$$

$$=\Big|-\beta\langle D^2d\times(d\times(V\cdot\nabla d+\nabla\times d))+2Dd\times(Dd\times(V\cdot\nabla d+\nabla\times d)),D^4d\rangle$$

$$-\beta\langle 2Dd\times(d\times(DV\cdot\nabla d+V\cdot\nabla Dd+\nabla\times(Dd)))$$

$$+d\times(D^2d\times(V\cdot\nabla d+\nabla\times d)),D^4d\rangle-\beta\langle 2d\times(Dd\times(DV\cdot\nabla d$$

$$+V\cdot\nabla Dd+\nabla\times(Dd))),D^4d\rangle-\beta\langle d\times d\times(D^2V\cdot\nabla d$$

$$+2DV\cdot\nabla Dd+V\cdot\nabla D^2d+\nabla\times(D^2d)),D^4d\rangle\Big|$$

$$\leq\beta\Big(6\|D^2d\|_{L^4(\Omega)}\|d\|_{L^\infty(\Omega)}\|Dd\|_{L^4(\Omega)}(1+\|V\|_{L^\infty(\Omega)})$$

$$+2\|Dd\|^2_{L^8(\Omega)}\|Dd\|_{L^4(\Omega)}(1+\|V\|_{L^\infty(\Omega)})+4\|Dd\|^2_{L^8(\Omega)}\|d\|_{L^\infty(\Omega)}\|DV\|_{L^4(\Omega)}$$

$$+\|d\|^2_{L^\infty(\Omega)}\|D^2V\|_{L^4(\Omega)}\|Dd\|_{L^4(\Omega)}+2\|d\|^2_{L^\infty(\Omega)}\|DV\|_{L^4(\Omega)}\|D^2d\|_{L^4(\Omega)}$$

$$+(1+\|V\|_{L^\infty(\Omega)})\|d\|^2_{L^\infty(\Omega)}\|D^3d\|_{L^2(\Omega)}\Big)\|D^4d\|_{L^2(\Omega)}$$

$$\leq \beta\Big(6C_G^2\|Dd\|^{\frac{5}{4}}_{H^3(\Omega)}\|Dd\|^{\frac{3}{4}}_{H^1(\Omega)}\|Dd\|^{\frac{1}{2}}_{L^2(\Omega)}(1+\|V\|_{L^\infty(\Omega)})$$

$$+2C_G^3\|Dd\|^{\frac{3}{2}}_{H^3(\Omega)}\|Dd\|^2_{H^1(\Omega)}\|Dd\|^{\frac{1}{2}}_{L^2(\Omega)}(1+\|V\|_{L^\infty(\Omega)})$$

$$+4C_G^2\|Dd\|^{\frac{3}{2}}_{H^3(\Omega)}\|Dd\|^{\frac{3}{2}}_{H^1(\Omega)}\|DV\|_{L^4(\Omega)}+C_G\|D^2V\|_{L^4(\Omega)}\|Dd\|_{H^3(\Omega)}\|Dd\|^{\frac{1}{2}}_{H^1(\Omega)}\|Dd\|^{\frac{1}{2}}_{L^2(\Omega)}$$

$$+2C_G\|DV\|_{L^4(\Omega)}\|Dd\|^{\frac{5}{4}}_{H^3(\Omega)}\|Dd\|^{\frac{3}{4}}_{H^1(\Omega)}+C_G(1+\|V\|_{L^\infty(\Omega)})\|Dd\|^{\frac{3}{2}}_{H^3(\Omega)}\|Dd\|^{\frac{1}{2}}_{H^1(\Omega)}\Big)$$

$$\leq \frac{\beta}{14}\|Dd\|^2_{H^3(\Omega)}+\tilde{C}\left(\frac{1}{756C_G^2}\right)\|Dd\|^{\frac{10}{3}}_{H^1(\Omega)}\|Dd\|^{\frac{4}{3}}_{L^2(\Omega)}(1+\|V\|^{\frac{8}{3}}_{L^\infty(\Omega)})$$

$$+\tilde{C}\left(\frac{1}{252C_G^3}\right)\|Dd\|^8_{H^1(\Omega)}\|Dd\|^2_{L^2(\Omega)}(1+\|V\|^4_{L^\infty(\Omega)})+\tilde{C}\left(\frac{1}{504C_G^2}\right)\|Dd\|^6_{H^1(\Omega)}\|DV\|^4_{L^4(\Omega)}$$

$$+\tilde{C}\left(\frac{1}{126C_G}\right)\|D^2V\|^2_{L^4(\Omega)}\|Dd\|_{H^1(\Omega)}\|Dd\|_{L^2(\Omega)}+\tilde{C}\left(\frac{1}{252C_G}\right)\|DV\|^{\frac{8}{3}}_{L^4(\Omega)}\|Dd\|^2_{H^1(\Omega)}$$

$$+\tilde{C}\left(\frac{1}{126C_G}\right)(1+\|V\|^4_{L^\infty(\Omega)})\|Dd\|^2_{H^1(\Omega)}$$

$$\leq \frac{\beta}{14}\|Dd\|^2_{H^3(\Omega)}+\tilde{C}\left(\frac{1}{756C_G^2}\right)\tilde{K}_1^{\frac{5}{3}}\tilde{K}^{\frac{2}{3}}(1+\|V\|^{\frac{8}{3}}_{L^\infty(\Omega)})+\tilde{C}\left(\frac{1}{252C_G^3}\right)\tilde{K}_1^4\tilde{K}(1+\|V\|^4_{L^\infty(\Omega)})$$

$$+\tilde{C}\left(\frac{1}{504C_G^2}\right)\tilde{K}_1^3\|DV\|^4_{L^4(\Omega)}+\tilde{C}\left(\frac{1}{126C_G}\right)\|D^2V\|^2_{L^4(\Omega)}\tilde{K}_1^{\frac{1}{2}}\tilde{K}^{\frac{1}{2}}$$

$$+\tilde{C}\left(\frac{1}{252C_G}\right)\|DV\|^{\frac{8}{3}}_{L^4(\Omega)}\tilde{K}_1+\tilde{C}\left(\frac{1}{126C_G}\right)(1+\|V\|^4_{L^\infty(\Omega)})\tilde{K}_1,$$

$$|\langle D^3(d\times f),D^3d\rangle|$$

$$=|-\langle D^2(d\times f),D^4d\rangle|$$

$$=|\langle D^2d\times f+2Dd\times Df+d\times D^2f,D^4d\rangle|$$

$$\leq \Big(\|D^2 d\|_{L^4(\Omega)}\|f\|_{L^4(\Omega)} + 2\|Dd\|_{L^4(\Omega)}\|Df\|_{L^4(\Omega)} + \|d\|_{L^\infty(\Omega)}\|D^2 f\|_{L^2(\Omega)}\Big)\|D^4 d\|_{L^2(\Omega)}$$

$$\leq C_G\|Dd\|_{H^3(\Omega)}^{\frac{5}{4}}\|Dd\|_{H^1(\Omega)}^{\frac{3}{4}}\|f\|_{L^4(\Omega)} + 2C_G\|Df\|_{L^4(\Omega)}\|Dd\|_{H^3(\Omega)}\|Dd\|_{H^1(\Omega)}^{\frac{1}{2}}\|Dd\|_{L^2(\Omega)}^{\frac{1}{2}}$$

$$+ \|d\|_{L^\infty(\Omega)}\|D^2 f\|_{L^2(\Omega)}\|Dd\|_{H^3(\Omega)}$$

$$\leq \frac{\beta}{14}\|Dd\|_{H^3(\Omega)}^2 + \tilde{C}\left(\frac{\beta}{42C_G}\right)\|Dd\|_{H^1(\Omega)}^2\|f\|_{L^4(\Omega)}^{\frac{8}{3}}$$

$$+ \frac{21}{2\beta}\|Dd\|_{H^1(\Omega)}\|Dd\|_{L^2(\Omega)}\|Df\|_{L^4(\Omega)}^2 + \frac{21}{2\beta}\|D^2 f\|_{L^2(\Omega)}^2$$

$$\leq \frac{\beta}{14}\|Dd\|_{H^3(\Omega)}^2 + \tilde{C}\left(\frac{\beta}{42C_G}\right)\tilde{K}_1\|f\|_{L^4(\Omega)}^{\frac{8}{3}} + \frac{21}{2\beta}\tilde{K}_1^{\frac{1}{2}}\tilde{K}^{\frac{1}{2}}\|Df\|_{L^4(\Omega)}^2 + \frac{21}{2\beta}\|D^2 f\|_{L^2(\Omega)}^2.$$

Differentiating the equation (10) by D^2, taking inner product with $D^2 u$ and integrating over Ω leads to

$$\langle D^2 u_t, D^2 u\rangle + \langle D^2(u\cdot\nabla u), D^2 u\rangle + \langle D^2\nabla P, D^2 u\rangle = \langle \mu D^2\Delta u, D^2 u\rangle - \lambda\langle D^2\nabla\cdot(\nabla d\odot\nabla d), D^2 u\rangle.$$

Each term in the above equation can be estamted. Indeed, it is known that

$$|\langle D^2(u\cdot\nabla u), D^2 u\rangle|$$

$$= |\langle D^2 u\cdot\nabla u + 2Du\cdot\nabla Du + u\cdot\nabla D^2 u, D^2 u\rangle|$$

$$= |-\langle Du\cdot\nabla u, D^3 u\rangle + \langle Du\cdot\nabla Du, D^2 u\rangle|$$

$$= |-\langle Du\cdot\nabla u, D^3 u\rangle - \langle D(\nabla\cdot u)Du, D^2 u\rangle - \langle Du\cdot\nabla D^2 u, Du\rangle|$$

$$\leq 2\|Du\|_{L^4(\Omega)}^2\|D^3 u\|_{L^2(\Omega)} \leq 2C_G^2\|u\|_{H^3(\Omega)}^{\frac{3}{2}}\|u\|_{H^1(\Omega)}^{\frac{3}{2}}$$

$$\leq \frac{\mu}{6}\|u\|_{H^3(\Omega)}^2 + \tilde{C}\left(\frac{\mu}{12C_G^2}\right)\|u\|_{H^1(\Omega)}^6 \leq \frac{\mu}{6}\|u\|_{H^3(\Omega)}^2 + \tilde{C}\left(\frac{\mu}{12C_G^2}\right)\tilde{K}_1^3,$$

$$
\begin{aligned}
& |\lambda \langle D^2\nabla\cdot(\nabla d \odot \nabla d),\, D^2 u\rangle| \\
= & \left|\lambda \sum_{i,\,j,\,k=1}^{2} \langle D^2\partial_i(\partial_i d_k \partial_j d_k),\, D^2 u_j\rangle\right| \\
= & \left|\lambda \sum_{i,\,j,\,k=1}^{2} \langle D^3(\partial_i d_k \partial_j d_k),\, D\partial_i u_j\rangle\right| \\
= & \left|\lambda \sum_{i,\,j,\,k=1}^{2} \langle D^3\partial_i d_k \partial_j d_k + \partial_i d_k D^3\partial_j d_k,\, D\partial_i u_j\rangle\right| \\
& + \left|3\lambda \sum_{i,\,j,\,k=1}^{2} \langle D^2\partial_i d_k \partial_j D d_k,\, D\partial_i u_j\rangle\right| + \left|3\lambda \sum_{i,\,j,\,k=1}^{2} \langle \partial_i D d_k \partial_j D^2 d_k,\, \partial_i D u_j\rangle\right| \\
= & \left|\lambda \sum_{i,\,j,\,k=1}^{2} \langle D^3\partial_i d_k \partial_j d_k + \partial_i d_k D^3\partial_j d_k,\, D\partial_i u_j\rangle\right| \\
& + \left|-3\lambda \sum_{i,\,j,\,k=1}^{2} \Big(\langle D^2\partial_i\partial_j d_k D d_k,\, D\partial_i u_j\rangle + \langle D^2\partial_i d_k D d_k,\, D\partial_i\partial_j u_j\rangle\Big)\right| \\
& + \left|3\lambda \sum_{i,\,j,\,k=1}^{2} \Big(-\langle D\partial_i d_k D^2 d_k,\, D\partial_i\partial_j u_j\rangle + \langle D\partial_i^2\partial_j d_k D^2 d_k,\, D u_j\rangle\Big)\right| \\
& + \left|3\lambda \sum_{i,\,j,\,k=1}^{2} \Big(-\langle D\partial_i d_k \partial_i D^2 d_k,\, D\partial_j u_j\rangle - \langle D\partial_i d_k \partial_i\partial_j D^2 d_k,\, D u_j\rangle\Big)\right| \\
\leq & \; 5|\lambda|\|D^4 d\|_{L^4(\Omega)}\|Dd\|_{L^4(\Omega)}\|D^2 u\|_{L^2(\Omega)} + 6|\lambda|\|D^4 d\|_{L^2(\Omega)}\|D^2 d\|_{L^4(\Omega)}\|Du\|_{L^4(\Omega)} \\
\leq & \; 5|\lambda|C_G^2\|Dd\|_{H^3(\Omega)}\|Dd\|^{\frac{1}{2}}_{H^1(\Omega)}\|Dd\|^{\frac{1}{2}}_{L^2(\Omega)}\|u\|^{\frac{3}{4}}_{H^3(\Omega)}\|u\|^{\frac{1}{4}}_{H^1(\Omega)} \\
& + 6|\lambda|C_G^2\|Dd\|^{\frac{5}{4}}_{H^3(\Omega)}\|Dd\|^{\frac{3}{4}}_{H^1(\Omega)}\|u\|^{\frac{1}{4}}_{H^3(\Omega)}\|u\|^{\frac{3}{4}}_{L^2(\Omega)} \\
\leq & \; \frac{\beta}{14}\|Dd\|^2_{H^3(\Omega)} + \tilde{C}\left(\frac{\beta}{140|\lambda|C_G^2}\right)\|u\|^{\frac{3}{2}}_{H^3(\Omega)}\|u\|^{\frac{1}{2}}_{H^1(\Omega)}\|Dd\|_{L^2(\Omega)}\|Dd\|_{H^1(\Omega)}
\end{aligned}
$$

$$+\tilde{C}\left(\frac{\beta}{168|\lambda|C_G^2}\right)\|u\|_{H^3(\Omega)}^{\frac{2}{3}}\|u\|_{H^1(\Omega)}^2\|Dd\|_{H^1(\Omega)}^2$$

$$\leq \frac{\beta}{14}\|Dd\|_{H^3(\Omega)}^2+\frac{\mu}{6}\|u\|_{H^3(\Omega)}^2+\tilde{C}\left(\frac{\mu}{12\tilde{C}\left(\frac{\beta}{140|\lambda|C_G^2}\right)}\right)\|Dd\|_{H^1(\Omega)}^4\|u\|_{H^1(\Omega)}^2\|Dd\|_{L^2(\Omega)}^4$$

$$+\tilde{C}\left(\frac{\mu}{12\tilde{C}\left(\frac{\beta}{168|\lambda|C_G^2}\right)}\right)\|u\|_{H^1(\Omega)}^3\|Dd\|_{H^1(\Omega)}^3$$

$$\leq \frac{\beta}{14}\|Dd\|_{H^3(\Omega)}^2+\frac{\mu}{6}\|u\|_{H^3(\Omega)}^2+\tilde{C}\left(\frac{\mu}{12\tilde{C}\left(\frac{\beta}{140|\lambda|C_G^2}\right)}\right)\tilde{K}_1^3\tilde{K}^2+\tilde{C}\left(\frac{\mu}{12\tilde{C}\left(\frac{\beta}{168|\lambda|C_G^2}\right)}\right)\tilde{K}_1^3.$$

