Contemporary Mathematics
https://ojs.wiserpub.com/index.php/CM/

UNIVERSAL WISER PUBLISHER

Research Article

Optical Solitons with the Concatenation Model Having Fractional Temporal Evolution with Depleted Self-Phase Modulation

Hanaa A. Eldidamony[1], Muhammad Amin S. Murad[2], Ahmed H. Arnous[3], Yakup Yildirim[4,5*], Anjan Biswas[6,7,8,9], Luminita Moraru[10,11]

[1]Department of Basic Science, Higher Technological Institute, 10th of Ramadan City, Egypt
[2]Department of Mathematics, College of Science, University of Duhok, Duhok, Iraq
[3]Department of Engineering Mathematics and Physics, Higher Institute of Engineering, El-Shorouk Academy, Cairo, Egypt
[4]Department of Computer Engineering, Biruni University, Istanbul, 34010, Turkey
[5]Mathematics Research Center, Near East University, 99138, Nicosia, Cyprus
[6]Department of Mathematics and Physics, Grambling State University, Grambling, LA, 71245-2715, USA
[7]Department of Physics and Electronics, Khazar University, Baku, AZ, 1096, Azerbaijan
[8]Department of Applied Sciences, Cross-Border Faculty of Humanities, Economics and Engineering, Dunarea de Jos University of Galati, 111 Domneasca Street, Galati, 800201, Romania
[9]Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Medunsa, 0204, South Africa
[10]Department of Chemistry, Physics and Environment, Faculty of Sciences and Environment, Dunarea de Jos University of Galati, 47 Domneasca Street, Galati, 800008, Romania
[11]Department of Physics, School of Science and Technology, Sefako Makgatho Health Sciences University, Medunsa, 0204, South Africa
E-mail: yyildirim@biruni.edu.tr

Received: 3 January 2025; **Revised:** 21 February 2025; **Accepted:** 3 March 2025

Abstract: This paper investigates the recovery of optical soliton solutions for the concatenation model, incorporating fractional temporal evolution while excluding self-phase modulation effects. To achieve this, we employ the enhanced direct algebraic method and the new projective Riccati equation approach, both of which prove effective in extracting a comprehensive spectrum of optical soliton solutions. The obtained soliton families include bright solitons, dark solitons, singular solitons, and straddled soliton structures, each characterized by distinct parameter constraints. Additionally, the impact of fractional temporal evolution on soliton behavior is analyzed, revealing how variations in the fractional order influence soliton amplitude, width, and stability. The derived parameter conditions governing the existence of these solitons provide deeper insight into the dynamics of optical pulse propagation in nonlinear media. These findings contribute to a broader understanding of soliton behavior in optical fiber systems and may offer potential applications in fiber-optic communication and photonic signal processing.

***Keywords*:** solitons, algebraic method, Riccati

MSC: 78A60, 81V80

DOI: https://doi.org/10.37256/cm.6220256345

1. Introduction

The year was 2014; that is exactly a decade ago from today. A new model for the transmission of solitons through optical fibers across transcontinental and transoceanic distances have surfaced [1–3]. This is the so-called concatenation model that is formulated after the conjoinment of three popular equations from nonlinear fiber optics [4–6]. They are the nonlinear Schrödinger's equation (NLSE), Lakshmanan-Porsezian-Daniel (LPD) equation and the Sasa-Satsuma equation (SSE). Later, during the year a version of the concatenation model was proposed that conjoined the well-known dispersive models. These are the Schrödinger-Hirota equation (SHE), LPD model and the fifth-order NLSE [1–5]. These gave way to the now-known dispersive concatenation model. The inclusion of SHE that carries third-order dispersion and the fifth-order NLSE that embeds fifth-order dispersive effects leads to the dispersive effects and hence the name.

Subsequently a plethora of research activities were conducted with these models [7–9]. Both such models were extended from Kerr-law of self-phase modulation (SPM) to power-law of SPM [10–12]. They were also addressed with polarization-mode dispersion [13–15]. The mobile soliton solutions were identified by the aid of undetermined coefficients [16–18]. Subsequently, the conservation laws were retrieved for the models [19–21]. Thereafter, quiescent optical solitons were recovered for the models with nonlinear chromatic dispersion (CD) with Kerr and power-law of SPM [22]. The models were also addressed numerically using the Laplace-Adomian decomposition where the numerical simulations were exhibited with impressively small error measure [23]. Recently, the bifurcation analysis for the model was also established [24]. Lately, the concatenation model was taken up with fractional temporal evolutions for the retrieval of slow soliton evolution as a measure to control the internet bottleneck effect [25]. The current paper addressees the concatenation model with fractional temporal evolution but in absence of SPM structure. This model addressed in this work thus carries unprecedented novelty.

There are two integration algorithms that made the soliton solutions retrieval possible. They are the enhanced direct algebraic method and the projective Riccati equation approach. Thee two approaches collectively yielded a full spectrum of soliton solutions to the model with fractional temporal evolution in absence of SPM. The solitons that emerged from the two integration schemes came with parameter constraints that are also enlisted in the paper. These constraints ensure the existence of such solitons with those restrictions in place. The details are all exhibited in the rest of the paper once the model is revisited along with the recapitulated integration schemes.

1.1 *Governing model*

The dimensionless form the concatenation model in absence of SPM and with fractional temporal evolution is structured as [1–5]:

$$i\frac{\partial^\alpha q}{\partial t^\alpha}+aq_{xx}+c_1\left[\tau_1 q_{xxxx}+\tau_2\left(q_x\right)^2q^*+\tau_3|q_x|^2q+\tau_4|q|^2q_{xx}+\tau_5q^2q_{xx}^*+\tau_6|q|^4q\right]$$

$$+ic_2\left[\tau_7q_{xxx}+\tau_8|q|^2q_x+\tau_9q^2q_x^*\right]=0, \tag{1}$$

where $\tau_i(i=1,\ ...,\ 9)$ are constants. This concatenation model given by (1) as mentioned is obtained after conjoining three familiar models from nonlinear optics, as mentioned earlier. The dependent variable is $q(x,\ t)$ that is a complex-valued function and represents the wave amplitude with the independent variables x and t representing the spatial and temporal variables respectively. Also, the complex number $i=\sqrt{-1}$ is the coefficient of the first term in (1) that represents the fractional temporal evolution with parameter α giving the fractional component of the temporal derivative with $0<\alpha\leq 1$. For $\alpha=1$, the fractional temporal evolutions reduces to linear temporal evolution. The first two terms comprise of the NLSE with linear CD and no SPM. In equation (1), the coefficient of c_1 comes from the LPD equation, while the coefficient of c_2 is from the SSE. Thus, this conjoined equation (1) represents the concatenation model stemming from the three basic equations in nonlinear optics.

The paper is organized as follows. Section 2 addresses mathematical preliminaries, where the fundamental equations governing the concatenation model with fractional temporal evolution are introduced. The underlying assumptions and the necessary transformations applied to simplify the governing equations are also discussed. Section 3 addresses integration algorithms, detailing the enhanced direct algebraic method and the new projective Riccati equation approach used to extract soliton solutions. The methodological framework, along with the step-by-step implementation of these techniques, is presented to demonstrate their effectiveness in retrieving a broad class of solitons. Section 4 addresses soliton solutions, where a full spectrum of optical solitons is recovered, including bright, dark, singular, and hybrid soliton structures. The parameter constraints ensuring soliton existence are derived, and the influence of fractional temporal evolution on soliton dynamics is analyzed. Section 5 addresses conclusions, summarizing the key findings and emphasizing the significance of the recovered soliton solutions in nonlinear optical systems. Possible extensions of the work, including the incorporation of additional perturbation effects and potential applications in optical communication, are also discussed.

2. Mathematical preliminaries

This sections reviews and rewrites a few of the basic concepts from fractional calculus [13–16]. Subsequently the governing model is recasted into workable formats of it after splitting into real and imaginary arts based on the phase-amplitude style.

2.1 *Conformable fractional derivative*

[Conformable fractional derivative] Given a function f: $[0,\ \infty) \to \mathbb{R}$ with $0 < \alpha \le 1$, then the conformable fractional derivative of f of order α is defined by

$$L_{\alpha}\left(f\right)(x) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon x^{1-\alpha}) - f(x)}{\varepsilon}, \tag{2}$$

for all $x > 0$ and $\alpha \in (0,\ 1]$.

Let $0 < \alpha \le 1$, $a,\ b,\ p \in \mathbb{R}$ and $f(x),\ g(x)$ be α-differentiable, at a point $x > 0$. Then

(1) $L_{\alpha}(af(x)+bg(x)) = aL_{\alpha}f(x)+bL_{\alpha}g(x)$

(2) $L_{\alpha}(x^{p}) = px^{p-\alpha}$

(3) $L_{\alpha}(\mu) = 0$, μ is constant

(4) $L_{\alpha}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{gL_{\alpha}(f(x)) - fL_{\alpha}(g(x))}{g(x)^{2}}$

(5) If in addition f is differentiable then $L_{\alpha}f(x) = x^{1-\alpha}\dfrac{df(x)}{dx}$ (3)

2.2 *Mathematical analysis*

The following solution structure is chosen to solve Eq. (1).

$$q(x,\ t) = U(\xi) e^{i\phi(x,\ t)}. \tag{4}$$

The wave variable ξ is denoted by:

$$\xi = k\left(x - v\frac{t^{\alpha}}{\alpha}\right). \tag{5}$$

In this case, $U(\xi)$ represents the amplitude component of the soliton solution, v is the soliton's speed, and the phase component $\phi(x,\ t)$ is defined as:

$$\phi(x,\ t) = -\kappa x + \omega\frac{t^{\alpha}}{\alpha} + \theta_0, \tag{6}$$

where κ is the solitons' frequency, ω is the wave number, and θ_0 is the phase constant.

Substituting Eq. (4) into Eq. (1), then decomposing into real and imaginary components, yields:

$$\begin{aligned} &k^2 U'' \left(a - 6c_1\kappa^2\tau_1 + 3c_2\kappa\tau_7\right) - \kappa^2 U \left(a - c_1\kappa^2\tau_1 + c_2\kappa\tau_7 + \omega\right) + c_1 k^4 \tau_1 U'''' + c_1 k^2 (\tau_4 + \tau_5) U^2 U'' \\ &+ c_1 k^2 (\tau_2 + \tau_3) U \left(U'\right)^2 + c_1\tau_6 U^5 + \kappa U^3 \left(c_2(\tau_8 - \tau_9) + \kappa \mathrm{c}_1(-\tau_2 + \tau_3 - \tau_4 - \tau_5)\right) = 0, \end{aligned} \tag{7}$$

and

$$\begin{aligned} &U' \left(-2a\kappa k + 4c_1\kappa^3 k\tau_1 - 3c_2\kappa^2 k\tau_7 - kv\right) + k^3 U^{(3)} (c_2\tau_7 - 4c_1\kappa\tau_1) \\ &+ U^2 U' \left(c_1\kappa k(-2\tau_2 - 2\tau_4 + 2\tau_5) + c_2 k(\tau_8 + \tau_9)\right) = 0. \end{aligned} \tag{8}$$

From the imaginary part, the soliton speed reaches

$$v = -2a\kappa + 4c_1\kappa^3\tau_1 - 3c_2\kappa^2\tau_7, \tag{9}$$

while the wave number reads

$$\kappa = \frac{c_2\tau_7}{4c_1\tau_1}, \tag{10}$$

with parametric restriction

$$2c_1\kappa(-\tau_2 - \tau_4 + \tau_5) + c_2(\tau_8 + \tau_9) = 0. \tag{11}$$

Equation (7) can be simplified as

$$h_3U^5+h_2U^3+h_6U^2U''+h_5U''+h_4U\left(U'\right)^2+h_1U+k^2U^{(4)}=0, \tag{12}$$

where

$$h_1=\frac{\kappa^2\left(-a+c_1\kappa^2\tau_1-c_2\kappa\tau_7-\omega\right)}{c_1k^2\tau_1},\quad h_2=\frac{\kappa\left(c_2\left(\tau_8-\tau_9\right)+\kappa c_1\left(-\tau_2+\tau_3-\tau_4-\tau_5\right)\right)}{c_1k^2\tau_1},$$

$$h_3=\frac{\tau_6}{k^2\tau_1},\quad h_4=\frac{\tau_2+\tau_3}{\tau_1},\quad h_5=\frac{a-6c_1\kappa^2\tau_1+3c_2\kappa\tau_7}{c_1\tau_1},\quad h_6=\frac{\tau_4+\tau_5}{\tau_1}. \tag{13}$$

3. An overview of integration algorithms

Suppose that we have a nonlinear evolution equation in the form:

$$F(u,\ u_t,\ u_x,\ u_{xx},\ u_{xt},\ \ldots)=0, \tag{14}$$

where $u=u(x,\ t)$ is an unknown function, F is a polynomial in u and its various partial derivatives $u_x,\ u_t$, with respect to $x,\ t$ respectively, in which the highest order derivatives and nonlinear terms are involved.

