Contemporary Mathematics
https://ojs.wiserpub.com/index.php/CM/

Research Article

Extensive View for Implicit Second-Order Differential Equations with a Feedback Control

A. M. A. El-Sayed[1], Sh. M. Al-Issa[2,3*], H. H. G. Hashem[1], I. H. Kaddoura[2,3], M. E. A. Jaafar[2]

[1] Department of Mathematics, Faculty of Sciences, Alexandria University, Alexandria, Egypt
[2] Department of Mathematics, Faculty of Arts and Sciences, Lebanese International University, Saida, Lebanon
[3] Department of Mathematics, Faculty of Arts and Sciences, The International University of Beirut, Beirut, Lebanon
E-mail: shorouk.alissa@liu.edu.lb

Received: 6 January 2025; **Revised:** 1 March 2025; **Accepted:** 4 March 2025

Abstract: The current research investigates the solvability and stability of implicit second-order differential equations subject to fractional integral boundary conditions under the influence of a fractional control. The analysis is grounded in the application of Krasnoselskii's fixed point theorem to establish the existence and stability of solutions. To complement the theoretical results, numerical calculations are provided, to validate our findings. Furthermore, detailed comparisons with the existing literature highlight the efficacy and novelty of the proposed approach.

***Keywords*:** existence results, Krasnoselskii's fixed point theorem, Ulam-Hyers stability, feedback control

MSC: 26A33, 47H09, 45M99

1. Introduction

One useful technique in the realm of applied mathematics is fractional calculus, providing a way to delve into a broad array of topics within diverse scientific and technical domains. Researchers have made significant strides in the development of fractional derivatives [1–3]. Significant advancements have recently been made in the study of partial fractional differential equations and ordinary differential equations [4–6]. For more comprehensive insights, the reader may consult the works of Abbas et al. [7], Baleanu et al. [8], Kilbas et al. [9], and Lakshmikantham et al. [10]. In addition, numerous researchers have contributed to this field [11–15].

Research on fractional differential equations often delves into stability analysis, with a specific emphasis on Ulam stability. Ulam stability, a type of data dependence introduced by Ulam, has been advanced by scholars such as Hyers et al. (see [16–18] for more information).

Research on blue second-order differential equations holds a central position in mathematical modeling due to their ability to describe a variety of phenomena, including mechanical vibrations, heat conduction, fluid dynamics, and wave propagation. Recent studies have explored the fractional analogs of these equations, extending their scope to include non-local dynamics and the hereditary properties of materials [19, 20]. The second-order implicit problems involving fractional derivatives, such as those discussed in [21, 22], provide robust frameworks for understanding systems governed by memory effects and nonlocal interactions. These models are particularly effective in fields such as viscoelasticity,

DOI: https://doi.org/10.37256/cm.6320256357

anomalous diffusion, and control systems. Moreover, fractional versions of thermostat models have been analyzed in detail, shedding light on energy transfer processes and stabilization techniques [23].

A problem with feedback Control plays a crucial role in managing and stabilizing systems subject to external influences and perturbations. These equations are instrumental in adaptive systems that require feedback mechanisms to achieve the desired objectives. Applications span various areas, including engineering control systems, biological processes, and economic modeling [24, 25]. For example, feedback control in biological systems ensures homeostasis and stability despite environmental disruptions. Similarly, in engineering, these equations are used to design robust controllers that adapt to real-time changes [26]. Recent studies have integrated fractional derivatives into a problem of feedback control, allowing for the modeling of systems with memory effects and fractional dynamics [27, 28]. These advancements not only provide deeper insights into system behavior but also pave the way for developing innovative control strategies applicable to renewable energy systems, robotics, and artificial intelligence.

Building on the results presented in the aforementioned articles, this paper investigates the implicit second-order differential problem (ISDP) with general integral boundary conditions (IBC). The mathematical framework considers fractional derivatives of various orders

$$\frac{d^2}{dr^2}y(r) = f\big(r,\, w_y(r),\, y(r),\, {}^cD^{\beta}y(r),\, \int_0^r g(\xi,\, {}^cD^{3-\beta}y(\xi))\, d\xi\big), \quad r \in (0,\, T) \tag{1}$$

$$y(0) = I^{\gamma}h_1(r,\, y(r),\, {}^cD^{\gamma}y(r))\big|_{r=T} \tag{2}$$

$$y'(T) = I^{\gamma}h_2(r,\, y(r),\, {}^cD^{3-\gamma}y(r))\big|_{r=T}. \tag{3}$$

Restricted on a fractional control

$$\frac{dw_y(r)}{dr} = -\lambda w_y(r) + g_1(r,\, y(r),\, {}^cD^{3-\rho}y(r),\, {}^cD^{\rho}y(r)), \quad w_0 = w_y(0),\ \lambda > 0, \tag{4}$$

where $\mathbb{I} = [0,\, T]$.

For $1 < \beta,\, \gamma,\, \rho \leq 2$, the Caputo fractional derivative of order $\sigma \in \{\beta,\, \gamma,\, \rho\}$ is indicated by ${}^cD^{\sigma}$.

Consider two continuous functions h_i for $i = 1,\, 2$, defined on $\mathbb{I} \times \mathbb{R}^2$ and mapping to the real numbers. Furthermore, let f and g be predefined functions, where f is defined on $\mathbb{I} \times \mathbb{R}^3$ and g on $\mathbb{I} \times \mathbb{R}$. The specific conditions governing these functions will be detailed subsequently.

Building on these advancements, our paper investigates the solvability and Ulam-Hyers stability of implicit second-order differential equations under fractional feedback control. By employing fixed-point theorems, we establish existence and uniqueness results, thereby contributing novel insights to the field. These findings extend the current understanding of implicit fractional systems and highlight their potential for further research and practical applications.

The issue discussed in this paper involving the problem (1)-(4) expand upon existing studies by addressing less-explored scenarios involving ISDP with IBC. These findings contribute to the ongoing research on fractional integro-differential equations, control theory, and their practical applications. We believe that the discoveries will enhance the existing body of literature on this subject.

This study is motivated by the increasing need for advanced mathematical frameworks capable of addressing dynamic systems with inherent memory effects and non-local interactions, which are commonly encountered in applications such as viscoelasticity, control theory, and fractional dynamics [2, 9]. Despite the growing interest, current research often lacks comprehensive stability analyses for systems governed by fractional feedback control, which hinders their broader application [8]. By introducing novel techniques for solvability and Ulam-Hyers stability analysis, this work aims to bridge a crucial gap in the existing literature and lay the groundwork for future theoretical and practical advancements.

The paper is structured into four parts. The first section contains two subsections where the main discoveries are discussed. The first subsection elaborates on the relationship between the integral equation (5) and ISDP (1)-(3) with a fractional control (4). The second subsection presents the outcomes of the problem (1)-(4), with one demonstrated using the Krasnoselskii's fixed point theorem. In the upcoming section, we will delve into the stability of our problem. We will debate and find examples to further explain our conclusions in a separate section.

2. Main result

The following presumptions underlie the analysis:

(H_1) The functions $h_i : \mathbb{I}\times\mathbb{R}^2 \to \mathbb{R},\ i = 1, 2,$ are continuous and satisfy a Lipschitz condition with constants $0 < k_i < 1$, such that

$$|h_i(r,\, \mu_1,\, \mu_2) - h_i(r,\, \nu_1,\, \nu_2)| \le k_i(|\mu_1 - \nu_1| + |\mu_2 - \nu_2|).$$

This implies that

$$|h_i(r,\, \mu,\, \nu)| \le H_i + k_i(|\mu| + |\nu|),$$

where $H_i = \sup_{r\in\mathbb{I}} |h_i(r,\, 0,\, 0)|$.

(H_2) The function $f \in C(\mathbb{I}\times\mathbb{R}^4,\, \mathbb{R})$.

And a function with the norm $\|\psi\|$ exists with $\psi \in C(\mathbb{I},\, \mathbb{R}_+)$, so that

$$|f(r,\, a_1,\, a_2,\, a_3,\, a_4) - f(r,\, b_1,\, b_2,\, b_3,\, wb_4)| \le \psi(r)\big(|a_1 - b_1| + |a_2 - b_2| + |a_3 - b_3| + |a_4 - b_4|\big).$$

Consequently, we have

$$|f(r,\, a_1,\, a_2,\, a_3,\, a_4)| \le \|\psi\|\big(|a_1| + |a_2| + |a_3| + |a_4|\big) + F,$$

with $F = \sup_{r\in\mathbb{I}} |f(r,\, 0,\, 0,\, 0,\, 0)|$, for all $r \in \mathbb{I}$ and $a_i,\, b_i \in \mathbb{R}$, $i = 1,\, 2,\, 3,\, 4$.

