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Research Article

Novel Approaches to Positive Solutions for Fractional Nonlinear Boundary Value Problems

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Abstract: This study explores a fractional boundary value problem based on Riemann-Liouville derivatives and integrals. New results are derived to begin the necessary and adequate circumstances for the existence and uniqueness of positive solutions, leveraging fixed-point theorems on right circular cones. A convergent iterative sequence for solving the problem is presented, along with a numerical scheme. The validity of the results is demonstrated through illustrative examples.

***Keywords*:** existence and uniqueness, fractional calculus, nonlinear, boundary value problem, operator

MSC: 26A03, 34A08, 26A09

1. Introduction

Fractional Calculus, which covers traditional calculus to non-integer orders, has gained attention for modeling complex physical processes in science and engineering. Fractional differential equations and Fractional order Boundary Value Problems (FBVPs) are charity to address real-world problems. Examples include a Caputo-type FBVP for modeling corneal shape [1–4], a heat conduction model [5–7], and a boundary value problem for glucose dynamics [8–14].

Bai in [15] studied confident results for nonlocal (FBVPs). The authors in [16] conducted simulations confirming the reality of single confident solutions for FBVPs. Research in [17–22] analyzed existence and uniqueness for coupled nonlinear FBVPs with anti-periodic boundary conditions, while [23] provided findings for m-point FBVPs and [24] for multi-point FBVPs involving the p-Laplacian operator. Theoretical improvements in [25–27] addressed unique positive solutions for nonlinear FBVPs with mixed-type limit conditions. Reality and individuality for nonlinear fractional q-difference balances with integral limit settings were studied in [28]. Erturk et al. [29] derived results on nonlocal FBVP stability, and Bekri et al. [30] explored the reality and individuality of nonlinear q-difference FBVPs. Finally, [31–33] analyzed existence and uniqueness for two Caputo-type FBVPs.

This study explores the reality and individuality of confident clarifications for the given FBVP

$$E_{0^+}^{\Delta}\,\left(v(t)+I_{0^+}^{\varepsilon}\sigma\left(t,\ v(t)\right)+f\left(t,\ v(t)\right)=0 \qquad (1)$$

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$$\lim_{t\to 0} t^{\Delta-3}v(t) = \lim_{t\to 0} t^{\Delta-3}v'(t) = v'(1) = 0$$

where $2 < \Delta \le \varepsilon \le 3$, $t \in [0,\ 1]$ and E^{Δ}_{0+} is the standard Riemann-Liouville (R-L) insignificant imitative of order Δ and I^{ε}_{0+} is the R-L insignificant integral of order ε. Also, the functions f and σ have some properties which will be presented later.

2. Basic definitions

Classification 1 The Riemann-Liouville (R-) insignificant integral is given by

$$I^{\Delta}_{a^+}y(t) = \frac{1}{\Gamma(\Delta)}\int_a^t (t-p)^{\Delta-1}y(p)\mathrm{d}p$$

here Γ signifies the Gamma function and a is a random fixed initial node. The function y is considered locally integrable and Δ is a real or complex number $Re(\Delta) > 0$.

Classification 2 The R-L insignificant derived of order $\Delta > 0$ of a continuous utility $y : (0,\ +\infty) \to \mathbb{R}$ is given by

$$E^{\Delta}_{a^+}y(t) = \frac{1}{\Gamma(m-\Delta)}\cdot\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_a^t (t-p)^{n-\Delta-1}y(p)\mathrm{d}p$$

where $m = [\Delta]+1$, considering right-hand is point-wise defined on $(0,\ +\infty)$.

Theorem 1 [34] If $v \in C(0,\ 1)\cap L(0,\ 1)$ with an insignificant derived of order $\Delta > 0$ that goes to $C(0,\ 1)\cap L(0,\ 1)$. Then,

$$I^{\Delta}_{0^+}E^{\Delta}_{0^+}v(t) = v(t) + C_1t^{\Delta-1} + C_2t^{\Delta-2} + \ldots.. + C_mt^{\Delta-m}.$$

Somewhere $m = [\Delta]+1$.

Entire manuscript, let $(F,\ \|\cdot\|)$ be a real Banach space and λ be a zero of F. A nonempty closed convex set Q is a right circular cone if fulfills the succeeding situations

i) $v \in Q,\ \mu \ge 0$ implies $\mu v \in Q$;

ii) $v_1 \le v_2 \iff v_2 - v_1 \in Q$.