Therefore, the Gronwall inequality (integral form) provides

$$\sup_{0\leq t\leq T}\|u\|_{H^2(\Omega)}^2+\sup_{0\leq t\leq T}\|Dd\|_{H^2(\Omega)}^2+\beta\int_0^T\|Dd\|_{H^3(\Omega)}^2\mathrm{d}t+\mu\int_0^T\|u\|_{H^3(\Omega)}^2\mathrm{d}t\leq \tilde{K}_2,$$

where

$$\tilde{K}_2=\left(1+\bar{\beta}T\exp(\bar{\beta}T)\right)\Big\{\|u_0\|_{H^2(\Omega)}^2+\|Dd_0\|_{H^2(\Omega)}^2+\tilde{K}_1$$

$$+2\int_0^T\Bigg\{\tilde{C}\left(\frac{\mu}{12\tilde{C}\left(\frac{\beta}{84C_G^2}\right)}\right)\tilde{K}_1^3\tilde{K}^2+\tilde{C}\left(\frac{\mu}{12\tilde{C}\left(\frac{\beta}{84C_G^2}\right)}\right)\tilde{K}_1^3+\tilde{C}\left(\frac{1}{210C_G^3}\right)\tilde{K}^6\tilde{K}_1^3$$

$$+\tilde{C}\left(\frac{1}{84C_G^2}\right)\tilde{K}_1^3+\tilde{C}\left(\frac{1}{84C_G^2}\right)\tilde{K}_1^3\tilde{K}^2+\tilde{C}\left(\frac{\beta}{112C_G^2|\alpha|}\right)\tilde{K}_1^3\tilde{K}^2+\tilde{C}\left(\frac{\beta}{84C_G^2|\alpha|}\right)\tilde{K}_1^3$$

$$+\tilde{C}\left(\frac{\beta}{294|\alpha|C_G^2}\right)\tilde{K}_1^{\frac{5}{3}}\tilde{K}^{\frac{2}{3}}(1+\|V\|_{L^\infty(\Omega)}^{\frac{8}{3}})+\tilde{C}\left(\frac{\beta}{196|\alpha|C_G^2}\right)\tilde{K}_1^3\|DV\|_{L^4(\Omega)}^4$$

$$+\tilde{C}\left(\frac{\beta}{98|\alpha|C_G}\right)\|D^2V\|_{L^2(\Omega)}^2\tilde{K}_1^{\frac{1}{2}}\tilde{K}^{\frac{1}{2}}+\tilde{C}\left(\frac{\beta}{196|\alpha|C_G}\right)\|DV\|_{L^4(\Omega)}^{\frac{8}{3}}\tilde{K}_1+\tilde{C}\left(\frac{\beta}{98|\alpha|C_G}\right)\tilde{K}_1(1+\|V\|_{L^\infty(\Omega)}^4)$$

$$+\tilde{C}\left(\frac{1}{756C_G^2}\right)\tilde{K}_1^{\frac{5}{3}}\tilde{K}^{\frac{2}{3}}(1+\|V\|_{L^\infty(\Omega)}^{\frac{8}{3}})+\tilde{C}\left(\frac{1}{252C_G^3}\right)\tilde{K}_1^4\tilde{K}(1+\|V\|_{L^\infty(\Omega)}^4)$$

$$+\tilde{C}\left(\frac{1}{504C_G^2}\right)\tilde{K}_1^3\|DV\|_{L^4(\Omega)}^4+\tilde{C}\left(\frac{\beta}{126\beta C_G}\right)\|D^2V\|_{L^4(\Omega)}^2\tilde{K}_1^{\frac{1}{2}}\tilde{K}^{\frac{1}{2}}+\tilde{C}\left(\frac{1}{252C_G}\right)\|DV\|_{L^4(\Omega)}^{\frac{8}{3}}\tilde{K}_1$$

$$+\tilde{C}\left(\frac{1}{126C_G}\right)(1+\|V\|_{L^\infty(\Omega)}^4)\tilde{K}_1+\tilde{C}\left(\frac{\beta}{42C_G}\right)\tilde{K}_1\|f\|_{L^4(\Omega)}^{\frac{8}{3}}+\frac{21}{2\beta}\tilde{K}_1^{\frac{1}{2}}\tilde{K}^{\frac{1}{2}}\|Df\|_{L^4(\Omega)}^2$$

$$+\frac{21}{2\beta}\|D^2f\|_{L^2(\Omega)}^2+\tilde{C}\left(\frac{\mu}{12C_G^2}\right)\tilde{K}_1^3+\tilde{C}\left(\frac{\mu}{12\tilde{C}\left(\frac{\beta}{140|\lambda|C_G^2}\right)}\right)\tilde{K}_1^3\tilde{K}^2$$

$$\left.\left.+\tilde{C}\left(\frac{\mu}{12\tilde{C}\left(\frac{\beta}{168|\lambda|C_G^2}\right)}\right)\tilde{K}_1^3\right\}(t)dt\right\}.$$

□

By the Gagliardo-Nirenberg inequality one may give

$$\|Dd\|_{L^\infty(\Omega)}\le C_G\|Dd\|_{H^2(\Omega)}^{\frac{1}{2}}\|Dd\|_{L^2(\Omega)}^{\frac{1}{2}}\le C_G\tilde{K}^{\frac{1}{4}}\tilde{K}_2^{\frac{1}{4}},$$

$$\|u\|_{L^\infty(\Omega)}\le C_G\|u\|_{H^2(\Omega)}^{\frac{1}{2}}\|u\|_{L^2(\Omega)}^{\frac{1}{2}}\le C_G\tilde{K}^{\frac{1}{4}}\tilde{K}_2^{\frac{1}{4}}.$$

Next $\|u\|_{L^\infty([0,T],H^3(\Omega))}$ and $\|Dd\|_{L^\infty([0,T],H^3(\Omega))}$ shall be estimated.

Lemma 7 Assume that u and d are the smooth local solution of the system (10)-(12) with $\nabla\cdot u=0$. Let $V\in L^2([0,T],H^3(\Omega))$, $f\in L^2([0,T],H^3(\Omega))\cap L^4([0,T];L^4(\Omega))$. One may obtain

$$\sup_{0\le t\le T}\|u\|_{H^3(\Omega)}^2+\sup_{0\le t\le T}\|Dd\|_{H^3(\Omega)}^2+\beta\int_0^T\|u\|_{H^4(\Omega)}^2\mathrm{d}t+\mu\int_0^T\|Dd\|_{H^4(\Omega)}^2\mathrm{d}t\le\tilde{K}_3,$$

here $\tilde{K}_3$ is defined in the proof.

Proof. Differentiating the equation (11) by D^4, taking inner product with D^4d and integrating over Ω yields

$$\langle D^4d_t,D^4d\rangle+\langle D^4(u\cdot\nabla d),D^4d\rangle+\alpha\langle D^4(d\times(\Delta d+V\cdot\nabla d+\nabla\times d)),D^4d\rangle$$

$$=\beta\langle D^4\Delta d,D^4d\rangle+\beta\langle D^4(|\nabla d|^2d),D^4d\rangle-\beta\langle D^4(d\times(d\times(V\cdot\nabla+\nabla\times d))),D^4d\rangle+\langle D^4(d\times f),D^4d\rangle.$$

The Gagliardo-Nirenberg inequality implies

$$\|Du\|_{L^4(\Omega)} \leq C_G \|u\|_{H^4(\Omega)}^{\frac{1}{6}} \|u\|_{L^\infty(\Omega)}^{\frac{5}{6}},$$

$$\|D^2u\|_{L^4(\Omega)} \leq C_G \|u\|_{H^4(\Omega)}^{\frac{1}{2}} \|u\|_{L^\infty(\Omega)}^{\frac{1}{2}},$$

$$\|D^3u\|_{L^2(\Omega)} \leq C_G \|u\|_{H^4(\Omega)}^{\frac{2}{3}} \|u\|_{L^\infty(\Omega)}^{\frac{1}{3}},$$

$$\|D^4d\|_{L^2(\Omega)} \leq C_G \|Dd\|_{H^4(\Omega)}^{\frac{2}{3}} \|Dd\|_{L^\infty(\Omega)}^{\frac{1}{3}},$$

$$\|D^3d\|_{L^4(\Omega)} \leq C_G \|Dd\|_{H^4(\Omega)}^{\frac{1}{2}} \|Dd\|_{L^\infty(\Omega)}^{\frac{1}{2}},$$

$$\|D^2d\|_{L^4(\Omega)} \leq C_G \|Dd\|_{H^4(\Omega)}^{\frac{1}{6}} \|Dd\|_{L^\infty(\Omega)}^{\frac{5}{6}},$$

$$\|D^3d\|_{L^2(\Omega)} \leq C_G \|Dd\|_{H^4(\Omega)}^{\frac{1}{3}} \|Dd\|_{L^\infty(\Omega)}^{\frac{2}{3}},$$

and thus

$$\begin{aligned}
&|\langle D^4(u\cdot\nabla d),\, D^4d\rangle| \\
&= |\langle D^4u\cdot\nabla d + 4D^3u\cdot\nabla Dd + 6D^2u\cdot\nabla D^2d + 4Du\cdot\nabla D^3d + u\cdot\nabla D^4d,\, D^4d\rangle| \\
&= |-\langle D^3u\cdot\nabla d,\, D^5d\rangle - \langle D^3u\cdot\nabla Dd,\, D^4d\rangle| \\
&\quad + |-4\langle D^3\nabla\cdot uDd,\, D^4d\rangle - 4\langle D^3u\cdot\nabla D^4d,\, Dd\rangle| \\
&\quad + |-6\langle D^2\nabla\cdot uD^2d,\, D^4d\rangle - 6\langle D^2u\cdot\nabla D^4d,\, D^2d\rangle| \\
&\quad + |-4\langle D\nabla\cdot uD^3d,\, D^4d\rangle - 4\langle Du\cdot\nabla D^4d,\, D^3d\rangle| \\
&= |-\langle D^3u\cdot\nabla d,\, D^5d\rangle + \langle D^3\nabla\cdot uDd,\, D^4d\rangle + \langle D^3u\cdot\nabla D^4d,\, Dd\rangle| \\
&\quad + |-4\langle D^3u\cdot\nabla D^4d,\, Dd\rangle - 6\langle D^2u\cdot\nabla D^4d,\, D^2d\rangle - 4\langle Du\cdot\nabla D^4d,\, D^3d\rangle| \\
&= |-\langle D^3u\cdot\nabla d,\, D^5d\rangle - 3\langle D^3u\cdot\nabla D^4d,\, Dd\rangle|
\end{aligned}$$

$$+|-6\langle D^2u\cdot\nabla D^4d,\, D^2d\rangle-4\langle Du\cdot\nabla D^4d,\, D^3d\rangle|$$

$$\le 4\|D^5d\|_{L^2(\Omega)}\|Dd\|_{L^\infty(\Omega)}\|D^3u\|_{L^2(\Omega)}+6\|D^5d\|_{L^2(\Omega)}\|D^2u\|_{L^4(\Omega)}\|D^2d\|_{L^4(\Omega)}$$

$$+4\|D^5d\|_{L^2(\Omega)}\|Du\|_{L^4(\Omega)}\|D^3d\|_{L^4(\Omega)}$$

$$\le 4C_G\|Dd\|_{H^4(\Omega)}\|Dd\|_{L^\infty(\Omega)}\|u\|^{\frac{2}{3}}_{H^4(\Omega)}\|u\|^{\frac{1}{3}}_{L^\infty(\Omega)}$$

$$+6C_G^2\|Dd\|^{\frac{7}{6}}_{H^4(\Omega)}\|Dd\|^{\frac{5}{6}}_{L^\infty(\Omega)}\|u\|^{\frac{1}{2}}_{H^4(\Omega)}\|u\|^{\frac{1}{2}}_{L^\infty(\Omega)}$$

$$+4C_G^2\|Dd\|^{\frac{3}{2}}_{H^4(\Omega)}\|Dd\|^{\frac{1}{2}}_{L^\infty(\Omega)}\|u\|^{\frac{1}{6}}_{H^4(\Omega)}\|u\|^{\frac{5}{6}}_{L^\infty(\Omega)}$$

$$\le \frac{\beta}{14}\|Dd\|^2_{H^4(\Omega)}+\tilde{C}\left(\frac{\beta}{168C_G}\right)\|Dd\|^2_{L^\infty(\Omega)}\|u\|^{\frac{4}{3}}_{H^4(\Omega)}\|u\|^{\frac{2}{3}}_{L^\infty(\Omega)}$$

$$+\tilde{C}\left(\frac{\beta}{252C_G^2}\right)\|Dd\|^2_{L^\infty(\Omega)}\|u\|^{\frac{6}{5}}_{H^4(\Omega)}\|u\|^{\frac{6}{5}}_{L^\infty(\Omega)}+\tilde{C}\left(\frac{\beta}{168C_G^2}\right)\|Dd\|^2_{L^\infty(\Omega)}\|u\|^{\frac{2}{3}}_{H^4(\Omega)}\|u\|^{\frac{10}{3}}_{L^\infty(\Omega)}$$

$$\le \frac{\beta}{14}\|Dd\|^2_{H^4(\Omega)}+\frac{\mu}{6}\|u\|^2_{H^4(\Omega)}+\tilde{C}\left(\frac{\mu}{18\tilde{C}\left(\frac{\beta}{168C_G}\right)}\right)\|Dd\|^6_{L^\infty(\Omega)}\|u\|^2_{L^\infty(\Omega)}$$