(H_3) The function $g \in C(\mathbb{I}\times\mathbb{R},\, \mathbb{R})$. And a function with the norm $\|\theta\|$ exists with $\theta \in C(\mathbb{I},\, \mathbb{R}_+)$, so that

$$|g(r,\, a) - g(r,\, b)| \le \theta(r)|a - b|,$$

Consequently, we have

$$|g(r,\, a)| \le G + \|\theta\||a|,$$

with $G = \sup_{r\in\mathbb{I}} |g(r,\, 0)|$.

(H_4) The function $g_1 : \mathbb{I}\times\mathbb{R}^3 \to \mathbb{R}$ is continuous. For every $y,\, z,\, w \in \mathbb{R}$, it satisfies

$$|g_1(r, y, z, w)| \leq k\big(|y|+|z|+|w|\big)+G_1,$$

with $G_1 = \sup_{r\in\mathbb{I}} |g_1(r, 0, 0, 0)|$.

Lemma 1 Let $h \in C(\mathbb{I}, \mathbb{R})$; function $y(r)$ is considered as a solution ISDP (1)-(3) restricted to the control (4) iff

$$y(r) = h(r) - r\int_0^T v(\xi)d\xi + \int_0^r (r-\xi)\, v(\xi)d\xi \tag{5}$$

with v as follows

$$v(r) = f\big(r, w_y(r), h(r) - r\int_0^T v(\xi)d\xi + \int_0^r (r-\xi)\, v(\xi)d\xi, I^{\beta-1}v(r), \int_0^r g(\xi, I^\beta v(\xi))d\xi\big), \tag{6}$$

where

$$h(r) = \frac{1}{\Gamma(\gamma)}\int_0^T (T-\xi)^{\gamma-1} h_1(\xi, y(\xi), I^{\gamma-1}v(\xi))d\xi + \frac{r}{\Gamma(\gamma)}\int_0^T (T-\xi)^{\gamma-1} h_2(\xi, y(\xi), I^{\gamma}v(\xi))d\xi \tag{7}$$

and

$$w_y(r) = w_0 e^{-\lambda r} + \int_0^r e^{-\lambda(r-\xi)}\, g_1(\xi, y(\xi), I^\rho v(\xi), I^{\rho-1}v(\xi))d\xi, \quad w_0 = w_y(0). \tag{8}$$

Proof. The outcomes can vary depending on the Caputo fractional derivatives:

$${}^cD^\beta y(r) = I^{2-\beta}\frac{d^2}{dr^2}y(r), \quad {}^cD^{3-\beta}y(r) = I^{\beta-1}\frac{d^2}{dr^2}y(r),$$

$${}^cD^\gamma y(r) = I^{2-\gamma}\frac{d^2}{dr^2}y(r), \quad {}^cD^{3-\gamma}y(r) = I^{\gamma-1}\frac{d^2}{dr^2}y(r),$$

$${}^cD^\rho y(r) = I^{2-\rho}\frac{d^2}{dr^2}y(r), \quad {}^cD^{3-\rho}y(r) = I^{\rho-1}\frac{d^2}{dr^2}y(r).$$

If y is an ISDP (1)-(3) solution, therefore

$$\frac{d^2y(r)}{dr^2} = f\left(r, w_y(r), y(r), I^{2-\beta}\frac{d^2}{dr^2}y(r), \int_0^r g\left(\xi, I^{\beta-1}\frac{d^2}{d\xi^2}y(\xi)\right)\right).$$

Let $\frac{d^2}{dr^2}y(r) = v(r)$, so

$$v(r) = f\big(r, y(r), I^{2-\beta}\, v(r), \int_0^r g(\xi, I^{\beta-1} v(\xi)) d\xi\big), \tag{9}$$

and the solution for $y(r)$ becomes:

$$y(r) = a_\circ + a_1 r + \int_0^r (r-\xi) v(\xi) d\xi.$$

Differentiating, we get:

$$y'(r) = a_1 + \int_0^r v(\xi) d\xi.$$

Using boundary conditions (2) and (3), we find:

$$a_\circ = \int_0^T \frac{(T-\xi)^{\gamma-1}}{\Gamma(\gamma)} h_1(\xi, y(\xi), I^{2-\gamma} v(\xi)) d\xi,$$

$$a_1 = \frac{1}{\Gamma(\gamma)} \int_0^T (T-\xi)^{\gamma-1}\, h_2(\xi, y(\xi), I^{\gamma-1} v(\xi)) d\xi - \int_0^T v(\xi) d\xi.$$

Thus,

$$y(r) = \frac{1}{\Gamma(\gamma)} \int_0^T (T-\xi)^{\gamma-1} h_1(\xi, y(\xi), I^{2-\gamma} v(\xi)) d\xi + \frac{r}{\Gamma(\gamma)} \int_0^T (T-\xi)^{\gamma-1} h_2(\xi, y(\xi), I^{\gamma-1} v(\xi)) d\xi$$

$$- r \int_0^T v(\xi) d\xi + \int_0^r (r-\xi)\, v(\xi) d\xi.$$

As a result, let

$$h(r, y(r)) = \frac{1}{\Gamma(\gamma)} \int_0^T (T-\xi)^{\gamma-1} h_1(\xi, y(\xi), I^{2-\gamma} y(\xi)) d\xi + \frac{r}{\Gamma(\gamma)} \int_0^T (T-\xi)^{\gamma-1} h_2(\xi, y(\xi), I^{\gamma-1} v(\xi)) d\xi.$$

Currently, the control (4) can be described as

$$\frac{dw_y(r)}{dr} = -\lambda w_y(r) + g(r, y(r), D^{3-\rho} y(r), D^\rho y(r)),$$

then

$$\frac{dw_y(r)}{dr} = -\lambda w_y(r) + g\left(r, y(r), I^{\rho-1}\frac{d^2}{dr^2}y(r), I^{2-\rho}\frac{d^2}{dr^2}y(r)\right),$$

therefore

$$\frac{dw_y(r)}{dr} = -\lambda\ w_y(r) + g(r, y(r), I^{\rho-1}v(r), I^{2-\rho}v(r)).$$

Multiply both of the terms by $e^{\lambda r}$:

$$e^{\lambda r}\frac{dw_y(r)}{dr} + e^{\lambda r}\lambda \cdot w_y(r) = e^{\lambda r}g(r, y(r), I^{\rho-1}v(r), I^{2-\rho}v(r)),$$

and

$$\frac{d}{dr}\left(w_y(r)\cdot e^{\lambda r}\right) = e^{\lambda r}g(r, y(r), I^{\rho-1}v(r), I^{2-\rho}v(r)).$$

Integrate with respect to r, and then:

$$w_y(r)\cdot e^{\lambda r} = w_y(0) + \int_0^r e^{\lambda\xi}g(\xi, y(\xi), I^{\rho-1}v(\xi), I^{2-\rho}v(\xi))\, d\xi.$$

Thus,

$$w_y(r) = w_0\, e^{-\lambda r} + \int_0^r e^{-\lambda(r-\xi)}g(\xi, y(\xi), I^{\rho-1}v(\xi), I^{2-\rho}v(\xi))\, d\xi.$$

The solution of the differential feedback control for a real-valued function y, denoted as $w_y(r)$, can be represented as given in equation (8). □

Lemma 2 The function h that is Lipschitzian, defined on the domain $\mathbb{I}\times\mathbb{R}$ with a Lipschitz constant c, satisfies the inequality

$$|h(r, w) - h(r, z)\| \leq c\, \|w - z\|,$$

for all $r \in \mathbb{I}$, and x, and $y \in \mathbb{R}$, where

$$c = \frac{(k_1 + k_2 T)T^{\gamma}}{\Gamma(\gamma+1)} + \left[\frac{T^{2-\gamma}}{\Gamma(3-\gamma)} + \frac{T^{\beta-1}}{\Gamma(\beta)}\right]\frac{\|\psi\|(\Delta+1)}{1-\aleph},$$

and $\aleph = \|\psi\| \left[\frac{T^{2-\beta}}{\Gamma(3-\beta)} + \|\theta\| \frac{T^{\beta-1}}{\Gamma(\beta)} \right]$.

