Contemporary Mathematics
https://ojs.wiserpub.com/index.php/CM/

Research Article

Parametrization Framework for the Deceleration Parameter in Scalar Field Dark Energy Model

Aroonkumar Beesham

Department of Mathematical Sciences, University of Zululand, Private Bag X1001, KwaDlangezwa, 3886, South Africa
E-mail: abeesham@yahoo.com

Received: 5 January 2025; **Revised:** 3 April 2025; **Accepted:** 16 April 2025

Abstract: The universe is currently experiencing acceleration. One way to explain this is via matter with negative energy, called dark energy. We propose a Friedmann-Lemaitre-Robertson-Walker cosmological model with a scalar field that represents dark energy. The deceleration parameter measures this acceleration. A novelty of this work is that a new parametrization of the deceleration parameter is introduced of the form $q = -1 + \eta/(1 + \mu a^{\eta})$ where η and μ are model parameters, and a is the scale factor or radius of the universe. We find the values of the parameters of the model by means of best-fit to recent observations, including cosmic chronometers, Pantheon+ and Baryon Acoustic Observations, using the Markov Chain Monte Carlo method. We address the expansion history of the universe and provide a viable description of the transition from deceleration to acceleration. A comparison is then made with the Lambda Cold Dark Matter (ΛCDM) model of general relativity. We examine the evolution of the main key cosmological parameters, such as the deceleration parameter, jerk parameter, snap parameter, density parameter, and equation-of-state parameter. By interpreting them, we obtain insights into what has been dubbed "dynamical dark energy" under the assumptions made above. Our method provides a framework that is independent of the model to explore dark energy. This leads to a deeper and more subtle understanding of the mechanisms driving late-time cosmic acceleration.

Keywords: cosmological model, dark energy, current acceleration of universe, variable deceleration parameter, observational constraints

MSC: 83F05, 85A40

1. Introduction

The accelerated expansion of the universe, first observed in the late 1990s through distant supernovae surveys, has revolutionized our understanding of the cosmos [1, 2]. For an explanation of this, there are two schools of thought by researchers in the field. The first one is the introduction of a mysterious form of exotic matter called Dark Energy (DE) [3–9], which is characterized by significant pressure that is negative, and causes the current acceleration. The second approach explores modified gravity theories by changing the gravitational action of general relativity [10]. Some modified gravity theories which have been used to interpret the acceleration of the expanding universe include $f(R)$ gravity [11–16], $f(T)$ gravity [17–21], and $f(Q)$ gravity, where functions of the Ricci scalar R, torsion T, and the non-metricity scalar Q are involved in modifying the gravitational action [22–27]. These models can provide alternative

DOI: https://doi.org/10.37256/cm.6320256354

explanations for the accelerated expansion of the universe due to changes in the expansion dynamics without invoking the need for exotic matter or dark energy. Besides these, there have been many dark energy models, such as quintessence [28], involving a dynamic scalar field; k-essence [29], which generalizes quintessence by considering more complex kinetic terms; phantom energy [30, 31], characterized by an equation of state parameter less than -1; and scalar-tensor theories, which couple a scalar field with gravity. Other relevant more recent references are [32–35]. Each of these models introduces different theoretical frameworks and predictions, which are very helpful in gaining varying insights into the evolution of the universe, and its likely solutions to cosmological puzzles such as the coincidence problem and the nature of dark energy.

The ΛCDM model is extremely successful in describing a large variety of observational data, but has several theoretical challenges [36–38]. One of the significant challenges is problem of fine tuning, arising from the cosmological constant (Λ) [39]. The other major problem is cosmic coincidence, the peculiar fact that the DE and matter densities are almost aligned presently [40–43]. Additionally, new observations indicate that the ΛCDM model does not accurately describe the latest low-redshift cosmological data. Although the ΛCDM is consistent with many observations, some data suggest that the DE density evolves over time, whereas it is constant in the standard model. These models are believed to fit observations better. It is important to carefully consider and test all proposed models with the help of cosmological observations. This is because the mechanisms that drive late-time acceleration are complex and observational data is becoming increasingly precise. For such analyses, proper parametrizations need to be used that permit a description that is independent of the model for explaining the present state of the universe [44–47].

Building on this framework, this study focuses on a particular parameterization of the deceleration parameter q, enabling a systematic analysis of the transition between the universe's deceleration and acceleration phases. By choosing a specific functional form for q, we seek to offer a flexible, yet physically grounded description of cosmic expansion that can be directly constrained by observational data. To achieve this, we employ a combination of diverse cosmological datasets, such as Cosmic Chronometers (CC), and Supernova Type Ia (SNeIa) from the Pantheon sample, as well as Baryon Acoustic Oscillations (BAO) datasets. The CC, SNIA, and BAO datasets help constrain key parameters like the Hubble constant, matter density, and dark energy density. These datasets align well with the ΛCDM model, with the Hubble constant typically ranging between 67-74 km/s/Mpc, though discrepancies exist between local and early universe measurements. Alternative models, such as modified gravity and quintessence, offer different approaches to explaining cosmic acceleration. Modified gravity models modify Einstein's equations and affect the cosmic expansion, which CC, SNIA, and BAO data can use to reveal discrepancies in the expansion rate and structure growth. Quintessence models, which allow for a dynamic dark energy equation of state, can produce distinct predictions for the universe's expansion compared to the constant dark energy in ΛCDM. These models also influence CC, SNIA, and BAO data, providing a potential distinction from the ΛCDM model. Earlier dark energy models proposed an earlier phase of dark energy dominance to explain discrepancies in H_0, influencing the CC, SNIA, and BAO data, though they have yet to provide a complete fit. Overall, while the ΛCDM model fits the datasets well, alternative models provide promising avenues, especially in resolving the Hubble tension. However, modified gravity, quintessence, and early dark energy models still face challenges in reconciling all observational constraints. The combination of CC, SNIA, and BAO data remains essential in distinguishing between models and advancing our understanding of dark energy and gravity.

In this paper, we discuss a Friedmann-Lemaitre-Robertson-Walker model with a scalar field dark energy and investigate its compatibility with recent observational datasets, including Cosmic Chronometers (CC), SNIa (Pantheon), and BAO. By considering a Variable Deceleration Parameter (VDP), we address the expansion history of the universe, providing a viable description of the transition from deceleration to acceleration. Using the Markov Chain Monte Carlo (MCMC) method, we constrain the model parameters and examine the cosmological quantities. In section 2, we provide the basics of the model, and relevant quantities. Then section 3 introduces the kinematic parameters, and section 4 entails the determination of the model parameters using the Markov Chain Monte CArlo (MCMC) technique. In section 5, we discuss our results, and finally, in section 6 is the conclusion.