Use the following traveling wave transformation

$$u(x,\ t)=U(\xi),\quad \xi=k(x-vt), \tag{15}$$

where v represents the wave speed. Then, Eq. (14) can be transformed to the following nonlinear ordinary differential equation:

$$F(U,\ U',\ U'',\ U''',\ \ldots)=0. \tag{16}$$

3.1 *Enhanced direct algebraic method*

Step-1: Suppose that the solution of Eq. (16) can be expressed in the form [17]:

$$U(\xi)=\alpha_0+\sum_{i=1}^{N}\left[\alpha_i\theta(\xi)^i+\beta_i\theta(\xi)^{-i}\right], \tag{17}$$

where θ satisfies

$$\theta'(\xi)^2=\sum_{l=0}^{4}T_l\theta(\xi)^l, \tag{18}$$

where T_l, $(l = 0,\ 1,\ 2,\ 3,\ 4)$ are constants provided that $T_4 \neq 0$. Also, $\alpha_N^2 + \beta_N^2 \neq 0$. ($\alpha_N$ and β_N should not be zero simultaneously) Eq. (18) provides several kinds of solutions of different types as follows:

Case-1: If we set $T_0 = T_1 = T_3 = 0$, we get bell-shaped soliton with $T_2 > 0,\ T_4 < 0$ and singular soliton with $T_2 > 0,\ T_4 > 0$

$$\theta(\xi) = \sqrt{-\frac{T_2}{T_4}}\ \mathrm{sech}\left[\sqrt{T_2}\xi\right],\ T_2 > 0,\ T_4 < 0, \tag{19}$$

$$\theta(\xi) = \sqrt{\frac{T_2}{T_4}}\ \mathrm{csch}\left[\sqrt{T_2}\xi\right],\ T_2 > 0,\ T_4 > 0. \tag{20}$$

Case-2: If we set $T_0 = \dfrac{T_2^2}{4\tau_4}$, $T_1 = T_3 = 0$, we have kink-shaped and singular solitons for $T_2 < 0,\ T_4 > 0$:

$$\theta(\xi) = \sqrt{-\frac{T_2}{2T_4}}\tanh\left[\sqrt{\frac{-T_2}{2}}\xi\right],\ T_2 < 0,\ T_4 > 0, \tag{21}$$

$$\theta(\xi) = \sqrt{-\frac{T_2}{2T_4}}\coth\left[\sqrt{\frac{-T_2}{2}}\xi\right],\ T_2 < 0,\ T_4 > 0. \tag{22}$$

Case-3: If we set $T_1 = T_3 = 0$, we get Jacobi elliptic doubly periodic type solution for different choices of T_0 as follows:

$$\theta(\xi) = \pm\sqrt{-\frac{m^2T_2}{(2m^2-1)T_4}}\mathrm{cn}\left(\sqrt{\frac{T_2}{(2m^2-1)}}\xi\ \middle|\ m\right);\ \ T_0 = \frac{m^2(1-m^2)T_2^2}{(2m^2-1)^2T_4}, \tag{23}$$

$$\theta(\xi) = \pm\sqrt{-\frac{m^2T_2}{(2-m^2)T_4}}\mathrm{dn}\left(\sqrt{\frac{T_2}{(2-m^2)}}\xi\ \middle|\ m\right);\ \ T_0 = \xi(1-m^2)T_2^2(2-m^2)^2T_4, \tag{24}$$

$$\theta(\xi) = \pm\sqrt{-\frac{m^2T_2}{(m^2+1)T_4}}\mathrm{sn}\left(\sqrt{-\frac{T_2}{(m^2+1)}}\xi\ \middle|\ m\right);\ \ T_0 = \frac{m^2T_2^2}{(m^2+1)^2T_4}. \tag{25}$$

Case-4: If we set $T_1 = T_3 = 0$, we get Weierstrass elliptic doubly periodic type solutions:

$$\theta(\xi) = \frac{3\wp'(\xi;\ g_2,\ g_3)}{\sqrt{T_4}[6\wp(\xi;\ g_2,\ g_3)+T_2]},\ \ T_4 > 0, \tag{26}$$

$$\theta(\xi) = \frac{\sqrt{T_0}[6\wp(\xi;\ g_2,\ g_3)+T_2]}{3\wp'(\xi;\ g_2,\ g_3)},\ \ T_0 > 0, \tag{27}$$

where $g_2=\dfrac{T_2^2}{12}+T_0T_4$ and $g_3=\dfrac{T_2}{216}(36T_0T_4-T_2^2)$ are called invariants of the Weierstrass elliptic function.

Case-5: If we set $T_0=T_1=0$, we get straddled soliton solutions with $T_2>0$ as follows:

$$\theta(\xi)=\frac{-T_2\operatorname{sech}^2\left[\frac{1}{2}\sqrt{T_2}\xi\right]}{\pm2\sqrt{T_2T_4}\tanh\left[\frac{1}{2}\sqrt{T_2}\xi\right]+T_3},\quad T_4>0, \tag{28}$$

$$\theta(\xi)=\frac{T_2\operatorname{csch}^2\left[\frac{1}{2}\sqrt{T_2}\xi\right]}{\pm2\sqrt{T_2T_4}\coth\left[\frac{1}{2}\sqrt{T_2}\xi\right]+T_3},\quad T_4>0, \tag{29}$$

$$\theta(\xi)=\frac{-T_2T_3\operatorname{sech}^2\left[\frac{1}{2}\sqrt{T_2}\xi\right]}{T_3^2-T_2T_4\left(1-\tanh\left[\frac{1}{2}\sqrt{T_2}\xi\right]\right)^2},\quad T_3\neq0, \tag{30}$$

$$\theta(\xi)=\frac{T_2T_3\operatorname{csch}^2\left[\frac{1}{2}\sqrt{T_2}\xi\right]}{T_3^2-T_2T_4\left(1-\coth\left[\frac{1}{2}\sqrt{T_2}\xi\right]\right)^2},\quad T_3\neq0. \tag{31}$$

Step-2: Determine the positive integer number N in Eq. (17) by balancing the highest order derivatives and the nonlinear terms in Eq. (21).

Step-3: Substitute Eq. (17) along with Eq. (18) into Eq. (16) The substitution yields a polynomial as $\theta(\xi)$. In polynomials, the process involves the collection of terms with similar powers and setting the resulting expression equal to zero. An over-determined system of algebraic equations is obtained, which can be solved using Mathematica to discover the unknown parameters in Eqs. (15) and (17). As a result, we derive the exact solutions of Eq. (14).

3.2 *New projective Riccati equation approach*

Step-1: Suppose that the solution of Eq. (16) can be expressed in the form [18]:

$$U(\xi)=A_0+\sum_{i=1}^{N}\chi(\xi)^{i-1}\left[A_i\chi(\xi)+B_i\phi(\xi)\right], \tag{32}$$

where $\chi(\xi)$ and $\phi(\xi)$ satisfies

$$\chi'(\xi)=-\chi(\xi)\phi(\xi),$$

$$\phi'(\xi)=1-\phi(\xi)^2-r\chi(\xi), \tag{33}$$

with

$$\phi(\xi)^2 = 1 - 2r\chi(\xi) + R(r)\chi(\xi)^2, \tag{34}$$

where r is nonzero constant and N a positive integer comes from the balancing principle in Eq. (16), which A_0, A_i, and B_i, $(i = 1,\ 2,\ ..,\ N)$ are constants. Also, $A_N^2 + B_N^2 \neq 0$. (A_N and B_N should not be zero simultaneously)

Step-2: The solutions of Eq. (33) are listed as follows:

Case-1: $R(r) = 0$

$$\chi(\xi) = \frac{1}{2r}\,\mathrm{sech}^2\left[\frac{\xi}{2}\right],\ \text{ and }\ \phi(\xi) = \tanh\left[\frac{\xi}{2}\right], \tag{35}$$

or

$$\chi(\xi) = -\frac{1}{2r}\,\mathrm{csch}^2\left[\frac{\xi}{2}\right],\ \text{ and }\ \phi(\xi) = \coth\left[\frac{\xi}{2}\right]. \tag{36}$$

Case-2: $R(r) = \frac{24}{25}r^2$

$$\chi(\xi) = \frac{1}{r}\frac{5\,\mathrm{sech}[\xi]}{5\,\mathrm{sech}[\xi] \pm 1},\ \text{ and }\ \phi(\xi) = \frac{\tanh[\xi]}{1 \pm 5\,\mathrm{sech}[\xi]}. \tag{37}$$

Case-3: $R(r) = \frac{5}{9}r^2$

$$\chi(\xi) = \frac{1}{r}\frac{3\,\mathrm{sech}[\xi]}{3\,\mathrm{sech}[\xi] \pm 2},\ \text{ and }\ \phi(\xi) = \frac{2}{2\coth[\xi] \pm 3\,\mathrm{csch}[\xi]}. \tag{38}$$

Case-4: $R(r) = r^2 - 1$

$$\chi(\xi) = \frac{4\,\mathrm{sech}[\xi]}{3\tanh[\xi] + 4r\,\mathrm{sech}[\xi] + 5},\ \text{ and }\ \phi(\xi) = \frac{5\tanh[\xi] + 3}{3\tanh[\xi] + 4r\,\mathrm{sech}[\xi] + 5}, \tag{39}$$

or

$$\chi(\xi) = \frac{\mathrm{sech}[\xi]}{r\,\mathrm{sech}[\xi] + 1},\ \text{ and }\ \phi(\xi) = \frac{\tanh[\xi]}{r\,\mathrm{sech}[\xi] + 1}. \tag{40}$$

Case-5: $R(r) = r^2 + 1$

$$\chi(\xi) = \frac{\mathrm{csch}[\xi]}{r\,\mathrm{csch}[\xi] + 1},\ \text{ and }\ \phi(\xi) = \frac{\coth[\xi]}{r\,\mathrm{csch}[\xi] + 1}. \tag{41}$$

4. Soliton solutions

This section will retrieve the soliton solutions to the given model represented by (1) after a skillful application of the two integration algorithms as revisited in the previous section. The details are presented in the subsequent two subsections.