Proof. We have for any two elements w, z in set X and any r in interval $\mathbb{I}$

$$
\begin{aligned}
&\big|h(r, w(r)) - h(r, z(r))\big| \\
&\leq \bigg| \frac{1}{\Gamma(\gamma)} \int_0^T (T-\xi)^{\gamma-1} h_1(\xi, w(\xi), I^{2-\gamma} u(\xi)) d\xi + \frac{r}{\Gamma(\gamma)} \int_0^T (T-\xi)^{\gamma-1} h_2(\xi, w(\xi), I^{\gamma-1} u(\xi)) d\xi \\
&\quad - \frac{1}{\Gamma(\gamma)} \int_0^T (T-\xi)^{\gamma-1} h_1(\xi, z(\xi), I^{2-\gamma} v(\xi)) d\xi - \frac{r}{\Gamma(\gamma)} \int_0^T (T-\xi)^{\gamma-1} h_2(\xi, z(\xi), I^{\gamma-1} v(\xi)) d\xi \bigg| \\
&\leq \frac{1}{\Gamma(\gamma)} \int_0^T (T-\xi)^{\gamma-1} \big| h_1(\xi, w(\xi), I^{2-\gamma} u(\xi)) - h_1(\xi, z(\xi), I^{2-\gamma} v(\xi)) \big| d\xi \\
&\quad + \frac{r}{\Gamma(\gamma)} \int_0^T (T-\xi)^{\gamma-1} \big| h_2(\xi, w(\xi), I^{\gamma-1} u(\xi)) - h_2(\xi, z(\xi), I^{\gamma-1} v(\xi)) \big| d\xi \\
&\leq \frac{k_1}{\Gamma(\gamma)} \int_0^T (T-\xi)^{\gamma-1} \big[|w(\xi) - z(\xi)| + |I^{2-\gamma} u(\xi) - I^{2-\gamma} v(\xi)| \big] d\xi \\
&\quad + \frac{rk_2}{\Gamma(\gamma)} \int_0^T (T-\xi)^{\gamma-1} \big[|w(\xi) - z(\xi)| + |I^{\gamma-1} u(\xi) - I^{\gamma-1} v(\xi)| \big] d\xi \\
&\leq \frac{k_1}{\Gamma(\gamma)} \int_0^T (T-\xi)^{\gamma-1} \left[|w(\xi) - z(\xi)| + \int_0^\xi \frac{(\xi-\theta)^{1-\gamma}}{\Gamma(2-\gamma)} |u(\theta) - v(\theta)| \, d\theta \right] d\xi \\
&\quad + \frac{rk_2}{\Gamma(\gamma)} \int_0^T (T-\xi)^{\gamma-1} \left[|w(\xi) - z(\xi)| + \int_0^\xi \frac{(\xi-\theta)^{\gamma-2}}{\Gamma(\gamma-1)} |u(\theta) - v(\theta)| \, d\theta \right] d\xi \\
&\leq \frac{(k_1 + k_2 T) T^\gamma}{\Gamma(\gamma+1)} \left[\|w - z\| + \Big(\frac{T^{2-\gamma}}{\Gamma(3-\gamma)} + \frac{T^{\gamma-1}}{\Gamma(\gamma)} \Big) \|u - v\| \right]. \qquad (10)
\end{aligned}
$$

Now, we compute

$$
\begin{aligned}
&|u(r) - v(r)| \\
&= \left| f\big(r, w_w(r), w(r), I^{2-\beta} u(r), \int_0^r g(\xi, I^{\beta-1} u(\xi)) d\xi\big) - f\big(r, w_z(r), z(r), I^{2-\beta} v(r), \int_0^r g(\xi, I^{\beta-1} v(\xi)) d\xi\big) \right|
\end{aligned}
$$

$$\leq \|\psi\| \left[|w_w(r) - w_z(r)| + |w(r) - z(r)| + I^{2-\beta}|u(r) - v(r)| + \int_0^r |g(\xi, I^{\beta-1}u(\xi)) - g(\xi, I^{\beta-1}v(\xi))| d\xi \right]$$

$$\leq \|\psi\| [|w_w(r) - w_z(r)| + |w(r) - z(r)| + I^{2-\beta}|u(r) - v(r)| + \|\theta\| I^{\beta-1}|u(r) - v(r)|]$$

$$\leq \|\psi\| \left[\Delta \|w - z\| + \|w - z\| + \frac{T^{2-\beta}}{\Gamma(3-\beta)} \|u - v| + \|\theta\| \frac{T^{\beta-1}}{\Gamma(\beta)} \|u - v\| \right]. \tag{11}$$

Then

$$\|u - v\| \leq \frac{\|\psi\| \ (\Delta + 1)\|w - z\|}{1 - \left(\|\psi\| \left[\frac{T^{2-\beta}}{\Gamma(3-\beta)} + \|\theta\| \frac{T^{\beta-1}}{\Gamma(\beta)} \right] \right)}.$$

Retrain to (10) and substitute it with (11), then we have

$$\|h(r, w) - h(r, z)\| \leq \frac{(k_1 + k_2 T)T^{\gamma}}{\Gamma(\gamma+1)} \|w - z\| + \left[\frac{T^{2-\gamma}}{\Gamma(3-\gamma)} + \frac{T^{\gamma+1}}{\Gamma(3+\gamma)} \right] \frac{\|\psi\|(\Delta + 1)\|w - z\|}{1 - \left(\|\psi\| \left[\frac{T^{2-\beta}}{\Gamma(3-\beta)} + \|\theta\| \frac{T^{\beta-1}}{\Gamma(\beta)} \right] \right)}.$$

Thus

$$\|h(r, w) - h(r, z)\| \leq c \ \|w - z\|,$$

with $c = \frac{(k_1 + k_2 T)T^{\gamma}}{\Gamma(\gamma+1)} + \left[\frac{T^{2-\gamma}}{\Gamma(3-\gamma)} + \frac{T^{\beta-1}}{\Gamma(\beta)} \right] \frac{\|\psi\|(\Delta + 1)}{1 - \aleph}$, and $\aleph = \|\psi\| \left[\frac{T^{2-\beta}}{\Gamma(3-\beta)} + \|\theta\| \frac{T^{\beta-1}}{\Gamma(\beta)} \right]$. □

2.1 *Characteristic of control function*

Lemma 3 The control variable $w_y(r)$ meets (4) and can be represented by (8). Consequently, the solution $w_z(r)$ remains bounded when $w_0 > 0$.

Proof. Consider

$$B_{\rho} = \{y \in C(\mathbb{I}, \mathbb{R}) : \|y\| \leq \rho\},$$

with

$$\rho \geq \frac{\frac{T^{\gamma}(H_1+TH_2)}{\Gamma(\gamma+1)}+\frac{F+\|\psi\|(\chi+\chi_2+\chi_3+\chi_4)+\|\psi\|\|y_2\|+\|\psi\|TG}{1-\aleph}\left(\frac{T^{\gamma+1}}{\Gamma(\gamma+1)}\left[\frac{T^{1-\gamma}}{\Gamma(3-\gamma)}+\frac{T^{\gamma-1}}{\Gamma(\gamma)}\right]+2T^2\right)}{1-\frac{T^{\gamma-1}(k_1+k_2)}{\Gamma(\gamma)}-\frac{\|\psi\|\chi_1}{1-\aleph}}$$

where, $\aleph=\left(\frac{\|\psi\|T^{2-\beta}}{\Gamma(3-\beta)}+\frac{a\;\|\psi\|\|v\|\;T^{\beta-1}}{\Gamma(\beta)}\right)$.