Also, right circular cone Q is a normal cone if there exests $M \in \mathbb{R}$ such that for all $v_1,\ v_2 \in Q$ with $\lambda \le v_1 \le v_2$ we have $\|v_1\| \le M\|v_2\|$ and M is known as normality constant.

$\forall\ v_1,\ v_2 \in F$, write $v_1 \sim\ v_2$ if there exist constants $\mu,\ \rho > 0$ such that $\mu v_1 \le v_2 \le \rho v_1$. If $r > \lambda$, then $Q_r = \{v \in Q : v \sim r\}$. It is clear that $Q_r \subset Q$.

Classification 3 Let $\zeta \in (0,\ 1)$. An operator $W : Q \to Q$ is called ζ-concave if for all $\mu \in (0,\ 1)$ and $v \in Q$ we have $W(\mu v) \ge \mu^{\zeta}W(v)$. Also, an operator $W : Q \to Q$ is called sub-homogeneous if for all $\lambda > 0$ and $v \in Q$ we have $W(\mu v) \ge \mu W(v)$.

Theorem 2 [35] Let P be a normal right circular cone in a real Banach space F, $W_1\ W_2 : Q \to Q$ be an increasing ζ-concave operator and a cumulative sub-homogeneous operator, respectively. If

i) For some $r > \lambda$ we have $W_1r \in Q_r$ and $W_2r \in Q_r$;

ii) For some constant ω_0 and all $v \in Q$ we have $W_1v \ge \omega_0W_2v$.

Then the operator $W = W_1 + W_2$ has single fixed point. In the other arguments, the operator equation $v = W_1 v + W_2 v$ has single solution $v^* \in Q_r$. Moreover, for any initial value v_0, the successive sequence $v_{m+1} = W_1 v_m + W_2 v_m$, for $m = 0,\ 1,\ 2,\ \ldots$ converges to the v^*.

3. Green function and foundaries

Lemma 1 Suppose $q,\ r : [0,\ 1] \to [0,\ +\infty)$ be the continuous functions, then the solution of the FBVP

$$E_{0^+}^{\Delta}\left[v(t) + I_{0^+}^{\varepsilon} q(t)\right] + r(t) = 0 \tag{2}$$

$$\lim_{t\to 0} t^{\Delta-3} v(t) = \lim_{t\to 0} t^{\Delta-3} v'(t) = v'(1) = 0$$

Is expressed by

$$v(t) = \int_0^1 G_1(t,\ p) r(p) \mathrm{d}p + \int_0^1 G_2(t,\ p) q(p) \mathrm{d}p \tag{3}$$

were

$$G_1(t,\ p) = \begin{cases} \dfrac{t^{\Delta-1}(1-p)^{\Delta-2} - (t-p)^{\Delta-1}}{\Gamma(\Delta)}, & 0 \le p \le t < 1, \\[2ex] \dfrac{t^{\Delta-1}(1-p)^{\Delta-2}}{\Gamma(\Delta)}, & 0 \le t \le p \le 1, \end{cases} \tag{4}$$

$$G_2(t,\ p) = \begin{cases} \dfrac{(\varepsilon-1)t^{\Delta-1}(1-p)^{\varepsilon-2} - (\Delta-1)(t-p)^{\varepsilon-1}}{(\Delta-1)\Gamma(\varepsilon)}, & 0 \le p \le t < 1, \\[2ex] \dfrac{(\varepsilon-1)t^{\Delta-1}(1-p)^{\varepsilon-2}}{(\Delta-1)\Gamma(\varepsilon)}, & 0 \le t \le p \le 1. \end{cases} \tag{5}$$

Proof. Evaluate the equation of (2), follows

$$v(t) + I_{0^+}^{\varepsilon} q(t) = -\frac{1}{\Gamma(\Delta)} \int_0^t (t-p)^{\Delta-1} r(p) \mathrm{d}p + c_1 t^{\Delta-1} + c_2 t^{\Delta-2} + c_3 t^{\Delta-3}$$

One can easily check that from the boundary conditions.
$\lim_{t\to 0} t^{\Delta-3} v(t) = \lim_{t\to 0} t^{\Delta-3} v'(t)$, we have $c_2 = c_3 = 0$. By derivation from the above relation, we have

$$v'(t) = -\frac{\varepsilon-1}{\Gamma(\varepsilon)} \int_0^t (t-p)^{\varepsilon-1} q(p) \mathrm{d}p - \frac{\Delta-1}{\Gamma(\Delta)} \int_0^t (t-p)^{\Delta-1} r(p) \mathrm{d}p + c_1 (\Delta-1) t^{\Delta-2}.$$