2. Cosmological model

For the $k = 0$ Friedmann-Lemaître-Robertson-Walker (FLRW) model, the metric is:

$$ds^2 = dt^2 - a^2(t)\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \tag{1}$$

where the symbols have their usual meanings. We take the model to be populated by two fluids that are perfect. One is matter m, with negligible pressure (dark and baryonic matter), and the other is a scalar field ϕ. The latter is believed to cause the present acceleration, and may be regarded as Dark Energy (DE). In this scenario, Einstein's field equations and the Klein-Gordon equation for the scalar field can be written as follows (assuming $8\pi G = c = 1$):

$$3H^2 = \rho_m + \rho_\phi = \rho_m + \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{2}$$

$$2\dot{H} + 3H^2 = -p_\phi = -\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right), \tag{3}$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0. \tag{4}$$

Here, $H = \frac{\dot{a}}{a}$ is the Hubble parameter, representing expansion or contraction. The parameters p_ϕ and ρ_ϕ are the pressure and energy density of the scalar field, respectively, and ρ_m is the energy density of the matter. The solution of the above equations with relevant initial conditions provides information about the evolution of the model. The evolution of ϕ with m is given by:

$$\rho_m = \rho_{m0}a^{-3} = \rho_{m0}(1+z)^3, \tag{5}$$

where the subscript 0 denotes the present time and since the redshift is $z = -1 + 1/a(t)$.

The energy density ρ_ϕ of the scalar field can be written as the familiar equation: $\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)$, whereas p_ϕ is: $p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)$. As usual, the potential function is $V(\phi)$. From Equations (2), (3) and (4), we can get the evolution of the the Hubble parameter H, and $V(\phi)$ in terms of H:

$$2\dot{H} = -\frac{\rho_{m0}}{a^3} - \dot{\phi}^2, \tag{6}$$

and

$$V(\phi) = +3H^2 + \dot{H} - \frac{\rho_{m0}}{2a^3}. \tag{7}$$

From Equation (6) we: $a\frac{d}{da}(H^2)+\frac{\rho_{m0}}{a^3}=-\dot{\phi}^2$. In addition, we can express $\dot{\phi}$ as $\dot{\phi}=aH\left(\frac{d\phi}{da}\right)$. The evolution of ϕ in terms of z is:

$$\frac{d\phi}{dz}=\left[\left(2E\frac{dE}{dz}-3\Omega_{m0}(1+z)^2\right)\frac{1}{E^2(1+z)}\right]^{1/2}, \tag{8}$$

where $E(z)=\frac{H(z)}{H_0}$ is the Hubble parameter without dimension, and $\Omega_{m0}=\frac{\rho_{m0}}{3H_0^2}$ is the current density of matter. Similarly, $V(z)$ is given by:

$$\frac{V(z)}{3H_0^2}=-(1+z)^3E\frac{dE}{dz}+E^2-\frac{1}{2}\Omega_{m0}(1+z)^3. \tag{9}$$

The definition of the deceleration parameter is:

$$q(z)=-\frac{\dot{H}}{H^2}-1=\frac{dE}{dz}\frac{(1+z)}{E}-1. \tag{10}$$

Deceleration is indicated by $q>0$, acceleration by $q<0$ and a constant expansion by $q=0$. For $q=-1$, we get de Sitter expansion (exponential expansion), and finally, $q<-1$ denotes expansion that is super-exponential.

The energy densitites Ω_m of matter and Ω_ϕ of the scalar field in terms of the redshift z are given by:

$$\Omega_m(z)=\frac{\rho_m}{3H^2}=\frac{\Omega_{m0}(1+z)^3}{E^2}, \tag{11}$$

$$\Omega_\phi(z)=1-\Omega_m(z)=1-\frac{\Omega_{m0}(1+z)^3}{E^2}. \tag{12}$$

Another parameter useful in the study of dark energy is the equation of state parameter (eos) denoted by $\omega_\phi(z)$, and defined by:

$$\omega_\phi(z)=\frac{p_\phi}{\rho_\phi}=\frac{-1-\frac{2\dot{H}}{3H^2}}{\Omega_\phi}. \tag{13}$$

Thus, we get:

$$\omega_\phi(z)=\frac{\frac{2}{3}(1+z)E\frac{dE}{dz}-E^2}{E^2-\Omega_{m0}(1+z)^3}. \tag{14}$$

3. Kinematical parameters

We now choose q as:

$$q = -1 + \frac{\eta}{1 + \mu a^{\eta}}, \tag{15}$$

for which

$$H = \kappa(\mu + a^{-\eta}). \tag{16}$$

The motivation for this form of q comes from a recent study by Pawde et al. [48] who considered $q = -1 + \eta/(1 + a^{\eta})$. As no observational constraints were provided via an MCMC analysis in that paper, we carried out this study [49], and found that the form for q did not fit the observations. Hence, in this investigation, we consider form (15). Another obvious motivation for the condition (15) is as follows: For large scale factor a, we have $q \to -1$, as is the case for the ΛCDM model. Hence the assumption of this condition ensures that the model will approach the ΛCDM model in the future. Additional motivation for our choice of deceleration parameter q as in Eq. (15) is as follows.

• The study of cosmological models within the climate of late time acceleration is expressed in terms of kinematic parameters such as q.

• Expressing q as $q = -(\ddot{a}/a)/(H^2)$, we see that it is essentially the acceleration divided by the expansion. In the past, the universe was decelerating, so $q > 0$, or $q = const > 0$. However, now we are experiencing acceleration, so $q < 0$. Hence, for continuity, we need a dynamic q as a function of time (or redshift), that changes sign from positive to negative.

• Hence, many forms of q have been adopted to try to explain this transition [50, 51] (and references therein).

• Since $a = 1/(1+z)$, for our choice of q, we can make some qualitative remarks about how q varies with redshift.

• Both parameters η and μ relate to q_0, i.e., $q_0 = -1 + \eta/(\mu + 1)$. Presently, the state of the model depends upon the value of these parameters. $\mu = \eta - 1 \implies q_0 = 0$, and the universe is undergoing constant expansion. $\mu > \eta - 1 \implies q_0 > 0$, and we have decelerated expansion. Finally, if $\mu < \eta - 1$, then $q_0 < 0$, and we have accelerated expansion.

• In the distant past, $z >> 1$, and $q(z) \to -1 + \eta$. For $\eta > 1$, we have $q > 0 \implies$ deceleration, i.e., the radiation and matter dominated eras.

• In the far future, from the form of $q(a)$ as in Eq. (15), and the discussion following that equation, we have seen that $q \to -1$, the asymptotic form for q for the ΛCDM model. Hence this model will asymptotically approach the ΛCDM model in the future.

