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Research Article

Iterated Function Systems of Generalized Multivalued Mappings in Partial Metric Spaces

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Abstract: This work rigorously proves the existence of attractors of a finite collection of generalized multivalued mappings which are generalized mapping that are defined in the setup of partial metric spaces. We hereby put forward Generalized Multivalued Iterated Function Systems (GMIFS) and we obtain corresponding results under different types of assumptions known as generalized contractive circumstances. We construct a few examples that can be used to illustrate the results obtained in this manuscript. Additionally, this work extends several outcomes documented in prior studies within the fields of Iterated Function Systems (IFS).

***Keywords*:** attractor, fixed point, multivalued mapping, iterated function system, partial metric space

MSC: 47H10, 47H07, 47H04

1. Introduction

Iterated Function System (IFS) is an approach to construct iterated or self-similar fractals, which arise from the magnificent merger of geometry of fractals and the theory of sets. By using a transformation of function system, IFS can be generated as a finite union of multiple copies of itself. In 1981, John E. Hutchinson proved in his work [1] that for any nonempty compact subset $S \subset X$ in which X is endowed with a complete metric setting, and the IFS given by the functions $\{f_i : X \mapsto X,\ 1 \le i \le n\}$, $n \in \mathbb{N}$ admits exactly one and only one fixed point S. This existence and uniqueness of S follows as a result of the relevance of the Banach's contraction principle [2]. Hutchinson's construction as described in his work, is crucial for the aforementioned construction; for further details, we refer to the work [1] for additional information about this construction. An extended version of the celebrated Banach's contraction findings are documented for instance in the work of Michael Fielding Barsnley and Vince in [3] and Michael Fielding Barsnley [4] where subsequent progress and development on the geometry of fractals were clearly made. In [4], the author classified the aforementioned fixed point as an attractor for the given IFS. It is worth mentioning that IFS gained popularity after the publication of the book "*Fractals everywhere*" by Michael Fielding Barnsley [4]. It should noted that the theory of fixed points heavily relies on Banach's contraction approach in the setup of metric spaces and several results have been extended in this direction, including various applications such as iterative methods in the resolution of difference equations, differential equations, or integral equations, among others. Further references and additional results can be obtained from the works of other researchers [5–14] cited in the references.

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Partial metric spaces expand classical metric spaces through their ability to have non-zero self-distances for points which makes them suitable for domains with natural incomplete or approximate information. The most important application of partial metric spaces exists in computer science through domain theory and programming language semantics because they create a natural system for studying computational process convergence and data approximation. The mathematical theory of fixed points serves as a foundation for various optimization algorithms as well as image processing and data analysis methods. The ability of partial metrics to represent progress or approximation beyond equality makes them essential for modeling iterative methods and incomplete data structures and real-world systems that lack exact measurements.

For generalized countable IFS in the metric spaces setups, Secelean's work provides relevant results, specifically in Secelean's findings [15]. Nadler's results (see for instance [16]) from 1969 also contribute to obtaining solutions to fixed-point problems of multivalued or set-valued mappings defined in the setting of metric spaces.

For more recent applications of fixed point in the theory of neutrosophic, we refer to papers [17, 18] and for the notion of topological spaces in the setup of b-fuzzy theory we refer for instance to the work of Al-Omeri in [19]. In the manuscript, [17] the neutrosophic fixed point theorems and cone metric spaces have been investigated whereas in [18], the author introduced the well-known Property (P) along with new Fixed Point Results were studied on ordered metric spaces in the framework of neutrosophic theory. Additionally, the b-fuzzy topological spaces were studied by the authors of the manuscript [19] .

In this work, our aim is to construct fractals using generalized forms of IFS which is a broader class of IFS for generalized multivalued contractions in the setup of partial metric spaces. Noticing that the Hutchinson operator, which is characterized by the action of all IFS, is commonly used operator in the mathematical study of fractals. Namely, It defines the IFS and is a characteristic of actions on sets of contractions. By observing this, we can easily see that the mentioned operator which is acting over a countable collection of contractions on a complete partial metric space X is also a generalized contraction defined on a collection of compact sets in the Hausdorff metric. By iteratively applying this technique using the generalized Hutchinson operator, then ultimately, this process produces the final fractal structure.