$$+\tilde{C}\left(\frac{\mu}{18\tilde{C}\left(\frac{\beta}{252C_G^2}\right)}\right)\|Dd\|^5_{L^\infty(\Omega)}\|u\|^3_{L^\infty(\Omega)}+\tilde{C}\left(\frac{\mu}{18\tilde{C}\left(\frac{\beta}{168C_G^2}\right)}\right)\|Dd\|^3_{L^\infty(\Omega)}\|u\|^5_{L^\infty(\Omega)}$$

$$\le \frac{\beta}{14}\|Dd\|^2_{H^4(\Omega)}+\frac{\mu}{6}\|u\|^2_{H^4(\Omega)}$$

$$+\left(\tilde{C}\left(\frac{\mu}{18\tilde{C}\left(\frac{\beta}{168C_G}\right)}\right)+\tilde{C}\left(\frac{\mu}{18\tilde{C}\left(\frac{\beta}{252C_G^2}\right)}\right)+\tilde{C}\left(\frac{\mu}{18\tilde{C}\left(\frac{\beta}{168C_G^2}\right)}\right)\right)C_G^8\tilde{K}^2\tilde{K}_2^2,$$

$$
\begin{aligned}
&|\alpha\langle D^4(d\times\Delta d),\, D^4 d\rangle|\\
&=|-\alpha\langle D^4(d\times\nabla d),\, D^4\nabla d\rangle|\\
&=|-\alpha\langle D^4 d\times\nabla d+4D^3 d\times\nabla Dd+6D^2 d\times\nabla D^2 d+4Dd\times\nabla D^3 d+d\times\nabla D^4 d,\, D^4\nabla d\rangle|\\
&\le 5|\alpha|\|D^4 d\|_{L^2(\Omega)}\|Dd\|_{L^\infty(\Omega)}\|D^5 d\|_{L^2(\Omega)}+10|\alpha|\|D^3 d\|_{L^4(\Omega)}\|D^2 d\|_{L^4(\Omega)}\|D^5 d\|_{L^2(\Omega)}\\
&\le 5C_G|\alpha|\|Dd\|_{H^4(\Omega)}^{\frac{5}{3}}\|Dd\|_{L^\infty(\Omega)}^{\frac{4}{3}}+10|\alpha|C_G^2\|Dd\|_{H^4(\Omega)}^{\frac{5}{3}}\|Dd\|_{L^\infty(\Omega)}^{\frac{4}{3}}\\
&\le \frac{\beta}{14}\|Dd\|_{H^4(\Omega)}^2+\tilde{C}\left(\frac{\beta}{140C_G|\alpha|}\right)\|Dd\|_{L^\infty(\Omega)}^8+\tilde{C}\left(\frac{\beta}{280C_G^2|\alpha|}\right)\|Dd\|_{L^\infty(\Omega)}^8\\
&\le \frac{\beta}{14}\|Dd\|_{H^4(\Omega)}^2+\left(\tilde{C}\left(\frac{\beta}{140C_G|\alpha|}\right)+\tilde{C}\left(\frac{\beta}{280C_G^2|\alpha|}\right)\right)C_G^8\tilde{K}^2\tilde{K}_2^2,\\
&|\beta\langle D^4(|\nabla d|^2 d),\, D^4 d\rangle|\\
&=|-\beta\langle D^3(|\nabla d|^2 d),\, D^5 d\rangle|\\
&=|-\beta\Big\langle 6\nabla Dd D^2\nabla dd+2\nabla d\nabla D^3 dd+6|\nabla Dd|^2 Dd\\
&\quad+6\nabla d\nabla D^2 dDd+6\nabla d\nabla DdD^2 d+|\nabla d|^2 D^3 d,\, D^5 d\rangle|\\
&\le|\beta|\Big(6\|D^2 d\|_{L^4(\Omega)}\|D^3 d\|_{L^4(\Omega)}\|d\|_{L^\infty(\Omega)}+12\|D^2 d\|_{L^4(\Omega)}^2\|Dd\|_{L^\infty(\Omega)}\\
&\quad+2\|Dd\|_{L^\infty(\Omega)}\|d\|_{L^\infty(\Omega)}\|D^4 d\|_{L^2(\Omega)}+7\|Dd\|_{L^\infty(\Omega)}^2\|D^3 d\|_{L^2(\Omega)}\Big)\|Dd\|_{H^4(\Omega)}\\
&\le 6|\beta|C_G^2\|Dd\|_{H^4(\Omega)}^{\frac{5}{3}}\|Dd\|_{L^\infty(\Omega)}^{\frac{4}{3}}+12|\beta|C_G^2\|Dd\|_{H^4(\Omega)}^{\frac{4}{3}}\|Dd\|_{L^\infty(\Omega)}^{\frac{8}{3}}\\
&\quad+2|\beta|C_G\|Dd\|_{H^4(\Omega)}^{\frac{5}{3}}\|Dd\|_{L^\infty(\Omega)}^{\frac{4}{3}}+7|\beta|C_G\|Dd\|_{H^4(\Omega)}^{\frac{4}{3}}\|Dd\|_{L^\infty(\Omega)}^{\frac{8}{3}}\\
&\le \frac{\beta}{14}\|Dd\|_{H^4(\Omega)}^2+\tilde{C}\left(\frac{1}{336C_G^2}\right)\|Dd\|_{L^\infty(\Omega)}^8+\tilde{C}\left(\frac{1}{672C_G^2}\right)\|Dd\|_{L^\infty(\Omega)}^8
\end{aligned}
$$

$$+\tilde{C}\left(\frac{1}{112C_G}\right)\|Dd\|_{L^\infty(\Omega)}^8+\tilde{C}\left(\frac{1}{392C_G}\right)\|Dd\|_{L^\infty(\Omega)}^8$$

$$\leq \frac{\beta}{14}\|Dd\|_{H^4(\Omega)}^2+\left(\tilde{C}\left(\frac{1}{336C_G^2}\right)+\tilde{C}\left(\frac{1}{672C_G^2}\right)+\tilde{C}\left(\frac{1}{112C_G}\right)+\tilde{C}\left(\frac{1}{392C_G}\right)\right)C_G^8\tilde{K}^2\tilde{K}_2^2,$$

$$|\alpha\langle D^4(d\times(V\cdot\nabla d+\nabla\times d)),\, D^4 d\rangle|$$

$$=|-\alpha\langle D^3(d\times(V\cdot\nabla d+\nabla\times d)),\, D^5 d\rangle|$$

$$=\Big|-\alpha\Big\langle D^3 d\times(V\cdot\nabla d+\nabla\times d)+3D^2 d\times(DV\cdot\nabla d+V\cdot\nabla Dd+\nabla\times Dd)$$

$$+3Dd\times(D^2V\cdot\nabla d+2DV\cdot\nabla Dd+V\cdot\nabla D^2 d+\nabla\times D^2 d)+d\times(D^3V\cdot\nabla d$$

$$+3D^2V\cdot\nabla Dd+3DV\cdot\nabla D^2 d+V\cdot\nabla D^3 d+\nabla\times D^3 d),\, D^5 d\Big\rangle\Big|$$

$$\leq |\alpha|\Big(4\|D^3 d\|_{L^2(\Omega)}\|Dd\|_{L^\infty(\Omega)}(\|V\|_{L^\infty(\Omega)}+1)+9\|D^2 d\|_{L^4(\Omega)}\|DV\|_{L^4(\Omega)}\|Dd\|_{L^\infty(\Omega)}$$

$$+3\|D^2 d\|_{L^4(\Omega)}^2(\|V\|_{L^\infty(\Omega)}+1)+3\|Dd\|_{L^\infty(\Omega)}^2\|D^2V\|_{L^2(\Omega)}$$

$$+\|D^3V\|_{L^2(\Omega)}\|Dd\|_{L^\infty(\Omega)}\|d\|_{L^\infty(\Omega)}+3\|DV\|_{L^4(\Omega)}\|D^3 d\|_{L^4(\Omega)}\|u\|_{L^\infty(\Omega)}$$

$$+\|d\|_{L^\infty(\Omega)}\|D^4 d\|_{L^2(\Omega)}(\|V\|_{L^\infty(\Omega)}+1)\Big)\|D^5 d\|_{L^2(\Omega)}$$

$$\leq |\alpha|\Big(4C_G\|Dd\|_{H^4(\Omega)}^{\frac{4}{3}}\|Dd\|_{L^\infty(\Omega)}^{\frac{5}{3}}(\|V\|_{L^\infty(\Omega)}+1)+9C_G\|Dd\|_{H^4(\Omega)}^{\frac{7}{6}}\|Dd\|_{L^\infty(\Omega)}^{\frac{11}{6}}\|DV\|_{L^4(\Omega)}$$

$$+3C_G^2\|Dd\|_{H^4(\Omega)}^{\frac{4}{3}}\|Dd\|_{L^\infty(\Omega)}^{\frac{5}{3}}(\|V\|_{L^\infty(\Omega)}+1)+3C_G\|D^2V\|_{L^2(\Omega)}\|Dd\|_{H^4(\Omega)}\|Dd\|_{L^\infty(\Omega)}^2$$

$$+\|D^3V\|_{L^2(\Omega)}\|Dd\|_{L^\infty(\Omega)}\|Dd\|_{H^4(\Omega)}+3C_G\|D^2V\|_{L^4(\Omega)}\|Dd\|_{H^4(\Omega)}^{\frac{7}{6}}\|Dd\|_{L^\infty(\Omega)}^{\frac{5}{6}}$$

$$+3C_G\|DV\|_{L^4(\Omega)}\|Dd\|_{H^4(\Omega)}^{\frac{3}{2}}\|Dd\|_{L^\infty(\Omega)}^{\frac{1}{2}}+C_G\|Dd\|_{H^4(\Omega)}^{\frac{5}{3}}\|Dd\|_{L^\infty(\Omega)}^{\frac{1}{3}}(\|V\|_{L^\infty(\Omega)}+1)\Big)$$

$$\leq \frac{\beta}{14}\|Dd\|_{H^4(\Omega)}^2+\tilde{C}\left(\frac{\beta}{616|\alpha|C_G}\right)\|Dd\|_{L^\infty(\Omega)}^5(\|V\|_{L^\infty(\Omega)}^3+1)+\tilde{C}\left(\frac{\beta}{1{,}386|\alpha|C_G}\right)\|Dd\|_{L^\infty(\Omega)}^{\frac{22}{5}}\|DV\|_{L^4(\Omega)}^{\frac{12}{5}}$$

$$+\tilde{C}\left(\frac{\beta}{462|\alpha|C_G^2}\right)\|Dd\|_{L^\infty(\Omega)}^5(\|V\|_{L^\infty(\Omega)}^3+1)+\tilde{C}\left(\frac{\beta}{462|\alpha|C_G}\right)\|Dd\|_{L^\infty(\Omega)}^4\|D^2V\|_{L^4(\Omega)}^2$$

$$+\tilde{C}\left(\frac{\beta}{154|\alpha|}\right)\|D^3V\|_{L^2(\Omega)}^2\|Dd\|_{L^\infty(\Omega)}^2+\tilde{C}\left(\frac{\beta}{462|\alpha|C_G}\right)\|Dd\|_{L^\infty(\Omega)}^2\|D^2V\|_{L^4(\Omega)}^{\frac{12}{5}}$$

$$+\tilde{C}\left(\frac{\beta}{462C_G}\right)\|Dd\|_{L^\infty(\Omega)}^2\|DV\|_{L^4(\Omega)}^4+\tilde{C}\left(\frac{\beta}{154|\alpha|C_G}\right)\|Dd\|_{L^\infty(\Omega)}^2(\|V\|_{L^\infty(\Omega)}^6+1)$$

$$\leq\frac{\beta}{14}\|Dd\|_{H^4(\Omega)}^2+\tilde{C}\left(\frac{\beta}{616|\alpha|C_G}\right)C_G^5\tilde{K}_2^{\frac{5}{4}}\tilde{K}^{\frac{5}{4}}(\|V\|_{L^\infty(\Omega)}^3+1)+\tilde{C}\left(\frac{\beta}{1{,}386|\alpha|C_G}\right)C_G^{\frac{22}{5}}\tilde{K}_2^{\frac{11}{10}}\tilde{K}^{\frac{11}{10}}\|DV\|_{L^4(\Omega)}^{\frac{12}{5}}$$

$$+\tilde{C}\left(\frac{\beta}{462|\alpha|C_G^2}\right)C_G^5\tilde{K}_2^{\frac{5}{4}}\tilde{K}^{\frac{5}{4}}(\|V\|_{L^\infty(\Omega)}^3+1)+\tilde{C}\left(\frac{\beta}{462|\alpha|C_G}\right)C_G^4\tilde{K}_2\tilde{K}\|D^2V\|_{L^4(\Omega)}^2$$

$$+\tilde{C}\left(\frac{\beta}{154|\alpha|}\right)\|D^3V\|_{L^2(\Omega)}^2C_G^2\tilde{K}_2^{\frac{1}{2}}\tilde{K}^{\frac{1}{2}}+\tilde{C}\left(\frac{\beta}{462|\alpha|C_G}\right)C_G^2\tilde{K}_2^{\frac{1}{2}}\tilde{K}^{\frac{1}{2}}\|D^2V\|_{L^4(\Omega)}^{\frac{12}{5}}$$

$$+\tilde{C}\left(\frac{\beta}{462C_G}\right)C_G^2\tilde{K}_2^{\frac{1}{2}}\tilde{K}^{\frac{1}{2}}\|DV\|_{L^4(\Omega)}^4+\tilde{C}\left(\frac{\beta}{152|\alpha|C_G}\right)C_G^2\tilde{K}_2^{\frac{1}{2}}\tilde{K}^{\frac{1}{2}}(\|V\|_{L^\infty(\Omega)}^6+1),$$

$$|\beta\langle D^4(d\times(d\times(V\cdot\nabla d+\nabla\times d))),D^4d\rangle|$$

$$=|-\beta\langle D^3(d\times(d\times(V\cdot\nabla d+\nabla\times d))),D^5d\rangle|$$

$$=\Big|-\beta\Big\langle D^3d\times d\times(V\cdot\nabla d+\nabla\times d)+3D^2u\times Dd\times(V\cdot\nabla d+\nabla\times d)$$