We can write in terms of redshift:

$$H = \frac{H_0}{1+\mu}[\mu + (1+z)^{\eta}]. \tag{17}$$

Consider the special form (15) of the deceleration parameter; Equations (10), (11), (12) and (14), and can be written as:

$$q = -1 + \frac{\eta(1+z)^{\eta}}{\mu + (1+z)^{\eta}}, \tag{18}$$

$$\Omega_m(z) = \frac{\Omega_{m0}(1+\mu)^2(1+z)^3}{[\mu+(1+z)^\eta]^2}, \tag{19}$$

$$\Omega_\phi(z) = 1 - \frac{\Omega_{m0}(1+\mu)^2(1+z)^3}{[\mu+(1+z)^\eta]^2}, \tag{20}$$

$$\omega_\phi(z) = \frac{(2\eta/3)(1+z)^\eta[\mu+(1+z)^\eta] - [\mu+(1+z)^\eta]^2}{[\mu+(1+z)^\eta]^2 - \Omega_{m0}(1+\mu)^2(1+z)^3}. \tag{21}$$

The jerk and snap parameters of cosmology give additional higher derivatives of the scale factor for the universe as opposed to the traditional two parameters (Hubble and deceleration). These higher order derivatives describe the cosmological expansion in further and finer detail.

$$j = \frac{1}{a}\frac{d^3a}{d\tau^3}\left(\frac{1}{a}\frac{da}{d\tau}\right)^{-3} = q(2q+1) + (1+z)\frac{dq}{dz}, \tag{22}$$

$$j(z) = \left(-1 + \frac{\eta(1+z)^\eta}{\mu+(1+z)^\eta}\right)\left(1 + \frac{2\eta(1+z)^\eta}{\mu+(1+z)^\eta}\right) + \frac{\eta^2\mu(1+z)^\eta}{[\mu+(1+z)^\eta]^2}, \tag{23}$$

$$s = \frac{j-1}{3\left(q - \frac{1}{2}\right)},$$

$$s = \frac{\left(-1 + \frac{\eta(1+z)^\eta}{\mu+(1+z)^\eta}\right)\left(1 + \frac{2\eta(1+z)^\eta}{\mu+(1+z)^\eta}\right) + \frac{\eta^2\mu(1+z)^\eta}{[\mu+(1+z)^\eta]^2} - 1}{3\left(-\frac{3}{2} + \frac{\eta(1+z)^\eta}{\mu+(1+z)^\eta}\right)}.$$

4. Determination of model parameters using MCMC technique

In cosmology, Bayesian methods are commonly used to estimate parameters by computing the posterior distribution of the parameters θ based on observed data D:

$$P(\theta \mid D) = \frac{L(D \mid \theta)P(\theta)}{P(D)}, \tag{24}$$

with $P(\theta)$ being the prior distribution, $P(D)$ the marginal likelihood and $L(D \mid \theta)$ the likelihood function. Bayesian parameter estimation involves exploring the parameter space θ, often with algorithms like Metropolis-Hastings [52], which helps guide a random walker through the space, preferring regions with higher likelihoods. The mean and uncertainty of each parameter are usually found by analyzing where the walker spends most of its time and how far it deviates within the parameter space. In situations where we have a nearly Gaussian posterior distribution, information criteria offers a simpler approach to model selection [53]. In this work, we investigate the form (18) for the deceleration parameter. A Markov Chain Monte Carlo (MCMC) analysis with the emcee package [54] is used to properly cover the parameter space in order to get reliable estimates. The GetDist package [55] is employed to visualize and plot the posterior distributions. This enables the proper determination of contraints for the parameters. The analysis was performed with 30 walkers and 30,000 iteration, allowing efficient sampling of the parameter space. The prior distributions were chosen based on observational constraints and set as: $H_0 \in [50,\ 100]$ km/s/Mpc, η, $\mu \in [0,\ 4]$, $M \in [-20,\ -18]$, $r_d \in [100,\ 200]$ Mpc. These priors ensure a broad but physically meaningful parameter exploration while preventing over-restriction of the search space. The GetDist package [56] is employed to visualize and plot the posterior distributions and the 1σ and 2σ confidence intervals for each parameter are presented in Table 1. The likelihood function was constructed by summing the χ^2 contributions from each dataset, ensuring a robust and unbiased estimation of cosmological parameters. To ensure consistency and reliability, the datasets were combined without additional weighting, treating each dataset equally in the likelihood function. This approach prevents any individual dataset from disproportionately influencing the constraints, thereby providing a well-balanced parameter estimation. The convergence of the MCMC chains was assessed by analyzing the posterior distributions and verifying that the results remained stable over multiple runs. Convergence was checked using Gelman-Rubin statistics.

Table 1. The calculated best-fit values for model using CC+SNIa+BAO datasets

MCMC results			
Model	Parameters	Prior	Joint
	H_0 (km s^{-1} Mpc^{-1})	[50, 100]	67.0 ± 1.6
	μ	[0, 4]	$1.89^{+0.22}_{-0.24}$
VDP model	η	[0, 4]	1.748 ± 0.063
	$\mathscr{M}$	[-20, -18]	-19.430 ± 0.052
	r_d (Mpc)	[100, 200]	146.7 ± 3.5

The selection of observational datasets in this study was motivated by their complementary roles in constraining different aspects of cosmic expansion (For the latest datasets that we used, we refer to the recent preprint by Du at al. [57] who gives a very comprehensive review of all the datasets, including Dark Energy Spectroscopic Instrument (DESI)). CC provides a direct, model-independent measurement of $H(z)$ based on galaxy age estimates, making them essential for tracking the universe's expansion history. SNIa - Pantheon+ data, on the other hand, offer precise luminosity distance measurements, playing a crucial role in constraining the equation of state of dark energy and the deceleration parameter $q(z)$. Finally, BAO serves as a standard ruler, allowing precise constraints on matter density Ω_m and the Hubble constant H_0. The combination of these datasets ensures that the parameter constraints obtained in this study are independent, robust, and cross-validated. By utilizing these three independent probes, we mitigate potential biases that might arise from relying on a single dataset. This approach enables a more comprehensive analysis of cosmic acceleration and dark energy evolution.

4.1 *CC*

CC are strong probes of cosmic expansion, and offer a model-independent way to estimate the Hubble parameter $H(z)$. We can calculate $H(z)$ at a given redshift z from the metallicity and age of passive nearby galaxies. This approximation comes from the formula: $H(z) \approx -(\Delta z/\Delta t)/(1+z)$. CC data is obtained from a number of sources [56, 58–61], over the redshift range $0.07 \lesssim z \lesssim 1.97$ [53]. This data provides the constraints on $H(z)$. To determine how well theoretical models fit in with CC data at any given redshift, we calculate the chi-squared statistic χ^2:

$$\chi^2_{\mathrm{CC}}(\theta) = \Delta H^T(z) C^{-1} \Delta H(z), \tag{25}$$

where $\Delta H(z)$ is the difference between the predicted expansion rate of the model, $H_{\mathrm{M}}(z)$, and that of cosmic chronometer data, $H_{\mathrm{D}}(z)$, at a given redshift z, and C the covariance matrix.