4.1 *Enhanced direct algebraic method*

Balancing U'''' and U^5 in Eq. (12) it implies $N = 1$ accordingly the solution takes the form:

$$U(\xi) = \alpha_0 + \alpha_1\theta(\xi) + \frac{\beta_1}{\theta(\xi)}, \tag{42}$$

Substitute Eq. (42) along with Eq. (18) into Eq. (12). The substitution yields a polynomial as $\theta(\xi)$. In polynomials, the process involves the collection of terms with similar powers and setting the resulting expression equal to zero. The following system of algebraic equations is obtained:

$$40\alpha_1\alpha_0^3\beta_1 h_3 + 60\alpha_1^2\alpha_0\beta_1^2 h_3 + 2h_2\left(6\alpha_1\alpha_0\beta_1 + \alpha_0^3\right) + 2\alpha_0^5 h_3 + 2\alpha_0 h_1 + \alpha_0^2\beta_1 h_6 T_3$$

$$+2\alpha_0\beta_1^2 h_4 T_4 - 4\alpha_1\alpha_0\beta_1 h_4 T_2 + 8\alpha_1\alpha_0\beta_1 h_6 T_2 - 2\alpha_1\beta_1^2 h_4 T_3 + 5\alpha_1\beta_1^2 h_6 T_3 - 2\alpha_1^2\beta_1 h_4 T_1 + 5\alpha_1^2\beta_1 h_6 T_1$$

$$+\alpha_1\alpha_0^2 h_6 T_1 + 2\alpha_1^2\alpha_0 h_4 T_0 + \alpha_1 h_5 T_1 + \beta_1 h_5 T_3 + \alpha_1 k^2 T_1 T_2 + 6\alpha_1 k^2 T_0 T_3 + \beta_1 k^2 T_2 T_3 + 6\beta_1 k^2 T_1 T_4 = 0,$$

$$2\beta_1\left(\beta_1^4 h_3 + \beta_1^2\left(h_4 + 2h_6\right) T_0 + 24k^2 T_0^2\right) = 0,$$

$$2\beta_1 T_0\left(\alpha_0\beta_1\left(h_4 + 4h_6\right) + 30k^2 T_1\right) + \beta_1^3\left(10\alpha_0\beta_1 h_3 + 2h_4 T_1 + 3h_6 T_1\right) = 0,$$

$$10\alpha_1\beta_1^4 h_3 + 20\alpha_0^2\beta_1^3 h_3 + 2\beta_1^3 h_2 + 2\alpha_0\beta_1^2 h_4 T_1 + 6\alpha_0\beta_1^2 h_6 T_1 - 2\alpha_1\beta_1^2 h_4 T_0 + 8\alpha_1\beta_1^2 h_6 T_0$$

$$+4\alpha_0^2\beta_1 h_6 T_0 + 2\beta_1^3 h_4 T_2 + 2\beta_1^3 h_6 T_2 + 4\beta_1 h_5 T_0 + 15\beta_1 k^2 T_1^2 + 40\beta_1 k^2 T_0 T_2 = 0,$$

$$20\alpha_0^3\beta_1^2 h_3 + 40\alpha_1\alpha_0\beta_1^3 h_3 + 6\alpha_0\beta_1^2 h_2 + 3\alpha_0^2\beta_1 h_6 T_1 + 2\alpha_0\beta_1^2 h_4 T_2 + 4\alpha_0\beta_1^2 h_6 T_2 - 4\alpha_1\alpha_0\beta_1 h_4 T_0$$

$$+8\alpha_1\alpha_0\beta_1 h_6 T_0 - 2\alpha_1\beta_1^2 h_4 T_1 + 7\alpha_1\beta_1^2 h_6 T_1 + 2\beta_1^3 h_4 T_3 + \beta_1^3 h_6 T_3 + 3\beta_1 h_5 T_1 + 15\beta_1 k^2 T_1 T_2 + 30\beta_1 k^2 T_0 T_3 = 0,$$

$$10\alpha_0^4\beta_1 h_3 + 60\alpha_1\alpha_0^2\beta_1^2 h_3 + 6\alpha_0^2\beta_1 h_2 + 20\alpha_1^2\beta_1^3 h_3 + 6\alpha_1\beta_1^2 h_2 + 2\beta_1 h_1 + 2\alpha_0^2\beta_1 h_6 T_2 + 2\alpha_0\beta_1^2 h_4 T_3$$

$$+2\alpha_0\beta_1^2 h_6 T_3 - 4\alpha_1\alpha_0\beta_1 h_4 T_1 + 8\alpha_1\alpha_0\beta_1 h_6 T_1 - 2\alpha_1\beta_1^2 h_4 T_2 + 6\alpha_1\beta_1^2 h_6 T_2 - 2\alpha_1^2\beta_1 h_4 T_0$$

$$+4\alpha_1^2\beta_1 h_6 T_0 + 2\beta_1^3 h_4 T_4 + 2\beta_1 h_5 T_2 + 2\beta_1 k^2 T_2^2 + 9\beta_1 k^2 T_1 T_3 + 24\beta_1 k^2 T_0 T_4 = 0,$$

$$60\alpha_1^2\alpha_0^2\beta_1 h_3+20\alpha_1^3\beta_1^2 h_3+6\alpha_1^2\beta_1 h_2+10\alpha_1\alpha_0^4 h_3+6\alpha_1\alpha_0^2 h_2+2\alpha_1 h_1-4\alpha_1\alpha_0\beta_1 h_4 T_3$$

$$+8\alpha_1\alpha_0\beta_1 h_6 T_3-2\alpha_1\beta_1^2 h_4 T_4+4\alpha_1\beta_1^2 h_6 T_4-2\alpha_1^2\beta_1 h_4 T_2+6\alpha_1^2\beta_1 h_6 T_2+2\alpha_1\alpha_0^2 h_6 T_2$$

$$+2\alpha_1^2\alpha_0 h_4 T_1+2\alpha_1^2\alpha_0 h_6 T_1+2\alpha_1^3 h_4 T_0+2\alpha_1 h_5 T_2+2\alpha_1 k^2 T_2^2+9\alpha_1 k^2 T_1 T_3+24\alpha_1 k^2 T_0 T_4=0,$$

$$40\alpha_1^3\alpha_0\beta_1 h_3+20\alpha_1^2\alpha_0^3 h_3+6\alpha_1^2\alpha_0 h_2-4\alpha_1\alpha_0\beta_1 h_4 T_4+8\alpha_1\alpha_0\beta_1 h_6 T_4-2\alpha_1^2\beta_1 h_4 T_3$$

$$+7\alpha_1^2\beta_1 h_6 T_3+3\alpha_1\alpha_0^2 h_6 T_3+2\alpha_1^2\alpha_0 h_4 T_2+4\alpha_1^2\alpha_0 h_6 T_2+2\alpha_1^3 h_4 T_1+\alpha_1^3 h_6 T_1+3\alpha_1 h_5 T_3$$

$$+15\alpha_1 k^2 T_2 T_3+30\alpha_1 k^2 T_1 T_4=0,$$

$$10\alpha_1^4\beta_1 h_3+20\alpha_0^2\alpha_1^3 h_3+2\alpha_1^3 h_2-2\alpha_1^2\beta_1 h_4 T_4+8\alpha_1^2\beta_1 h_6 T_4+2\alpha_1^3 h_4 T_2+2\alpha_1^3 h_6 T_2+2\alpha_0\alpha_1^2 h_4 T_3$$

$$+6\alpha_0\alpha_1^2 h_6 T_3+4\alpha_0^2\alpha_1 h_6 T_4+4\alpha_1 h_5 T_4+15\alpha_1 k^2 T_3^2+40\alpha_1 k^2 T_2 T_4=0,$$

$$10\alpha_0\alpha_1^4 h_3+2\alpha_1^3 h_4 T_3+3\alpha_1^3 h_6 T_3+2\alpha_0\alpha_1^2 h_4 T_4+8\alpha_0\alpha_1^2 h_6 T_4+60\alpha_1 k^2 T_3 T_4=0,$$

$$2\alpha_1^5 h_3+2\alpha_1^3 h_4 T_4+4\alpha_1^3 h_6 T_4+48\alpha_1 k^2 T_4^2=0,$$

which can be solved using Mathematica to discover the unknown parameters in Eq. (15) and Eq. (42). As a result, we derive the exact solutions of Eq. (1).

Case-1: If we set $T_0=T_1=T_3=0$

$$\alpha_0=\beta_1=0,\ \ \alpha_1=\pm\sqrt{\frac{2T_4\left(9h_5T_2+10h_1\right)}{T_2\left(\left(h_4+h_6\right)T_2+h_2\right)}},\ \ k=\frac{\sqrt{-h_5T_2-h_1}}{T_2},$$

$$h_3=\frac{\left(\left(h_4+h_6\right)T_2+h_2\right)\left(3h_5T_2\left(\left(h_4-2h_6\right)T_2+4h_2\right)+2h_1\left(\left(h_4-4h_6\right)T_2+6h_2\right)\right)}{2\left(9h_5T_2+10h_1\right)^2}, \tag{43}$$

$$q(x,\ t)=\pm\sqrt{\frac{-2\left(9h_5T_2+10h_1\right)}{\left(\left(h_4+h_6\right)T_2+h_2\right)}}\,\mathrm{sech}\left[\sqrt{\frac{-h_5T_2-h_1}{T_2}}\left(x-v\frac{t^\alpha}{\alpha}\right)\right]\ e^{i\left(-\kappa x+\omega\frac{t^\alpha}{\alpha}+\theta_0\right)}, \tag{44}$$

$$q(x,\ t)=\pm\sqrt{\frac{2\left(9h_5T_2+10h_1\right)}{\left(\left(h_4+h_6\right)T_2+h_2\right)}}\,\mathrm{csch}\left[\sqrt{\frac{-h_5T_2-h_1}{T_2}}\left(x-v\frac{t^\alpha}{\alpha}\right)\right]\ e^{i\left(-\kappa x+\omega\frac{t^\alpha}{\alpha}+\theta_0\right)}. \tag{45}$$

Solutions (44) and (45) are bright and singular solitons with $h_5T_2+h_1<0$ and $T_2>0$.

Case-2: If we set

$$T_0 = \frac{T_2^2}{4T_4}, \; T_1 = T_3 = 0$$

Result-1:

$$\alpha_0 = \beta_1 = 0, \; \alpha_1 = \pm 2\sqrt{\frac{\mu_1 T_4}{\mu_2 T_2}}, \; h_3 = \frac{\mu_2 \mu_3}{4\mu_1^2}, \; k = \sqrt{\frac{\mu_4}{2\mu_2 T_2^2}}, \tag{46}$$

where

$$\mu_1 = 3h_5T_2 + 5h_1, \; \mu_2 = 4h_2 - (h_4 - 4h_6)T_2, \; \mu_3 = 6h_2h_5T_2 + h_1((h_4 - 4h_6)T_2 + 6h_2),$$

$$\mu_4 = -2h_1((h_4 + h_6)T_2 + h_2) - h_5T_2((h_4 + 2h_6)T_2 + 2h_2),$$

$$q(x,\,t) = \pm\sqrt{\frac{-2\mu_1}{\mu_2}} \tanh\left[\frac{1}{2}\sqrt{\frac{-\mu_4}{2\mu_2 T_2}}\left(x - v\frac{t^\alpha}{\alpha}\right)\right] \; e^{i\left(-\kappa x + \omega \frac{t^\alpha}{\alpha} + \theta_0\right)}, \tag{47}$$

$$q(x,\,t) = \pm\sqrt{\frac{-2\mu_1}{\mu_2}} \coth\left[\frac{1}{2}\sqrt{\frac{-\mu_4}{2\mu_2 T_2}}\left(x - v\frac{t^\alpha}{\alpha}\right)\right] \; e^{i\left(-\kappa x + \omega \frac{t^\alpha}{\alpha} + \theta_0\right)}. \tag{48}$$

Solutions (47) and (48) are dark and singular solitons with $\mu_4 > 0, \; \mu_2 > 0, \; \mu_1 < 0$ and $T_2 < 0$.
Result-2:

$$\alpha_0 = 0, \; \alpha_1 = \pm\sqrt{\frac{2\mu_5 T_4}{\mu_6 T_2}}, \; \beta_1 = \pm\sqrt{\frac{\mu_5 T_2}{2\mu_6 T_4}}, \; h_3 = \frac{\mu_7}{2\mu_5^2}, \; k = \frac{1}{2}\sqrt{-\frac{\mu_8}{T_2^2}}, \tag{49}$$

where

$$\mu_5 = 9h_5T_2 - 5h_1, \; \mu_6 = h_2 - 2(h_4 + h_6)T_2,$$

$$\mu_7 = (h_2 - 2(h_4 + h_6)T_2)(3h_5T_2((h_4 - 2h_6)T_2 - 2h_2) + h_1(3h_2 - (h_4 - 4h_6)T_2)), \; \mu_8 = h_1 - 2h_5T_2,$$

$$\mu_9 = 12h_5T_2(2(2h_4 - 13h_6)T_2 - 5h_2) + 5h_1(9h_2 - 10(h_4 - 3h_6)T_2),$$

$$q(x,\,t) = \pm\sqrt{\frac{-\mu_5}{\mu_6}}\left[\tanh\left[\frac{1}{2}\sqrt{\frac{\mu_8}{2T_2}}\left(x - v\frac{t^\alpha}{\alpha}\right)\right] + \coth\left[\frac{1}{2}\sqrt{\frac{\mu_8}{2T_2}}\left(x - v\frac{t^\alpha}{\alpha}\right)\right]\right] \; e^{i\left(-\kappa x + \omega \frac{t^\alpha}{\alpha} + \theta_0\right)}. \tag{50}$$