$$+3D^2d\times d\times(DV\cdot\nabla d+V\cdot\nabla Dd+\nabla\times Dd)+3Dd\times D^2d\times(V\cdot\nabla d+\nabla\times d)$$

$$+6Dd\times Dd\times(DV\cdot\nabla d+V\cdot\nabla Dd+\nabla\times Dd)+d\times D^3$$

$$+3Dd\times d\times(D^2V\cdot\nabla d+2DV\cdot\nabla d+V\cdot\nabla D^2d+\nabla\times D^2d)$$

$$+d\times D^3d\times(V\cdot\nabla d+\nabla\times d)+3d\times D^2d\times(DV\cdot\nabla d+V\cdot\nabla Dd+\nabla\times Dd)$$

$$+3d\times Dd\times(D^2V\cdot\nabla d+2DV\cdot\nabla Dd+V\cdot\nabla D^2d+\nabla\times D^2d)$$

$$+d\times d\times(D^3V\cdot\nabla d+3D^2V\cdot\nabla Dd+3DV\cdot\nabla D^2d+V\cdot\nabla D^3d+\nabla\times D^3d),D^5d\Big\rangle\Big|$$

$$
\begin{aligned}
&\leq \beta\Big(8\|d\|_{L^\infty(\Omega)}\|Dd\|_{L^\infty(\Omega)}\|D^3d\|_{L^2(\Omega)}(\|V\|_{L^\infty(\Omega)}+1)+12\|D^2d\|_{L^2(\Omega)}\|Dd\|^2_{L^\infty(\Omega)}(\|V\|_{L^\infty(\Omega)}+1)\\
&\quad+18\|D^2d\|_{L^4(\Omega)}\|d\|_{L^\infty(\Omega)}\|DV\|_{L^4(\Omega)}\|Dd\|_{L^\infty(\Omega)}+6\|D^2d\|^2_{L^4(\Omega)}\|d\|_{L^\infty(\Omega)}(\|V\|_{L^\infty(\Omega)}+1)\\
&\quad+6\|Dd\|^3_{L^\infty(\Omega)}\|DV\|_{L^2(\Omega)}+6\|Dd\|^2_{L^\infty(\Omega)}\|d\|_{L^\infty(\Omega)}\|D^2V\|_{L^2(\Omega)}\\
&\quad+\|d\|_{L^\infty(\Omega)}\|D^3V\|_{L^2(\Omega)}\|Dd\|_{L^\infty(\Omega)}+3\|d\|_{L^\infty(\Omega)}\|DV\|_{L^4(\Omega)}\|D^3d\|_{L^4(\Omega)}\\
&\quad+3\|d\|_{L^\infty(\Omega)}\|D^2V\|_{L^4(\Omega)}\|D^2d\|_{L^4(\Omega)}+\|d\|^2_{L^\infty(\Omega)}\|D^4d\|_{L^2(\Omega)}(\|V\|_{L^\infty(\Omega)}+1)\Big)\|D^5d\|_{L^2(\Omega)}\\
&\leq \beta\Big(8C_G\|Dd\|^{\frac{4}{3}}_{H^4(\Omega)}\|Dd\|^{\frac{5}{3}}_{L^\infty(\Omega)}(\|V\|_{L^\infty(\Omega)}+1)+12\|Dd\|_{H^4(\Omega)}\|Dd\|_{H^1(\Omega)}\|Du\|^2_{L^\infty(\Omega)}(\|V\|_{L^\infty(\Omega)}+1)\\
&\quad+18C_G\|Dd\|^{\frac{7}{6}}_{H^4(\Omega)}\|Dd\|^{\frac{11}{6}}_{L^\infty(\Omega)}\|DV\|_{L^4(\Omega)}+6C_G^2\|Dd\|^{\frac{4}{3}}_{H^4(\Omega)}\|Dd\|^{\frac{5}{3}}_{L^\infty(\Omega)}(\|V\|_{L^\infty(\Omega)}+1)\\
&\quad+6\|Dd\|_{H^4(\Omega)}\|Dd\|^3_{L^\infty(\Omega)}\|DV\|_{L^2(\Omega)}+6\|Dd\|_{H^4(\Omega)}\|Dd\|^2_{L^\infty(\Omega)}\|D^2V\|_{L^2(\Omega)}\\
&\quad+\|Dd\|_{H^4(\Omega)}\|Dd\|_{L^\infty(\Omega)}\|D^3V\|_{L^2(\Omega)}+3C_G\|Dd\|^{\frac{3}{2}}_{H^4(\Omega)}\|Dd\|^{\frac{1}{2}}_{L^\infty(\Omega)}\|DV\|_{L^4(\Omega)}\\
&\quad+3C_G\|Dd\|^{\frac{7}{6}}_{H^4(\Omega)}\|Dd\|^{\frac{5}{6}}_{L^\infty(\Omega)}\|D^2V\|_{L^4(\Omega)}+C_G\|Dd\|^{\frac{5}{3}}_{H^4(\Omega)}\|Dd\|^{\frac{1}{3}}_{L^\infty(\Omega)}(\|V\|_{L^\infty(\Omega)}+1)\Big)\\
&\leq \frac{\beta}{14}\|Dd\|^2_{H^4(\Omega)}+\tilde{C}\left(\frac{1}{1{,}568C_G}\right)\|Dd\|^5_{L^\infty(\Omega)}(\|V\|^3_{L^\infty(\Omega)}+1)\\
&\quad+\tilde{C}\left(\frac{1}{2{,}352}\right)\|Dd\|^2_{H^1(\Omega)}\|Dd\|^4_{L^\infty(\Omega)}(\|V\|^2_{L^\infty(\Omega)}+1)+\tilde{C}\left(\frac{1}{3{,}528C_G}\right)\|Dd\|^{\frac{22}{5}}_{L^\infty(\Omega)}\|DV\|^{\frac{12}{5}}_{L^4(\Omega)}\\
&\quad+\tilde{C}\left(\frac{1}{1{,}176C_G^2}\right)\|Dd\|^5_{L^\infty(\Omega)}(\|V\|^3_{L^\infty(\Omega)}+1)+\tilde{C}\left(\frac{1}{1{,}176}\right)\|Dd\|^6_{L^\infty(\Omega)}\|DV\|^2_{L^2(\Omega)}\\
&\quad+\tilde{C}\left(\frac{1}{1{,}176}\right)\|Dd\|^4_{L^\infty(\Omega)}\|D^2V\|^2_{L^2(\Omega)}+\tilde{C}\left(\frac{1}{196}\right)\|Dd\|^2_{L^\infty(\Omega)}\|D^3V\|^2_{L^2(\Omega)}\\
&\quad+\tilde{C}\left(\frac{1}{588C_G}\right)\|Dd\|^2_{L^\infty(\Omega)}\|DV\|^4_{L^4(\Omega)}+\tilde{C}\left(\frac{1}{588C_G}\right)\|Dd\|^2_{L^\infty(\Omega)}\|D^2V\|^{\frac{12}{5}}_{L^4(\Omega)}
\end{aligned}
$$

$$
\begin{aligned}
&+\tilde{C}\left(\frac{1}{196C_G}\right)\|Dd\|_{L^\infty(\Omega)}^2(\|V\|_{L^\infty(\Omega)}+1)\\
&\leq \frac{\beta}{14}\|Dd\|_{H^4(\Omega)}^2+\tilde{C}\left(\frac{1}{1{,}568C_G}\right)C_G^5\tilde{K}_2^{\frac{5}{4}}\tilde{K}^{\frac{5}{4}}(\|V\|_{L^\infty(\Omega)}^3+1)\\
&+\tilde{C}\left(\frac{1}{2{,}352}\right)C_G^4\tilde{K}_1\tilde{K}_2\tilde{K}(\|V\|_{L^\infty(\Omega)}^2+1)+\tilde{C}\left(\frac{1}{3{,}528C_G}\right)C_G^{\frac{22}{5}}\tilde{K}_2^{\frac{11}{10}}\tilde{K}^{\frac{11}{10}}\|DV\|_{L^4(\Omega)}^{\frac{12}{5}}\\
&+\tilde{C}\left(\frac{1}{1{,}176C_G^2}\right)C_G^5\tilde{K}_2^{\frac{5}{4}}\tilde{K}^{\frac{5}{4}}(\|V\|_{L^\infty(\Omega)}^3+1)+\tilde{C}\left(\frac{1}{1{,}176}\right)C_G^6\tilde{K}_2^{\frac{3}{2}}\tilde{K}^{\frac{3}{2}}\|DV\|_{L^2(\Omega)}^2\\
&+\tilde{C}\left(\frac{1}{1{,}176}\right)C_G^4\tilde{K}_2\tilde{K}\|D^2V\|_{L^2(\Omega)}^2+\tilde{C}\left(\frac{1}{196}\right)C_G^2\tilde{K}_2^{\frac{1}{2}}\tilde{K}^{\frac{1}{2}}\|D^3V\|_{L^2(\Omega)}^2\\
&+\tilde{C}\left(\frac{1}{588C_G}\right)C_G^2\tilde{K}_2^{\frac{1}{2}}\tilde{K}^{\frac{1}{2}}\|DV\|_{L^4(\Omega)}^4+\tilde{C}\left(\frac{1}{588C_G}\right)C_G^2\tilde{K}_2^{\frac{1}{2}}\tilde{K}^{\frac{1}{2}}\|D^2V\|_{L^4(\Omega)}^{\frac{12}{5}}\\
&+\tilde{C}\left(\frac{1}{196C_G}\right)C_G^2\tilde{K}_2^{\frac{1}{2}}\tilde{K}^{\frac{1}{2}}(\|V\|_{L^\infty(\Omega)}+1),
\end{aligned}
$$

$$
\begin{aligned}
&|\langle D^4(d\times f),\, D^4d\rangle|\\
&=|-\langle D^3(d\times f),\, D^5d\rangle|\\
&=|-\langle D^3d\times f+3D^2d\times Df+3Dd\times D^2f+d\times D^3f,\, D^5d\rangle|\\
&\leq \Big(\|D^3d\|_{L^2(\Omega)}\|f\|_{L^\infty(\Omega)}+3\|D^2d\|_{L^4(\Omega)}\|Df\|_{L^4(\Omega)}+3\|Dd\|_{L^\infty(\Omega)}\|D^2f\|_{L^2(\Omega)}\\
&+\|d\|_{L^\infty(\Omega)}\|D^3f\|_{L^2(\Omega)}\Big)\|D^5d\|_{L^2(\Omega)}\\
&\leq C_G\|Dd\|_{H^4(\Omega)}^{\frac{4}{3}}\|Dd\|_{L^\infty(\Omega)}^{\frac{2}{3}}\|f\|_{L^\infty(\Omega)}+C_G\|Dd\|_{H^4(\Omega)}^{\frac{7}{6}}\|Dd\|_{L^\infty(\Omega)}^{\frac{5}{6}}\|Df\|_{L^4(\Omega)}\\
&+3\|Dd\|_{H^4(\Omega)}\|Dd\|_{L^\infty(\Omega)}\|D^2f\|_{L^2(\Omega)}+\|Dd\|_{H^4(\Omega)}\|D^3f\|_{L^2(\Omega)}\\
&\leq \frac{\beta}{14}\|Dd\|_{H^4(\Omega)}^2+\tilde{C}\left(\frac{\beta}{56C_G}\right)\|Dd\|_{L^\infty(\Omega)}^2\|f\|_{L^\infty(\Omega)}^3+\tilde{C}\left(\frac{\beta}{56C_G}\right)\|Dd\|_{L^\infty(\Omega)}^2\|Df\|_{L^4(\Omega)}^{\frac{12}{5}}
\end{aligned}
$$

$$+\frac{14}{\beta}\|Dd\|^2_{L^\infty(\Omega)}\|D^2 f\|^2_{L^2(\Omega)}+\frac{14}{\beta}\|D^3 f\|^2_{L^2(\Omega)}$$

$$\leq \frac{\beta}{14}\|Dd\|^2_{H^4(\Omega)}+\tilde{C}\left(\frac{\beta}{56C_G}\right)C_G^2\tilde{K}_2^{\frac{1}{2}}\tilde{K}^{\frac{1}{2}}\|f\|^3_{L^\infty(\Omega)}+\tilde{C}\left(\frac{\beta}{56C_G}\right)C_G^2\tilde{K}_2^{\frac{1}{2}}\tilde{K}^{\frac{1}{2}}\|Df\|^{\frac{12}{5}}_{L^4(\Omega)}$$

$$+\frac{14}{\beta}C_G^2\tilde{K}_2^{\frac{1}{2}}\tilde{K}^{\frac{1}{2}}\|D^2 f\|^2_{L^2(\Omega)}+\frac{14}{\beta}\|D^3 f\|^2_{L^2(\Omega)}.$$

Differentiating the equation (10) by D^3, taking inner product with D^3u and integrating over Ω, we obtain

$$\langle D^3u_t,\, D^3u\rangle+\langle D^3(u\cdot\nabla u),\, D^3u\rangle+\langle D^3\nabla P,\, D^3u\rangle$$

$$=\mu\langle D^3\Delta u,\, D^3u\rangle-\lambda\langle D^3\nabla\cdot(\nabla d\odot\nabla d),\, D^3u\rangle.$$