4.2 *SNIa*

The apparent magnitude of a star is given by

$$m(z) = 5\log_{10}\left(\frac{d_L(z)}{\mathrm{Mpc}}\right) + \mathscr{M} + 25, \tag{26}$$

where $\mathscr{M}$ is the absolute magnitude of the star and $d_L(z)$ its luminosity distance given by:

$$d_L(z) = (1+z)\int_0^z \frac{dz'}{H(z')H_0}. \tag{27}$$

For the SNe Ia data,

$$\chi^2_{\mathrm{S}} = \Delta D^T C_{\mathrm{t}}^{-1} \Delta D, \tag{28}$$

where the total covariance matrix $C_{\mathrm{t}} = C_{\mathrm{sys}} + C_{\mathrm{stat}}$ is the sum of the systematic and statistical covariance matrices, respectively. The deviation of the distance modulus of a star is given by:

$$\Delta D = \mu(z_i) - \mu_{\mathrm{model}}(z_i, \theta), \tag{29}$$

where $\mu(z_i) = m(z_i) - \mathscr{M}$ is the observed distance modulus. We project $\mathscr{M}$ up to a normalization constant in the likelihood function $L \propto e^{-\chi^2/2}$ [62].

4.3 *BAO*

The sound horizon, r_d, defined at the epoch of baryon decoupling ($z_d \approx 1{,}060$), is given by:

$$r_d = \frac{1}{H_0}\int_{z_d}^{\infty} \frac{c_s(z)}{E(z)} dz, \tag{30}$$

where $c_s(z)$ is the sound speed, a function of the baryon-to-photon density ratio, and $E(z) \equiv H(z)/H_0$ is the dimensionless Hubble parameter. Now, in cosmology, measurements of Baryon Acoustic Oscillations (BAO) depend on r_d. In this work, we take r_d as a free parameter [63–66], rather than as a prior based on CMB Planck data.

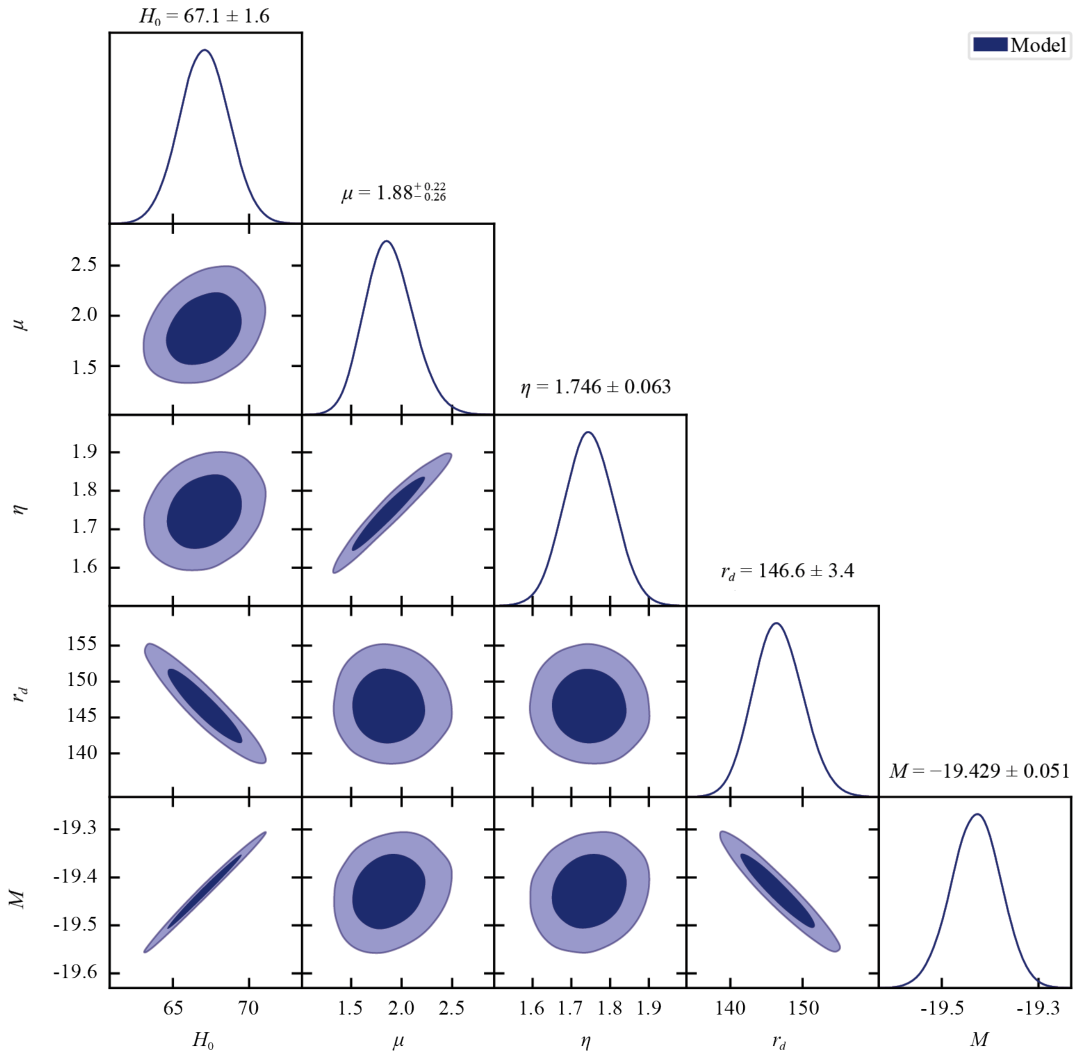

Figure 1. Confidence contours at 1σ and 2σ levels for best fit of each parameter of our model from MCMC analysis from combined CC+SNIa+BAO dataset

Data from the completed Sloan Digital Sky Survey (SDSS-IV) [67] and the BAO catalogs from the first-year observations of the Dark Energy Spectroscopic Instrument (DESI Y1) [68] are utilised in the analysis. The distance measures that are used are the Hubble distance $D_H(z) = c/H(z)$, the comoving angular diameter distance $D_M(z)/r_d$:

$$D_M(z) = \frac{c}{H_0}\int_0^z \frac{dx'}{E(x')}, \tag{31}$$

and the volume-averaged distance $D_V(z)/r_d$, which encodes the position of the BAO peak:

$$D_V(z) = \left[zD_M^2(z)D_H(z)\right]^{1/3}. \tag{32}$$

The χ^2 statistic for the distance measurements scaled by the sound horizon, D_X/r_d, is:

$$\chi^2_{D_X/r_d} = \Delta D_X^T C_{D_X}^{-1} \Delta D_X, \tag{33}$$

where $\Delta D_X = D_{X,\,\text{Model}}/r_d - D_{X,\,\text{Data}}/r_d$ for $X = H, M, V$, and $C_{D_X}^{-1}$ is the inverse covariance matrix for each X. In Figure 1, we have plotted the confidence contours for each parameter for our choice of the varying deceleration parameter (18). Table 1 contains the best-fit values for the parameters of our model.