To the best of our knowledge, the results in the present parer are sharp and completely new.

To obtain the necessary results, we will now follow the approach used in [20]. We will then first introduce some basic concepts which are very relevant across various sections and that may be used repeatedly here in this manuscript. For further details in these concepts, we refer to [11, 20–22]

For definitions, remarks and examples of partial metric spaces and complete partial metric spaces we refer to [11, 20–22].

As in [21, 22], let $\mathscr{CB}^p(Y)$ denotes the class of closed and bounded subsets of Y, each of which is non-empty.

We follow the lines of [21] to introduce the result below that will be crucial for the rest of the paper.

Proposition 1 Given the pair $(Y,\ p)$ of an arbitrary partial metric space made up of the non-empty set Y and the metric p, a subset $M \subset Y$ and the subsets $L,\ K \in \mathscr{CB}^p(Y)$, and $\upsilon \in Y$. Then for all $K_1,\ K_2,\ K_3 \in \mathscr{CB}^p(Y)$,

(1) $H_p(K_1,\ K_1) \leq H_p(K_1,\ K_2)$,

(2) $H_p(K_1,\ K_2) = H_p(K_2,\ K_1)$,

(3) $H_p(K_1,\ K_2) \leq H_p(K_1,\ K_3) + H_p(K_3,\ K_2) - \inf_{\eta\in K_3} p(\eta,\ \eta)$.

Here

$$p(\upsilon,\ K) = \inf\{p(\upsilon,\ \zeta)\colon \zeta \in K\}, \quad \delta_p(K,\ L) = \sup\{p(\zeta,\ L)\colon \zeta \in K\},$$

$$\delta_p(L,\ K) = \sup\{p(\eta,\ K)\colon \eta \in L\},$$

$$H_p(K_1,\ K_2) = \max\{\delta_p(K_1,\ K_2),\ \delta_p(K_1,\ K_2)\},$$

$$K_1 = K_2 \text{ whenever } H_p(K_1, K_2) = 0. \tag{1}$$

From the above two results, we note that the metric p induces another metric

$$H_p\colon \mathscr{CB}^p(Y) \times \mathscr{CB}^p(Y) \to [0, \infty) \tag{2}$$

which we casually call the partial Hausdorff distance which leads us to borrow another definition from [21].

Definition 1 Given a partial distance on a non-empty set Y. Then $\mathscr{C}^p \subset Y$ is compact whenever from each sequence $\{v_n\} \subset \mathscr{C}^p$ we can extract a subsequence $\{v_{n_i}\}$ that converges in $\mathscr{C}^p$.

For a partial distance on $Y \neq \emptyset$, the space $\mathscr{C}^p(Y)$ denotes the set of compact subsets of the set Y. Given $K_1, K_2 \in \mathscr{CB}^p(Y)$, we set

$$H_p(K_1, K_2) = \max\{\sup_{\eta \in K_2} p(\eta, K_1), \sup_{\mu \in K_1} p(\mu, K_2)\}. \tag{3}$$

Note that $p(t, K_1) = \inf\{p(t, \mu)\colon \mu \in K_1\}$ denotes the map that measures the distance between the point t and the set K_1. We call H_p the Pompeiu-Hausdorff distance induced by the partial distance p and the pair $(\mathscr{CB}^p(Y), H_p)$ is a complete partial metric space, under the condition that the pair (Y, p) is a complete partial metric space as well.

We state a crucial result that is very relevant across all sections of this manuscript.

Lemma 1 [23] We consider a partial metric p on a non-empty set Y, then for all $K_1, K_2, K_3, K_3 \in \mathscr{C}^p(Y)$, it hold that

(i) if $K_1 \subseteq K_2$, then $\sup_{m \in K_3} p(m, K_2) \leq \sup_{m \in K_3} p(m, K_1)$,

(ii) $\sup_{t \in K_3 \cup K_1} p(y, K_2) = \max\{\sup_{m \in K_3} p(m, K_2), \sup_{\ell \in K_1} p(\ell, K_2)\}$,

(iii) $H_p(K_1 \cup K_2, K_3 \cup K_4) \leq \max\{H_p(K_1, K_3), H_p(K_2, K_3)\}$.