Calculating each term of the above equation gets

$$|\langle D^3(u\cdot\nabla u),\, D^3u\rangle|$$

$$=|\langle D^3u\cdot\nabla u+3D^2u\cdot\nabla Du+3Du\cdot\nabla D^2u+u\cdot\nabla D^3u,\, D^3u\rangle|$$

$$=|-\langle D^3\nabla\cdot uu,\, D^3u\rangle-\langle D^3u\cdot\nabla D^3u,\, u\rangle$$

$$-3\langle D^2\nabla\cdot uDu,\, D^3u\rangle-3\langle D^2u\cdot\nabla D^3u,\, Du\rangle$$

$$-3\langle D\nabla\cdot uD^2u,\, D^3u\rangle-3\langle Du\cdot\nabla D^3u,\, D^2u\rangle|$$

$$=|-\langle D^3u\cdot\nabla D^3u,\, u\rangle-3\langle D^2u\cdot\nabla D^3u,\, Du\rangle-3\langle Du\cdot\nabla D^3u,\, D^2u\rangle|$$

$$\leq\|D^4u\|_{L^2(\Omega)}\|D^3u\|_{L^2(\Omega)}\|u\|_{L^\infty(\Omega)}+6\|D^4u\|_{L^2(\Omega)}\|Du\|_{L^4(\Omega)}\|D^2u\|_{L^4(\Omega)}$$

$$\leq C_G\|u\|^{\frac{5}{3}}_{H^4(\Omega)}\|u\|^{\frac{4}{3}}_{L^\infty(\Omega)}+6C_G^2\|u\|^{\frac{5}{3}}_{H^4(\Omega)}\|u\|^{\frac{4}{3}}_{L^\infty(\Omega)}$$

$$\leq\frac{\mu}{6}\|u\|^2_{H^4(\Omega)}+\tilde{C}\left(\frac{\mu}{12C_G}\right)\|u\|^8_{L^\infty(\Omega)}+\tilde{C}\left(\frac{\mu}{72C_G^2}\right)\|u\|^8_{L^\infty(\Omega)}$$

$$\leq\frac{\mu}{6}\|u\|^2_{H^4(\Omega)}+\left(\tilde{C}\left(\frac{\mu}{12C_G}\right)+\tilde{C}\left(\frac{\mu}{72C_G^2}\right)\right)C_G^8\tilde{K}^2\tilde{K}_2^2,$$

$$
\begin{aligned}
&|\lambda\langle D^3\nabla\cdot(\nabla d\odot\nabla d),\, D^3u\rangle| \\
=&\left|\lambda\sum_{i,\,j,\,k=1}^{2}\langle D^3\partial_i(\partial_i d_k\partial_j d_k),\, D^3u_j\rangle\right| \\
=&\left|\lambda\sum_{i,\,j,\,k=1}^{2}\langle D^4(\partial_i d_k\partial_j d_k),\, D^2\partial_i u_j\rangle\right| \\
=&\left|\lambda\sum_{i,\,j,\,k=1}^{2}\langle D^4\partial_i d_k\partial_j d_k+4D^3\partial_i d_k\partial_j Dd_k+6D_i^{\partial} d_k\partial_j D^2d_k+4D\partial_i d_k\partial_j D^3d_k+\partial_i d_k\partial_j D^4d_k,\, D^2\partial_i u_j\rangle\right| \\
=&\left|\lambda\sum_{i,\,j,\,k=1}^{2}\langle D^4\partial_i d_k\partial_j d_k+\partial_i d_k\partial_j D^4d_k,\, D^2\partial_i u_j\rangle\right| \\
&+\left|-4\lambda\sum_{i,\,j,\,k=1}^{2}\Big(\langle D^3\partial_i\partial_j d_k Dd_k,\, D^2\partial_i u_j\rangle+\langle D^3\partial_i d_k Dd_k,\, D^2\partial_i\partial_j u_j\rangle\Big)\right| \\
&+\left|-6\lambda\sum_{i,\,j,\,k=1}^{2}\Big(\langle D^2\partial_i^2 d_k\partial_j D^2d_k,\, D^2u_j\rangle+\langle D^2\partial_i d_k\partial_j\partial_i D^2d_k,\, D^2u_j\rangle\Big)\right| \\
&+\left|-4\lambda\sum_{i,\,j,\,k=1}^{2}\Big(\langle D\partial_i^2 d_k D^3\partial_j d_k,\, D^2u_j\rangle+\langle D\partial_i d_k\partial_i\partial_j D^3d_k,\, D^2u_j\rangle\Big)\right| \\
=&\left|\lambda\sum_{i,\,j,\,k=1}^{2}\langle D^4\partial_i d_k\partial_j d_k+\partial_i d_k\partial_j D^4d_k,\, D^2\partial_i u_j\rangle\right| \\
&+\left|-4\lambda\sum_{i,\,j,\,k=1}^{2}\Big(\langle D^3\partial_i\partial_j d_k Dd_k,\, D^2\partial_i u_j\rangle+\langle D\partial_i d_k\partial_i\partial_j D^3d_k,\, D^2u_j\rangle\Big)\right| \\
&+\left|6\lambda\sum_{i,\,j,\,k=1}^{2}\Big(\langle D^2\partial_i^2\partial_j d_k D^2d_k,\, D^2u_j\rangle+\langle D^2\partial_i^2 d_k D^2d_k,\, D^2\partial_j u_j\rangle\Big)\right| \\
&+\left|6\lambda\sum_{i,\,j,\,k=1}^{2}\Big(\langle D^3\partial_i d_k\partial_j\partial_i D^2d_k,\, Du_j\rangle+\langle D^2\partial_i d_k\partial_i\partial_j D^3d_k,\, Du_j\rangle\Big)\right|
\end{aligned}
$$

$$+\left|4\lambda\sum_{i,\,j,\,k=1}^{2}\Big(\langle D^2\partial_i^2 d_k D^3\partial_j d_k,\, Du_j\rangle+\langle D\partial_i^2 d_k D^4\partial_j d_k,\, Du_j\rangle\Big)\right|$$

$$=\left|\lambda\sum_{i,\,j,\,k=1}^{2}\langle D^4\partial_i d_k\partial_j d_k+\partial_i d_k\partial_j D^4 d_k,\, D^2\partial_i u_j\rangle\right|$$

$$+\left|-4\lambda\sum_{i,\,j,\,k=1}^{2}\Big(\langle D^3\partial_i\partial_j d_k D d_k,\, D^2\partial_i u_j\rangle+\langle D\partial_i d_k\partial_i\partial_j D^3 d_k,\, D^2 u_j\rangle\Big)\right|$$

$$+\left|\lambda\sum_{i,\,j,\,k=1}^{2}\Big(6\langle D^2\partial_i^2\partial_j d_k D^2 d_k,\, D^2 u_j\rangle+6\langle D^2\partial_i d_k\partial_i\partial_j D^3 d_k,\, Du_j\rangle+4\langle D\partial_i^2 d_k D^4\partial_j d_k,\, Du_j\rangle\Big)\right|$$

$$+\left|-6\lambda\sum_{i,\,j,\,k=1}^{3}\Big(\langle D^3\partial_i\partial_j d_k\partial_i D^2 d_k,\, Du_j\rangle+\langle D^3\partial_i d_k\partial_i D^2 d_k,\, D\partial_j u_j\rangle\Big)\right|$$

$$+\left|-4\lambda\sum_{i,\,j,\,k=1}^{2}\Big(\langle\partial_j D^2\partial_i^2 d_k D^3 d_k,\, Du_j\rangle+\langle D^2\partial_i^2 d_k D^3 d_k,\, D\partial_j u_j\rangle\Big)\right|$$

$$=\left|\lambda\sum_{i,\,j,\,k=1}^{2}\langle D^4\partial_i d_k\partial_j d_k+\partial_i d_k\partial_j D^4 d_k,\, D^2\partial_i u_j\rangle\right|$$

$$+\left|-4\lambda\sum_{i,\,j,\,k=1}^{2}\Big(\langle D^3\partial_i\partial_j d_k D d_k,\, D^2\partial_i u_j\rangle+\langle D\partial_i d_k\partial_i\partial_j D^3 d_k,\, D^2 u_j\rangle\Big)\right|$$

$$+\left|\lambda\sum_{i,\,j,\,k=1}^{2}\Big(6\langle D^2\partial_i^2\partial_j d_k D^2 d_k,\, D^2 u_j\rangle+6\langle D^2\partial_i d_k\partial_i\partial_j D^3 d_k,\, Du_j\rangle+4\langle D\partial_i^2 d_k D^4\partial_j d_k,\, Du_j\rangle\Big)\right|$$

$$+\left|\lambda\sum_{i,\,j,\,k=1}^{2}\Big(-6\langle D^3\partial_i\partial_j d_k\partial_i D^2 d_k,\, Du_j\rangle-4\langle\partial_j D^2\partial_i^2 d_k D^3 d_k,\, Du_j\rangle\Big)\right|$$

$$\leq 6|\lambda|\|D^5 d\|_{L^2(\Omega)}\|Dd\|_{L^\infty(\Omega)}\|D^3 u\|_{L^2(\Omega)}+10|\lambda|\|D^5 d\|_{L^2(\Omega)}\|D^2 u\|_{L^4(\Omega)}\|D^2 d\|_{L^4(\Omega)}$$

$$+20|\lambda|\|D^5 d\|_{L^2(\Omega)}\|Du\|_{L^4(\Omega)}\|D^3 d\|_{L^4(\Omega)}$$

$$\leq 6|\lambda|C_G\|Dd\|_{H^4(\Omega)}\|Dd\|_{L^\infty(\Omega)}\|u\|_{H^4(\Omega)}^{\frac{2}{3}}\|u\|_{L^\infty(\Omega)}^{\frac{1}{3}}$$

$$
+10C_G^2|\lambda|\|Dd\|_{H^4(\Omega)}^{\frac{7}{6}}\|Dd\|_{L^\infty(\Omega)}^{\frac{5}{6}}\|u\|_{H^4(\Omega)}^{\frac{1}{2}}\|u\|_{L^\infty(\Omega)}^{\frac{1}{2}}
$$

$$
+20C_G^2|\lambda|\|Dd\|_{H^4(\Omega)}^{\frac{3}{2}}\|Dd\|_{L^\infty(\Omega)}^{\frac{1}{2}}\|u\|_{H^4(\Omega)}^{\frac{1}{6}}\|u\|_{L^\infty(\Omega)}^{\frac{5}{6}}
$$

$$
\leq \frac{\beta}{14}\|Dd\|_{H^4(\Omega)}+\tilde{C}\left(\frac{\beta}{252|\lambda|C_G}\right)\|Dd\|_{L^\infty(\Omega)}^2\|u\|_{H^4(\Omega)}^{\frac{4}{3}}\|u\|_{L^\infty(\Omega)}^{\frac{3}{2}}
$$

$$
+\tilde{C}\left(\frac{\beta}{420|\lambda|C_G^2}\right)\|Dd\|_{L^\infty(\Omega)}^2\|u\|_{H^4(\Omega)}^{\frac{6}{5}}\|u\|_{L^\infty(\Omega)}^{\frac{6}{5}}+\tilde{C}\left(\frac{\beta}{840|\lambda|C_G^2}\right)\|Dd\|_{L^\infty(\Omega)}^2\|u\|_{H^4(\Omega)}^{\frac{2}{3}}\|u\|_{L^\infty(\Omega)}^{\frac{10}{3}}
$$

$$
\leq \frac{\beta}{14}\|Dd\|_{H^4(\Omega)}^2+\frac{\mu}{6}\|u\|_{H^4(\Omega)}^2+\tilde{C}\left(\frac{\mu}{18\tilde{C}\left(\frac{\beta}{252|\lambda|C_G}\right)}\right)\|Dd\|_{L^\infty(\Omega)}^6\|u\|_{L^\infty(\Omega)}^2
$$

$$
+\tilde{C}\left(\frac{\mu}{18\tilde{C}\left(\frac{\beta}{420|\lambda|C_G^2}\right)}\right)\|Dd\|_{L^\infty(\Omega)}^5\|u\|_{L^\infty(\Omega)}^3+\tilde{C}\left(\frac{\mu}{18\tilde{C}\left(\frac{\beta}{840|\lambda|C_G^2}\right)}\right)\|Dd\|_{L^2(\Omega)}^3\|u\|_{L^\infty(\Omega)}^5
$$

$$
\leq \frac{\beta}{14}\|Dd\|_{H^4(\Omega)}^2+\frac{\mu}{6}\|u\|_{H^4(\Omega)}^2
$$

$$
+\left(\tilde{C}\left(\frac{\mu}{18\tilde{C}\left(\frac{\beta}{252|\lambda|C_G}\right)}\right)+\tilde{C}\left(\frac{\mu}{18\tilde{C}\left(\frac{\beta}{420|\lambda|C_G^2}\right)}\right)+\tilde{C}\left(\frac{\mu}{18\tilde{C}\left(\frac{\beta}{840|\lambda|C_G^2}\right)}\right)\right)C_G^8\tilde{K}^2\tilde{K}_2^2.
$$

Therefore, recalling the Gronwall inequality (integral form) finds

$$
\sup_{0\leq t\leq T}\|u\|_{H^3(\Omega)}^2+\sup_{0\leq t\leq T}\|Dd\|_{H^3(\Omega)}^2+\mu\int_0^T\|u\|_{H^4(\Omega)}^2dt+\beta\int_0^T\|Dd\|_{H^4(\Omega)}^2dt\leq\tilde{K}_3,
$$

where

$$
\tilde{K}_3=(1+\bar{\beta}T\exp(\bar{\beta}T))\left\{\|u_0\|_{H^3(\Omega)}^2+\|Dd_0\|_{H^3(\Omega)}^2+\tilde{K}_2+2\int_0^T\left\{\left(\tilde{C}\left(\frac{\mu}{18\tilde{C}\left(\frac{\beta}{168C_G}\right)}\right)+\tilde{C}\left(\frac{\mu}{18\tilde{C}\left(\frac{\beta}{252C_G^2}\right)}\right)\right.\right.\right.
$$

$$+\tilde{C}\left(\frac{\mu}{18\tilde{C}\left(\frac{\beta}{168C_G^2}\right)}\right)+\tilde{C}\left(\frac{\frac{\beta}{14}}{140C_G|\alpha|}\right)+\tilde{C}\left(\frac{\beta}{280C_G^2|\alpha|}\right)+\tilde{C}\left(\frac{1}{336C_G^2}\right)+\tilde{C}\left(\frac{1}{672C_G^2}\right)+\tilde{C}\left(\frac{1}{112C_G}\right)$$

$$+\tilde{C}\left(\frac{1}{392C_G}\right)+\tilde{C}\left(\frac{\mu}{12C_G}\right)+\tilde{C}\left(\frac{\mu}{72C_G^2}\right)+\tilde{C}\left(\frac{\mu}{18\tilde{C}\left(\frac{\beta}{252|\lambda|C_G}\right)}\right)+\tilde{C}\left(\frac{\mu}{18\tilde{C}\left(\frac{\beta}{420|\lambda|C_G^2}\right)}\right)$$

$$+\tilde{C}\left(\frac{\mu}{18\tilde{C}\left(\frac{\beta}{840|\lambda|C_G^2}\right)}\right)C_G^8\tilde{K}^2\tilde{K}_2^2+\tilde{C}\left(\frac{\beta}{616|\alpha|C_G}\right)C_G^5\tilde{K}_2^{\frac{5}{4}}\tilde{K}^{\frac{5}{4}}\left(\|V\|_{L^\infty(\Omega)}^3+1\right)$$