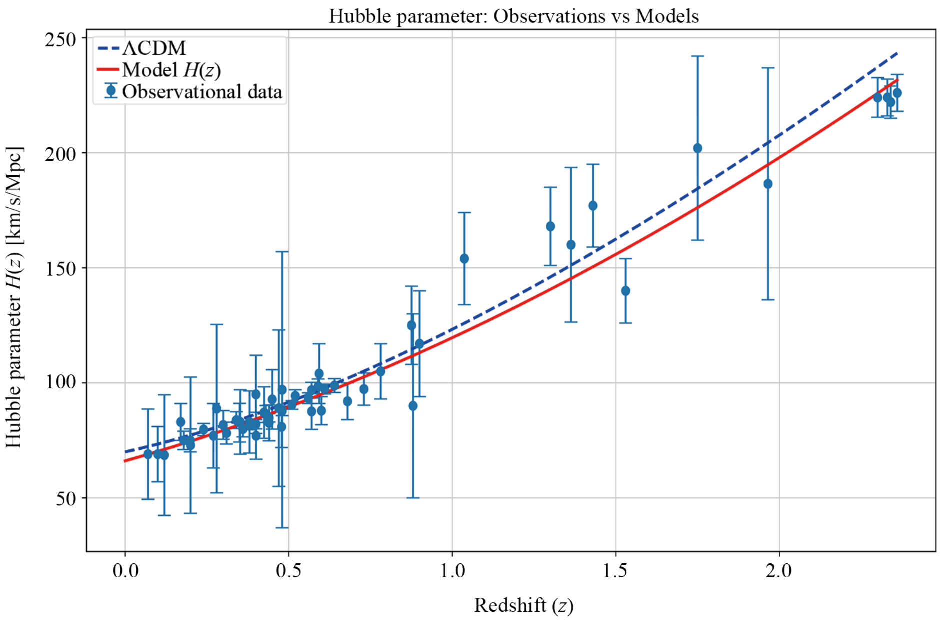

Figure 2. Comparison of the Hubble parameter curve with that of the ΛCDM model, showing the closeness of both curves, supporting our model strongly

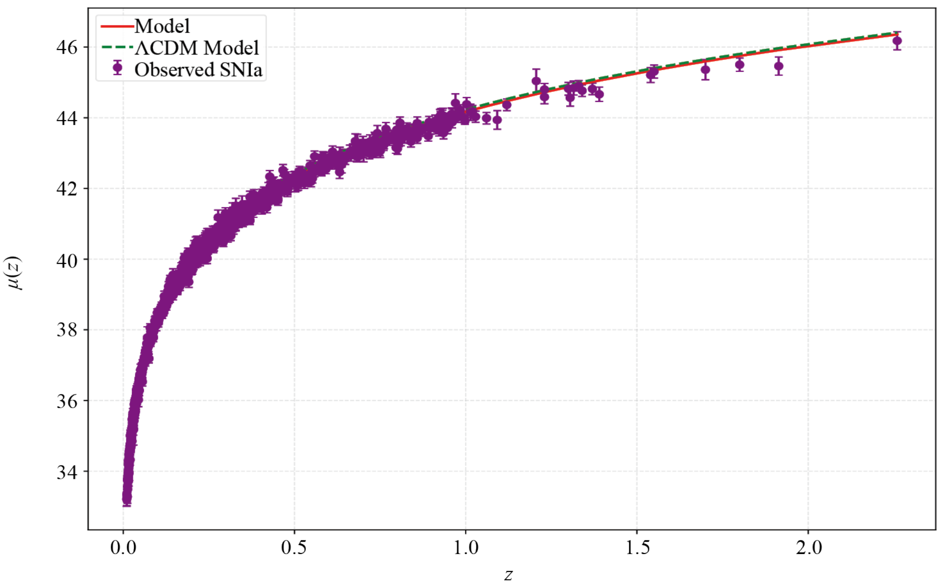

Figure 3. Comparison of the distance modulus curve with that of the ΛCDM model, showing that they almost coincide, supporting our model strongly

Additionally, Figure 2 compares the Hubble parameter curve predicted by the model with that of the ΛCDM model, demonstrating consistency with the observational data, and Figure 3 gives a comparison of the distance modulus curve with the ΛCDM model.

5. Results and discussion

We have obtained the best fit values of the parameters H_0, η and μ in section 4. Now, we discuss the cosmological evolution of the model variables as constrained by the observations.

5.1 *Deceleration parameter*

Based on the best-fit values for H_0, η and μ from Table 1, we continue our evaluation in this section of the various cosmographic and physical parameters of our model. The deceleration parameter q was reconstructed, and this is illustrated in Figure 4, together with the 1σ error bounds. The figure illustrates that at a best-fit transition redshift of $z_{\mathrm{tr}} = 0.748^{+0.4}_{-0.4}$, $q(z)$ changes sign from positive to negative. This means a transition from deceleration (required for structure formation, etc) to acceleration, and this signifies the physical significance of the deceleration parameter. The current value of $q(z)$ is $q_0 = -0.57^{+0.46}_{-0.46}$, and z_{tr} and q_0 are within observational constraints. Apart from these key values of q, the figure also illustrates the very close correspondence between the two models. A distinct advantage of our model is that late-time acceleration is achieved without the need for postulating exotic matter as in the case with the standard ΛCDM model [69, 70].

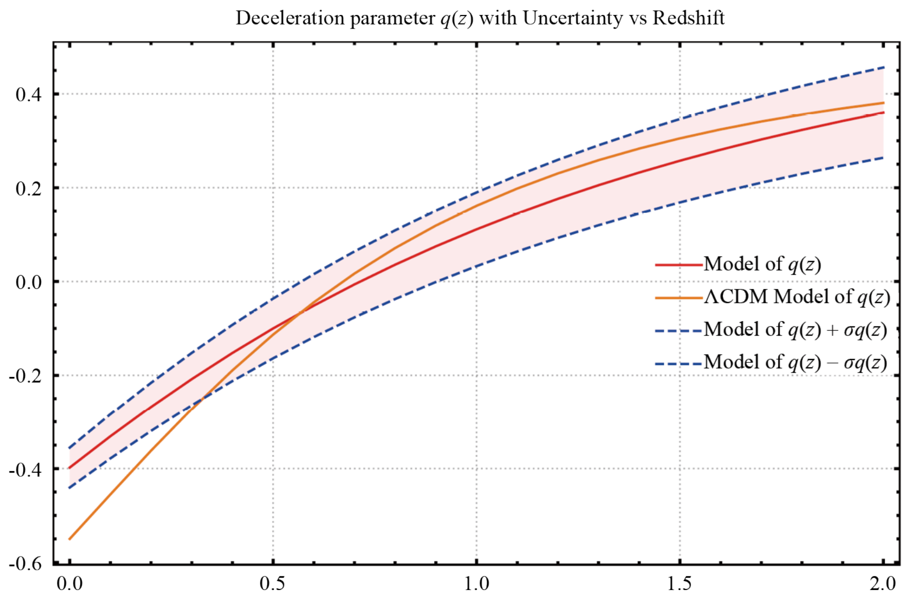

Figure 4. $q(z)$ vs z for our model and ΛCDM model with 1σ error bounds. The curves are close to each other, supporting our model