Solution (50) are dark-singular solitons with $\mu_8 < 0,\ \mu_5 < 0,\ \mu_6 > 0$ and $T_2 < 0$.
Case 3-1: If we set

$$T_1 = T_3 = 0,\ T_0 = \frac{m^2(1-m^2)T_2^2}{(2m^2-1)^2 T_4}$$

Result-1:

$$\alpha_0 = \beta_1 = 0,\ \ \alpha_1 = \pm\sqrt{\frac{2\mu_{10}T_4}{\mu_{11}T_2}},\ \ h_3 = \frac{\mu_{11}\mu_{12}}{2\mu_{10}^2},\ \ k = \sqrt{-\frac{\mu_{13}}{\mu_{11}T_2^2}}, \tag{51}$$

where

$$\mu_{10} = 10h_1\left(1-2m^2\right)^2 + 3h_5\left(16m^4-16m^2+3\right)T_2,$$

$$\mu_{11} = T_2\left(h_4\left(12m^4-12m^2+1\right)+h_6\left(-8m^4+8m^2+1\right)\right)+h_2\left(-8m^4+8m^2+1\right),$$

$$\mu_{12} = 2h_1\left(1-2m^2\right)^2\left(\left(h_4-4h_6\right)T_2+6h_2\right)+3h_5T_2\left(4h_2\left(1-2m^2\right)^2+\left(h_4-2h_6\right)\left(8m^4-8m^2+1\right)T_2\right),$$

$$\mu_{13} = h_1\left(1-2m^2\right)^2\left(\left(h_4+h_6\right)T_2+h_2\right)+h_5T_2\left(h_2\left(1-2m^2\right)^2+T_2\left(h_6\left(1-2m^2\right)^2+h_4\left(6m^4-6m^2+1\right)\right)\right),$$

$$q(x,\ t) = \sqrt{-\frac{2\mu_{10}m^2}{\mu_{11}(2m^2-1)}}\,\mathrm{cn}\left(\sqrt{-\frac{\mu_{13}}{\mu_{11}T_2(2m^2-1)}}\left(x-v\frac{t^\alpha}{\alpha}\right)\middle|\, m\right)\ e^{i\left(-\kappa x+\omega\frac{t^\alpha}{\alpha}+\theta_0\right)}. \tag{52}$$

For $m \to 1^-$, we get

$$q(x,\ t) = \sqrt{-\frac{2\mu_{10}}{\mu_{11}}}\,\mathrm{sech}\left(\sqrt{-\frac{\mu_{13}}{\mu_{11}T_2}}\left(x-v\frac{t^\alpha}{\alpha}\right)\right)\ e^{i\left(-\kappa x+\omega\frac{t^\alpha}{\alpha}+\theta_0\right)}. \tag{53}$$

Solutions (52) and (53) are Jacobi elliptic doubly periodic type and bright soliton solutions.
Result-2:

$$\alpha_0 = \alpha_1 = 0,\ \ \beta_1 = \pm\sqrt{\frac{2\mu_{10}m^2\left(m^2-1\right)T_2}{\mu_{14}\left(1-2m^2\right)^2T_4}},\ \ h_3 = \frac{\mu_{11}\mu_{12}}{2\mu_{10}^2},\ \ k = \sqrt{-\frac{\mu_{13}}{\mu_{11}T_2^2}}, \tag{54}$$

where

$$\mu_{14}=h_2\left(8m^4-8m^2-1\right)-T_2\left(h_4\left(12m^4-12m^2+1\right)+h_6\left(-8m^4+8m^2+1\right)\right),$$

$$q(x,\,t)=\sqrt{\frac{2\mu_{10}\left(m^2-1\right)}{\mu_{14}\left(1-2m^2\right)}}\,\text{nc}\left(\sqrt{-\frac{\mu_{13}}{\mu_{11}T_2(2m^2-1)}}\left(x-v\frac{t^\alpha}{\alpha}\right)\middle|\,m\right)\,e^{i\left(-\kappa x+\omega\frac{t^\alpha}{\alpha}+\theta_0\right)}. \tag{55}$$

Solution (55) is are Jacobi elliptic doubly periodic type solution.
Case 3-2: If we set

$$T_1=T_3=0,\;\; T_0=\frac{(1-m^2)T_2^2}{(2-m^2)^2T_4}$$

Result-1:

$$\alpha_0=\beta_1=0,\;\;\alpha_1=\pm\sqrt{\frac{2\mu_{15}T_4}{\mu_{16}T_2}},\;\;h_3=\frac{\mu_{16}\mu_{17}}{2\mu_{15}^2},\;\;k=\sqrt{\frac{\mu_{18}}{\mu_{16}T_2^2}}, \tag{56}$$

where

$$\mu_{15}=10h_1\left(m^2-2\right)^2+3h_5\left(3m^4-8m^2+8\right)T_2,$$

$$\mu_{16}=T_2\left(h_4\left(m^4+4m^2-4\right)+h_6\left(m^4-16m^2+16\right)\right)+h_2\left(m^4-16m^2+16\right),$$

$$\mu_{17}=2h_1\left(m^2-2\right)^2\left(\left(h_4-4h_6\right)T_2+6h_2\right)+3h_5T_2\left(\left(h_4-2h_6\right)m^4T_2+4h_2\left(m^2-2\right)^2\right),$$

$$\mu_{18}=h_1\left(-\left(m^2-2\right)^2\right)\left(\left(h_4+h_6\right)T_2+h_2\right)-h_5T_2\left(h_2\left(m^2-2\right)^2+T_2\left(h_6\left(m^2-2\right)^2+h_4\left(m^4-2m^2+2\right)\right)\right),$$

$$q(x,\,t)=\sqrt{-\frac{2\mu_{15}m^2}{\mu_{16}(2-m^2)}}\,\text{dn}\left(\sqrt{\frac{\mu_{18}}{\mu_{16}T_2(2-m^2)}}\left(x-v\frac{t^\alpha}{\alpha}\right)\middle|\,m\right)\,e^{i\left(-\kappa x+\omega\frac{t^\alpha}{\alpha}+\theta_0\right)}. \tag{57}$$

For $m\to 1^-$, we get

$$q(x,\,t)=\sqrt{-\frac{2\mu_{15}}{\mu_{16}}}\,\text{sech}\left(\sqrt{\frac{\mu_{18}}{\mu_{16}T_2}}\left(x-v\frac{t^\alpha}{\alpha}\right)\right)\,e^{i\left(-\kappa x+\omega\frac{t^\alpha}{\alpha}+\theta_0\right)} \tag{58}$$

Solutions (57) and (58) are Jacobi elliptic doubly periodic type and a bright soliton solutions.

Result-2:

$$\alpha_0=\alpha_1=0,\ \ \beta_1=\pm\sqrt{\frac{2\mu_{15}\left(1-m^2\right)T_2}{\mu_{16}\left(m^2-2\right)^2T_4}},\ \ h_3=\frac{\mu_{16}\mu_{17}}{2\mu_{15}^2},\ \ k=\sqrt{\frac{\mu_{18}}{\mu_{16}T_2^2}}, \tag{59}$$

$$q(x,\ t)=\sqrt{\frac{2\mu_{15}\left(1-m^2\right)}{\mu_{16}\left(m^2-2\right)m^2}}\,\mathrm{nd}\left(\sqrt{\frac{\mu_{18}}{\mu_{16}T_2(2-m^2)}}\left(x-v\frac{t^\alpha}{\alpha}\right)\middle|\, m\right)\ e^{i\left(-\kappa x+\omega\frac{t^\alpha}{\alpha}+\theta_0\right)}. \tag{60}$$

Solution (60) is Jacobi elliptic doubly periodic type.

Case 3-3: If we set

$$T_1=T_3=0,\ T_0=\frac{m^2T_2^2}{(m^2+1)^2T_4}$$

Result-1:

$$\alpha_0=\beta_1=0,\ \ \alpha_1=\pm\sqrt{\frac{2\mu_{19}T_4}{\mu_{20}T_2}},\ \ h_3=\frac{\mu_{20}\mu_{21}}{2\mu_{19}^2},\ \ k=\sqrt{\frac{\mu_{22}}{\mu_{20}T_2^2}}, \tag{61}$$

where

$$\mu_{19}=10h_1\left(m^2+1\right)^2+3h_5\left(3m^4+2m^2+3\right)T_2,$$

$$\mu_{20}=T_2\left(h_4\left(m^4-6m^2+1\right)+h_6\left(m^4+14m^2+1\right)\right)+h_2\left(m^4+14m^2+1\right),$$

$$\mu_{21}=2h_1\left(m^2+1\right)^2\left(\left(h_4-4h_6\right)T_2+6h_2\right)+3h_5T_2\left(\left(h_4-2h_6\right)\left(m^2-1\right)^2T_2+4h_2\left(m^2+1\right)^2\right),$$

$$\mu_{22}=h_1\left(-\left(m^2+1\right)^2\right)\left(\left(h_4+h_6\right)T_2+h_2\right)-h_5T_2\left(h_2\left(m^2+1\right)^2+T_2\left(h_4\left(m^4+1\right)+h_6\left(m^2+1\right)^2\right)\right),$$

$$q(x,\ t)=\sqrt{-\frac{2\mu_{19}m^2}{\mu_{20}(m^2+1)}}\,\mathrm{sn}\left(\sqrt{-\frac{\mu_{22}}{(m^2+1)\mu_{20}T_2}}\left(x-v\frac{t^\alpha}{\alpha}\right)\middle|\, m\right)\ e^{i\left(-\kappa x+\omega\frac{t^\alpha}{\alpha}+\theta_0\right)}. \tag{62}$$

For $m\to 1^-$, we get

$$q(x,\ t)=\sqrt{-\frac{\mu_{19}}{\mu_{20}}}\tanh\left(\sqrt{-\frac{\mu_{22}}{2\mu_{20}T_2}}\left(x-v\frac{t^\alpha}{\alpha}\right)\right)\ e^{i\left(-\kappa x+\omega\frac{t^\alpha}{\alpha}+\theta_0\right)}. \tag{63}$$

Solutions (62) and (63) are Jacobi elliptic doubly periodic type and dark soliton solutions.

Result-2:

$$\alpha_0=\alpha_1=0,\ \ \beta_1=\pm\frac{m}{m^2+1}\sqrt{\frac{2\mu_{19}T_2}{\mu_{20}T_4}},\ \ h_3=\frac{\mu_{20}\mu_{21}}{2\mu_{19}^2},\ \ k=\sqrt{\frac{\mu_{22}}{\mu_{20}T_2^2}}, \tag{64}$$

$$q(x,\ t)=\sqrt{-\frac{2\mu_{19}}{\mu_{20}(m^2+1)}}\,\mathrm{ns}\left(\sqrt{-\frac{\mu_{22}}{(m^2+1)\mu_{20}T_2}}\left(x-v\frac{t^{\alpha}}{\alpha}\right)\bigg|\, m\right)\ e^{i\left(-\kappa x+\omega\frac{t^{\alpha}}{\alpha}+\theta_0\right)}. \tag{65}$$

For $m\to 1^-$, we get

$$q(x,t)=\sqrt{-\frac{\mu_{19}}{\mu_{20}}}\coth\left(\sqrt{-\frac{\mu_{22}}{2\mu_{20}T_2}}\left(x-v\frac{t^{\alpha}}{\alpha}\right)\right)\ e^{i\left(-\kappa x+\omega\frac{t^{\alpha}}{\alpha}+\theta_0\right)} \tag{66}$$