$$+\tilde{C}\left(\frac{\beta}{1{,}386|\alpha|C_G}\right)C_G^{\frac{22}{2}}\tilde{K}_2^{11}\tilde{K}^{111}\|DV\|_{L^4(\Omega)}^{\frac{12}{2}}+\tilde{C}\left(\frac{\beta}{462|\alpha|C_G^2}\right)C_G^5\tilde{K}_2^{\frac{5}{4}}\tilde{K}^{\frac{5}{4}}\left(\|V\|_{L^\infty(\Omega)}^3+1\right)$$

$$+\tilde{C}\left(\frac{\beta}{462|\alpha|C_G}\right)C_G^4\tilde{K}_2\tilde{K}\left\|D^2V\right\|_{L^4(\Omega)}^2+\tilde{C}\left(\frac{\beta}{154|\alpha|}\right)\left\|D^3V\right\|_{L^2(\Omega)}^2C_G^2\tilde{K}_2^{\frac{1}{2}}\tilde{K}^{\frac{1}{2}}$$

$$+\tilde{C}\left(\frac{\beta}{462|\alpha|C_G}\right)C_G^2\tilde{K}_2^{\frac{1}{2}}\tilde{K}^{\frac{1}{2}}\left\|D^2V\right\|_{L^4(\Omega)}^{\frac{12}{5}}+\tilde{C}\left(\frac{\beta}{462C_G}\right)C_G^2\tilde{K}_2^{\frac{1}{2}}\tilde{K}^{\frac{1}{2}}\|DV\|_{L^4(\Omega)}^4$$

$$+\tilde{C}\left(\frac{\beta}{154|\alpha|C_G}\right)C_G^2\tilde{K}_2^{\frac{1}{2}}\tilde{K}^{\frac{1}{2}}\left(\|V\|_{L^\infty(\Omega)}^6+1\right)+\tilde{C}\left(\frac{1}{1{,}568C_G}\right)C_G^5\tilde{K}_2^{\frac{5}{4}}\tilde{K}^{\frac{5}{4}}\left(\|V\|_{L^\infty(\Omega)}^3+1\right)$$

$$+\tilde{C}\left(\frac{1}{2{,}352}\right)C_G^4\tilde{K}_1\tilde{K}_2\tilde{K}\left(\|V\|_{L^\infty(\Omega)}^2+1\right)$$

$$+\tilde{C}\left(\frac{1}{3{,}528C_G}\right)C_G^{\frac{22}{5}}\tilde{K}_2^{11}\tilde{K}^{11}\|DV\|_{L^4(\Omega)}^{\frac{12}{5}}+\tilde{C}\left(\frac{1}{1{,}176C_G^2}\right)C_G^5\tilde{K}_2^{\frac{5}{4}}\tilde{K}^{\frac{5}{4}}\left(\|V\|_{L^\infty(\Omega)}^3+1\right)$$

$$+\tilde{C}\left(\frac{1}{1{,}176}\right)C_G^6\tilde{K}_2^{\frac{3}{2}}\tilde{K}^{\frac{3}{2}}\|DV\|_{L^2(\Omega)}^2+\tilde{C}\left(\frac{1}{1{,}176}\right)C_G^4\tilde{K}_2\tilde{K}\left\|D^2V\right\|_{L^2(\Omega)}^2$$

$$+\tilde{C}\left(\frac{1}{196}\right)C_G^2\tilde{K}_2^{\frac{1}{2}}\tilde{K}^{\frac{1}{2}}\left\|D^3V\right\|_{L^2(\Omega)}^2+\tilde{C}\left(\frac{1}{588C_G}\right)C_G^2\tilde{K}_2^{\frac{1}{2}}\tilde{K}^{\frac{1}{2}}\|DV\|_{L^4(\Omega)}^4$$

$$+\tilde{C}\left(\frac{1}{688C_G}\right)C_G^2\tilde{K}_2^{\frac{1}{2}}\tilde{K}^{\frac{1}{2}}\left\|D^2V\right\|_{L^4(\Omega)}^{\frac{12}{5}}+\tilde{C}\left(\frac{1}{196C_G}\right)C_G^2\tilde{K}_2^{\frac{1}{2}}\tilde{K}^{\frac{1}{2}}\left(\|V\|_{L^\infty(\Omega)}+1\right)$$

$$+\tilde{C}\left(\frac{\beta}{56C_G}\right)C_G^2\tilde{K}_2^{\frac{1}{2}}\tilde{K}^{\frac{1}{2}}\|f\|_{L^\infty(\Omega)}^3+\tilde{C}\left(\frac{\beta}{56C_G}\right)C_G^2\tilde{K}_2^{\frac{1}{2}}\tilde{K}^{\frac{1}{2}}\|Df\|_{L^4(\Omega)}^{\frac{12}{3}}$$

$$+\frac{14}{\beta}C_G^2\tilde{K}_2^{\frac{1}{2}}\tilde{K}^{\frac{1}{2}}\left\|D^2f\right\|_{L^2(\Omega)}^2+\frac{14}{\beta}\left\|D^3f\right\|_{L^2(\Omega)}^2\Bigg\}(t)dt\Bigg\}$$

□

Similarly, it is easy to see that

$$\|Du\|_{L^\infty(\Omega)}\leq C_G\|u\|_{H^3(\Omega)}^{\frac{2}{3}}\|u\|_{L^2(\Omega)}^{\frac{1}{3}}\leq C_G\tilde{K}_3^{\frac{1}{3}}\tilde{K}^{\frac{1}{6}},$$

$$\|u\|_{L^\infty(\Omega)}\leq C_G\|u\|_{H^3(\Omega)}^{\frac{1}{3}}\|u\|_{L^2(\Omega)}^{\frac{2}{3}}\leq C_G\tilde{K}_3^{\frac{1}{6}}\tilde{K}^{\frac{1}{3}},$$

$$\|D^2d\|_{L^\infty(\Omega)}\leq C_G\|Dd\|_{H^3(\Omega)}^{\frac{2}{3}}\|Dd\|_{L^2(\Omega)}^{\frac{1}{3}}\leq C_G\tilde{K}_3^{\frac{1}{3}}\tilde{K}^{\frac{1}{6}},$$

$$\|Dd\|_{L^\infty(\Omega)}\leq C_G\|Dd\|_{H^3(\Omega)}^{\frac{1}{3}}\|Dd\|_{L^2(\Omega)}^{\frac{2}{3}}\leq C_G\tilde{K}_3^{\frac{1}{6}}\tilde{K}^{\frac{1}{3}}.$$

Next, the estimates of $\sup_{0\leq t\leq T}\|u\|_{H^m(\Omega)}^2$ and $\sup_{0\leq t\leq T}\|Dd\|_{H^m(\Omega)}^2$ $(m\geq 2)$ shall be shown in the following lamma.

Lemma 8 Assume that u and d are the smooth local solution of the system (10)-(12) with $\nabla\cdot u=0$, one may see that

$$\sup_{0\leq t\leq T}\|u\|_{H^m(\Omega)}^2+\sup_{0\leq t\leq T}\|Dd\|_{H^m(\Omega)}^2+\beta\int_0^T\|Dd\|_{H^{m+1}(\Omega)}^2\mathrm{d}t+\mu\int_0^T\|u\|_{H^{m+1}(\Omega)}^2\mathrm{d}t\leq\tilde{K}_m,$$

here $\tilde{K}_m$ are defined in the proof.

Proof. Differentiating the equation (11) with D^{m+1}, taking inner product with $D^{m+1}d$ and integrating over Ω leads to

$$\langle D^{m+1}d_t,D^{m+1}d\rangle+\langle D^{m+1}(u\cdot\nabla d),D^{m+1}d\rangle+\alpha\langle D^{m+1}(d\times\Delta d),D^{m+1}d\rangle$$

$$+\alpha\langle D^{m+1}(d\times(V\cdot\nabla d+\nabla\times d)),D^{m+1}d\rangle=\beta\langle D^{m+1}\Delta d,D^{m+1}d\rangle+\beta\langle D^{m+1}(|\nabla d|^2d),D^{m+1}d\rangle$$

$$-\beta\langle D^{m+1}(d\times(d\times(V\cdot\nabla d+\nabla\times d))),D^{m+1}d\rangle+\langle D^{m+1}(d\times f),D^{m+1}d\rangle.$$

Let us recall the Gagliardo-Nirenberg-Moser inequality (see [41] Proposition 3.7)

$$\|D^s(fg)-fD^sg\|_{L^2(\Omega)}\leq C_K\Big(\|f\|_{H^k(\Omega)}\|g\|_{L^\infty(\Omega)}+\|Df\|_{L^\infty(\Omega)}\|g\|_{H^{k-1}(\Omega)}\Big),\quad (k=|s|\geq 2),$$

the following estimates can be obtained

$$
\begin{aligned}
&|\langle D^{m+1}(u\cdot\nabla d), D^{m+1}d\rangle| \\
&= |-\langle D^m(u\cdot\nabla d), D^{m+2}d\rangle| \\
&\leq \Big(\|u\|_{L^\infty(\Omega)}\|D^m\nabla d\|_{L^2(\Omega)}+C_K\|u\|_{H^m(\Omega)}\|Dd\|_{L^\infty(\Omega)}+C_K\|Du\|_{L^\infty(\Omega)}\|Dd\|_{H^{m-1}(\Omega)}\Big)\|D^{m+2}d\|_{L^2(\Omega)} \\
&\leq \frac{\beta}{12}\|Dd\|^2_{H^{m+1}(\Omega)}+\frac{9}{\beta}\|u\|^2_{L^\infty(\Omega)}\|Dd\|^2_{H^m(\Omega)}+\frac{9C_K^2}{\beta}\|Dd\|^2_{L^\infty(\Omega)}\|u\|^2_{H^m(\Omega)}+\frac{9C_K^2}{\beta}\|Du\|^2_{L^\infty(\Omega)}\|Dd\|^2_{H^{m-1}(\Omega)} \\
&\leq \frac{\beta}{12}\|Dd\|^2_{H^{m+1}(\Omega)}+\frac{9}{\beta}C_G^2\tilde{K}_3^{\frac{1}{3}}\tilde{K}^{\frac{2}{3}}\|Dd\|^2_{H^m(\Omega)}+\frac{9C_K^2}{\beta}C_G^2\tilde{K}_3^{\frac{1}{3}}\tilde{K}^{\frac{2}{3}}\|u\|^+_{H^m(\Omega)}\frac{9C_K^2}{\beta}C_G^2\tilde{K}_3^{\frac{2}{3}}\tilde{K}^{\frac{1}{3}}\tilde{K}_{m-1}, \\
&|\beta\langle D^{m+1}(|\nabla d|^2 d), D^{m+1}d\rangle| \\
&= |-\beta\langle D^m(|\nabla d|^2 d), D^{m+2}d\rangle| \\
&\leq \beta\|D^m(|Dd|^2 d)\|_{L^2(\Omega)}\|D^{m+2}d\|_{L^2(\Omega)} \\
&\leq \beta\Big(\|Dd\|^2_{L^\infty(\Omega)}\|D^m d\|_{L^2(\Omega)}+C_K(\||\nabla d|^2\|_{H^m}\|d\|_{L^\infty(\Omega)} \\
&\quad +C_K\|Dd\|_{L^\infty(\Omega)}\|D^2 d\|_{L^\infty(\Omega)}\|d\|_{H^{m-1}})\Big)\|D^{m+2}d\|_{L^2(\Omega)} \\
&\leq \beta\Big(\|Dd\|^2_{L^\infty(\Omega)}\|Dd\|_{H^{m-1}(\Omega)}+C_K\|Dd\|_{L^\infty(\Omega)}\|Dd\|_{H^m(\Omega)}+C_K^2 m\|Dd\|_{L^\infty(\Omega)}\|Dd\|_{H^m(\Omega)} \\
&\quad +C_K^2 m\|D^2 d\|_{L^\infty(\Omega)}\|Dd\|_{H^{m-1}(\Omega)}+C_K\|Dd\|_{L^\infty(\Omega)}\|D^2 d\|_{L^\infty(\Omega)}\|d\|_{H^{m-1}(\Omega)}\Big)\|Dd\|_{H^{m+1}(\Omega)} \\
&\leq \frac{\beta}{12}\|Dd\|^2_{H^{m+1}(\Omega)}+15\beta(\|Dd\|^4_{L^\infty(\Omega)}+C_K^4 m^2\|D^2 d\|^2_{L^\infty(\Omega)})\|Dd\|^2_{H^{m-1}(\Omega)} \\
&\quad +15\beta(C_K^2+C_K^4 m^2)\|Dd\|^2_{L^\infty(\Omega)}\|Dd\|^2_{H^m(\Omega)}+15\beta C_K^2\|Dd\|^2_{L^\infty(\Omega)}\|D^2 d\|^2_{L^\infty(\Omega)}\|d\|^2_{H^{m-1}}, \\
&\leq \frac{\beta}{12}\|Dd\|^2_{H^{m+1}(\Omega)}+15\beta(C_G^4\tilde{K}_3^{\frac{2}{3}}\tilde{K}^{\frac{4}{3}}+C_K^4 m^2 C_G^2\tilde{K}_3^{\frac{3}{3}}\tilde{K}^{\frac{1}{3}})\tilde{K}_{m-1} \\
&\quad +15\beta(C_K^2+C_K^4 m^2)C_G^2\tilde{K}_3^{\frac{1}{3}}\tilde{K}^{\frac{2}{3}}\|Dd\|^2_{H^m(\Omega)}+15\beta C_K^2 C_G^4\tilde{K}_3\tilde{K}\|d\|^2_{H^{m-1}},
\end{aligned}
$$

$$
\begin{aligned}
&|\alpha\langle D^{m+1}(d\times(\Delta d)), D^{m+1}d\rangle|\\
=\ &\left|-\alpha\left\langle D^{m+1}(d\times(\nabla d)), D^{m+1}\nabla d\right\rangle\right|\\
=\ &\left|-\alpha\left\langle\left(D^{m+1}d\times(\nabla d)+d\times(D^{m+1}\nabla d)+\sum_{i=1}^{m}C_iD^id\times D^{m+1-i}\nabla d\right), D^{m+1}\nabla d\right\rangle\right|\\
\leq\ &C_G^2|\alpha|(2^{m+1}-1)\|Dd\|_{L^\infty(\Omega)}\|Dd\|_{H^m(\Omega)}\|D^{m+1}\nabla d\|_{L^2(\Omega)}\\
\leq\ &\frac{\beta}{12}\|Dd\|^2_{H^{m+1}(\Omega)}+\frac{3(2^{m+1}-1)^2C_G^4|\alpha|^2}{\beta}\|Dd\|^2_{L^\infty(\Omega)}\|Dd\|^2_{H^m(\Omega)}\\
\leq\ &\frac{\beta}{12}\|Dd\|^2_{H^{m+1}(\Omega)}+\frac{3(2^{m+1}-1)^2|\alpha|^2}{\beta}C_G^6\tilde{K}_3^{\frac{1}{3}}\tilde{K}^{\frac{2}{3}}\|Dd\|^2_{H^m(\Omega)},
\end{aligned}
$$