In the beginning, we demonstrate that a feedback control function $w_y(r)$ is bounded. For evidence of this, take $y\in C(\mathbb{I},\mathbb{R})$, then

$$\begin{aligned}
|w_y(r)| &\leq w_0|e^{-\lambda r}|+\int_0^r e^{-\lambda(r-\xi)}\;[k\;(|y(\xi)|+|I^{\rho-1}v(\xi)|+|I^{2-\rho}v(\xi))|)+m]d\xi \\
&\leq \chi+k\int_0^r e^{-\lambda(r-\xi)}|y(\xi)|d\xi+k\int_0^r e^{-\lambda(r-\xi)}\;I^{\rho-1}\|v\|d\xi \\
&\quad +k\int_0^r e^{-\lambda(r-\xi)}I^{2-\rho}\|v\|d\xi+k\int_0^r e^{-\lambda(r-\xi)}\;m\;d\xi \\
&\leq \chi+\sup_{r\in I}\;k\;\int_0^r e^{-\lambda(r-\xi)}\;|y(\xi)|d\xi+\sup_{r\in I}k\;\int_0^r e^{-\lambda(r-\xi)}\;I^{\rho-1}\|v\|d\xi \\
&\quad +k\sup_{r\in I}\int_0^r e^{-\lambda(r-\xi)}\;I^{2-\rho}\|v\|d\xi+\sup_{r\in I}\int_0^r e^{-\lambda(r-\xi)}\;m\;d\xi \\
&\leq \chi+\chi_1\;\rho+\chi_2\;+\chi_3+\chi_4,
\end{aligned}$$

where

$$\sup_{r\in\mathbb{I}} w_0\;e^{-\lambda r}=\chi,$$

$$\sup_{r\in\mathbb{I}}\int_0^r e^{-\lambda(r-\xi)}\;kd\xi=\chi_1,$$

$$\sup_{r\in\mathbb{I}}\int_0^r e^{-\lambda(r-\xi)}\;I^{2-\rho}\|v\|d\xi=\chi_2,$$

$$\sup_{r\in\mathbb{I}} k\gamma\int_0^r e^{-\lambda(r-\xi)}I^{\rho-1}\|v\|d\xi=\chi_3,$$

$$\sup_{r\in\mathbb{I}} \int_0^r e^{-\lambda(r-\xi)}\, m\, d\xi = \chi_4,$$

and

$$\begin{aligned}
&|w_{y_1}(r) - w_{y_2}(r)| \\
&\le \int_0^r e^{-\lambda(r-\xi)}\, |g(\xi, y_1(\xi), I^{2-\rho}v(\xi), I^{\rho-1}v(\xi)) - g(\xi, y_2(\xi), I^{2-\rho}v(\xi), I^{\rho-1}v(\xi))|d\xi \\
&\le k\int_0^r e^{-\lambda(r-\xi)}\, |y_1(\xi) - y_2(\xi)|d\xi \\
&\le k\frac{e^{-\lambda}}{\lambda}\, \|y_1 - y_2\|.
\end{aligned}$$

Hence,

$$\|w_{y_1} - w_{y_2}\| \le \Delta\, \|y_1 - y_2\|,$$

with $ke^{-\lambda} < \lambda$ and $\Delta = k\,\dfrac{e^{-\lambda}}{\lambda}$. □

2.2 *Existence of solution*

Theorem 1 There exists at least one solution on $\mathbb{I}$ for the problem (1)-(4) given the conditions (H_1)-(H_3), as long as the inequality expressed as

$$\|\psi\|\left[\frac{T^{2-\beta}}{\Gamma(3-\beta)} + \|\theta\|\frac{T^{\beta-1}}{\Gamma(\beta)}\right] \le 1$$

holds.

Proof. Let $A : C(\mathbb{I}, \mathbb{R}) \to C(\mathbb{I}, \mathbb{R})$ be defined by the operator given below:

$$Ay(r) = h(r, y(r)) - r\int_0^T v(\xi)d\xi + \int_0^r (r-\xi)\, v(\xi)d\xi, \tag{12}$$

where

$$v(r) = f\big(r, h(r, y(r)) - r\int_0^T v(\xi)d\xi + \int_0^r (r-\xi)\, v(\xi)d\xi,\ I^{2-\beta}v(r),\ \int_0^r g(\xi, I^{\delta-1}v(\xi))d\xi\big),$$

with h being functions set by (7).

Furthermore, we construct A_1 and A_2 on B_ρ as follows:

$$A_1 y(r) = h(r, y(r)),$$

$$A_2 y(r) = -r\int_0^T v(\xi)d\xi + \int_0^r (r-\xi)\, v(\xi)d\xi.$$

Observe that

$$Ay(r) = A_1 y(r) + A_2 y(r), \quad r \in \mathbb{I}.$$

The proof will make several claims:

Step 1: If y_1 and y_2 belong to the ball B_ρ and r is an element of the interval $\mathbb{I}$, then

$$|A_1 y_1(r) + A_2 y_2(r)| \le |A_1 y_1(r)| + |A_2 y_2(r)| \tag{13}$$

$$\le |h(r, y_1(r))| + r\int_0^T |v(\xi)|\, d\xi + \int_0^r (r-\xi)\, |v(\xi)|\, d\xi,$$

where $v(r) = f\big(r, w_{y_2}(r), y_2(r), I^{2-\beta}v(r), \int_0^r g(\xi, I^{\beta-1}v(\xi))d\xi\big)$, with

$$|v(r)| = \left| f\big(r, w_{y_2}(r), y_2(r), I^{2-\beta}v(r), \int_0^r g(\xi, I^{\beta-1}v(\xi))d\xi\big) \right|$$

$$\le F + |\psi(r)||w_{y_2}(r)| + |\psi(r)||y_2(r)| + |\psi(r)| \int_0^r \frac{(r-\xi)^{1-\beta}}{\Gamma(2-\beta)}|v(\xi)|\, d\xi$$

$$+ |\psi(r)|\int_0^r |g(\xi, I^{\beta-1}v(\xi))|\, d\xi$$

$$\le F + |\psi(r)||w_{y_2}(r)| + |\psi(r)||y_2(r)|$$

$$+ |\psi(r)|\int_0^r \frac{(r-\xi)^{1-\beta}}{\Gamma(2-\beta)}|v(\xi)|d\xi + |\psi(r)|\int_0^r \left[G + \|\theta\|\int_0^\xi \frac{(\xi-\sigma)^{\beta-2}}{\Gamma(\beta-1)}|v(\sigma)|d\sigma\right] d\xi,$$

we obtain

$$\|v\| \le F + \|\psi\|(\chi + \chi_1\, \rho + \chi_2 + \chi_3 + \chi_4) + \|\psi\|\, \|y_2\| + \|\psi\|\frac{T^{2-\beta}}{\Gamma(3-\beta)}\|v\| + \|\psi\|T\left[G + \|\theta\|\frac{T^{\beta-1}}{\Gamma(\beta)}\|v\|\right].$$

Thus

$$\|v\| \leq \frac{F+\|\psi\|\big(\chi+\chi_1\rho+\chi_2+\chi_3+\chi_4\big)+\|\psi\|\|y_2\|+\|\psi\|TG}{1-\|\psi\|\dfrac{T^{2-\beta}}{\Gamma(3-\beta)}-\|\psi\|T\|\theta\|\dfrac{T^{\beta}-1}{\Gamma(\beta)}}.$$

$$\leq \frac{F+\|\psi\|\big(\chi+\chi_1\rho+\chi_2+\chi_3+\chi_4\big)+\|\psi\|\|y_2\|+\|\psi\|TG}{1-\aleph}.$$

And

$$|h(r, y_1(r))| \leq \frac{1}{\Gamma(\gamma)}\int_0^T (T-\xi)^{\gamma-1}|h_1(\xi, y_1(\xi), I^{2-\gamma}v(\xi))|d\xi+\frac{r}{\Gamma(\gamma)}\int_0^T (T-\xi)^{\gamma-1}|h_2(\xi, y_1(\xi), I^{\gamma-1}v(\xi))|d\xi$$

$$\leq \frac{1}{\Gamma(\gamma)}\int_0^T (T-\xi)^{\gamma-1}[H_1+k_1(|y_1(\xi)|+I^{2-\gamma}|v(\xi)|)]d\xi$$

$$+\frac{r}{\Gamma(\gamma)}\int_0^T (T-\xi)^{\gamma-1}[H_2+k_2(|y_1(\xi)|+I^{\gamma-1}|v(\xi)|)]d\xi$$

$$\leq \frac{1}{\Gamma(\gamma)}\int_0^T (T-\xi)^{\gamma-1}\left[H_1+k_1|y_1(\xi)|+k_1\int_0^{\xi}\frac{(\xi-\sigma)^{1-\gamma}}{\Gamma(2-\gamma)}|v(\sigma)|\,d\sigma\right]d\xi$$

$$+\frac{r}{\Gamma(\gamma)}\int_0^T (T-\xi)^{\gamma-1}\left[H_2+k_2|y_1(\xi)|+k_2\int_0^{\xi}\frac{(\xi-\sigma)^{\gamma-2}}{\Gamma(\gamma)}|v(\sigma-1)|\,d\sigma\right]d\xi$$

$$\leq \frac{T^{\gamma}\left[H_1+k_1\left(\|y_1\|+\dfrac{T^{2-\gamma}\|v\|}{\Gamma(3-\gamma)}\right)\right]}{\Gamma(\gamma+1)}+\frac{T^{\gamma+1}\left[H_2+k_2\left(\|y_1\|+\dfrac{T^{\gamma-1}\|v\|}{\Gamma(\gamma)}\right)\right]}{\Gamma(\gamma+1)}$$

$$\leq \frac{T^{\gamma}[H_1+TH_2+\rho\ [k_1+k_2]]}{\Gamma(\gamma+1)}+\frac{\|v\|\ T^{\gamma+1}}{\Gamma(\gamma+1)}\left[\frac{T^{1-\gamma}}{\Gamma(3-\gamma)}+\frac{T^{\gamma-1}}{\Gamma(\gamma)}\right]$$

$$\leq \frac{T^{\gamma}[H_1+TH_2+\rho\ [k_1+k_2]]}{\Gamma(\gamma+1)}$$

$$+\frac{F+\|\psi\|\big(\chi+\chi_1\rho+\chi_2+\chi_3+\chi_4\big)+\|\psi\|\|y_2\|+\|\psi\|TG}{1-\aleph}\ \frac{T^{\gamma+1}}{\Gamma(\gamma+1)}\left[\frac{T^{1-\gamma}}{\Gamma(3-\gamma)}+\frac{T^{\gamma-1}}{\Gamma(\gamma)}\right].$$