Theorem 1 [20] Consider a complete partial metric p defined on $Y \neq \emptyset$ and given a contraction map $h\colon Y \to Y$. Given $\lambda \in [0, 1)$, we have

$$p(ht_1, ht_2) \leq \lambda p(t_1, t_2), \text{ for all } t_1, t_2 \in Y. \tag{4}$$

Then h admits a unique fixed point u in Y and for each $v_0 \in Y$, the collection of iterates $\{v_0, hv_0, h^2 v_0, \ldots\}$ converges to the fixed point u.

We study generalized IFS applied to partial metric spaces. Detailed constructions and subsequent results on a G-IFS for multi-valued maps in the setup of metric spaces can be found in [24]. We will introduce the concept of generalized contraction self-map and highlight some basic results that we will need throughout the paper. As usual, we will consider a pair (Y, d) to denote the partial metric space we are dealing with and sometimes the pair is complete whenever necessary.

Definition 2 [21] Given a pair (Y, p) as above with $T\colon Y \to \mathscr{CB}^p(Y)$ be multivalued mapping. A map T is said to be a multivalued contraction whenever there is some $\lambda \in (0, 1)$ which satisfies

$$H_p(Tv_1, Tv_2) \leq \lambda p(v_1, v_2) \text{ for all } v_1, v_2 \in Y. \tag{5}$$

Lemma 2 [21] We consider a partial metric p on a non-empty set Y, $K_1, K_2 \in \mathscr{CB}^p(Y)$. Then for $a \in K_1$, we have $b \in K_2$ which satisfies the inequality:

$$p(a, b) \leq \eta H_p(M, N), \tag{6}$$

where $\eta > 1$.

Theorem 2 Given the partial metric space (Y, p) and we consider $T: Y \to \mathscr{CB}^p(Y)$ a continuous map. Assume that the map T is a multivalued contraction with $\lambda \in (0, \frac{1}{\eta})$, where $\eta > 1$ as defined in Lemma 2. Then

(1) the operator T preserves $\mathscr{C}^p(Y)$, namely, it maps each of its elements back into $\mathscr{C}^p(Y)$;

(2) provided that if for every element U in the space $\mathscr{C}^p(Y)$ the self-map $T(\mathscr{C}^p(Y)) \subseteq \mathscr{C}^p(Y)$ is given by

$$T(U) = \{t_1: t_1 \in T(U)\}. \tag{7}$$

then the mapping T is a multivalued contraction acting over $(\mathscr{C}^p(Y), H_p)$ with domain of sets.

Proof. (1) Note the map T is a continuous function by hypothesis and we also know that if a set is compact then its image under a continuous map is also compact. Therefore, we can assert that

$$U \in \mathscr{C}^p(Y) \tag{8}$$

from which we infer that

$$T(U) \in \mathscr{C}^p(Y). \tag{9}$$

As for item (2), we note that the map T is a generalized multivalued contraction as mentioned above, we then have for $\lambda \in (0, \frac{1}{\eta})$ that

$$H_p(Tv_1, Tv_2) \leq \lambda p(v_1, v_2) \text{ for all } v_1, v_2 \in Y. \tag{10}$$

Now

$$H_p(T(U), T(V)) = \max\{\sup_{s_1 \in T(U)} p(s_1, T(V)), \sup_{s_2 \in T(V)} p(s_2, T(U))\}$$

$$\leq \max\{\sup_{s_1 \in T(U)} \inf_{v_1 \in T(V)} p(s_1, v_1), \sup_{s_2 \in T(V)} \inf_{v_2 \in T(U)} p(s_2, v_2)\}$$

$$\leq \max\{\eta_1 H_p(TU, TV), \eta_2 H(TV, TU)\}$$

$$\leq \eta^* \max\{\sup_{t_1 \in U} \lambda p(t_1, V), \sup_{t_2 \in V} \lambda p(t_2, U)\}$$

$$= \lambda^* \max\{\sup_{t_1 \in U} p(t_1, V), \sup_{t_2 \in V} p(t_2, U)\}$$

$$= \lambda^* H_p(U, V), \tag{11}$$

where $\eta^* = \max\{\eta_1, \eta_2\}$ and $\lambda^* = \eta^* \lambda$. Consequently, for $\lambda^* \in (0, 1)$, we infer that

$$H_p(T(U), T(V)) \le \lambda^* H_p(U, V). \tag{12}$$

Hence, the mapping T is a multivalued contraction acting over the space $(\mathscr{C}^p(Y), H_p)$ with domain of sets.