Solutions (65) and (66) are Jacobi elliptic doubly periodic type and singular soliton solutions.
Case-4: If we set $T_1=T_3=0$
Result-1:

$$\alpha_0=\beta_1=0,\ \ \alpha_1=\pm\sqrt{\frac{2\mu_{23}T_4}{\mu_{24}}},\ \ h_3=\frac{\mu_{24}\mu_{25}}{2\mu_{23}^2},\ \ k=\sqrt{\frac{\mu_{26}}{\mu_{24}}}, \tag{67}$$

where

$$\mu_{23}=10h_1T_2+3h_5\left(3T_2^2-4T_0T_4\right),$$

$$\mu_{24}=h_2\left(T_2^2+12T_0T_4\right)+T_2\left(h_6\left(T_2^2+12T_0T_4\right)+h_4\left(T_2^2-8T_0T_4\right)\right),$$

$$\mu_{25}=2h_1\left(\left(h_4-4h_6\right)T_2+6h_2\right)+3h_5\left(4h_2T_2+\left(h_4-2h_6\right)\left(T_2^2-4T_0T_4\right)\right),$$

$$\mu_{26}=-h_1\left(\left(h_4+h_6\right)T_2+h_2\right)-h_5\left(h_6T_2^2+h_2T_2+h_4\left(T_2^2-2T_0T_4\right)\right),$$

$$q(x,\ t)=\pm\sqrt{\frac{2\mu_{23}}{\mu_{24}}}\left(\frac{3\wp'\left(\sqrt{\frac{\mu_{26}}{\mu_{24}}}\left(x-v\frac{t^{\alpha}}{\alpha}\right);\ g_2,\ g_3\right)}{\left[6\wp\left(\sqrt{\frac{\mu_{26}}{\mu_{24}}}\left(x-v\frac{t^{\alpha}}{\alpha}\right);\ g_2,\ g_3\right)+T_2\right]}\right)\ e^{i\left(-\kappa x+\omega\frac{t^{\alpha}}{\alpha}+\theta_0\right)}. \tag{68}$$

For $\tau_0=0$, we obtain

$$q(x,\ t) = \pm\sqrt{\frac{2\mu_{23}T_2}{\mu_{24}}}\operatorname{csch}\left(\sqrt{\frac{T_2\mu_{26}}{\mu_{24}}}\left(x - v\frac{t^\alpha}{\alpha}\right)\right)\ e^{i\left(-\kappa x+\omega\frac{t^\alpha}{\alpha}+\theta_0\right)}, \tag{69}$$

$$q(x,\ t) = \sqrt{\frac{2\mu_{23}T_4T_0}{\mu_{24}}}\left(\frac{\left[6\wp\left(\sqrt{\frac{\mu_{26}}{\mu_{24}}}\left(x - v\frac{t^\alpha}{\alpha}\right);\ g_2,\ g_3\right)+T_2\right]}{3\wp'\left(\sqrt{\frac{\mu_{26}}{\mu_{24}}}\left(x - v\frac{t^\alpha}{\alpha}\right);\ g_2,\ g_3\right)}\right)\ e^{i\left(-\kappa x+\omega\frac{t^\alpha}{\alpha}+\theta_0\right)}, T_0 > 0. \tag{70}$$

Solutions (68) and (70) are Weierstrass elliptic doubly periodic-type solutions and (69) is a singular soliton solution, where

$$g_2 = \frac{T_2^2}{12}+T_0T_4 \ \text{ and }\ g_3 = \frac{T_2}{216}(36T_0T_4 - T_2^2).$$

Case-5: If we set $T_0 = T_1 = 0$

$$\alpha_0 = \beta_1 = 0,\ \ \alpha_1 = \pm 2\sqrt{\frac{3h_5T_4}{(2h_4+3h_6)\,T_2}},\ \ h_1 = \frac{1}{5}(-4)h_5T_2,\ \ k = \sqrt{-\frac{h_5}{5T_2}},$$

$$h_2 = \frac{(2h_4+3h_6)\,T_3^2}{8T_4} - \frac{1}{6}\,(4h_4+3h_6)\,T_2,\ \ h_3 = -\frac{(2h_4+3h_6)\,(h_4+4h_6)\,T_2}{60h_5}, \tag{71}$$

$$q(x,\ t) = \pm 2\sqrt{\frac{3h_5T_4T_2}{(2h_4+3h_6)}}\left(\frac{-\operatorname{sech}^2\left[\sqrt{-\frac{h_5}{20}}\left(x - v\frac{t^\alpha}{\alpha}\right)\right]}{\pm 2\sqrt{T_2T_4}\tanh\left[\sqrt{-\frac{h_5}{20}}\left(x - v\frac{t^\alpha}{\alpha}\right)\right]+T_3}\right)\ e^{i\left(-\kappa x+\omega\frac{t^\alpha}{\alpha}+\theta_0\right)},\ \ T_4 > 0, \tag{72}$$

$$q(x,\ t) = \pm 2\sqrt{\frac{3h_5T_4T_2}{(2h_4+3h_6)}}\left(\frac{\operatorname{csch}^2\left[\sqrt{-\frac{h_5}{20}}\left(x - v\frac{t^\alpha}{\alpha}\right)\right]}{\pm 2\sqrt{T_2T_4}\coth\left[\sqrt{-\frac{h_5}{20}}\left(x - v\frac{t^\alpha}{\alpha}\right)\right]+T_3}\right)\ e^{i\left(-\kappa x+\omega\frac{t^\alpha}{\alpha}+\theta_0\right)},\ \ T_4 > 0, \tag{73}$$

$$q(x,\ t) = \pm 2\sqrt{\frac{3h_5T_4T_2}{(2h_4+3h_6)}}\left(\frac{-T_3\operatorname{sech}^2\left[\sqrt{-\frac{h_5}{20}}\left(x - v\frac{t^\alpha}{\alpha}\right)\right]}{T_3^2 - T_2T_4\left(1-\tanh\left[\sqrt{-\frac{h_5}{20}}\left(x - v\frac{t^\alpha}{\alpha}\right)\right]\right)^2}\right)\ e^{i\left(-\kappa x+\omega\frac{t^\alpha}{\alpha}+\theta_0\right)},\ \ T_3 \neq 0, \tag{74}$$

$$q(x,\ t) = \pm 2\sqrt{\frac{3h_5T_4T_2}{(2h_4+3h_6)}}\left(\frac{T_3\,\mathrm{csch}^2\left[\sqrt{-\frac{h_5}{20}}\left(x-v\frac{t^\alpha}{\alpha}\right)\right]}{T_3^2 - T_2T_4\left(1-\coth\left[\sqrt{-\frac{h_5}{20}}\left(x-v\frac{t^\alpha}{\alpha}\right)\right]\right)^2}\right)e^{i\left(-\kappa x+\omega\frac{t^\alpha}{\alpha}+\theta_0\right)},\ T_3 \neq 0. \tag{75}$$

Solutions (72) and (74) are straddled bright-dark solitons, while solutions (73) and (75) are straddled singlar-singular solitons with $h_5 < 0$.

4.2 *Projective Riccati equation method*

Balancing U'''' and U^5 in Eq. (12) it implies $N = 1$ accordingly the solution takes the form:

$$U(\xi) = A_0 + A_1\chi(\xi) + B_1\phi(\xi), \tag{76}$$

where A_0, A_1 and B_1 are constants to be determined such that $A_1 \neq 0$ or $B_1 \neq 0$. (A_1 and B_1 should not be zero simultaneously.)

Substituting the solution Eq. (76) which satisfies the condition Eq. (33), Eq. (34) into Eq. (12), leads to a set of the following nonlinear equations:

$$A_0\left(A_0^2\left(10B_1^2h_3+h_2\right)+A_0^4h_3+3B_1^2h_2+5B_1^4h_3+h_1\right)=0,$$

$$A_1\left(A_0^2\left(30B_1^2h_3+3h_2+h_6\right)+5A_0^4h_3+3B_1^2h_2+5B_1^4h_3+B_1^2h_6+h_1+h_5+k^2\right)$$

$$-2A_0B_1^2r\left(10A_0^2h_3+10B_1^2h_3+3h_2+h_6\right)=0,$$

$$-A_1r\left(B_1^2\left(60A_0^2h_3+6h_2+2h_4+7h_6\right)+3\left(A_0^2h_6+h_5+5k^2\right)+20B_1^4h_3\right)$$

$$+A_0B_1^2\left(R(r)\left(10A_0^2h_3+10B_1^2h_3+3h_2+4h_6\right)+r^2\left(20B_1^2h_3+h_4+4h_6\right)\right)$$

$$+A_0A_1^2\left(10A_0^2h_3+30B_1^2h_3+3h_2+h_4+2h_6\right)=0,$$

$$20A_1B_1^4h_3r^2+5A_1B_1^2h_4r^2+10A_1B_1^2h_6r^2-20A_0B_1^4h_3rR(r)+10A_1B_1^4h_3R(r)+3A_1B_1^2h_2R(r)$$

$$+30A_0^2A_1B_1^2h_3R(r)-2A_0B_1^2h_4rR(r)+2A_1B_1^2h_4R(r)-10A_0B_1^2h_6rR(r)+7A_1B_1^2h_6R(r)$$

$$-60A_0A_1^2B_1^2h_3r+10A_1^3B_1^2h_3+2A_1h_5R(r)+2A_0^2A_1h_6R(r)-2A_0A_1^2h_4r-6A_0A_1^2h_6r+A_1^3h_2$$

$$+10A_0^2A_1^3h_3+A_1^3h_4+A_1^3h_6+30A_1k^2r^2+20A_1k^2R(r)=0,$$

$$A_1R(r)\left(A_0A_1\left(30B_1^2h_3+h_4+4h_6\right)-r\left(20B_1^4h_3+B_1^2\left(8h_4+17h_6\right)+60k^2\right)\right)$$

$$+A_0B_1^2\left(5B_1^2h_3+h_4+4h_6\right)R(r)^2+A_1^3\left(5A_0A_1h_3-r\left(20B_1^2h_3+2h_4+3h_6\right)\right)=0,$$

$$A_1R(r)^2\left(5B_1^4h_3+3B_1^2\left(h_4+2h_6\right)+24k^2\right)+A_1^3\left(10B_1^2h_3+h_4+2h_6\right)R(r)+A_1^5h_3=0,$$

$$B_1\left(A_0^2\left(10B_1^2h_3+3h_2\right)+5A_0^4h_3+B_1^2\left(B_1^2h_3+h_2\right)+h_1\right)=0,$$

$$-20A_0^2B_1^3h_3r-A_0^2B_1h_6r+20A_0A_1B_1^3h_3+6A_0A_1B_1h_2+20A_0^3A_1B_1h_3+2A_0A_1B_1h_6$$

$$-4B_1^5h_3r-2B_1^3h_2r-B_1^3h_6r-B_1h_5r-B_1k^2r=0,$$

$$10A_0^2B_1^3h_3R(r)+2A_0^2B_1h_6R(r)-40A_0A_1B_1^3h_3r-2A_0A_1B_1h_4r-8A_0A_1B_1h_6r+10A_1^2B_1^3h_3$$

$$+3A_1^2B_1h_2+30A_0^2A_1^2B_1h_3+A_1^2B_1h_4+2A_1^2B_1h_6+4B_1^5h_3r^2+B_1^3h_4r^2+2B_1^3h_6r^2+2B_1^5h_3R(r)$$

$$+B_1^3h_2R(r)+2B_1^3h_6R(r)+2B_1h_5R(r)+6B_1k^2r^2+8B_1k^2R(r)=0,$$

$$20A_0A_1B_1^3h_3R(r)+2A_0A_1B_1h_4R(r)+8A_0A_1B_1h_6R(r)-20A_1^2B_1^3h_3r-4A_1^2B_1h_4r-7A_1^2B_1h_6r$$

$$+20A_0A_1^3B_1h_3-4B_1^5h_3rR(r)-2B_1^3h_4rR(r)-5B_1^3h_6rR(r)-36B_1k^2rR(r)=0,$$

$$B_1\left(A_1^2\left(10B_1^2h_3+3h_4+6h_6\right)R(r)+5A_1^4h_3+R(r)^2\left(B_1^4h_3+B_1^2\left(h_4+2h_6\right)+24k^2\right)\right)=0,$$

which can be solved with the aid of Mathematica software tool. Then, the following results are raised:

Case-1: $R(r)=0$

$$A_0=A_1=0,\ \ B_1=\pm\sqrt{\frac{6h_5-20h_1}{8h_2+h_4-4h_6}},\ \ h_3=\frac{\left(8h_2+h_4-4h_6\right)\left(h_1\left(12h_2-h_4+4h_6\right)-6h_2h_5\right)}{4\left(10h_1-3h_5\right)^2},$$

$$k=\sqrt{\frac{4h_1\left(-2h_2+h_4+h_6\right)+h_5\left(4h_2-h_4-2h_6\right)}{8h_2+h_4-4h_6}}, \tag{77}$$

$$q(x,\,t)=\pm\sqrt{\frac{6h_5-20h_1}{8h_2+h_4-4h_6}}\tanh\left[\sqrt{\frac{4h_1\left(-2h_2+h_4+h_6\right)+h_5\left(4h_2-h_4-2h_6\right)}{2(8h_2+h_4-4h_6)}}\left(x-v\frac{t^\alpha}{\alpha}\right)\right]\ e^{i\left(-\kappa x+\omega\frac{t^\alpha}{\alpha}+\theta_0\right)}, \tag{78}$$

or

$$q(x,\ t)=\pm\sqrt{\frac{6h_5-20h_1}{8h_2+h_4-4h_6}}\coth\left[\sqrt{\frac{4h_1\left(-2h_2+h_4+h_6\right)+h_5\left(4h_2-h_4-2h_6\right)}{2(8h_2+h_4-4h_6)}}\left(x-v\frac{t^{\alpha}}{\alpha}\right)\right]e^{i\left(-\kappa x+\omega\frac{t^{\alpha}}{\alpha}+\theta_0\right)}. \quad (79)$$

Solutions (78) and (79) are dark and singular solitons with $4h_1\left(-2h_2+h_4+h_6\right)+h_5\left(4h_2-h_4-2h_6\right)>0$, and $2(8h_2+h_4-4h_6)>0$, and $6h_5-20h_1>0$.