$$
\begin{aligned}
&|\beta\left\langle D^{m+1}\big(d\times(d\times(V\cdot\nabla d+\nabla\times d))\big), D^{m+1}d\right\rangle|\\
=\ &\left|-\beta\left\langle D^{m}\big(d\times(d\times(V\cdot\nabla u+\nabla\times d))\big), D^{m+2}d\right\rangle\right|\\
\leq\ &\beta\|D^m(d\times(d\times(V\cdot\nabla d+\nabla\times d)))\|_{L^2(\Omega)}\|D^{m+2}d\|_{L^2(\Omega)}\\
\leq\ &\beta\Big(\|d\|_{L^\infty(\Omega)}\|D^m(d\times(V\cdot\nabla d+\nabla\times d))\|_{L^2(\Omega)}\\
&+C_K\|d\|_{H^m(\Omega)}\|Dd\|_{L^\infty(\Omega)}\|d\|_{L^\infty(\Omega)}(\|V\|_{L^\infty(\Omega)}+1)\\
&+C_K\|Dd\|_{L^\infty(\Omega)}\|d\times(V\cdot\nabla d+\nabla\times d)\|_{H^{m-1}}\Big)\|D^{m+2}d\|_{L^2(\Omega)}\\
\leq\ &\beta\Big(\|d\|^2_{L^\infty(\Omega)}(\|V\|_{L^\infty(\Omega)}+1)\|Dd\|_{H^m(\Omega)}+C_K\|d\|^2_{L^\infty(\Omega)}\|Dd\|_{L^\infty(\Omega)}\|V\|_{H^m(\Omega)}\\
&+\|Dd\|_{H^{m-1}(\Omega)}(C_K\|d\|^2_{L^\infty(\Omega)}\|DV\|_{L^\infty(\Omega)}+C_K\|Dd\|_{L^\infty(\Omega)}\|d\|_{L^\infty(\Omega)}(\|V\|_{L^\infty(\Omega)}+1))\\
&+2C_K\|d\|_{H^m(\Omega)}(\|V\|_{L^\infty(\Omega)}+1)\|d\|_{L^\infty(\Omega)}\|Du\|_{L^\infty(\Omega)}+C_K^2m\|Dd\|^2_{L^\infty(\Omega)}\|d\|_{L^\infty(\Omega)}\|V\|_{H^{m-1}(\Omega)}\\
&+\|Dd\|_{H^{m-2}(\Omega)}(C_K^2m\|Dd\|_{L^\infty(\Omega)}\|d\|_{L^\infty(\Omega)}\|DV\|_{L^\infty(\Omega)}+C_K^2m\|Dd\|^2_{L^\infty(\Omega)}\|d\|_{L^\infty(\Omega)}(\|V\|_{L^\infty(\Omega)}+1))\\
&+C_K^2m\|d\|_{H^{m-1}(\Omega)}\|Dd\|^2_{L^\infty(\Omega)}(\|V\|_{L^\infty(\Omega)}+1)+C_K^3m^2\|Dd\|^3_{L^\infty(\Omega)}\|d\|_{L^\infty(\Omega)}\|V\|_{H^{m-2}(\Omega)}
\end{aligned}
$$

$$+C_K^3 m^2 \|Dd\|_{L^2(\Omega)}^2 \|DV\|_{L^\infty(\Omega)} \|Dd\|_{H^{m-3}(\Omega)}\Big) \|D^{m+2}d\|_{L^2(\Omega)}$$

$$\leq \frac{\beta}{12}\|Dd\|_{H^{m+1}(\Omega)}^2 + 48\beta\Big((\|V\|_{L^\infty(\Omega)}^2+1)\|Dd\|_{H^m(\Omega)}^2 + C_K^2\|Dd\|_{L^\infty(\Omega)}^2\|V\|_{H^m(\Omega)}^2$$

$$+\|Dd\|_{H^{m-1}(\Omega)}^2 (C_K^2\|DV\|_{L^\infty(\Omega)}^2 + C_K^2\|Dd\|_{L^\infty(\Omega)}^2(\|V\|_{L^\infty(\Omega)}^2+1))$$

$$+4C_K^2\|d\|_{H^m(\Omega)}^2(\|V\|_{L^\infty(\Omega)}^2+1)\|Dd\|_{L^\infty(\Omega)}^2 + C_K^4 m^2\|Dd\|_{L^\infty(\Omega)}^4\|V\|_{H^{m-1}(\Omega)}^2$$

$$+\|Dd\|_{H^{m-2}(\Omega)}^2(C_K^4 m^2\|Dd\|_{L^\infty(\Omega)}^2\|DV\|_{L^\infty(\Omega)}^2 + C_K^4 m^2\|Dd\|_{L^\infty(\Omega)}^4(\|V\|_{L^\infty(\Omega)}^2+1))$$

$$+C_K^4 m^2\|d\|_{H^{m-1}(\Omega)}^2\|Dd\|_{L^\infty(\Omega)}^4(\|V\|_{L^\infty(\Omega)}^2+1) + C_K^6 m^4\|Dd\|_{L^\infty(\Omega)}^6\|V\|_{H^{m-2}(\Omega)}^2$$

$$+C_K^6 m^4\|Dd\|_{L^2(\Omega)}^4\|DV\|_{L^\infty(\Omega)}^2\|Dd\|_{H^{m-3}(\Omega)}^2\Big)$$

$$\leq \frac{\beta}{12}\|Dd\|_{H^{m+1}(\Omega)}^2 + 48\beta\Big((\|V\|_{L^\infty(\Omega)}^2+1)\|Dd\|_{H^m(\Omega)}^2 + C_K^2 C_G^2 \tilde{K}_3^{\frac{1}{3}}\tilde{K}^{\frac{2}{3}}\|V\|_{H^m(\Omega)}^2$$

$$+\tilde{K}_{m-1}(C_K^2\|DV\|_{L^\infty(\Omega)}^2 + C_G^2\tilde{K}_3^{\frac{1}{3}}\tilde{K}^{\frac{2}{3}}(\|V\|_{L^\infty(\Omega)}^2+1))$$

$$+4C_K^2\|d\|_{H^m(\Omega)}^2(\|V\|_{L^\infty(\Omega)}^2+1)C_G^2\tilde{K}_3^{\frac{1}{3}}\tilde{K}^{\frac{2}{3}} + C_K^4 m^2 C_G^4\tilde{K}_3^{\frac{2}{3}}\tilde{K}^{\frac{4}{3}}\|V\|_{H^{m-1}(\Omega)}^2$$

$$+\tilde{K}_{m-2}(C_K^4 m^2 C_G^2\tilde{K}_3^{\frac{1}{3}}\tilde{K}^{\frac{2}{3}}\|DV\|_{L^\infty(\Omega)}^2 + C_K^4 m^2 C_G^4\tilde{K}_3^{\frac{2}{3}}\tilde{K}^{\frac{4}{3}}(\|V\|_{L^\infty(\Omega)}^2+1))$$

$$+C_K^4 m^2\|d\|_{H^{m-1}(\Omega)}^2 C_G^4\tilde{K}_3^{\frac{2}{3}}\tilde{K}^{\frac{4}{3}}(\|V\|_{L^\infty(\Omega)}^2+1) + C_K^6 m^4 C_G^6\tilde{K}_3\tilde{K}^2\|V\|_{H^{m-2}(\Omega)}^2$$

$$+C_K^6 m^4 C_G^4\tilde{K}_3^{\frac{2}{3}}\tilde{K}^{\frac{4}{3}}\|DV\|_{L^\infty(\Omega)}^2\tilde{K}_{m-3}\Big)$$

$$|\alpha\left\langle D^{m+1}\big(d\times(V\cdot\nabla d+\nabla\times d)\big),\, D^{m+1}d\right\rangle|$$

$$= |-\alpha\left\langle D^{m}\big(d\times(V\cdot\nabla d+\nabla\times d)\big),\, D^{m+2}d\right\rangle|$$

$$\leq |\alpha|\|D^m(d\times(V\cdot\nabla d+\nabla\times d))\|_{L^2(\Omega)}\|D^{m+2}d\|_{L^2(\Omega)}$$

$$\leq |\alpha|\Big(\|d\|_{L^\infty(\Omega)}\|D^m(V\cdot\nabla d+\nabla\times d)\|_{L^2(\Omega)} + C_K\|d\|_{H^m(\Omega)}\|Dd\|_{L^\infty(\Omega)}(\|V\|_{L^\infty(\Omega)}+1)$$

$$+C_K\|Dd\|_{L^\infty(\Omega)}\|V\cdot\nabla d+\nabla\times d\|_{H^{m-1}(\Omega)}\Big)\|D^{m+2}d\|_{L^2(\Omega)}$$

$$\leq|\alpha|\Big(\|d\|_{L^\infty(\Omega)}(\|V\|_{L^\infty(\Omega)}+1)\|Dd\|_{H^m(\Omega)}+C_K\|V\|_{H^m(\Omega)}\|Dd\|_{L^\infty(\Omega)}\|d\|_{L^\infty(\Omega)}$$

$$+\|Dd\|_{H^{m-1}(\Omega)}(C_K\|DV\|_{L^\infty(\Omega)}\|d\|_{L^\infty(\Omega)}+C_K\|Dd\|_{L^\infty(\Omega)}(\|V\|_{L^\infty(\Omega)}+1))$$

$$+C_K\|d\|_{H^m(\Omega)}(\|V\|_{L^\infty(\Omega)}+1)\|Dd\|_{L^\infty(\Omega)}+C_K^2m\|Dd\|^2_{L^\infty(\Omega)}\|V\|_{H^{m-1}(\Omega)}$$

$$+C_K^2m\|Dd\|_{L^\infty(\Omega)}\|DV\|_{L^\infty(\Omega)}\|Dd\|_{H^{m-2}(\Omega)}\Big)\|D^{m+2}u\|_{L^2(\Omega)}$$

$$\leq\frac{\beta}{12}\|Dd\|^2_{H^{m+1}(\Omega)}+\frac{30\alpha^2}{\beta}\Big((\|V\|^2_{L^\infty(\Omega)}+1)\|Dd\|^2_{H^m(\Omega)}+C_K^2\|V\|^2_{H^m(\Omega)}\|Dd\|^2_{L^\infty(\Omega)}$$

$$+\|Dd\|^2_{H^{m-1}(\Omega)}(C_K^2\|DV\|^2_{L^\infty(\Omega)}+C_K^2\|Dd\|^2_{L^\infty(\Omega)}(\|V\|^2_{L^\infty(\Omega)}+1))$$

$$+C_K^2\|d\|^2_{H^m(\Omega)}(\|V\|^2_{L^\infty(\Omega)}+1)\|Dd\|^2_{L^\infty(\Omega)}+C_K^4m^2\|Dd\|^4_{L^\infty(\Omega)}\|V\|^2_{H^{m-1}(\Omega)}$$

$$+C_K^4m^2\|Dd\|^2_{L^\infty(\Omega)}\|DV\|^2_{L^\infty(\Omega)}\|Dd\|^2_{H^{m-2}(\Omega)}\Big)$$

$$\leq\frac{\beta}{12}\|Dd\|^2_{H^{m+1}(\Omega)}+\frac{30\alpha^2}{\beta}\Big((\|V\|^2_{L^\infty(\Omega)}+1)\|Dd\|^2_{H^m(\Omega)}+C_K^2\|V\|^2_{H^m(\Omega)}C_G^2\tilde{K}_3^{\frac{1}{3}}\tilde{K}^{\frac{2}{3}}$$

$$+\tilde{K}_{m-1}(C_K^2\|DV\|^2_{L^\infty(\Omega)}+C_K^2C_G^2\tilde{K}_3^{\frac{1}{3}}\tilde{K}^{\frac{2}{3}}(\|V\|^2_{L^\infty(\Omega)}+1))$$

$$+C_K^2\|d\|^2_{H^m(\Omega)}(\|V\|^2_{L^\infty(\Omega)}+1)C_G^2\tilde{K}_3^{\frac{1}{3}}\tilde{K}^{\frac{2}{3}}+C_K^4m^2C_G^4\tilde{K}_3^{\frac{2}{3}}\tilde{K}^{\frac{4}{3}}\|V\|^2_{H^{m-1}(\Omega)}$$

$$+C_K^4m^2C_G^2\tilde{K}_3^{\frac{1}{3}}\tilde{K}^{\frac{2}{3}}\|DV\|^2_{L^\infty(\Omega)}\tilde{K}_{m-2}\Big),$$

$$
\begin{aligned}
&|\langle D^{m+1}(d\times f), D^{m+1}d\rangle|\\
&=|-\langle D^{m}(d\times f), D^{m+2}d\rangle|\\
&\le \|D^{m}(d\times f)\|_{L^2(\Omega)}\|D^{m+2}d\|_{L^2(\Omega)}\\
&\le \Big(\|d\|_{L^\infty(\Omega)}\|D^m f\|_{L^2(\Omega)}+C_K(\|d\|_{H^m(\Omega)}\|f\|_{L^\infty(\Omega)}+\|Dd\|_{L^\infty(\Omega)}\|f\|_{H^{m-1}(\Omega)})\Big)\|D^{m+2}u\|_{L^2(\Omega)}\\
&\le \frac{\beta}{12}\|Dd\|^2_{H^{m+1}(\Omega)}+\frac{9}{\beta}\|f\|^2_{H^m(\Omega)}+\frac{9C_K^2}{\beta}\|f\|^2_{L^\infty(\Omega)}\|d\|^2_{H^m(\Omega)}+\frac{9C_K^2}{\beta}\|Dd\|^2_{L^\infty(\Omega)}\|f\|^2_{H^{m-1}(\Omega)}\\
&\le \frac{\beta}{12}\|Dd\|^2_{H^{m+1}(\Omega)}+\frac{9}{\beta}\|f\|^2_{H^m(\Omega)}+\frac{9C_K^2}{\beta}\|f\|^2_{L^\infty(\Omega)}\|d\|^2_{H^m(\Omega)}+\frac{9C_K^2}{\beta}C_G^2\tilde{K}_3^{\frac{1}{3}}\tilde{K}^{\frac{2}{3}}\|f\|^2_{H^{m-1}(\Omega)}.
\end{aligned}
$$