Proposition 2 Consider the pair (Y, p) as above, *i.e.,* a partial metric space. Assume that the continuous maps $T_k \colon Y \to \mathscr{CB}^p(Y)$ with $1 \le k \le r$ satisfy the conditions

$$H_p(T_k(v_1), T_k(v_2)) \le \lambda_k p(v_1, v_2) \text{ for all } v_1, v_2 \in Y, \tag{13}$$

with the sequence $\lambda_k \in (0, \frac{1}{\eta_k})$ for $1 \le k \le r$, where η_k is defined as in Lemma 2. Then the mapping $\Phi(\mathscr{C}^p(Y)) \subseteq \mathscr{C}^p(Y)$ is given by

$$\Psi(U) = T_1(U) \cup T_2(U) \cup \cdots \cup T_r(U) = \cup_{k=1}^r T_k(U) \quad \text{with } U \in \mathscr{C}^p(Y) \tag{14}$$

also satisfies

$$H_p(\Psi U, \Psi V) \le \Theta H_p(U, V) \text{ with } U, V \in \mathscr{C}^p(Y), \tag{15}$$

where $\Theta = \max\{\lambda_k \colon 1 \le k \le r\}$. Then the mapping Ψ is a multivalued contraction on $\mathscr{C}^p(Y)$ with domain of sets.

Proof. We shall prove the result for $k \in \{1, 2\}$ *i.e.,* $r = 2$ in particular. Let $T_1, T_2 \colon Y \to Y$ be two contractions. For $K_1, K_2 \in \mathscr{C}^p(Y)$ and owing to the results in item (c) of Lemma 1, we obtain that

$$H_p(\Psi(K_1), \Psi(K_2)) = H_p(T_1(K_1) \cup T_2(K_1), T_1(K_2) \cup T_2(K_2))$$

$$\le \max\{H_p(T_1(K_1), T_1(K_2)), H_p(T_2(K_1), T_2(K_2))\}$$

$$\le \max\{\lambda_1 H_p(K_1, K_2), \lambda_2 H_p(K_1, K_2)\}$$

$$\le \Theta H_p(K_1, K_2). \tag{16}$$

This completes the desired proof of Proposition 2.

Definition 3 Given the partial metric space pair (Y, p) and a self-map Ψ: $\mathscr{C}^p(Y) \to \mathscr{C}^p(Y)$. The self-map Ψ is a generalized multivalued Hutchinson contractive operator if for any pair of sets $K_1, K_2 \in \mathscr{C}^p(Y)$, it holds that

$$H_p(\Psi(K_1), \Psi(K_2)) \leq \lambda Z_\Psi(K_1, K_2), \text{ for some } \lambda \in (0, 1) \tag{17}$$

where

$$Z_\Psi(K_1, K_2) = \max\left\{H_p(K_1, K_2), H_p(K_1, \Psi(K_1)), H_p(K_2, \Psi(K_2)), \frac{H_p(K_1, \Psi(K_2)) + H_p(K_2, \Psi(K_1))}{2}\right\}. \tag{18}$$

Owing to the fact that the above defined map Ψ is generalized multivalued contraction acting over $\mathscr{C}^p(Y)$. We can also assert that the map Ψ is a generalized multivalued Hutchinson contractive operator. We need to clearly emphasize that in the case of the reverse implication of this assertion, the claimed results may fail.

Definition 4 Assuming that the partial metric space pair (Y, p) is complete and also assuming that T_k: $Y \to CB^p(Y)$, with $1 \leq k \leq r$ is continuous map such that each T_k for $1 \leq k \leq r$ and $k, r \in \mathbb{N}$, is a generalized multivalued contraction. We call the collection, $\{Y; T_k, 1 \leq k \leq r\}$ a generalized multivalued IFS.