Earlier in section 3, we had pointed out that our motivation for our particular choice (15) for the deceleration parameter stems from that of Pawde et al. [48] who considered $q = -1 + \eta/(1 + a^{\eta})$. There were no observational constraints placed on the parameters from an MCMC analysis. Hence, we did this study [49], and found that that form of q did not match with that of the ΛCDM model. Hence we came up with an improvement of q of the form (15), in which we added an additional parameter. The results that we find with our parametrization fits observations well as pointed out previously. Another parametrization of q as a function of a that is closely related to ours is that of Banerjee and Das [71], who chose

$$q = -1 - \frac{pa^p}{1+a^p} \tag{34}$$

They found that for $-2 < p < -1$, the model exhibits a transition from deceleration to acceleration. Equation (34) can be integrated to yield $a = (e^{-Apt} - 1)^{-1/p}$, where A is a constant of integration. Now this is identical to the solution found in [48] if we choose $p = -\eta$, and put $A = 1$. However, Banerjee and Das [71] also did not carry out an MCMC analysis to determine the parameters of their model. Again, we point out that their solution does not match the observations [49]. Regarding the transition redshift z_{tr}, in refs [48, 49], they were found to depend on the parameter of the model since no MCMC analysis was carried. This is in contrast to our model where we subjected our model to an MCMC analysis, and found that $z_{\text{tr}} = 0.748^{+0.4}_{-0.4}$, which is within observational constraints [69, 70].

5.2 *Jerk parameter*

The jerk parameter [72] is significant in cosmology, as it is essentially the third order term in the Taylor series expansion of $a(t)$. It is a unique characterization of cosmic dynamics. It contains useful information related to the evolution of the cosmos and distinguishes between various dark energy models. It acts as an indispensable bridge between dark energy and normal cosmological models. The different values of j establish a relationship between several theories of dark energy and the ΛCDM model. A grasp of the jerk parameter is fundamental to study the dynamics in cosmic expansion and the transitions between different eras of evolution. The jerk parameter for our model is sketched in Figure 5, and it may be seen that j is not constant, in contrast to the ΛCDM model. It is a monotonically decreasing function.

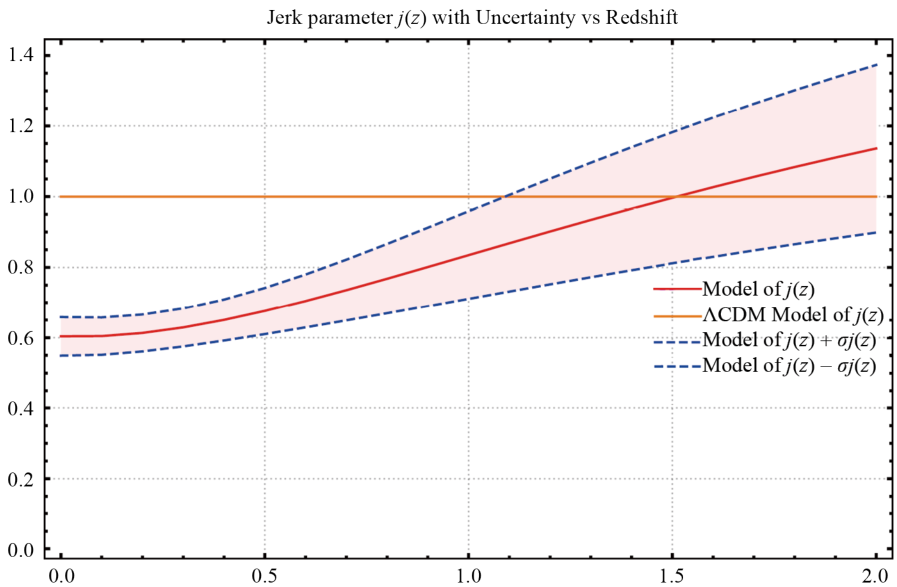

Figure 5. Jerk parameter $j(z)$ vs redshift z with 1σ error bounds, showing the dynamical feature of our model

5.3 *Snap parameter*

The snap parameter [72], which is represented by s, is a cosmological parameter that involves the fourth time derivative of the scale factor. Hence, it provides insight into how the curvature and expansion dynamics of the universe are set. It plays an important role in the Taylor series expansion that describes the growth of the Universe. $j = 1$ for the

ΛCDM model, and so we get $s = -(2+3q)$. The snap parameter for our model is plotted in Figure 6, from which it can be seen that it closely follows that of the ΛCDM model.

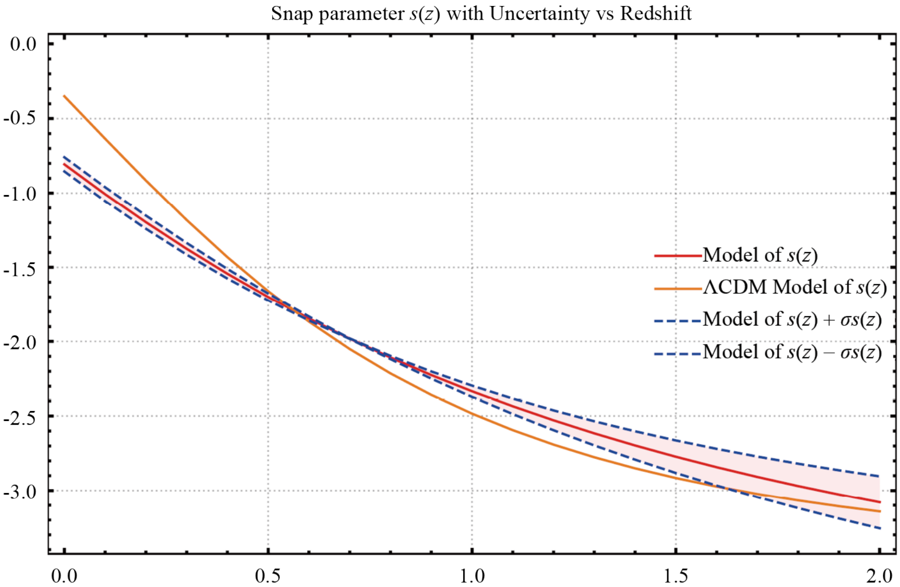

Figure 6. Snap parameter $s(z)$ versus redshift z, showing the closeness of our model to the ΛCDM model

5.4 *Matter and scalar field energy densities*

Figure 7 and Figure 8 are the plots of the density parameters for m and ϕ, respectively. During early times, m dominates, while ϕ is negligible. With time, the influence of m decreases due to expansion, whilst that of ϕ increases. However, in the course of time, the scalar field density parameter becomes dominant and overshoots that of matter. This leads to acceleration of the expansion of the universe, which is the critical point in cosmic evolution. Furthermore, the density parameters at present time have been determined to be $\Omega_{m0} = 0.33$, $\Omega_{\phi 0} = 0.7$ for the $\mathscr{CC}+\mathscr{SNI}\dashv+\mathscr{BAO}$ datasets. In the ΛCDM model, $\Omega_{m0} = 0.315 \pm 0.007$ [73–75], and the values for our model are in agreement with those of the ΛCDM model.