Now from the third boundary condition, we have

$$c_1 = \frac{\varepsilon - 1}{(\Delta - 1)\Gamma(\varepsilon)} \int_0^1 (1-p)^{\varepsilon - 2} q(p) \mathrm{d}p + \frac{1}{\Gamma(\Delta)} \int_0^1 (1-p)^{\Delta - 2} r(p) \mathrm{d}p.$$

Hence,

$$v(t) = -\frac{1}{\Gamma(\Delta)} \int_0^t (t-p)^{\Delta-1} r(p) \mathrm{d}p + \frac{1}{\Gamma(\Delta)} \int_0^1 t^{\Delta-1} (1-p)^{\Delta-1} r(p) \mathrm{d}p - \frac{1}{\Gamma(\varepsilon)} \int_0^t (t-p)^{\varepsilon-1} q(p) \mathrm{d}p$$

$$+ \frac{\varepsilon - 1}{(\Delta - 1)\Gamma(\varepsilon)} \int_0^1 t^{\Delta-1} (1-p)^{\varepsilon-2} q(p) \mathrm{d}p$$

$$= \int_0^1 G_1(t,\ p) r(p) \mathrm{d}p + \int_0^1 G_2(t,\ p) q(p) \mathrm{d}p.$$

□

4. Results

In this segment, we establish existence and uniqueness results for the FBVP (1) using Theorem 1. For convenience, we present the following hypotheses:

(A1) $f,\ \sigma \in C([0,\ 1]) \times [0,\ +\infty))$ and they are increasing functions with respect to the second variable, also $\sigma(t,\ 0)0$;

(A2) For $0 < \rho < 1,\ \ (t,\ v) \in [0,\ 1] \times [0,\ +\infty),$ we have $\sigma(t,\ \rho v) \geq \rho \sigma(t,\ v)$;

(A3) For $0 < \rho,\ \zeta < 1,\ \ (t,\ v) \in [0,\ 1] \times [0,\ +\infty)$, we have $f(t,\ \rho v) \geq \rho^{\zeta} f(t,\ v)$;

(A4) There exists a constant $\omega_0 > 0$ such that $f(t,\ v) \geq \omega_0 \sigma(t,\ v),\ \ t \in [0,\ 1],\ \ u \geq 0.$

Now we set

$$A_1 = \int_0^1 G_1(1,\ p) f(p,\ 0) \mathrm{d}p, A_2 = \int_0^1 \frac{(1-s)^{\Delta-2}}{\Gamma(s)} f(p,\ 1) \mathrm{d}p,$$

$$B_1 = \int_0^1 G_2(1,\ p) \sigma(p,\ 0) \mathrm{d}p, B_2 = \int_0^1 \frac{(\varepsilon-1)(1-p)^{\varepsilon-2}}{(\Delta-1)\Gamma(s)} \sigma(p,\ 1) \mathrm{d}p.$$

Theorem 3 Assume that (A1)-(A4) hold. Then, insignificant boundary value problem (1) has single positive solution. In fact, the problem has single solution v in Q_r, with $r(t) = t^{\Delta-1}$, $t \in [0,\ 1]$. Also, for all initial value $v_0 \in Q_r$, the successive sequence

$$v_{m+1}(t) = \int_0^1 G_1(t,\ p) f(p,\ v_m(p)) \mathrm{d}p + \int_0^1 G_2(t,\ p) \sigma(p,\ v_m(p)) \mathrm{d}p,\ m = 0,\ 1,\ \ldots.$$

converges to the solution v^*.

Proof. The proof for the statement can be constructed under the assumption that (A1)-(A4) hold. These assumptions typically relate to conditions like continuity, compactness, monotonicity, and the structure of the Green's functions $G_1(t,\ p)$ and $G_2(t,\ p)$ associated with the problem.

We confirm that T maps Q_r into itself and is a compact operator.

By Schauder's fixed point theorem, there at least one fixed point $v_*(t) \in Q_r$. This fixed point satisfies the boundary value problem and is a solution.

If $LM < 1$ the operator T is a contraction. By the Banach fixed-point theorem, the solution is unique.