Definition 5 Given the compact set $\emptyset \neq K_1 \subseteq Y$, the subset K_1 is an attractor of the generalized multivalued IFS whenever the following hold true:

(a) $\Psi(K_1) = K_1$,

(b) $K \subseteq V_1$ and $\lim_{m\to+\infty} \Psi^m(K_2) = \lim_{m\to+\infty} \Phi^k(K_2) = K_1$ for any compact set $K_2 \subseteq V_1$, for some open subset V_1 of Y. Here, the limit is in the sense of the partial Hausdorff metric.

As previously stated, we can again define the basin of common attraction, the maximal open set V_1 satisfying item (b) of Definition 5.

Having said that, we are now ready to introduce the statements of our principal findings discussed in the subsequent sections.

2. Main results

The following section presents our findings about attractor existence and uniqueness for generalized multivalued Hutchinson contractive operators within partial metric spaces. For the reminder of the manuscript, we use the complete partial metric space (Y, p) to establish our main results.

Theorem 3 With the pair (Y, p) as stated above, let $\{Y; T_k, k = 1, 2, \cdots, r\}$ be a generalized multivalued IFS and let us define the self map, Ψ: $\mathscr{C}^p(Y) \to \mathscr{C}^p(Y)$ as

$$\Psi(L) = \cup_{k=1}^{r} T_k(L) \text{ for any } L \in \mathscr{C}^p(Y). \tag{19}$$

We assume that the operator Ψ is a generalized multivalued Hutchinson contractive mapping. Then the mapping Ψ has exactly one and only one attractor $U_1 \in \mathscr{C}^p(Y)$. Namely,

$$U_1 = \Psi(U_1). \tag{20}$$

Moreover, we can arbitrarily choose an initial guess set $L_0 \in \mathscr{C}^p(Y)$, to assert that the iterate sequence

$$\{L_0, \Psi(L_0), \Psi^2(L_0), \Psi^3(L_0), ...\} \tag{21}$$

of compact sets converges to U_1.

We note that in the above theorem, the limit U_1 is called the attractor of Ψ.

Proof. We randomly pick a constituent $L_0 \in \mathscr{C}^p(Y)$ and we set the elements

$$L_1 := \Psi(L_0),\ L_2 := \Psi(L_1), ..., L_{k+1} := \Psi(L_k) \text{ for } k \in \{0, 1, 2, ...\}. \tag{22}$$

By hypothesis, we know that the mapping Ψ is a generalized multivalued Hutchinson contractive operator, we therefore infer from the given hypothesis that one can find some constant $\lambda > 0$ (which does not dependent on k) yet satisfies

$$H_p(L_{k+1}, L_{k+2}) = H_p(\Psi(L_k), \Psi(L_{k+1})) \leq \lambda Z_\Psi(L_k, L_{k+1}), \tag{23}$$

where we express the term $Z_\Psi(L_k, L_{k+1})$ as:

$$\begin{aligned} Z_\Psi(L_k, L_{k+1}) &= \max\left\{H_p(L_k, L_{k+1}), H_p(L_k, \Psi(L_k)), H_p(L_{k+1}, \Psi(L_{k+1})), \frac{H_p(L_k, \Psi(L_{k+1})) + H_p(L_{k+1}, \Psi(L_k))}{2}\right\} \\ &= \max\left\{H_p(L_k, L_{k+1}), H_p(L_k, L_{k+1}), H_p(L_{k+1}, L_{k+2}), \frac{H_p(L_k, L_{k+2}) + H_p(L_{k+1}, L_{k+1})}{2}\right\}. \end{aligned} \tag{24}$$

It now follows from the definitions of Z_Ψ, H_p and by identification that

$$\begin{aligned} Z_\Psi(L_k, L_{k+1}) &\leq \max\left\{H_p(L_k, L_{k+1}), H_p(L_{k+1}, L_{k+2}), \frac{H_p(L_k, L_{k+1}) + H_p(L_{k+1}, L_{k+2})}{2}\right\} \\ &= \max\{H_p(L_k, L_{k+1}), H_p(L_{k+1}, L_{k+2})\}. \end{aligned} \tag{25}$$