Case-2: $R(r)=\dfrac{24r^2}{25}$

Result-1:

$$A_0=0,\ A_1=\frac{4\sqrt{3}}{5\sqrt{2}}\mathrm{B}_1 r,\ B_1=\pm\sqrt{\frac{6h_5-20h_1}{8h_2+h_4-4h_6}},\ h_3=\frac{(8h_2+h_4-4h_6)\left(h_1\left(12h_2-h_4+4h_6\right)-6h_2h_5\right)}{4\left(10h_1-3h_5\right)^2},$$

$$k=\sqrt{\frac{4h_1\left(-2h_2+h_4+h_6\right)+h_5\left(4h_2-h_4-2h_6\right)}{8h_2+h_4-4h_6}}, \quad (80)$$

$$q(x,\ t)=\pm\sqrt{\frac{6h_5-20h_1}{8h_2+h_4-4h_6}}\times\left(\frac{2\sqrt{6}\operatorname{csch}\left[\sqrt{\frac{4h_1\left(-2h_2+h_4+h_6\right)+h_5\left(4h_2-h_4-2h_6\right)}{8h_2+h_4-4h_6}}\left(x-v\frac{t^{\alpha}}{\alpha}\right)\right]+1}{\coth\left[\sqrt{\frac{4h_1\left(-2h_2+h_4+h_6\right)+h_5\left(4h_2-h_4-2h_6\right)}{8h_2+h_4-4h_6}}\left(x-v\frac{t^{\alpha}}{\alpha}\right)\right]\pm5\operatorname{csch}\left[\sqrt{\frac{4h_1\left(-2h_2+h_4+h_6\right)+h_5\left(4h_2-h_4-2h_6\right)}{8h_2+h_4-4h_6}}\left(x-v\frac{t^{\alpha}}{\alpha}\right)\right]}\right)$$

$$\times e^{i\left(-\kappa x+\omega\frac{t^{\alpha}}{\alpha}+\theta_0\right)}. \quad (81)$$

Solution (81) is a straddled singular-singular soliton with $4h_1\left(-2h_2+h_4+h_6\right)+h_5\left(4h_2-h_4-2h_6\right)>0,\ 8h_2+h_4-4h_6>0$, and $6h_5-20h_1>0$.

Result-2:

$$A_0=B_1=0,\ A_1=\pm\frac{12\sqrt{2}}{5}\sqrt{\frac{h_5}{2h_4+3h_6}}r, h_1=-\frac{1}{5}\left(4h_5\right),$$

$$h_2=\frac{1}{16}\left(6h_4+17h_6\right),\ h_3=-\frac{2h_4^2+11h_6h_4+12h_6^2}{60h_5},\ k=\sqrt{-\frac{h_5}{5}}, \quad (82)$$

$$q(x,\ t)=\pm\left(\frac{12\sqrt{\frac{2h_5}{2h_4+3h_6}}}{5\pm\cosh\left[\sqrt{-\frac{h_5}{5}}\left(x-v\frac{t^\alpha}{\alpha}\right)\right]}\right)e^{i\left(-\kappa x+\omega\frac{t^\alpha}{\alpha}+\theta_0\right)}. \tag{83}$$

Solution (83) is a bright soliton with $h_5<0$, and $2h_4+3h_6<0$.

Result-3:

$$A_0=A_1=0,\ \ B_1=\pm2\sqrt{\frac{6h_5}{4h_4+7h_6}},\ \ h_1=-\frac{h_5\,(32h_4+43h_6)}{40h_4+70h_6},\ \ h_2=\frac{h_4}{3}+\frac{47h_6}{48},$$

$$h_3=-\frac{4h_4^2+23h_6h_4+28h_6^2}{120h_5},\ \ k=\sqrt{-\frac{2h_5\,(2h_4+3h_6)}{5\,(4h_4+7h_6)}}, \tag{84}$$

$$q(x,\ t)=\pm\left(\frac{2\sqrt{\frac{6h_5}{4h_4+7h_6}}}{\coth\left[\sqrt{-\frac{2h_5\,(2h_4+3h_6)}{5\,(4h_4+7h_6)}}\left(x-v\frac{t^\alpha}{\alpha}\right)\right]\pm5\,\mathrm{csch}\left[\sqrt{-\frac{2h_5\,(2h_4+3h_6)}{5\,(4h_4+7h_6)}}\left(x-v\frac{t^\alpha}{\alpha}\right)\right]}\right)e^{i\left(-\kappa x+\omega\frac{t^\alpha}{\alpha}+\theta_0\right)}. \tag{85}$$

Solution (85) is a straddled singular-singular soliton with $h_5>0$, $(2h_4+3h_6)<0$, and $(4h_4+7h_6)>0$.

Case-3: $R(r)=\frac{5}{9}r^2$

Result-1:

$$A_0=0,\ \ A_1=\frac{1}{3}\sqrt{5}B_1r,\ \ B_1=\pm\sqrt{\frac{6h_5-20h_1}{8h_2+h_4-4h_6}},$$

$$h_3=\frac{(8h_2+h_4-4h_6)\,(h_1\,(12h_2-h_4+4h_6)-6h_2h_5)}{4\,(10h_1-3h_5)^2},$$

$$k=\sqrt{\frac{4h_1\,(-2h_2+h_4+h_6)+h_5\,(4h_2-h_4-2h_6)}{8h_2+h_4-4h_6}}, \tag{86}$$

$$q(x,\ t)=\pm\sqrt{\frac{6h_5-20h_1}{8h_2+h_4-4h_6}}\times\left(\frac{\sqrt{5}\operatorname{csch}\left[\sqrt{\frac{4h_1(-2h_2+h_4+h_6)+h_5(4h_2-h_4-2h_6)}{8h_2+h_4-4h_6}}\left(x-v\frac{t^\alpha}{\alpha}\right)\right]+2}{3\operatorname{csch}\left[\sqrt{\frac{4h_1(-2h_2+h_4+h_6)+h_5(4h_2-h_4-2h_6)}{8h_2+h_4-4h_6}}\left(x-v\frac{t^\alpha}{\alpha}\right)\right]\pm 2\coth\left[\sqrt{\frac{4h_1(-2h_2+h_4+h_6)+h_5(4h_2-h_4-2h_6)}{8h_2+h_4-4h_6}}\left(x-v\frac{t^\alpha}{\alpha}\right)\right]}\right)$$

$$\times\ e^{i\left(-\kappa x+\omega\frac{t^\alpha}{\alpha}+\theta_0\right)}. \tag{87}$$

Solution (87) is a straddled singular-singular soliton with $4h_1(-2h_2+h_4+h_6)+h_5(4h_2-h_4-2h_6)>0$, $8h_2+h_4-4h_6>0$, and $6h_5-20h_1>0$.

Result-2:

$$A_0=B_1=0,\ A_1=\pm2\sqrt{\frac{5h_5}{3(2h_4+3h_6)}}r,\ h_1=-\frac{1}{5}(4h_5),\ h_2=\frac{1}{15}(17h_4+33h_6),$$

$$h_3=-\frac{2h_4^2+11h_6h_4+12h_6^2}{60h_5},\ k=\sqrt{-\frac{h_5}{5}}, \tag{88}$$

$$q(x,\ t)=\pm\left(\frac{2\sqrt{\frac{15h_5}{(2h_4+3h_6)}}}{3\pm2\cosh\left[\sqrt{-\frac{h_5}{5}}\left(x-v\frac{t^\alpha}{\alpha}\right)\right]}\right)e^{i\left(-\kappa x+\omega\frac{t^\alpha}{\alpha}+\theta_0\right)}. \tag{89}$$

Solution (89) is a bright soliton with $h_5<0$, and $(2h_4+3h_6)<0$.

Result-3:

$$A_0=A_1=0,\ B_1=\pm2\sqrt{\frac{15h_5}{10h_4+63h_6}},\ h_1=\frac{4h_5(3h_6-2h_4)}{10h_4+63h_6},\ h_2=\frac{h_4}{3}+\frac{3h_6}{5},$$

$$h_3=-\frac{10h_4^2+103h_6h_4+252h_6^2}{300h_5},\ k=\sqrt{-\frac{h_5(2h_4+3h_6)}{10h_4+63h_6}}, \tag{90}$$

$$q(x,\ t)=\pm\left(\frac{4\sqrt{\frac{15h_5}{10h_4+63h_6}}}{2\coth\left[\sqrt{-\frac{h_5(2h_4+3h_6)}{10h_4+63h_6}}\left(x-v\frac{t^\alpha}{\alpha}\right)\right]\pm3\operatorname{csch}\left[\sqrt{-\frac{h_5(2h_4+3h_6)}{10h_4+63h_6}}\left(x-v\frac{t^\alpha}{\alpha}\right)\right]}\right)e^{i\left(-\kappa x+\omega\frac{t^\alpha}{\alpha}+\theta_0\right)}. \tag{91}$$

Solution (91) is a straddled singular-singular soliton with $h_5 > 0$, $10h_4 + 63h_6 > 0$ and $(2h_4 + 3h_6) < 0$.
Case-4: $R(r) = r^2 - 1$
Result-1:

$$A_0 = 0, \; A_1 = B_1\sqrt{r^2-1}, \; B_1 = \pm\sqrt{\frac{6h_5 - 20h_1}{8h_2 + h_4 - 4h_6}}, \; h_3 = \frac{(8h_2 + h_4 - 4h_6)\left(h_1\left(12h_2 - h_4 + 4h_6\right) - 6h_2h_5\right)}{4\left(10h_1 - 3h_5\right)^2},$$

$$k = \sqrt{\frac{4h_1\left(-2h_2 + h_4 + h_6\right) + h_5\left(4h_2 - h_4 - 2h_6\right)}{8h_2 + h_4 - 4h_6}}, \tag{92}$$

$$q(x,\, t) = \pm\sqrt{\frac{6h_5 - 20h_1}{8h_2 + h_4 - 4h_6}} \times \left(\frac{\begin{array}{c} 4\sqrt{r^2-1}\,\text{sech}\left[\sqrt{\frac{4h_1\left(-2h_2 + h_4 + h_6\right) + h_5\left(4h_2 - h_4 - 2h_6\right)}{8h_2 + h_4 - 4h_6}}\left(x - v\frac{t^\alpha}{\alpha}\right)\right] \\ +\, 5\tanh\left[\sqrt{\frac{4h_1\left(-2h_2 + h_4 + h_6\right) + h_5\left(4h_2 - h_4 - 2h_6\right)}{8h_2 + h_4 - 4h_6}}\left(x - v\frac{t^\alpha}{\alpha}\right)\right] + 3 \end{array}}{\begin{array}{c} 3\tanh\left[\sqrt{\frac{4h_1\left(-2h_2 + h_4 + h_6\right) + h_5\left(4h_2 - h_4 - 2h_6\right)}{8h_2 + h_4 - 4h_6}}\left(x - v\frac{t^\alpha}{\alpha}\right)\right] \\ +\, 4r\text{sech}\left[\sqrt{\frac{4h_1\left(-2h_2 + h_4 + h_6\right) + h_5\left(4h_2 - h_4 - 2h_6\right)}{8h_2 + h_4 - 4h_6}}\left(x - v\frac{t^\alpha}{\alpha}\right)\right] + 5 \end{array}}\right)$$

$$\times\; e^{i\left(-\kappa x + \omega\frac{t^\alpha}{\alpha} + \theta_0\right)}, \tag{93}$$

or

$$q(x,\, t) = \pm\sqrt{\frac{6h_5 - 20h_1}{8h_2 + h_4 - 4h_6}} \times \left(\frac{\sqrt{r^2-1}\,\text{csch}\left[\sqrt{\frac{4h_1\left(-2h_2 + h_4 + h_6\right) + h_5\left(4h_2 - h_4 - 2h_6\right)}{8h_2 + h_4 - 4h_6}}\left(x - v\frac{t^\alpha}{\alpha}\right)\right] + 1}{\begin{array}{c} r\text{csch}\left[\sqrt{\frac{4h_1\left(-2h_2 + h_4 + h_6\right) + h_5\left(4h_2 - h_4 - 2h_6\right)}{8h_2 + h_4 - 4h_6}}\left(x - v\frac{t^\alpha}{\alpha}\right)\right] \\ +\, \coth\left[\sqrt{\frac{4h_1\left(-2h_2 + h_4 + h_6\right) + h_5\left(4h_2 - h_4 - 2h_6\right)}{8h_2 + h_4 - 4h_6}}\left(x - v\frac{t^\alpha}{\alpha}\right)\right] \end{array}}\right)$$

$$\times\; e^{i\left(-\kappa x + \omega\frac{t^\alpha}{\alpha} + \theta_0\right)}. \tag{94}$$