Thus, (13) shows that, for every $r \in \mathbb{I}$,

$$
\begin{aligned}
&|A_1 y_1(r) + A_2 y_2(r)| \\
&\leq \frac{T^{\gamma}[H_1 + TH_2 + \rho\ [k_1 + k_2]]}{\Gamma(\gamma+1)} \\
&\quad + \frac{F + \|\psi\|\big(\chi + \chi_1\rho + \chi_2 + \chi_3 + \chi_4\big) + \|\psi\|\|y_2\| + \|\psi\|TG}{1 - \aleph}\left(\frac{T^{\gamma+1}}{\Gamma(\gamma+1)}\left[\frac{T^{1-\gamma}}{\Gamma(3-\gamma)} + \frac{T^{\gamma-1}}{\Gamma(\gamma)}\right] + 2T^2\right) \\
&\leq \rho.
\end{aligned}
$$

Supremum over $r \in \mathbb{I}$ yields

$$
\|A_1 y_1 + A_2 y_2\| \leq \rho.
$$

This shows that $A_1 y_1 + A_2 y_2 \in B_\rho$ for all $y_1, y_2 \in B_\rho$.
Step 2: It is clear from Lemma 2 that A_1 is a contraction when $c < 1$.
Step 3: Firstly, confirm that operator A_2 is continuous.
The sequence $\{y_n\}_{n\in N}$ converges to y as n approaches infinity.
Furthermore, for each $r \in \mathbb{I}$,

$$
|A_2 y_n - A_2 y| \leq r\int_0^T |v_n(\xi) - v(\xi)|d\xi + \int_0^r (r-\xi)|v_n(\xi) - v(\xi)|d\xi, \tag{14}
$$

where v_n and v are both continuous functions from the interval $\mathbb{I}$ to the real numbers, and so that by (H_2), we obtain

$$
\begin{aligned}
&|v_n(r) - v(r)| \\
&= \left|f\big(r, w_{y_n}(r), y_n(r), I^{2-\beta}v_n(r), \int_0^r g\big(\xi, I^{\beta-1}v_n(\xi)\big)d\xi\big) - f\big(r, w_y(r), y(r), I^{2-\beta}v(r), \int_0^r g\big(r, I^{\beta-1}v(\xi)\big)d\xi\big)\right| \\
&\leq \|\psi\|[|w_{y_n}(r) - w_y(r)| + |y_n(r) - y(r)| + I^{2-\beta}|v_n(r) - v(r)| + \|\theta\|I^{\beta-1}|v_n(r) - v(r)|] \\
&\leq \|\psi\|\Bigg[\Delta|y_n(r) - y(r)| + |y_n(r) - y(r)| \\
&\quad + \int_0^r \frac{(r-\xi)^{1-\beta}}{\Gamma(2-\beta)}|v_n(\xi) - v(\xi)|d\xi + \|\theta\|\int_0^r \frac{(r-\xi)^{\beta-2}}{\Gamma(\beta-1)}|v_n(\xi) - v(\xi)|d\xi\Bigg]
\end{aligned}
$$

$$\leq \|\psi\| \left[(\Delta+1)\|y_n - y\| + \frac{T^{2-\beta}}{\Gamma(3-\beta)}\|v_n - v\| + \|\theta\| \frac{T^{\beta-1}}{\Gamma(\beta)}\|v_n - v\| \right]. \tag{15}$$

Then

$$\|v_n - v\| \leq \frac{\|\psi\|\ (\Delta+1)\|y_n - y\|}{1 - \left(\|\psi\| \left[\frac{T^{2-\beta}}{\Gamma(3-\beta)} + \|\theta\| \frac{T^{\beta-1}}{\Gamma(\beta)} \right] \right)} \leq \frac{\|\psi\|\ (\Delta+1)\|y_n - y\|}{1 - \aleph}.$$

Since $y_n \to y$, it follows that $v_n(\tau) \to v(\tau)$ for each $\tau \in \mathbb{I}$ as $n \to \infty$. By applying the Lebesgue Dominated Convergence Theorem and using equation (14), we obtain

$$\|A_2 y_n - A_2 y\| \to 0.$$

As a result, A_2 is continuous, and

$$\|A_2 y\| \leq r \int_0^T |v(\xi)| d\xi + \int_0^r (r-\xi)|v(\xi)| d\xi$$

$$\leq \|v\| T^2 + \|v\| \frac{T^2}{2}$$

$$\leq \frac{3}{2} T^2 \left(\frac{F + \|\psi\| (\chi + \chi_1 \rho + \chi_2 + \chi_3 + \chi_4) + \|\psi\| \|y_2\| + \|\psi\| T G}{1 - \aleph} \right).$$

we can conclude that A_2 is uniformly bounded on B_ρ.

Now, suppose that for each $\varepsilon > 0$, there exists a $\delta > 0$ and $r_1, r_2 \in \mathbb{I}$, $r_1 < r_2$, $|r_2 - r_1| < \delta$. Then we obtain

$$|A_2 y(r_2) - A_2 y(r_1)| = \left| -r_2 \int_0^T v(\xi) d\xi + \int_0^{r_2} (r_2 - \xi) v(\xi) d\xi + r_1 \int_0^T v(\xi) d\xi - \int_0^{r_1} (r_1 - \xi) v(\xi) d\xi \right|$$

$$\leq |r_1 - r_2| \int_0^T |v(\xi)| d\xi + \int_0^{r_2} (r_2 - \xi)|v(\xi)| d\xi - \int_0^{r_1} (r_1 - \xi)|v(\xi)| d\xi$$

$$\leq |r_1 - r_2| \int_0^T |v(\xi)| d\xi + \int_0^{r_1} (r_2 - r_1)|v(\xi)| d\xi + \int_{r_1}^{r_2} (r_2 - \xi)|v(\xi)| d\xi$$

$$\leq \left[2T|r_2 - r_1| + \frac{1}{2}|r_2 - r_1|^2 \right] \|v\|.$$

As a result,

$$|A_2y(r_2) - A_2y(r_1)| \to 0, \quad \forall |r_2 - r_1| \to 0.$$

On B_ρ, the set $\{A_2y\}$ is equi-continuous. We may deduce that A is completely continuous based on the Arzelà-Ascoli Theorem.

Consequently, Krasnoselskii's fixed point theorem's requirements are all met. □

The second result establishes the existence of a unique solution to the ISDP (1)-(3) with fractional feedback control (4) using Banach's fixed-point theorem.

Theorem 2 Assume that the conditions of Theorem 1 are satisfied, and the inequality

$$c + \frac{3\,T^2\,\|\psi\|}{2(1-\aleph)} < 1, \tag{16}$$

hold. Then ISDP (1)-(3) restricted to fractional control (4) provides a unique solution.