The boundary value problem has a unique positive solution $v_*(t) \in Q_r$. Starting from any initial value $v_0 \in Q\ _r$, the iterative sequence defined by the operator T converges to $v_*(t)$.

Let us add the following hypothesis to the previous hypothesis (A1)-(A4).

(A5) $f,\ \sigma\ :\ [0,\ 1]\ \times\ [0,\ +\infty)\ \to\ [0,\ +\infty)$ are respectively increasing and decreasing function with reverence to the second variable and $f(t,\ 0)0,\ \sigma(t,\ 1)0$.

(A6) For any $\rho \in (0,\ 1)$, there exist $y(\rho),\ g(\rho) \in (\mu,\ 1)$ such that for all $t \in [0,\ 1]$ we have $f(t,\ \rho v) \geq y(\rho) f(t,\ u)$, $\sigma(t,\ \rho v) \leq \dfrac{1}{g(\rho)\ \Psi(t,\ \rho)}$. □

Theorem 4 Assume (A5) and (A6) hold, then FBVP has unique solution u^* in Q_r with $r(t) = t^{\Delta-1}$, $t \in [0,\ 1]$. Also, for any first significance problem v_0 and w_0 in Q_r building sequentially the structures

$$v_{m+1}(t) = \int_0^1 G_1(t,\ p) f(p,\ v_m(p))\,\mathrm{d}p + \int_0^1 G_2(t,\ p)\sigma(p,\ w_m(p))\,\mathrm{d}p,\ m = 0,\ 1,\ \ldots.$$

$$w_{m+1}(t) = \int_0^1 G_1(t,\ p) f(p,\ w_m(p))\,\mathrm{d}p + \int_0^1 G_2(t,\ p)\sigma(p,\ v_m(p))\,\mathrm{d}p,\ m = 0,\ 1,\ \ldots.$$

we have $v_m(t) \to v^*(t),\ \ w_m(t) \to v^*(t)$ as $m \to +\infty$, where $G_1(t,\ p)$ and $G_2(t,\ p)$ are given in (4) and (5).

Proof. The proof extends the analysis to two iterative sequences, $\{v_m(t)\}$ and $\{w_m(t)\}$ defined using the given forms, and aims to show that both sequences converge to the unique solution u^* of the fractional boundary value problem (FBVP). Below is the proof under assumptions (A5) and (A6).

By the Banach fixed-point theorem, T has a unique fixed $u^*(t) \in Q_r$ which the unique solution of the FBVP is under the assumptions (A5) and (A6) the FBVP has a unique solution $u^*(t) \in Q_r$ moreover, for any initial values $v_0(t),\ w_0(t) \in Q_r$ the sequences $\{v_m(t)\}$ and $\{w_m(t)\}$ converge to $v^*(t)$, which coincides with$u^*(t)$. □

5. Illustrations

Illustration 1 Let us consider the following FBVP

$$E_{0^+}^{\frac{5}{2}}\ (v(t) + I_{0^+}^{\frac{8}{3}} \sigma(t,\ v(t)) + f(t,\ v(t)) = 0,$$

$$\lim_{t\to 0} t^{-\frac{1}{2}} v(t) = \lim_{t\to 0} t^{\Delta-3} v'(t) = v'(1) = 0,$$

where

$$f(t,\ v) = v^{\frac{1}{3}} + k\frac{t^{\frac{7}{2}}}{\Gamma\left(\frac{7}{2}\right)}$$

and $\sigma(t,\ v) = \frac{v}{1+v}e^t + \frac{(l-k)t^{\frac{7}{2}}}{\Gamma\left(\frac{7}{2}\right)}$, with $k > l > 0$. Now

$$f(t,\ \rho v) = \rho^{\frac{1}{3}} v^{\frac{1}{3}} + k\frac{t^{\frac{7}{2}}}{\Gamma\left(\frac{7}{2}\right)} \geq \rho^{\frac{1}{3}}\left(v^{\frac{1}{3}}NN + k\frac{t^{\frac{7}{2}}}{\Gamma\left(\frac{7}{2}\right)}\right) = \rho^n f(t,\ v)$$

$$\sigma(t,\ \rho v) = \frac{\rho v}{1+\rho v}e^t + \frac{(l-k)t^{\frac{7}{2}}}{\Gamma\left(\frac{7}{2}\right)} \geq \rho\left[\frac{v}{1+v}e^t + \frac{(l-k)t^{\frac{7}{2}}}{\Gamma\left(\frac{7}{2}\right)}\right].$$