Solutions (93) and (94) are straddled bright-dark soliton and straddled singular-singular soliton with $6h_5 - 20h_1 > 0$, $8h_2 + h_4 - 4h_6 > 0$, and $4h_1\left(-2h_2 + h_4 + h_6\right) + h_5\left(4h_2 - h_4 - 2h_6\right)$.
Result-2:

$$A_0 = B_1 = 0,\ \ A_1 = \pm 2\sqrt{\frac{3h_5\left(r^2-1\right)}{2h_4+3h_6}},\ \ h_1 = -\frac{1}{5}\left(4h_5\right),\ \ h_2 = \frac{2h_4r^2+6h_6r^2+4h_4+3h_6}{6\left(r^2-1\right)},$$

$$h_3 = \frac{-2h_4^2-11h_6h_4-12h_6^2}{60h_5},\ \ k = \sqrt{-\frac{h_5}{5}}, \tag{95}$$

$$q(x,\ t) = \pm\left(\frac{8\sqrt{\frac{3h_5\left(r^2-1\right)}{2h_4+3h_6}}}{3\sinh\left[\sqrt{-\frac{h_5}{5}}\left(x-v\frac{t^\alpha}{\alpha}\right)\right]+4r+5\cosh\left[\sqrt{-\frac{h_5}{5}}\left(x-v\frac{t^\alpha}{\alpha}\right)\right]}\right)e^{i\left(-\kappa x+\omega\frac{t^\alpha}{\alpha}+\theta_0\right)}, \tag{96}$$

or

$$q(x,\ t) = \pm\left(\frac{2\sqrt{\frac{3h_5\left(r^2-1\right)}{2h_4+3h_6}}}{r+\cosh\left[\sqrt{-\frac{h_5}{5}}\left(x-v\frac{t^\alpha}{\alpha}\right)\right]}\right)e^{i\left(-\kappa x+\omega\frac{t^\alpha}{\alpha}+\theta_0\right)}. \tag{97}$$

Solutions (96) and (97) are straddled bright-singular soliton and bright soliton with $h_5 < 0$, $r > 1$, and $2h_4+3h_6 < 0$.
Result-3:

$$A_0 = A_1 = 0,\ \ B_1 = \pm 2\sqrt{-\frac{3h_5\left(r^2-1\right)}{-2h_4\left(r^2-1\right)-3h_6\left(r^2+3\right)}},$$

$$h_1 = -\frac{2h_5\left(4h_4\left(r^2-1\right)+3h_6\left(2r^2-7\right)\right)}{5\left(2h_4\left(r^2-1\right)+3h_6\left(r^2+3\right)\right)},\ \ h_2 = \frac{h_6\left(2r^2-3\right)}{2\left(r^2-1\right)}+\frac{h_4}{3},$$

$$h_3 = -\frac{\left(h_4+4h_6\right)\left(2h_4\left(r^2-1\right)+3h_6\left(r^2+3\right)\right)}{60h_5\left(r^2-1\right)},\ \ k = \sqrt{-\frac{h_5\left(2h_4+3h_6\right)\left(r^2-1\right)}{5\left(2h_4\left(r^2-1\right)+3h_6\left(r^2+3\right)\right)}}, \tag{98}$$

$$q(x,\ t)=\pm 2\sqrt{-\frac{3h_5\left(r^2-1\right)}{-2h_4\left(r^2-1\right)-3h_6\left(r^2+3\right)}}\times\left(\frac{5\tanh\left[\sqrt{-\frac{h_5\left(2h_4+3h_6\right)\left(r^2-1\right)}{5\left(2h_4(r^2-1)+3h_6(r^2+3)\right)}}\left(x-v\frac{t^{\alpha}}{\alpha}\right)\right]+3}{3\tanh\left[\sqrt{-\frac{h_5\left(2h_4+3h_6\right)\left(r^2-1\right)}{5\left(2h_4(r^2-1)+3h_6(r^2+3)\right)}}\left(x-v\frac{t^{\alpha}}{\alpha}\right)\right]+4r\operatorname{sech}\left[\sqrt{-\frac{h_5\left(2h_4+3h_6\right)\left(r^2-1\right)}{5\left(2h_4(r^2-1)+3h_6(r^2+3)\right)}}\left(x-v\frac{t^{\alpha}}{\alpha}\right)\right]+5}\right)$$

$$\times\ e^{i\left(-\kappa x+\omega\frac{t^{\alpha}}{\alpha}+\theta_0\right)},\tag{99}$$

or

$$q(x,\ t)=\pm\left(\frac{2\sqrt{\frac{3h_5\left(r^2-1\right)}{2h_4\left(r^2-1\right)+3h_6\left(r^2+3\right)}}}{r\operatorname{csch}\left[\sqrt{-\frac{h_5\left(2h_4+3h_6\right)\left(r^2-1\right)}{5\left(2h_4\left(r^2-1\right)+3h_6\left(r^2+3\right)\right)}}\left(x-v\frac{t^{\alpha}}{\alpha}\right)\right]+\coth\left[\sqrt{-\frac{h_5\left(2h_4+3h_6\right)\left(r^2-1\right)}{5\left(2h_4\left(r^2-1\right)+3h_6\left(r^2+3\right)\right)}}\left(x-v\frac{t^{\alpha}}{\alpha}\right)\right]}\right)e^{i\left(-\kappa x+\omega\frac{t^{\alpha}}{\alpha}+\theta_0\right)}.\tag{100}$$

Solutions (99) and (100) are straddled bright-dark soliton and straddled dark-singular soliton with $h_5\left(r^2-1\right)>0$, $2h_4\left(r^2-1\right)+3h_6\left(r^2+3\right)>0$, and $(2h_4+3h_6)<0$.

Case-5: $R(r)=r^2+1$

Result-1:

$$A_0=0,\ \ A_1=B_1\sqrt{r^2+1},\ \ B_1=\pm\sqrt{\frac{6h_5-20h_1}{8h_2+h_4-4h_6}},$$

$$h_3=\frac{\left(8h_2+h_4-4h_6\right)\left(h_1\left(12h_2-h_4+4h_6\right)-6h_2h_5\right)}{4\left(10h_1-3h_5\right)^2},\ \ k=\sqrt{\frac{4h_1\left(-2h_2+h_4+h_6\right)+h_5\left(4h_2-h_4-2h_6\right)}{8h_2+h_4-4h_6}},\tag{101}$$

$$q(x,\ t) = \pm\sqrt{\frac{6h_5-20h_1}{8h_2+h_4-4h_6}} \times \left(\frac{\sqrt{r^2+1}\operatorname{sech}\left[\sqrt{\frac{4h_1\left(-2h_2+h_4+h_6\right)+h_5\left(4h_2-h_4-2h_6\right)}{8h_2+h_4-4h_6}}\left(x-v\frac{t^\alpha}{\alpha}\right)\right]+1}{r\operatorname{sech}\left[\sqrt{\frac{4h_1\left(-2h_2+h_4+h_6\right)+h_5\left(4h_2-h_4-2h_6\right)}{8h_2+h_4-4h_6}}\left(x-v\frac{t^\alpha}{\alpha}\right)\right]+\tanh\left[\sqrt{\frac{4h_1\left(-2h_2+h_4+h_6\right)+h_5\left(4h_2-h_4-2h_6\right)}{8h_2+h_4-4h_6}}\left(x-v\frac{t^\alpha}{\alpha}\right)\right]} \right)$$

$$\times\ e^{i\left(-\kappa x+\omega\frac{t^\alpha}{\alpha}+\theta_0\right)}. \tag{102}$$

Solution (102) is a straddled bright-dark soliton with $6h_5-20h_1>0$, $8h_2+h_4-4h_6>0$, and $4h_1\left(-2h_2+h_4+h_6\right)+h_5\left(4h_2-h_4-2h_6\right)>0$.

Result-2:

$$A_0=B_1=0,\ A_1=\pm2\sqrt{\frac{3h_5\left(r^2+1\right)}{2h_4+3h_6}},\ h_1=-\frac{1}{5}\left(4h_5\right),$$

$$h_2=\frac{2h_4\left(r^2-2\right)+3h_6\left(2r^2-1\right)}{6\left(r^2+1\right)},\ h_3=-\frac{2h_4^2+11h_6h_4+12h_6^2}{60h_5},\ k=\sqrt{-\frac{h_5}{5}}, \tag{103}$$

$$q(x,\ t)=\pm\left(\frac{2\sqrt{\frac{3h_5\left(r^2+1\right)}{2h_4+3h_6}}}{r+\sinh\left[\sqrt{-\frac{h_5}{5}}\left(x-v\frac{t^\alpha}{\alpha}\right)\right]}\right)e^{i\left(-\kappa x+\omega\frac{t^\alpha}{\alpha}+\theta_0\right)}. \tag{104}$$

Solution (104) is a singular soliton with $h_5<0$, and $2h_4+3h_6<0$.

Result-3:

$$A_0=A_1=0,\ B_1=\pm2\sqrt{\frac{3h_5\left(r^2+1\right)}{2h_4\left(r^2+1\right)+3h_6\left(r^2-3\right)}},$$

$$h_1=-\frac{2h_5\left(4h_4\left(r^2+1\right)+3h_6\left(2r^2+7\right)\right)}{5\left(2h_4\left(r^2+1\right)+3h_6\left(r^2-3\right)\right)},\ h_2=\frac{h_6\left(2r^2+3\right)}{2\left(r^2+1\right)}+\frac{h_4}{3},$$

$$h_3=-\frac{\left(h_4+4h_6\right)\left(2h_4\left(r^2+1\right)+3h_6\left(r^2-3\right)\right)}{60h_5\left(r^2+1\right)},\ k=\sqrt{-\frac{h_5\left(2h_4+3h_6\right)\left(r^2+1\right)}{5\left(2h_4\left(r^2+1\right)+3h_6\left(r^2-3\right)\right)}}, \tag{105}$$

$$q(x,\ t)=\pm\left(\frac{2\sqrt{\dfrac{3h_5\left(r^2+1\right)}{2h_4\left(r^2+1\right)+3h_6\left(r^2-3\right)}}}{\begin{array}{c} r\operatorname{sech}\left[\sqrt{-\dfrac{h_5\left(2h_4+3h_6\right)\left(r^2+1\right)}{5\left(2h_4\left(r^2+1\right)+3h_6\left(r^2-3\right)\right)}}\left(x-v\dfrac{t^\alpha}{\alpha}\right)\right] \\ +\tanh\left[\sqrt{-\dfrac{h_5\left(2h_4+3h_6\right)\left(r^2+1\right)}{5\left(2h_4\left(r^2+1\right)+3h_6\left(r^2-3\right)\right)}}\left(x-v\dfrac{t^\alpha}{\alpha}\right)\right]\end{array}}\right)e^{i\left(-\kappa x+\omega\frac{t^\alpha}{\alpha}+\theta_0\right)}. \tag{106}$$

Solution (106) is a straddled singular-singular soliton with

$$h_5>0,\ 2h_4\left(r^2+1\right)+3h_6\left(r^2-3\right)>0,$$

and

$$\left(2h_4+3h_6\right)<0.$$

Figure 1 illustrates the behavior of bright soliton (44) under the influence of fractional temporal evolution. It comprises subfigures that depict soliton profiles through surface plots, with the fractional temporal evolution parameter α varying from 0.3 to 0.9. The parameters are fixed at $c_2=1$, $c_1=1$, $\tau_1=1$, $\tau_7=1$, $a=1$, $T_2=1$, $k=1$, $\omega=1$, $\tau_2=1$, $\tau_3=1$, $\tau_4=1$, $\tau_5=1$, $\tau_8=1$, and $\tau_9=1$. This visualization offers a comprehensive analysis of how fractional temporal evolution influences the solitons' structure and dynamics.