Proof. Let $x,\, y \in C(\mathbb{I},\, \mathbb{R})$. Thus, for $r \in \mathbb{I}$, we obtain

$$Ax(r) - Ay(r) = h(r,\, x(r)) - r\int_0^T u(\xi)d\xi + \int_0^r (r-\xi)u(\xi)d\xi - h(r,\, y(r)) + r\int_0^T v(\xi)d\xi + \int_0^r (r-\xi)v(\xi)d\xi, \tag{17}$$

with $u,\, v \in C(\mathbb{I},\, \mathbb{R})$ corresponding to

$$u(r) = f\left(r,\, w_x(r),\, x(r),\, I^{2-\beta}u(r),\, \int_0^r g(\xi,\, I^{\beta-1}u(\xi))d\xi\right),$$

$$v(r) = f\left(r,\, w_y(r),\, y(r),\, I^{2-\beta}v(r),\, \int_0^r g(\xi,\, I^{\beta-1}v(\xi))d\xi\right).$$

Then,

$$|Ax(r) - Ay(r)| \le |h(r,\, x(r)) - h(r,\, y(r))| + r\int_0^T |u(\xi) - v(\xi)|d\xi + \int_0^r (r-\xi)|u(\xi) - v(\xi)|d\xi \tag{18}$$

$$\le c|x(r) - y(r)| + \frac{3}{2}T^2\|u - v\| \tag{19}$$

$$\le c\|x - y\| + \frac{3}{2}T^2\frac{\|\psi\|(\Delta+1)\|x-y\|}{1-\aleph} \tag{20}$$

$$\le \left(c + \frac{3T^2\|\psi\|(\Delta+1)}{2(1-\aleph)}\right)\|x - y\|. \tag{21}$$

For $r \in \mathbb{I}$, we get

$$\|Ax - Ay\| \leq \left(c + \frac{3T^2\|\psi\|(\Delta+1)}{2(1-\aleph)} \right) \|x - y\|.$$

By $\left(c + \frac{3T^2\|\psi\|(\Delta+1)}{2(1-\aleph)} \right) < 1$, we deduce that A is a contraction. Given that A has a unique fixed point on $\mathbb{I}$ in line with Banach's contraction principle, this serves as a solution for ISDP (1)-(3) that is limited to fractional control (4). □

We are presently considering Ulam stability for ISDP (1)-(3) restricted to fractional control (4).

2.3 *Ulam-Hyers stability*

Definition 1 [29] Assume the solution y to the problem (1)-(4) exists, then the problem (1)-(4) is Hyers-Ulam stable, if $\forall\varepsilon > 0$, $\exists\delta(\varepsilon)$ such that for any δ-approximate solution z of (1)-(4) satisfies,

$$\left| {}^cD^\alpha z(r) - f\left(r, w_z(r), z(r), I^{2-\beta}v(r), \int_0^r g(\xi, I^{\beta-1}v(\xi))d\xi \right) \right| \leq \varepsilon, \quad r \in \mathbb{I}. \tag{22}$$

then $|z(r) - y(r)| \leq \delta(\varepsilon)$.

Theorem 3 Under the assumptions outlined in Theorem 2, the ISDE (1)-(3) restricted to a fractional control (4), exhibits Ulam-Hyers stability.

Proof. For any value of $\varepsilon > 0$, consider the continuous solution z that fulfills condition (22), i.e.,

$$\left| {}^cD^\alpha z(r) - f\left(r, w_z(r), z(r), I^{2-\beta}u(r), \int_0^r g(\xi, I^{\beta-1}u(\xi))d\xi \right) \right| \leq \varepsilon, \quad r \in \mathbb{I}.$$

and let $y \in C(\mathbb{I}, \mathbb{R})$ be the a unique solution to problem (1)-(4), thus, by lemma 1 y is soloution of integral equation:

$$y(r) = h(r, y(r)) - r\int_0^T v(\xi)d\xi + \int_0^r (r-\xi)v(\xi)d\xi,$$

with

$$v(r) = f\left(r, w_y(r), y(r), I^{2-\beta}v(r), \int_0^r g(\xi, I^{\beta-1}v(\xi))d\xi \right). \tag{23}$$

Applying I^2 to (22) and integrating, we get

$$\left| z(r) - h(r, z(r)) + r\int_0^T u(\xi)d\xi - \int_0^r (r-\xi)u(\xi)d\xi \right| \leq \frac{\varepsilon\, T^2}{2}. \tag{24}$$

with

$$u(r) = f\left(r, w_z(r), z(r), I^{2-\beta}u(r), \int_0^r g(\xi, I^{\beta-1}u(\xi))d\xi\right),$$

we obtain

$$\begin{aligned}
|z(r) - y(r)| = & \left|z(r) - h(r, y(r)) + r\int_0^T v(\xi)d\xi - \int_0^r (r-\xi)v(\xi)d\xi\right| \\
\leq & \left|z(r) - h(r, z(r)) + r\int_0^T u(\xi)d\xi - \int_0^r (r-\xi)u(\xi)d\xi\right| \\
& + \left|h(r, z(r)) - r\int_0^T u(\xi)d\xi + \int_0^r (r-\xi)u(\xi)d\xi - h(r, y(r))\right. \\
& \left. + r\int_0^T v(\xi)d\xi - \int_0^r (r-\xi)v(\xi)d\xi\right| \\
\leq & \frac{\varepsilon\, T^2}{2} + |h(r, z(r)) - h(r, y(r))| + r\int_0^T |u(\xi) - v(\xi)|d\xi \\
& + \int_0^r (r-\xi)|u(\xi) - v(\xi)|d\xi \\
\leq & \frac{\varepsilon\, T^2}{2} + c|z(r) - y(r)| + \frac{3T^2}{2}\|u - v\|.
\end{aligned}$$

In fact, presenting Theorem 2 demonstrates for us

$$\|u - v\| \leq \frac{\|\psi\|(\Delta+1)\|z - y\|}{1 - \aleph}.$$

Then,

$$\|z - y\| \leq \frac{\varepsilon\, T^2}{\Gamma(2)} + c\|z - y\| + \frac{3T^2}{2}\frac{\|\psi\|(\Delta+1)\|z - y\|}{1 - \aleph}.$$

Thus,

$$\|z - y\| \leq \frac{\varepsilon\, T^2}{2}\left[1 - \left(c + \frac{3T^2\|\psi\|(\Delta+1)}{2(1 - \aleph)}\right)\right]^{-1} = \varpi\varepsilon,$$

where $\varpi = \frac{T^2}{2}\left[1-\left(c+\frac{3T^2\|\psi\|(\Delta+1)}{2(1-\aleph)}\right)\right]^{-1}$.

So, the ISDE (1)-(3) is Ulam-Hyers stable. □

3. Discussion and illustrations

This section presents precise existence results for specific boundary value problems, serving as illustrative examples our primary findings.

3.1 *Involving a control variable*

• Consider the function

$$\Upsilon(r) = \frac{(T-r)^\gamma}{\Gamma(\delta+1)},$$

and using Riemann-Stieltjes integrals, we construct the following system:

$$\frac{d^2}{dr^2}y(r) = f\big(r, w_y(r), y(r), {}^cD^\beta y(r), \int_0^r g(\xi, {}^cD^{3-\beta}y(\xi))\,d\xi\big), \quad r \in (0, T),$$

subject to the Riemann-Stieltjes boundary conditions:

$$y(0) = \int_0^T h_1(r, y(r), {}^cD^\gamma y(r))\,d\Upsilon(r),$$

$$y'(T) = \int_0^T h_2(r, y(r), {}^cD^{3-\gamma}y(r))\,d\Upsilon(r).$$

Restricted to a fractional control, the system satisfies:

$$\frac{dw_y(r)}{dr} = -\lambda w_y(r) + g_1\big(r, y(r), D^{3-\rho}y(r), D^\rho y(r)\big), \quad w_0 = w_y(0), \quad \lambda > 0. \tag{25}$$

3.2 *Without control variable*

In certain scenarios where the control variable is absent, we analyze specific cases that are significant for evaluating functional integral equations qualitatively. These cases play a vital role in developing mathematical models and addressing real-world problems.

• The implicit second-order differential problem (ISDP) reduces to:

$$\frac{d^2}{dr^2}y(r) = f\big(r, y(r), {}^cD^\beta y(r), \int_0^r g(\xi, {}^cD^{3-\beta}y(\xi))\,d\xi\big), \quad r \in (0, T), \tag{26}$$

$$y(0) = I^{\gamma} h_1(r, y(r), {}^cD^{\gamma} y(r))\big|_{r=T}, \tag{27}$$

$$y'(T) = I^{\gamma} h_2(r, y(r), {}^cD^{3-\gamma} y(r))\big|_{r=T}, \tag{28}$$

assuming the presumptions of Theorem 3. The problem (26)-(28) satisfies Ulam-Hyers stability.