Differentiating the equation (10) by D^m, multiplying by $D^m u$ and integrating with respect to x over Ω, one gets

$$
\begin{aligned}
&\langle D^m u_t, D^m u\rangle+\langle D^m(u\cdot\nabla d), D^m u\rangle+\langle D^m\nabla P, D^m u\rangle\\
&=\mu\langle D^m\Delta u, D^m u\rangle-\lambda\langle D^m\nabla\cdot(\nabla d\odot\nabla d), D^m u\rangle.
\end{aligned}
$$

Calculations yield

$$
\begin{aligned}
&|\langle D^m(u\cdot\nabla u), D^m u\rangle|\\
&=|-\langle D^{m-1}(u\cdot\nabla u), D^{m+1}u\rangle|\\
&\le \Big(\|u\|_{L^\infty(\Omega)}\|D^{m-1}\nabla u\|_{L^2(\Omega)}+C_K\|u\|_{H^{m-1}(\Omega)}\|Du\|_{L^\infty(\Omega)}\\
&\quad+\|Du\|_{L^\infty(\Omega)}\|Du\|_{H^{m-2}(\Omega)}\Big)\|D^{m+1}u\|_{L^2(\Omega)}\\
&\le \frac{\mu}{4}\|u\|^2_{H^{m+1}(\Omega)}+\frac{3}{\mu}\|u\|^2_{L^\infty(\Omega)}\|u\|^2_{H^m(\Omega)}+\frac{6C_K^2}{\mu}\|Du\|^2_{L^\infty(\Omega)}\|u\|^2_{H^{m-1}(\Omega)}\\
&\le \frac{\mu}{4}\|u\|^2_{H^{m+1}(\Omega)}+\frac{3}{\mu}C_G^2\tilde{K}_3^{\frac{1}{3}}\tilde{K}^{\frac{2}{3}}\|u\|^2_{H^m(\Omega)}+\frac{6C_K^2}{\mu}C_G^2\tilde{K}_3^{\frac{2}{3}}\tilde{K}^{\frac{1}{3}}\tilde{K}_{m-1},
\end{aligned}
$$

$$
\begin{aligned}
&|\lambda\langle D^m\nabla\cdot(\nabla d\odot\nabla d),\, D^m u\rangle| \\
=& |-\lambda\langle D^{m-1}\nabla\cdot(\nabla d\odot\nabla d),\, D^{m+1}u\rangle| \\
\le& |\lambda|\|D^{m-1}\nabla\cdot(\nabla d\odot\nabla d)\|_{L^2(\Omega)}\|D^{m+1}u\|^2_{L^2(\Omega)} \\
\le& |\lambda|\Big(\|Dd\|_{L^\infty(\Omega)}\|D^m\nabla d\|_{L^2(\Omega)}+C_K\|Dd\|_{H^m(\Omega)}\|Dd\|_{L^\infty(\Omega)} \\
&+C_K\|D^2d\|_{L^\infty(\Omega)}\|Dd\|_{H^{m-1}(\Omega)}\Big)\|D^{m+1}u\|_{L^2(\Omega)} \\
\le& \frac{\mu}{4}\|u\|^2_{H^{m+1}(\Omega)}+\left(\frac{3\lambda^2}{\mu}+\frac{3\lambda^2C_K^2}{\mu}\right)\|Dd\|^2_{H^m(\Omega)}\|Dd\|^2_{L^\infty(\Omega)}+\frac{3\lambda^2C_K^2}{\mu}\|D^2d\|^2_{L^\infty(\Omega)}\|Dd\|^2_{H^{m-1}(\Omega)} \\
\le& \frac{\mu}{4}\|u\|^2_{H^{m+1}(\Omega)}+\left(\frac{3\lambda^2}{\mu}+\frac{3\lambda^2C_K^2}{\mu}\right)C_G^2\tilde{K}_3^{\frac{1}{3}}\tilde{K}^{\frac{2}{3}}\|Dd\|^2_{H^m(\Omega)}+\frac{3\lambda^2C_K^2}{\mu}C_G^2\tilde{K}_3^{\frac{2}{3}}\tilde{K}^{\frac{1}{3}}\tilde{K}_{m-1}.
\end{aligned}
$$

Employing the Gronwall inequality(integral form), it is shown that

$$
\sup_{0\le T}\|u\|^2_{H^m(\Omega)}+\sup_{0\le t\le T}\|\nabla d\|^2_{H^m(\Omega)}+\mu\int_0^T\|u\|^2_{H^{m+1}(\Omega)}dt+\beta\int_0^T\|\nabla d\|^2_{H^{m+1}(\Omega)}dt\le\tilde{K}_{m-1},
$$

where

$$
\begin{aligned}
\tilde{K}_m=&\Bigg\{1+2\Bigg\{\beta+\mu+\frac{9}{\beta}C_G^2\tilde{K}_3^{\frac{1}{3}}\tilde{K}^{\frac{2}{3}}+15\beta\left(C_K^2+C_K^4m^2\right)C_G^2\tilde{K}_3^{\frac{1}{3}}\tilde{K}^{\frac{2}{3}}+\frac{3\left(2^{m+1}-1\right)^2|\alpha|^2}{\beta}C_G^6\tilde{K}_3^{\frac{1}{3}}\tilde{K}^{\frac{2}{3}} \\
&+48\beta\left(\|V\|^2_{L^\infty(\Omega)}+1\right)+\frac{30\alpha^2}{\beta}\left(\|V\|^2_{L^\infty(\Omega)}+1\right)+\left(\frac{3\lambda^2}{\mu}+\frac{3\lambda^2C_K^2}{\mu}\right)C_G^2\tilde{K}_3^{\frac{1}{3}}\tilde{K}^{\frac{2}{3}}+\frac{9C_K^2}{\beta}C_G^2\tilde{K}_3^{\frac{1}{3}}\tilde{K}^{\frac{2}{3}} \\
&+\frac{3}{\mu}C_G^2\tilde{K}_3^{\frac{1}{3}}\tilde{K}^{\frac{2}{3}}\Bigg\}T\times\exp\Bigg\{2\Bigg(\beta+\mu+\frac{9}{\beta}C_G^2\tilde{K}_3^{\frac{1}{3}}\tilde{K}^{\frac{2}{3}}+15\beta\left(C_K^2+C_K^4m^2\right)C_G^2\tilde{K}_3^{\frac{1}{3}}\tilde{K}^{\frac{2}{3}} \\
&+\frac{3\left(2^{m+1}-1\right)^2|\alpha|^2}{\beta}C_G^6\tilde{K}_3^{\frac{1}{3}}\tilde{K}^{\frac{2}{3}}+48\beta\left(\|V\|^2_{L^\infty(\Omega)}+1\right)+\frac{30\alpha^2}{\beta}\left(\|V\|^2_{L^\infty(\Omega)}+1\right) \\
&+\left(\frac{3\lambda^2}{\mu}+\frac{3\lambda^2C_K^2}{\mu}\right)C_G^2\tilde{K}_3^{\frac{1}{3}}\tilde{K}^{\frac{2}{3}}+\frac{9C_K^2}{\beta}C_G^2\tilde{K}_3^{\frac{1}{3}}\tilde{K}^{\frac{2}{3}}+\frac{3}{\mu}C_G^2\tilde{K}_3^{\frac{1}{3}}\tilde{K}^{\frac{2}{3}}\Bigg)T\Bigg\}\Bigg\}
\end{aligned}
$$

$$
\times \Big\{ \|u_0\|^2_{H^m(\Omega)} + \|\nabla d_0\|^2_{H^m(\Omega)} + \tilde{K}_{m-1} + 2\int_0^T \Big\{ \frac{9C_K^2}{\beta} C_G^2 \tilde{K}_3^{\frac{2}{3}} \tilde{K}^{\frac{1}{3}} \tilde{K}_{m-1} + 15\beta \big(C_G^4 \tilde{K}_3^{\frac{2}{3}} \tilde{K}^{\frac{4}{3}}
$$

$$
+ C_K^4 m^2 C_G^2 \tilde{K}_3^{\frac{3}{3}} \tilde{K}^{\frac{1}{3}} \big) \tilde{K}_{m-1} + 15\beta C_K^2 C_G^4 \tilde{K}_3 \tilde{K} \|d\|^2_{H^{m-1}} + 48\beta \Big(C_K^2 C_G^2 \tilde{K}_3^{\frac{1}{3}} \tilde{K}^{\frac{2}{3}} \|V\|^2_{H^m(\Omega)}
$$

$$
+ \tilde{K}_{m-1} (C_K^2 \|DV\|^2_{L^\infty(\Omega)} + C_G^2 \tilde{K}_3^{\frac{1}{3}} \tilde{K}^{\frac{2}{3}} (\|V\|^2_{L^\infty(\Omega)} + 1))
$$

$$
+ 4C_K^2 \|d\|^2_{H^m(\Omega)} (\|V\|^2_{L^\infty(\Omega)} + 1) C_G^2 \tilde{K}_3^{\frac{1}{3}} \tilde{K}^{\frac{2}{3}} + C_K^4 m^2 C_G^4 \tilde{K}_3^{\frac{2}{3}} \tilde{K}^{\frac{4}{3}} \|V\|^2_{H^{m-1}(\Omega)}
$$

$$
+ \tilde{K}_{m-2} (C_K^4 m^2 C_G^2 \tilde{K}_3^{\frac{1}{3}} \tilde{K}^{\frac{2}{3}} \|DV\|^2_{L^\infty(\Omega)} + C_K^4 m^2 C_G^4 \tilde{K}_3^{\frac{2}{3}} \tilde{K}^{\frac{4}{3}} (\|V\|^2_{L^\infty(\Omega)} + 1))
$$

$$
+ C_K^4 m^2 \|d\|^2_{H^{m-1}(\Omega)} C_G^4 \tilde{K}_3^{\frac{2}{3}} \tilde{K}^{\frac{4}{3}} (\|V\|^2_{L^\infty(\Omega)} + 1) + C_K^6 m^4 C_G^6 \tilde{K}_3 \tilde{K}^2 \|V\|^2_{H^{m-2}(\Omega)}
$$

$$
+ C_K^6 m^4 C_G^4 \tilde{K}_3^{\frac{2}{3}} \tilde{K}^{\frac{4}{3}} \|DV\|^2_{L^\infty(\Omega)} \tilde{K}_{m-3} \Big) + \frac{30\alpha^2}{\beta} \Big((C_K^2 \|V\|^2_{H^m(\Omega)} C_G^2 \tilde{K}_3^{\frac{1}{3}} \tilde{K}^{\frac{2}{3}}
$$

$$
+ \tilde{K}_{m-1} (C_K^2 \|DV\|^2_{L^\infty(\Omega)} + C_K^2 C_G^2 \tilde{K}_3^{\frac{1}{3}} \tilde{K}^{\frac{2}{3}} (\|V\|^2_{L^\infty(\Omega)} + 1))
$$

$$
+ C_K^2 \|d\|^2_{H^m(\Omega)} (\|V\|^2_{L^\infty(\Omega)} + 1) C_G^2 \tilde{K}_3^{\frac{1}{3}} \tilde{K}^{\frac{2}{3}} + C_K^4 m^2 C_G^4 \tilde{K}_3^{\frac{2}{3}} \tilde{K}^{\frac{4}{3}} \|V\|^2_{H^{m-1}(\Omega)}
$$

$$
+ C_K^4 m^2 C_G^2 \tilde{K}_3^{\frac{1}{3}} \tilde{K}^{\frac{2}{3}} \|DV\|^2_{L^\infty(\Omega)} \tilde{K}_{m-2} \Big) + \frac{9}{\beta} \|f\|^2_{H^m(\Omega)} + \frac{9C_K^2}{\beta} \|f\|^2_{L^\infty(\Omega)} \|d\|^2_{H^m(\Omega)}
$$

$$
+ \frac{9C_K^2}{\beta} C_G^2 \tilde{K}_3^{\frac{1}{3}} \tilde{K}^{\frac{2}{3}} \|f\|^2_{H^{m-1}(\Omega)} + \frac{6C_K^2}{\mu} C_G^2 \tilde{K}_3^{\frac{2}{3}} \tilde{K}^{\frac{1}{3}} \tilde{K}_{m-1} + \frac{3\lambda^2 C_K^2}{\mu} C_G^2 \tilde{K}_3^{\frac{2}{3}} \tilde{K}^{\frac{1}{3}} \tilde{K}_{m-1} \Big\} dt \Big\}.
$$

Here $\|d\|_{H^m}(\Omega) \leq \tilde{K}_{m-1} + |\Omega|$ and $\|d\|_{H^{m-1}(\Omega)} \leq \tilde{K}_{m-2} + |\Omega|$ when $\Omega = \mathbb{T}^2$, while $\Omega = \mathbb{R}^2$, $\|d\|_{H^m_Q(\Omega)}$ and $\|d\|_{H^{m-1}_Q(\Omega)}$ should be replaced by $\tilde{K}_{m-1}$ and $\tilde{K}_{m-2}$, respectively. □

Invoking Lemma 5-Lemma 8, Theorem 1 is valid.

5. Conclusions

In this paper, we have established the global existence of smooth solutions for the incompressible Navier-Stokes-Landau-Lifshitz (NSLL) equations with the Dzyaloshinskii-Moriya (DM) interaction and V-flow term in two-dimensional domains. By employing rigorous energy estimates and a priori bounds, we demonstrated that under small initial data conditions, the system admits global smooth solutions in both periodic and whole-space settings. This work extends previous results by incorporating additional physical effects arising from the DM interaction and anisotropic V-flow, which significantly influence magnetization dynamics.

Although our study focuses on a two-dimensional setting, extending these results to three dimensions remains a formidable challenge due to the stronger nonlinear interactions and the limitations of conventional analytical techniques such as Gronwall-type inequalities. A project for future investigation should explore alternative analytical approaches, computational methods, and experimental validation to better understand the impact of these interactions in three-dimensional magnetorheological fluids and spintronic applications.

The findings of this study contribute to the broader understanding of magnetoviscoelastic interactions and provide a theoretical foundation for further research on coupled fluid-magnetization models. Potential future directions include investigating more complex boundary conditions, incorporating temperature-dependent effects, and extending the framework to non-homogeneous and non-isotropic media.

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Conflict of interest

The authors declare no competing financial interest.

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