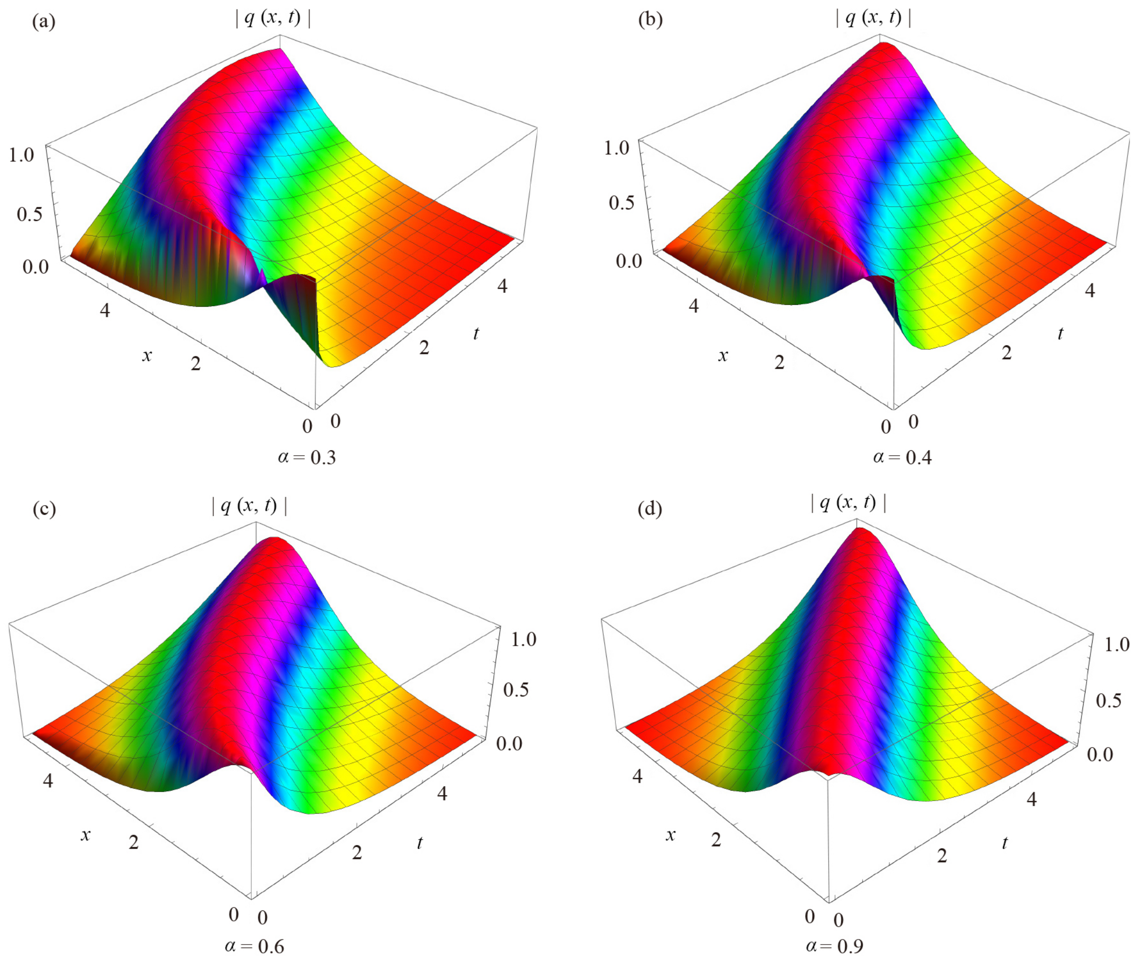

Figure 1. Profile of a bright soliton

5. Conclusions

This paper successfully retrieved optical soliton solutions to the concatenation model with fractional temporal evolution in absence of SPM. Two integration algorithms have made this possible. A full spectrum of solitons have emerged by the collective application of the integration architectures. This shows that it is possible to recover soliton solutions to the model when he SPM effect had been depleted. The results are thus indeed encouraging to move forward with he analysis. One can therefore have a look at the model with polarization mode dispersion as well as with dispersion-flattened fibers. The results of such research activities will be reported once available after they are all confirmed to be in conjunction with the previously reported ones [26–28].

Acknowledgment

This work for the fifth author (AB) was funded by the budget of Grambling State University for the Endowed Chair of Mathematics. AB also thankfully acknowledges the financial support received from Universal Wiser Publisher Private Limited.

Conflict of interest

The authors claim that there is no conflict of interest.

References

[1] Ankiewicz A, Akhmediev N. Higher-order integrable evolution equation and its soliton solutions. *Physics Letters A*. 2014; 378(4): 358-361.

[2] Ankiewicz A, Wang Y, Wabnitz S, Akhmediev N. Extended nonlinear Schrödinger equation with higher-order odd and even terms and its rogue wave solutions. *Physical Review E*. 2014; 89(1): 012907.

[3] Chowdury A, Kedziora D, Ankiewicz A, Akhmediev N. Breather-to-soliton conversions described by the quintic equation of the nonlinear Schrödinger hierarchy. *Physical Review E*. 2015; 91(3): 032928.

[4] Chowdury A, Kedziora D, Ankiewicz A, Akhmediev N. Soliton solutions of an integrable nonlinear Schrödinger equation with quintic terms. *Physical Review E*. 2014; 90(3): 032922.

[5] Chowdury A, Kedziora D, Ankiewicz A, Akhmediev N. Breather solutions of the integrable quintic nonlinear Schrödinger equation and their interactions. *Physical Review E*. 2015; 91(2): 022919.

[6] Ekici M, Sarmaşık CA. Certain analytical solutions of the concatenation model with a multiplicative white noise in optical fibers. *Nonlinear Dynamics*. 2024; 112(11): 9459-9476.

[7] Ekici M, Sarmaşık CA. Various dynamic behaviors for the concatenation model in birefringent fibers. *Optical and Quantum Electronics*. 2024; 56(8): 1342.

[8] Akram U, Tang Z, Althobaiti S, Althobaiti A. Dynamics of optical dromions in concatenation model. *Nonlinear Dynamics*. 2024; 112(16): 14321-14341.

[9] Khan MAU, Akram G, Sadaf M. Dynamics of novel exact soliton solutions of concatenation model using effective techniques. *Optical and Quantum Electronics*. 2024; 56(3): 385.

[10] Rabie WB, Khalil TA, Badra N, Hashemi M, Ahmed HM, Mirzazadeh M. Diverse new solitons and other exact solutions for concatenation model using modified extended mapping method. *Optical and Quantum Electronics*. 2023; 55(11): 952.

[11] Tang L. Optical solitons perturbation for the concatenation system with power law nonlinearity. *Journal of Optics*. 2024: 1-10. Available from: https://doi.org/10.1007/s12596-024-01757-6.

[12] Jawad AJM, Biswas A, Yildirim Y, Alshomrani AS. A pen-picture of optical solitons for the concatenation model with power-law of self-phase modulation by Sardar's sub-equation method and tanh-coth approach. *Optoelectronics and Advanced Materials-Rapid Communications*. 2024; 18(7-8): 346-352.

[13] Murad MAS, Faridi WA, Iqbal M, Arnous AH, Shah NA, Chung JD. Analysis of Kudryashov's equation with conformable derivative via the modified Sardar sub-equation algorithm. *Results in Physics*. 2024; 60: 107678.

[14] Murad MAS, Arnous AH, Faridi WA, Iqbal M, Nisar KS, Kumar S. Two distinct algorithms for conformable time-fractional nonlinear Schrödinger equations with Kudryashov's generalized non-local nonlinearity and arbitrary refractive index. *Optical and Quantum Electronics*. 2024; 56(8): 1320.

[15] Murad MAS, Iqbal M, Arnous AH, Biswas A, Yildirim Y, Alshomrani AS. Optical dromions with fractional temporal evolution by enhanced modified tanh expansion approach. *Journal of Optics*. 2024; 1-10. Available from: https://doi.org/10.1007/s12596-024-01979-8.

[16] Murad MAS, Arnous AH, Biswas A, Yildirim Y, Alshomrani AS. Suppressing internet bottleneck with Kudryashov's extended version of self-phase modulation and fractional temporal evolution. *Journal of Optics*. 2024; 1-14. Available from: https://doi.org/10.1007/s12596-024-01937-4.

[17] Arnous AH, Elsherbeny AM, Secer A, Ozisik M, Bayram M, Shah NA, et al. Optical solitons for the dispersive concatenation model with spatio-temporal dispersion having multiplicative white noise. *Results in Physics*. 2024; 56: 107299.

[18] Arnous AH, Hashemi MS, Nisar KS, Shakeel M, Ahmad J, Ahmad I, et al. Investigating solitary wave solutions with enhanced algebraic method for new extended Sakovich equations in fluid dynamics. *Results in Physics*. 2024; 57: 107369.

[19] Jawad AJM, Abu-AlShaeer MJ. Highly dispersive optical solitons with cubic law and cubic-quintic-septic law nonlinearities by two methods. *Al-Rafidain Journal of Engineering Sciences*. 2023; 1(1): 1-8.

[20] Jihad N, Almuhsan MAA. Evaluation of impairment mitigations for optical fiber communications using dispersion compensation techniques. *Al-Rafidain Journal of Engineering Sciences*. 2023; 1(1): 81-92.

[21] Ozkan YS, Yassar E. Three efficient schemes and highly dispersive optical solitons of perturbed fokas-lenells equation in stochastic form. *Ukrainian Journal of Physical Optics*. 2024; 25(5): S1017-S1038.

[22] Li N, Chen Q, Triki H, Liu F, Sun Y, Xu S, et al. Bright and dark solitons in a (2 + 1)-dimensional spin-1 Bose-Einstein condensates. *Ukrainian Journal of Physical Optics*. 2024; 25(5): S1060-S1074.

[23] Gao X, Shi J, Belic MR, Chen J, Li J, Zeng L, et al. W-shaped solitons under inhomogeneous self-defocussing Kerr nonlinearity. *Ukrainian Journal of Physical Optics*. 2024; 25(5): S1075-S1085.

[24] Dakova-Mollova A, Miteva P, Slavchev V, Kovachev K, Kasapeteva Z, Dakova D, et al. Propagation of broadband optical pulses in dispersionless media. *Ukrainian Journal of Physical Optics*. 2024; 25(5): S1102-S1110.

[25] Wazwaz AM. Pure-quartic stationary optical bullets for (3 + 1)-dimensional nonlinear Schrödinger's equation with fourth-order dispersive effects and parabolic law of nonlinearity. *Ukrainian Journal of Physical Optics*. 2024; 25(5): S1131-S1136.

[26] Sun Y, Hu Z, Triki H, Mirzazadeh M, Liu W, Biswas A, et al. Analytical study of three-soliton interactions with different phases in nonlinear optics. *Nonlinear Dynamics*. 2023; 111: 18391-18400.

[27] Liu FY, Triki H, Zhou Q. Optical nondegenerate solitons in a birefringent fiber with a 35 degree elliptical angle. *Optics Express*. 2024; 32(2): 2746-2765.

[28] Yang A, Xu S, Liu H, Li N, Sun Y. Predicting the soliton dynamics and system parameters in optical fiber couplers. *Nonlinear Dynamics*. 2024; 113: 1523-1537.