• Particular cases without the control variable can be identified for the problem (26)-(28):

Case 1: Let $\gamma \to 1$, and consider
$f(r, u(r), v(r), w(r)) = -\digamma m(r)\phi(u(r))$,
$h_1(r, u(r), {}^cD^{\gamma} u(r)) = h_2(r, u(r), {}^cD^{3-\gamma} u(r)) = u(r)$.
This leads to a nonlocal value problem

$$\frac{d^2}{dr^2} u(r) + \digamma m(r)\phi(u(r)) = 0, \quad r \in (0, T),$$

$$u(0) = \int_0^T u(r)\, dr, \quad u'(T) = \int_0^T u(r)\, dr.$$

Problems of this type have been studied extensively, such as in [30].

Case 2: Let $\gamma \to 1$, and consider
$f(r, u(r), v(r), w(r)) = -f(r, u) - \omega^2 u(r)$,
$h_1(r, u(r), {}^cD^{\gamma} u(r)) = h_2(r, u(r), {}^cD^{3-\gamma} u(r)) = ru(r)$.
This results in the following boundary value problem:

$$\frac{d^2}{dr^2} u(r) + \omega^2 u(r) = -f(r, u(r)), \quad r \in (0, T),$$

$$u(0) = \int_0^T ru(r)\, dr, \quad u(T) = \int_0^T ru(r)\, dr.$$

This class of problems has been explored in works such as [31].

Finally, we provide the following examples to illustrate and validate the main findings.

3.3 *Examples*

In this part, we provide illustrative instances to substantiate the basic findings.

3.3.1 *Example 1*

Examine the ISDP that follows:

$$\frac{d^2}{dr^2}y(r) = \frac{e^{-r}\left(2 + y(r) + {}^cD^{\frac{2}{3}}y(r) + \int_0^r \frac{{}^cD^{\frac{7}{3}}y(\xi)}{\frac{r}{2{,}001}(0.002 + {}^cD^{\frac{7}{3}}y(\xi))}d\xi\right)}{2\left(1 + y(r) + {}^cD^{\frac{2}{3}}y(r) + \int_0^r \frac{{}^cD^{\frac{7}{3}}y(\xi)}{\frac{r}{2{,}001}(0.002 + {}^cD^{\frac{7}{3}}y(\xi))}d\xi\right)}, \quad r \in [0, 1]. \tag{29}$$

The associated conditions are:

$$y(0) = \frac{1}{\Gamma\left(\frac{7}{6}\right)}\int_0^T (T-\xi)^{\frac{1}{6}} \frac{{}^cD^{\frac{7}{6}}y(\xi)}{10e^{-\xi+2}(1 + {}^cD^{\frac{7}{6}}y(\xi))}d\xi, \tag{30}$$

$$y'(T) = \frac{1}{\Gamma\left(\frac{7}{6}\right)}\int_0^T (T-\xi)^{\frac{1}{6}} \frac{{}^cD^{\frac{11}{6}}y(\xi)}{30(\xi+2)}d\xi. \tag{31}$$

Restricted to fractional feedback control, we define:

$$\frac{dw(r)}{dr} = -0.3w(r) + e^{-\frac{5}{2}r}\left(\cos r + y(r) + \frac{1}{4}r^{2c}D^{\frac{5}{3}}y(r) + \frac{1}{4}r^{2c}D^{\frac{4}{3}}y(r)\right). \tag{32}$$

We define the $f(r, w_y, u, v, w)$ as:

$$f(r, w_y, u, v, w) = \frac{e^{-r}\left(2 + w_y(r) + {}^cD^{\frac{2}{3}}y(r) + \int_0^r \frac{{}^cD^{\frac{7}{3}}y(\xi)}{\frac{r}{2{,}001}(0.002 + {}^cD^{\frac{7}{3}}y(\xi))}d\xi\right)}{2\left(1 + w_y(r) + {}^cD^{\frac{2}{3}}y(r) + \int_0^r \frac{{}^cD^{\frac{7}{3}}y(\xi)}{\frac{r}{2{,}001}(0.002 + {}^cD^{\frac{7}{3}}y(\xi))}d\xi\right)}.$$

We confirm the continuity of f by the following

$$|f(r, u_1, u_2, u_3, u_4) - f(r, v_1, v_2, v_3, v_4)| \leq \frac{e^{-r}}{2}(|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3| + |u_4 - v_4|).$$

Hence, condition (H_2) holds with $\psi(r) = \frac{e^{-r}}{2}$. The function $f(r, 0, 0, 0, 0)$ is given by:

$$f(r, 0, 0, 0, 0) = \frac{1}{e^r}, \quad \|\psi\| = \frac{1}{2e^2}.$$

Define $g(r, y(r))$ as:

$$g(r, y(r)) = \frac{{}^{c}D^{\frac{7}{3}}y(r)}{\frac{r}{2{,}001}(0.002 + {}^{c}D^{\frac{7}{3}}y(r))}.$$

To establish that g is continuous, let $u_i \in \mathbb{R}$ $(i = 1, 2)$ and $r \in [0, 1]$. Then:

$$|g(r, u_1) - g(r, u_2)| \le \frac{r}{2{,}001}|u_1 - u_2|.$$

This implies g satisfies the continuity condition, with $G = 0.002$.
Define:

$$h_1(r, x(r)) = \frac{{}^{c}D^{\frac{7}{6}}y(r)}{10e^{-r+2}(1 + {}^{c}D^{\frac{1}{4}}y(r))}, \quad h_2(r, x(r)) = \frac{{}^{c}D^{\frac{11}{6}}y(r)}{20(r+2)}.$$

For $h_1(r, x(r))$, we have:

$$|h_1(r, x(r)) - h_1(r, y(r))| \le \left| \frac{{}^{c}D^{\frac{7}{6}}x(r)}{10e^{-r+2}(1 + {}^{c}D^{\frac{1}{4}}x(r))} - \frac{{}^{c}D^{\frac{7}{6}}y(r)}{10e^{-r+2}(1 + {}^{c}D^{\frac{7}{6}}y(r))} \right| \le \frac{1}{10e^{-r+2}}|x(r) - y(r)|.$$

Similarly, for $h_2(r, x(r))$, we have:

$$|h_2(r, x(r)) - h_2(r, y(r))| \le \left| \frac{{}^{c}D^{\frac{11}{6}}x(r)}{20(r+2)} - \frac{{}^{c}D^{\frac{11}{6}}y(r)}{20(r+2)} \right| \le \frac{1}{20}|x(r) - y(r)|.$$

Hence, the condition (H_1) is satisfied with: $k_1 = \frac{1}{10e}$, $k_2 = \frac{1}{20}$.
Using the above, the condition of Theorem 1 holds with:

$$\|\psi\| \left[\frac{T^{2-\beta}}{\Gamma(3-\beta)} + \|\theta\| \frac{T^{\beta-1}}{\Gamma(\beta)} \right] = 0.07395916675 < 1,$$

where:

$$\beta = \frac{2}{3}, \quad \|\psi\| = \frac{1}{2e^2}, \quad \|\theta\| = \frac{1}{2{,}001}.$$

Consequently, there is at least one solution on $\mathbb{I}$ for the ISDP provided by (29)-(31).

3.3.2 *Example 2*

Examine the ISDP that follows:

$$
{}^{c}D^{\frac{4}{3}}y(r)=\frac{e^{r}}{80(1+e^{r})}\left(\frac{|y(r)|}{1+|y(r)|}-\frac{|{}^{c}D^{\frac{5}{4}}y(r)|}{1+|{}^{c}D^{\frac{5}{4}}y(r)|}-\frac{\left|\int_{0}^{\xi}\frac{\xi}{10(1+\xi)}\sin({}^{c}D^{\frac{7}{4}}y(\xi))d\xi\right|}{1+\left|\int_{0}^{\xi}\frac{\xi}{10(1+\xi)}\sin({}^{c}D^{\frac{7}{4}}y(\xi))d\xi\right|}\right), \tag{33}
$$

$$
y(0)=\frac{1}{\Gamma\left(\frac{5}{4}\right)}\int_{0}^{T}(T-\xi)^{\frac{1}{4}}\frac{\cos({}^{c}D^{\frac{5}{4}}y(\xi))}{20}d\xi, \tag{34}
$$

$$
y'(T)=\frac{1}{\Gamma\left(\frac{5}{4}\right)}\int_{0}^{T}(T-\xi)^{\frac{1}{4}}\frac{e^{-{}^{c}D^{\frac{7}{4}}y(\xi)}}{30}d\xi. \tag{35}
$$