If we set $\omega_0 \in \left[0,\ \frac{a}{e+b-a}\right]$, then

$$f(t,\ \rho v) = v^{\frac{1}{3}} + k\frac{t^{\frac{7}{2}}}{\Gamma\left(\frac{7}{2}\right)} \geq k\frac{t^{\frac{7}{2}}}{\Gamma\left(\frac{7}{2}\right)(e+b-a)}(e+b-a)$$

$$\geq \omega_0\left[\frac{v}{1+v}e^t + \frac{(l-k)t^{\frac{7}{2}}}{\Gamma\left(\frac{7}{2}\right)}\right] = \omega_0\sigma(t,\ v).$$

Since all conditions of Theorem 3 are met, the problem (Illustration 1) with $f,\ \sigma$ has positive solution.

6. Mathematical results

Having established the reality and individuality of a result to (1), we now turn our attention to its numerical solution. This approach is relatively straightforward, relying on Theorems 3 and 4. The recurrence relation, derived from operator (4.1) and presented in Theorem 3, can be applied with ease using an initial trial solution, say, for example, $v_0(t) \equiv 0$, and then the programme iterates to find sequential $v_n(t)$ stopping when the maximum difference in two successive iterations drops below a given tolerance value. The iterative scheme is implemented using the computer algebra system *Mathematica.* Transitions between iterations are performed both symbolically and numerically. Numerical computation is applied when approximating the integral in the equation from Theorem 3c, utilizing cubic spline interpolation.

Firstly, we consider Illustration 1 to confirm the validity of the presented numerical method.

We derive the following algorithm using the Green's function method.

Phase 1 The Mesh points $t_0,\ t_1,\ \ldots,\ t_N$ are considered for effectively large number of N.

Phase 2 Cubic spline interpolation is used to get $v_n(p)$'s.
Phase 3 The next approximate solution is obtained by the numerical integration:

$$v_{m+1}(t_i) = \frac{1}{\Gamma(\Delta)} \int_0^{t_i} \left[t_i^{\Delta-1}(1-p)^{\Delta-1} - (t_i-p)^{\Delta-1}\right] \left[v_m^{\frac{1}{3}}(p) + k\frac{p^{\frac{7}{2}}}{\Gamma\left(\frac{7}{2}\right)}\right] \mathrm{d}p$$

$$+\frac{t_i^{\Delta-1}}{\Gamma(\Delta)} \int_{t_j}^{1} (1-p)^{\Delta-2} \left[v_m^{\frac{1}{3}}(p) + k\frac{p^{\frac{7}{2}}}{\Gamma\left(\frac{7}{2}\right)}\right] \mathrm{d}p + \frac{1}{(\Delta-1)\Gamma(\varepsilon)} \int_0^{t_i} \left[(\varepsilon-1)t_i^{\Delta-1}(1-p)^{\varepsilon-2}\right.$$

$$\left.-(\Delta-1)(t_i-p)^{\varepsilon-1}\right] \times \left[\frac{v_m(p)}{1+v_m(p)}e^p + \frac{(l-k)p^{\frac{7}{2}}}{\Gamma\left(\frac{7}{2}\right)}\right] \mathrm{d}p$$

$$+\frac{(\varepsilon-1)t_i^{\Delta-1}}{(\Delta-1)\Gamma(\varepsilon)} \int_{t_i}^{1} \left[(1-p)^{\varepsilon-2}\right] \left[\frac{v_m(p)}{1+v_m(p)}e^p + \frac{(l-k)p^{\frac{7}{2}}}{\Gamma\left(\frac{7}{2}\right)}\right] \mathrm{d}p, \ m = 0,\ 1,\ \ldots$$

Phase 4 Phase 1, 2, 3 are iterated to find consecutive $v_m(v)$ stopping when $|v_{m+1} - v_m| < TOL$.