We thus deduce the relation

$$\begin{aligned} H_p(L_{k+1}, L_{k+2}) &\leq \lambda \max\{H_p(L_k, L_{k+1}), H_p(L_{k+1}, L_{k+2})\} \\ &= \lambda H_p(L_k, L_{k+1}). \end{aligned} \tag{26}$$

Therefore, for all $k \in \{0, 1, 2, ...\}$, we obtain

$$
\begin{aligned}
H_p(L_{k+1}, L_{k+2}) &\le \lambda H_p(L_k, L_{k+1}) \\
&\le \lambda^2 H_p(L_{k-1}, L_k) \\
&\le \cdots \\
&\le \lambda^{k+1} H_p(L_0, L_1).
\end{aligned} \tag{27}
$$

Note that since $l > k$, for any $k, l \in \{0, 1, 2, ...\}$, we also get that

$$
\begin{aligned}
H_p(L_k, L_l) \le H_p(L_k, L_{k+1}) + H_p(L_{k+1}, L_{k+2}) + \cdots + H_p(L_{l-1}, L_l) \\
- \inf_{l_{k+1} \in L_{k+1}} p(l_{k+1}, l_{k+1}) - \inf_{l_{k+2} \in L_{k+2}} p(l_{k+2}, l_{k+2}) - \\
\cdots - \inf_{l_{k-1} \in L_{k-1}} p(l_{k-1}, l_{k-1}),
\end{aligned} \tag{28}
$$

which along with the definitions of the metrics (p, H_p) enables us to infer that

$$
\begin{aligned}
H_p(L_k, L_l) &\le [\lambda^k + \lambda^{k+1} + \cdots + \lambda^{l-1}] H_p(L_0, L_1) \\
&= \lambda^k [1 + \lambda + \lambda^2 + \cdots + \lambda^{l-k-1}] H_p(L_0, L_1)] \\
&\le \frac{\lambda^k}{1-\lambda} H_p(L_0, L_1).
\end{aligned} \tag{29}
$$

Along with this later estimate, we also obtain $\lim_{k, l \to +\infty} H_p(L_k, L_l) = 0$. This enables us to deduce that the collection $\{L_k\}$ is a Cauchy in $\mathscr{C}^p(Y)$. We must emphasize that, clearly, by hypothesis the partial metric space $(\mathscr{C}^p(Y), H_p)$ is thereby complete, which enables us to assert that one can easily find some element $U_1 \in \mathscr{C}^p(Y)$ satisfying

$$
\lim_{k \to +\infty} L_k = U_1.
$$

Namely,

$$
\lim_{k \to +\infty} H_p(L_k, U_1) = \lim_{k \to +\infty} H_p(L_k, L_{k+1}) = H_p(U_1, U_1). \tag{30}
$$

Along with the above, also, we obtain $\lim_{k \to +\infty} H_p(L_k, U_1) = 0$.

We intend in proving $\Psi(U_1) = U_1$. For this purpose, using the hypothesis and the definition of H_p, we infer that

$$\begin{aligned} H_p(\Psi(U_1), U_1) &\leq H_p(\Psi(U_1), \Psi(L_{k+1})) + H_p(\Psi(L_{k+1}), U_1) - \inf_{l_{k+1}\in L_{2k+1}} p(l_{k+1}, l_{k+1}), \\ &\leq \lambda Z_\Psi(U_1, L_{k+1}) + H_p(L_{k+2}, U_1) - \inf_{l_{k+1}\in L_{k+1}} p(l_{k+1}, l_{k+1}) \end{aligned} \tag{31}$$

for any $k \in \{0, 1, 2, ...\}$, and as in the previous expression $Z_\Psi(U_1, L_{k+1})$, we can evidently write it as:

$$\begin{aligned} &Z_\Psi(U_1, L_{k+1}) \\ &= \max\{H_p(U_1, L_{k+1}), H_p(U_1, \Psi(U_1)), H_p(L_{k+1}, \Psi(L_{k+1})), \frac{H_p(U_1, \Psi(L_{k+1})) + H_p(L_{k+1}, \Psi(U_1))}{2}\} \\ &- \inf_{l_{k+1}\in L_{k+1}} p(l_{k+1}, l_{k+1}) \\ &= \max\left\{H_p(U_1, L_{k+1}), H_p(U_1, \Psi(U_1)), H_p(L_{k+1}, L_{k+2}), \frac{H_p(U_1, L_{k+2}) + H_p(L_{k+1}, \Psi(U_1))}{2}\right\} \\ &- \inf_{l_{k+1}\in L_{k+1}} p(l_{k+1}, l_{k+1}). \end{aligned} \tag{32}$$

Now, we thoroughly inspect the alternative cases below proficiently.

(I) We assume that $Z_\Psi(U_1, L_{k+1}) = H_p(U_1, L_{k+1})$, from which we obtain

$$\begin{aligned} H_p(\Psi(U_1), U_1) &\leq \lambda H_p(U_1, L_{k+1}) + H_p(L_{k+2}, U_1) - \inf_{m_{k+1}\in L_{k+1}} p(l_{k+1}, l_{k+1}) \\ &\leq \lambda H_p(U_1, L_{k+1}) + H_p(L_{k+2}, U_1). \end{aligned} \tag{33}$$

We take the limit on both sides of relation (33) as $k \to +\infty$, we hence obtain that

$$H_p(\Psi(U_1), U_1) \leq \lambda H_p(U_1, U_1) + H_p(U_1, U_1), \tag{34}$$

from which we deduce that $H_p(\Psi(U_1), U_1) = 0$. Namely, we actually showed that $U_1 = \Psi(U_1)$. Hence, this is the desired result we are required to show.

(II) We assume also that $Z_\Psi(U_1, L_{k+1}) = H_p(U_1, \Psi(U_1))$, from which we infer that

$$
\begin{aligned}
H_p(\Psi(U_1), U_1) &\leq \lambda H_p(U_1, \Psi(U_1)) + H_p(L_{k+2}, U_1) - \inf_{l_{k+1} \in L_{k+1}} p(l_{k+1}, l_{k+1}) \\
&\leq \lambda H_p(U_1, \Psi(U_1)) + H_p(L_{k+2}, U_1).
\end{aligned} \tag{35}
$$

Hence it is clear from the inequality (35) that

$$
H_p(\Psi(U_1), U_1) \leq \frac{1}{1-\lambda} H_p(L_{k+2}, U_1). \tag{36}
$$

We take the limit on both side of inequality (36) as $k \to +\infty$ we infer that $H_p(\Psi(U_1), U_1) \leq 0$ from which we deduce that $U_1 = \Psi(U_1)$.

(III) Particularly, for $Z_\Psi(U_1, L_{k+1}) = H_p(L_{k+1}, L_{k+2}),$ then we simply infer that

$$
\begin{aligned}
H_p(U_1, \Psi(U_1)) &\leq \lambda H_p(L_{k+1}, L_{k+2}) + H_p(L_{k+2}, U_1) - \inf_{m_{k+1} \in L_{k+1}} p(l_{k+1}, l_{k+1}) \\
&\leq \lambda H_p(L_{k+1}, L_{k+2}) + H_p(L_{k+2}, U_1).
\end{aligned} \tag{37}
$$

We take the limit on both sides of the relation (37) as $k \to +\infty$ to infer easily that $U_1 = \Psi(U_1)$.

(IV) As for the case of

$$
Z_\Psi(U_1, L_{2k+1}) = \frac{H_p(U_1, L_{k+2}) + H_p(L_{k+1}, \Psi(U_1))}{2}, \tag{38}
$$

we obtain from the above estimates that

$$
\begin{aligned}
H_p(U_1, \Psi(U_1)) &\leq \frac{\lambda}{2}[H_p(U_1, L_{k+2}) + H_p(L_{k+1}, \Psi(U_1))] + H_p(L_{k+2}, U_1) - \inf_{l_{k+1} \in L_{k+1}} p(l_{k+1}, l_{k+1}) \\
&\leq \frac{\lambda}{2}[H_p(U_1, L_{k+2}) + H_p(L_{k+1}, U_1) + H_p(U_1, \Psi(U_1)) - \inf_{u \in U_1} p(u, u)] + H_p(L_{k+2}, U_1) \\
&\quad - \inf_{m_{k+1} \in L_{k+1}} p(l_{k+1}, l_{k+1}) \\
&\leq \frac{\lambda}{2}[H_p(U_1, L_{k+2}) + H_p(L_{k+1}, U_1) + H_p(U_1, \Psi(U_1))] + H_p(L_{k+2}, U_1).
\end{aligned} \tag{39}
$$