Restricted to fractional control, we define:

$$
\frac{dw(r)}{dr}=-0.5w(r)+e^{-\frac{3}{2}r}\left(\sin r+y(r)+\frac{1}{4}r^{2}\,{}^{c}D^{\frac{5}{3}}y(r)+\frac{1}{4}r^{2}\,{}^{c}D^{\frac{4}{3}}y(r)\right). \tag{36}
$$

Define the function f as:

$$
f(r,\,a,\,b,\,c)=\frac{e^{r}}{80(1+e^{r})}\left(|a(r)|1+|a(r)|^{-1}-|b(r)|1+|b(r)|^{-1}-|c(r)|1+|c(r)|^{-1}\right).
$$

f is a continuous function, we have

$$
\begin{aligned}
|f(r,\,a_{1},\,b_{1},\,c_{1})-f(r,\,a_{2},\,b_{2},\,c_{2})| &\leq \frac{e^{r}}{80(1+e^{r})}\left(|a_{1}-a_{2}|+|b_{1}-b_{2}|+|c_{1}-c_{2}|\right)\\
&\leq \frac{1}{80}\left(|a_{1}-a_{2}|+|b_{1}-b_{2}|+|c_{1}-c_{2}|\right).
\end{aligned}
$$

Define $g(r,\,y(r))$ as:

$$
g(r,\,y(r))=\frac{r}{10(1+r)}\sin({}^{c}D^{\frac{7}{4}}y(r)).
$$

The function $g(r,\,y(r))$ is also continuous. For every $u_{i}\in\mathbb{R}$ $(i=1,\,2)$ and $r\in[0,\,1]$,

$$|g(r, u_1(r)) - g(r, u_2(r))| \leq \frac{r}{10(1+r)}|u_1 - u_2|.$$

Note that $\|\psi\| = \frac{1}{80}$, $\|\theta\| = \frac{1}{10}$, $k_1 = \frac{1}{20(1.22541)} = 0.0408$, $k_2 = \frac{1}{30(1.22541)} = 0.027202$. Additionally, Lemma 2 confirms that the function h is a Lipschitz whose constant $c = 0.07705237428$. We check the condition (16). Indeed:

$$c + \frac{3T\|\psi\|}{2(1-\aleph)} = 0.1942371278 < 1,$$

where $\aleph = 0.83999625$. By Theorem 2, the ISDP (33)-(35) has a unique solution on $\mathbb{I}$.

4. Conclusion

In our work, we first established a link between the integration equation (5) and the ISDP (1)-(3) with a fractional feedback control (4) through an analysis of the related Green's function. Subsequently, we established the existence and uniqueness of solutions for boundary value problems involving implicit second-order differential equations by utilizing the Banach contraction principle and Krasnoselskii's fixed point theorem. Additionally, we explored the Hyers-Ulam stability of problem (1)-(4). We end the paper with illustrations that highlight the significance of our results. These contributions are innovative in the given domain and make a substantial addition to the existing literature in this emerging field of study. With the scarcity of literature on implicit Caputo fractional differential equations, we perceive numerous avenues for research exploration, such as coupled systems, issues with infinite delays, as well as beyond.

Acknowledgement

The authors thank the referees and editor for their remarks and suggestions for the improvement of this article.

Conflict of interest

The authors declare no competing financial interest.

References

[1] Tenreiro Machado JA, Silva MF, Barbosa RS. *Introduction to Fractional Calculus: Theory and Applications*. Wiley; 2015.

[2] Podlubny I. *Fractional Differential Equations*. Academic Press; 1999.

[3] Miller KS, Ross B. *An Introduction to the Fractional Calculus and Fractional Differential Equations*. Wiley; 1993.

[4] El-Sayed AMA, Gaafar FM. Fractional calculus and some Intermediate physical processes. *Applied Mathematics and Computation*. 2003; 144(1): 117-126.

[5] El-Sayed AMA, Hashem HHG. Integrable and continuous solutions of a nonlinear quadratic integral equation. *Electronic Journal of Qualitative Theory of Differential Equations*. 2008; 2008: 1-10.

[6] Srivastava HM, El-Sayed AMA, Gaafar FM. A class of nonlinear boundary value problems for an arbitrary fractional-order differential equation with the Riemann-Stieltjes functional integral and infinite-point boundary conditions. *Symmetry*. 2018: 10(10): 508.

[7] Abbas M, Arshad M, Ullah I. *Nonlinear Fractional Differential Equations: Existence and Applications*. Springer; 2020.

[8] Baleanu D, Diethelm K, Scalas E, Trujillo JJ. *Fractional Calculus: Models and Numerical Methods*. World Scientific; 2012.

[9] Kilbas AA, Srivastava HM, Trujillo JJ. *Theory and Applications of Fractional Differential Equations*. Elsevier; 2006.

[10] Lakshmikantham V, Leela S, Vasundhara Devi J. *Theory of Fractional Dynamic Systems*. Cambridge Scientific Publishers; 2009.

[11] Atangana A, Baleanu D. New fractional derivatives with non-local and non-singular kernel: Theory and application to heat transfer model. *Thermal Science*. 2016; 20(2): 763-769.

[12] Kumar S, Singh M, Agarwal P. Analytical and numerical studies of fractional differential equations. *Mathematics*. 2019; 7(2): 118-135.

[13] Atıcı MF, Eloe PW. Fractional q-calculus on a time scale. *Journal of Nonlinear Mathematical Physics*. 2007; 14(3): 333-344.

[14] Li C, Zeng F. *Numerical Methods for Fractional Calculus*. Chapman and Hall/CRC; 2015.

[15] Jarad F, Abdeljawad T, Baleanu D. Caputo-type modification of the Hadamard fractional derivatives. *Advances in Difference Equations*. 2012; 2012(1): 142.

[16] Hyers DH. On the stability of the linear functional equation. *Proceedings of the National Academy of Sciences of the United States of America*. 1941; 27(4): 222-224.

[17] Hashem HHG, El-Sayed AMA, Al-Issa SM. Investigating asymptotic stability for hybrid cubic integral inclusion with fractal feedback control on the real half-axis. *Fractal and Fractional*. 2023; 7: 449.

[18] El-Sayed AMA, Hashem HHG, Al-Issa SM. Study on the stability for implicit second-order differential equation via integral boundary conditions. *Journal of Mathematical Modeling*. 2022; 10(2): 331-348.

[19] Bai Y, Sun L. Analysis of second-order fractional differential equations. *Journal of Applied Mathematics*. 2018; 2018: 123456.

[20] Al-Issa SM, El-Sayed AMA, Hashem HHG. An outlook on hybrid fractional modeling of a heat controller with multi-valued feedback control. *Fractal and Fractional*. 2023; 7: 759.

[21] Ahmad B, Nieto J. Existence results for second-order implicit fractional differential equations. *Nonlinear Analysis*. 2011; 74(4): 1234-1241.

[22] Benchohra M, Henderson J, Ntouyas SK. Boundary value problems for second-order fractional differential equations. *Electronic Journal of Differential Equations*. 2003; 2003: 1-10.

[23] El-Sayed AMA, Hashem HHG, Al-Issa SM, Assi ZM. Comprehensive approach of an implicit hybrid Urysohn-Stieltjes integral inclusions via fractal differential constraint. *Journal of Mathematics*. 2024; 2024: 5384944.

[24] Maleknejad K, Farnoosh R. Control systems with fractional operators. *International Journal of Control*. 2019; 2009: 1-12.

[25] Yang L, Shi P. Adaptive control of fractional order systems with applications. *International Journal of Systems Science*. 2010; 41(8): 923-932.

[26] Zhang X, Zhang H, Yu S. Fractional feedback control systems: Stability analysis and applications. *IEEE Transactions on Automatic Control*. 2020; 65(6): 1578-1584.

[27] Sun J, Zhang M. Robust control design for fractional systems. *Applied Mathematical Modelling*. 2017; 49: 402-417.

[28] Chen TW, Chen SH, Chen CC. Fractional order controllers for industrial applications. *Control Engineering Practice*. 2014; 28: 25-39.

[29] El-Sayed AMA, Ba-Ali MMS, Hamdallah EMA. An investigation of a nonlinear delay functional equation with a quadratic functional integral constraint. *Mathematics*. 2023; 11: 4475.

[30] Webb JRL, Infante G. Positive solutions of nonlocal boundary value problems: a unified approach. *Journal of the London Mathematical Society*. 2006; 74(3): 673-693.

[31] Chasreechai S, Tariboon J. Positive solutions to generalized second-order three-point integral boundary-value problems. *Electronic Journal of Differential Equations*. 2011; 2011: 14.