The exact solution is unknown in fact, but the iteration stopping criteria used is set $|v_{m+1} - v_m| < 10^{-09}$, and then the numerical solution is obtained. For the step size of the node points, $r = 0.05$, the number of iterations, $M = 15$, and $TOL = 10^{-09}$, the errors are of order 10^{-09}. The solution curve $v(t)$ is shown graphically in Figure 1 for $\Delta = 2.6$ and $\varepsilon = 2.7$ when $a = 1$ and $b = 2$. For other graphical simulations, $(\Delta,\ \varepsilon)$ are taken as $(2.2,\ 2.5)$, $(2.5,\ 2.5)$, $(2.5,\ 2.8)$, $(2.8,\ 2.9)$, and $(3,\ 3)$. The solution curves $v(t)$ are displayed in Figures respectively. For $\Delta = 2.6$ and $\varepsilon = 2.7$ when $a = 1$ and $b = 2$, the convergence is plotted in Figure 1-8 and the error is plotted in

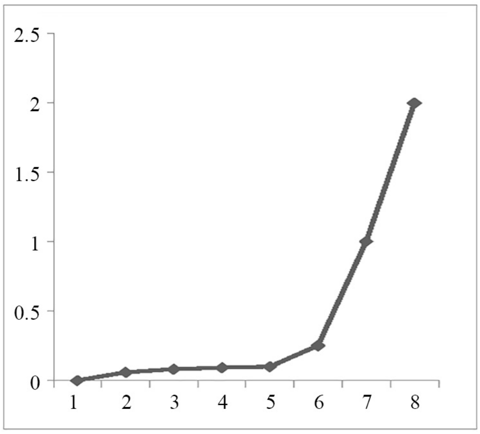

Figure 1. Solution curve $v(t)$ for $\Delta = 2.6$ and $\varepsilon = 2.7$

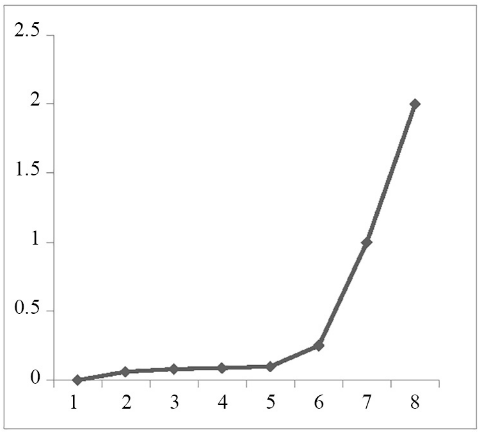

Figure 2. Solution curve $v(t)$ for $\Delta = 2.2$ and $\varepsilon = 2.5$

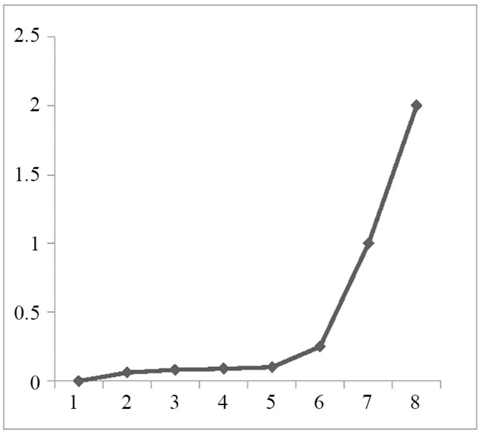

Figure 3. Solution curve $v(t)$ for $\Delta = 2.5$ and $\varepsilon = 2.5$

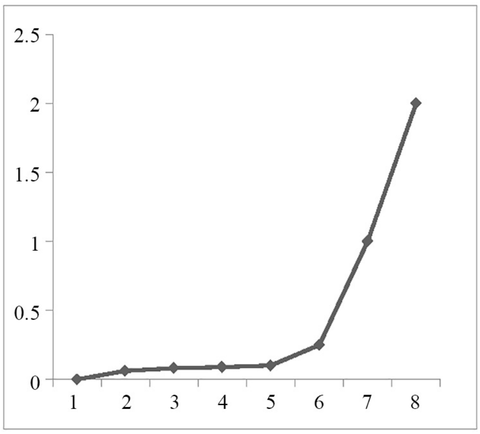

Figure 4. Solution curve $\mathbf{v}(t)$ for $\Delta = 2.5$ and $\varepsilon = 2.8$