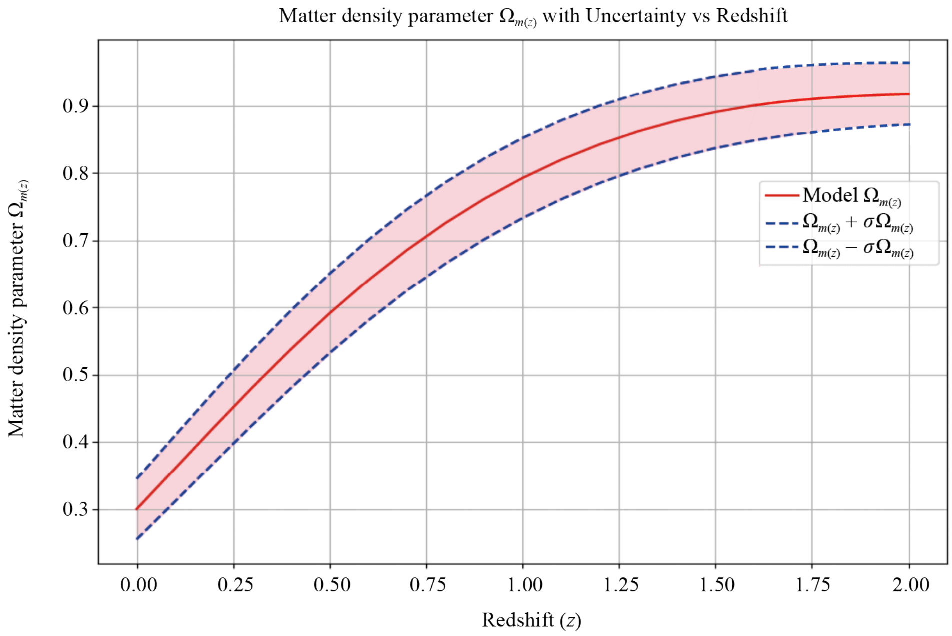

Figure 7. Matter energy density Ω_m versus redshift z, showing evolution to the present value of 0.33, very close to that of ΛCDM model

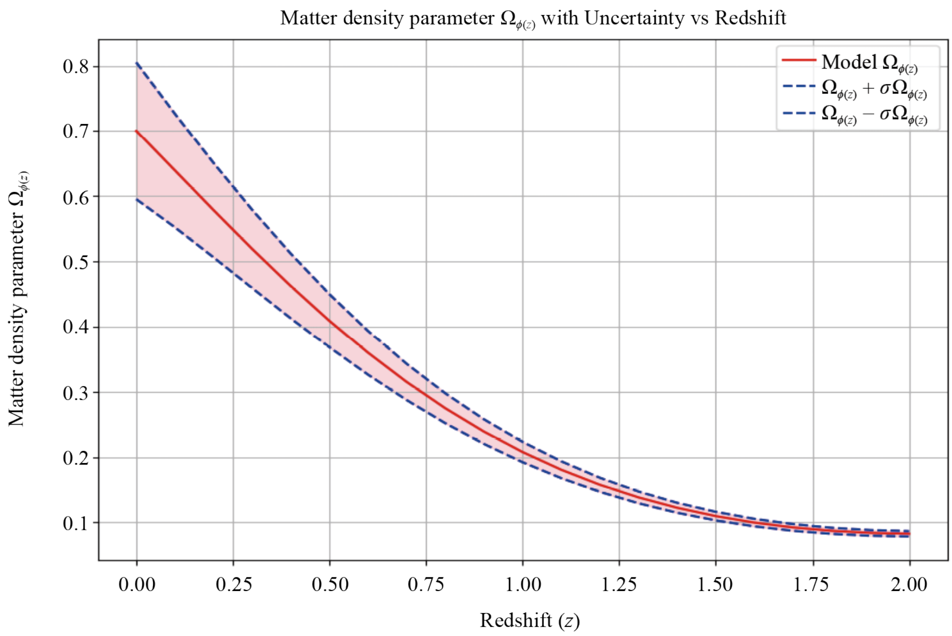

Figure 8. Matter energy density Ω_ϕ versus redshift z, showing evolution to 0.7, as Ω_ϕ denotes dark energy (almost same value as that of ΛCDM)

5.5 *Equation of state*

Figure 9 shows ω_ϕ versus z. It starts from the quintessence region, moves into the phantom phase for $1.45 > z > 0.35$, and then reverts back into the quintessence region ($\omega_\phi \geq -1$) at present, and in future. Furthermore, $\omega_{\phi 0} = -0.99$, in excellent agreement with observations [76, 77].

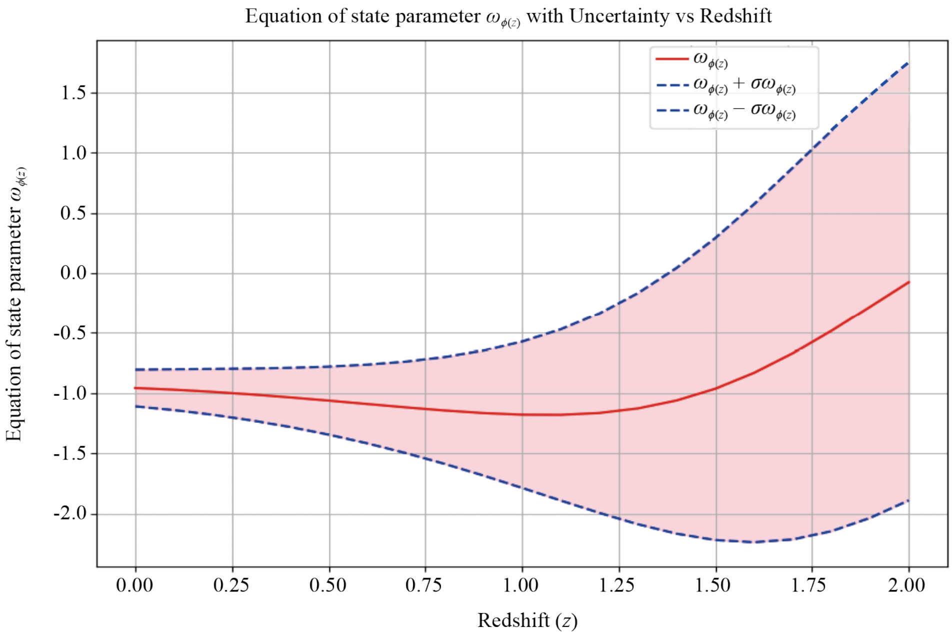

Figure 9. Scalar field energy density ω_ϕ versus redshift z showing late time evolution to $\omega_{\phi 0} = -0.99$ (-1 for ΛCDM)

5.6 *Summary of results*

We summarise our results in the Table 2 below showing also the latest observational findings:

Table 2. Our results

Parameter	Our model	Observations
H_0	67.0 ± 1.6	67.4 ± 0.5
q_0	-0.57 ± 0.46	-0.581 ± 0.03
z_{tr}	0.748 ± 0.4	0.724 ± 0.047
Ω_{m0}	0.33	0315 ± 0.007
$\Omega_{\phi 0}$	0.67	0.685 ± 0.007
ω_0	-0.99	-1.03 ± 0.03

6. Conclusion

In this paper, we investigated a scalar field dark energy model with a specific form (15) of the deceleration parameter. The motivation for this form was a simpler form studied by Pawde et al. [48] who considered $q = -1 + \eta/(1 + a^\eta)$. This was without the parameter μ as compared to our choice. We found that that choice by Pawde et al. was not compatible with observations [49]. We began by giving a background to the dark energy cosmological model, together with all the relevant equations. We considered baryonic and dark matter together with a scalar field which represents dark energy. Then we motivated for our choice of deceleration parameter, and gave all the kinematic parameters in terms of z, such as the Hubble, jerk and snap parameters. In addition, we gave the energy parameters for the matter and scalar field.

Then in section 3, we subjected our model to observational constraints to determine the parameters of our model. Using a combination of cosmic chronometers, Pantheon and baryon acoustic oscillation datasets, we found the following parameters: the present value of the Hubble parameter H_0, the constants μ and ν in our form (15) of the deceleration parameter, along with their corresponding 1-σ and 2-σ confidence regions, which are given in Table 1 and Figure 1, respectively. The absolute magnitude M and the sound horizon r_d were also constrained by observations. We then compared our Hubble parameter $H(z)$ and distance modulus curve with that of the ΛCDM model, finding a good fit. We then plotted the cosmographic parameters $q(z)$, $j(z)$ and $s(z)$. The current value $q_0 = -0.427$ of the deceleration parameter, and transition redshift $z_{tr} = 0.748$ are well within observational constraints.

We now comment upon the physical significance of our results. Regarding the deceleration parameter, at a best-fit transition redshift of $z_{\text{tr}} = 0.748^{+0.4}_{-0.4}$, $q(z)$ changes sign from positive to negative. This means a transition from deceleration (required for structure formation, etc) to acceleration, and this signifies the physical significance of the deceleration parameter. Regarding the scalar field ω_ϕ, we pointed out earlier that for redshift $1.45 > z > 0.35$, we have a phantom phase ($\omega_\phi < -1$). It is fortunate that the universe is not in a phantom phase at present for the following reason. It could cause the expansion of the universe to accelerate so quickly that a scenario known as the Big Rip occurs. This signifies a possible end to the universe. The expansion of the universe reaches an infinite degree in finite time, causing the expansion to accelerate without bounds. This acceleration necessarily passes the speed of light (since it involves expansion of the universe itself, not particles moving within it), causing more and more objects to leave our observable universe faster than its expansion, as light and information emitted from distant stars and other cosmic sources cannot "catch up" with the expansion. As the observable universe expands, objects will be unable to interact with each other via fundamental forces, and finally, the expansion will prevent any action of forces between any particles, even within atoms, "ripping apart" the universe, making distances between individual particles infinite. For the redshift $z > 0.35$, the universe is in quintessence, hence avoiding the undesirable features of the phantom phase. The varying scalar field for our model is in contrast to the ΛCDM for which $\omega_\phi = -1$. Hence our model is dynamic which is a novel feature, but it tends to $\omega_\phi = -1$ in future. Thus our model is viable, and provides an alternative to the standard ΛCDM model. In view of the importance of scalar fields, we feel that this model has the potential to contribute to the knowledge of dark energy, and towards a better understanding of the current acceleration of the universe.

We would like to emphasize that, systematic uncertainties in observational datasets can influence the accuracy of parameter estimations. CC data rely on galaxy age dating, which depends on stellar population models and can be affected by metallicity variations. Uncertainties in stellar evolution models can introduce systematic biases in $H(z)$ measurements. SNIa - Pantheon+ data, while highly precise, may be affected by selection biases and possible evolution in the intrinsic brightness of supernovae, leading to redshift-dependent errors. BAO measurements are sensitive to assumptions about the sound horizon scale (r_d), which is influenced by early-universe physics and can affect the inferred cosmic expansion rate. To mitigate these biases, we incorporated the full statistical and systematic error estimates associated with each dataset. Additionally, treating these datasets independently in our likelihood analysis reduces the risk of correlated systematics affecting the results. The agreement between our model and observational constraints further reinforces the robustness of our findings, demonstrating that the estimated parameters remain stable even when accounting for systematic uncertainties.

We summarize our key findings, stating how the proposed model improves upon and complements existing models like the ΛCDM model:

• In ref [48], a choice for the deceleration parameter was made, which did not fit observations. We improved upon their choice (Equation (15)), and found that our model fitted observational constraints, in contrast to their model.

• The MCMC technique was used to determine the parameters of the model.

• We plotted the Hubble parameter and distance modulus curves, and found excellent correspondence with the ΛCDM model.

• The deceleration parameter was plotted and the values of q_0 and q_{tr} were found and and all these showed close correspondence with the ΛCDM model.

• A transition from deceleration to acceleration is achieved without the need to postulate the existence of any kind of exotic matter as is the case with the ΛCDM model.

• The jerk parameter illustrated in Figure 5 shows that it is varying in contrast to the ΛCDM model which is constant ($j = 1$).

• The snap parameter closely follows that of the ΛCDM model, but moves away slightly from it at late times.

• The energy densities of the matter and scalar field are illustrated in Figures 7 and 8. These closely resemble those of the ΛCDM model.

• The equation of state parameter of the scalar field shown in Figure 9 indicates that it is varying whereas that of the ΛCDM model is constant at $\omega = -1$. Our model has $\omega_{\phi 0} = -0.99$, in excellent agreement.

What are possible extensions of this work? There are many modified theories of gravity and cosmological models in the literature that satisfy current observational constraints. No doubt, what is required is a systematic study of these to determine which can be ruled out either from the theoretical point of view, or from improved astronomical observations. Another extension of this work is to look at improved forms for the deceleration parameter in Equation (15) that could better describe the late-time acceleration of the universe. The next obvious extension is to get a form that will describe all eras of cosmological evolution, viz., radiation, matter and late time acceleration eras. Another area of future research is to look at other modified theories that again could improve upon our description. Then comes the consideration of different initial assumptions instead of a form of the deceleration parameter, e.g, en evolving equation of state, scale factor and Hubble parameter.

Acknowledgement

The author is grateful to the referees for their inciteful comments which have led to an improvement of the manuscript.

Conflict of interest

The author declares no competing financial interest.

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