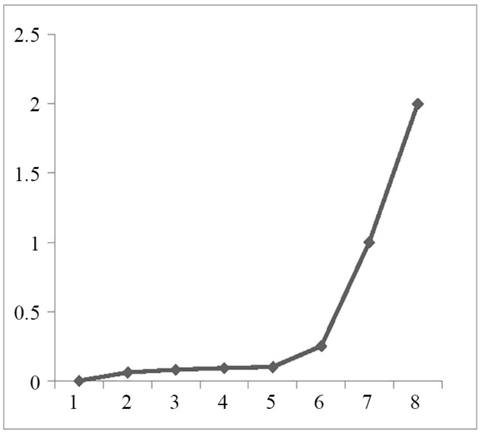

Figure 5. Solution curve $v(t)$ for $\Delta = 2.8$ and $\varepsilon = 2.9$

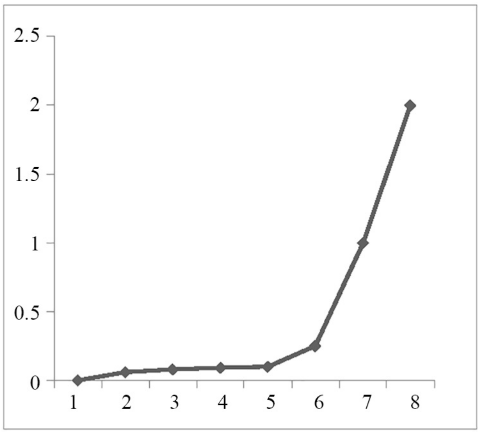

Figure 6. Solution curve $v(t)$ for $\Delta = 3$ and $\varepsilon = 3$

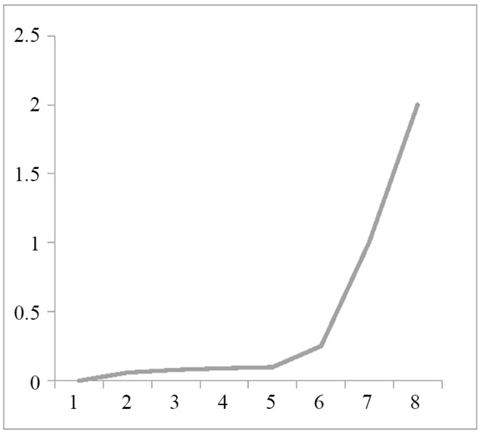

Figure 7. Convergence curve $m = 5$ (level bar), $m = 10$ (vertical bar), $m = 15$ (x), and $m = 20$ (solid) for $\Delta = 2.6$ and $\varepsilon = 2.7$

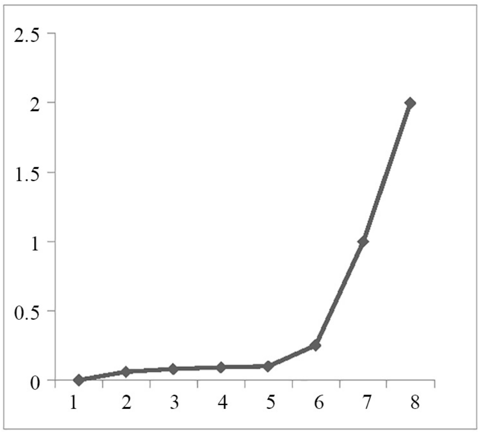

Figure 8. Error curve for $\Delta = 2.6$ and $\varepsilon = 2.7$

Table 1 illustrations the mathematical results and absolute residual errors of the present method for $M = 10$, $\Delta = 2.3$, and $\varepsilon = 2.2$.

Table 1. Numerical solution and absolute residual error of example 5.1 for $N = 10$, $\Delta = 2.3$ and $\varepsilon = 2.2$

t	Numerical solution	Absolute residual error
0.0	0	
0.1	0.1146	4.6×10^{-11}
0.2	0.2543	1.0×10^{-10}
0.3	0.4838	1.5×10^{-10}
0.4	0.5929	1.7×10^{-10}
0.5	0.6736	1.6×10^{-10}
0.6	0.6295	1.4×10^{-10}
0.7	0.7678	1.1×10^{-10}
0.8	0.7744	7.0×10^{-10}
0.9	0.6521	3.3×10^{-10}
1.0	0.6256	2.1×10^{-10}

7. Conclusions

This article explores a period of FBVPs linking Riemann-Liouville derivatives and integrals, establishing novel essential and adequate circumstances for the reality then individuality of optimistic results. By fixed-point theorems on right circular cones, we derive a convergent iterative scheme for solving the FBVPs. The validity of the results is demonstrated through numerical examples. These findings provide a foundation for future studies on modeling real-world problems and conducting both qualitative and quantitative analyses of similar FBVPs.

Conflict of interest

The authors declare no competing financial interest.

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