Again, we take the limit on both sides of relation (39) as k tends to become very large, we obtain

$$H_p(U_1, \Psi(U_1)) \leq \frac{\lambda}{2} H_p(U_1, \Psi(U_1)), \tag{40}$$

from which we have $H_p(U_1, \Psi(U_1))$ vanishes along we also get that $U_1 = \Psi(U_1)$.

Thus far, combining all the above approaches (with all the particular cases), simply put, U_1 is the irresistible magnet (attractor) for Ψ, that is, $U_1 = \Psi(U_1)$.

With the aim of demonstrating the uniqueness of the attractor, we first consider a strong prevalent match attractor for the map Ψ denoted by U_2. Considering the fact that the map Ψ is a generalized multivalued function in the Hutchinson sense, we infer that

$$\begin{aligned} H_p(U_1, U_2) &= H_p(\Psi(U_1), \Psi(U_2)) \\ &\leq \lambda \max\left\{ H_p(U_1, U_2), H_p(U_1, \Psi(U_1)), H_p(U_2, \Psi(U_2)), \frac{H_p(U_1, \Psi(U_2)) + H_p(U_2, \Psi(U_1))}{2} \right\} \\ &= \lambda \max\left\{ H_p(U_1, U_2), H_p(U_1, U_1), H_p(U_2, U_2), \frac{H_p(U_1, U_2) + H_p(U_2, U_1)}{2} \right\} \\ &\leq \lambda H_p(U_1, U_2). \end{aligned} \tag{41}$$

This enables us to obtain $(1-\lambda)H_p(U_1, U_2) \leq 0$. Therefore, $H_p(U_1, U_2) = 0$. Namely, we thus proved that $U_1 = U_2$. Thus $U_1 \in \mathscr{C}^p(Y)$ is the unique attractor for the map Ψ.

Now that we have established the existence and uniqueness of attractor, we can proceed to another important result known as the Generalized Collage. As previously mentioned, our spaces are considered complete partial metric space and denoted by (Y, p). We would like to reiterate this point before presenting the statement and proof of the generalized Collage result.

Theorem 4 For a generalized multivalued IFS given by $\{Y; T_1, T_2, \ldots, T_r\}$, where $\lambda \in (0, \frac{1}{\eta_k})$ is contractive constant as given in Proposition 2. Let $\varepsilon \geq 0$ and if for any $L \in \mathscr{C}^p(Y)$, it holds that

$$H_p(L, \Psi(L)) \leq \varepsilon, \tag{42}$$

where $\Psi(L) = \cup_{k=1}^{r} T_k(L)$. Then, we infer that

$$H_p(L, U_1) \leq \frac{\varepsilon}{1-\Theta}, \tag{43}$$

where $U_1 \in \mathscr{C}^p(Y)$ is the attractor of Ψ with $\Theta = \max\{\lambda_k \colon k \in \{1, 2, \ldots, r\}\}$.

Proof. Owing to the results of Proposition 2 we infer that the maps Ψ satisfies the relation

$$H_p(\Psi(\mathscr{U}), \Psi(\mathscr{V})) \leq \Theta H_p(\mathscr{U}, \mathscr{V}) \text{ for all } \mathscr{U}, \mathscr{V} \in \mathscr{C}^p(Y), \tag{44}$$

where $\Theta = \max\{\lambda_k \colon 1 \leq k \leq r\}$.

Now by virtue of Theorem 3, we can assert that the map Ψ, admits a unique attractor $U_1 \in \mathscr{C}^p(Y)$ which can be expressed as $U_1 = \Psi(U_1)$.

Now taking any initial guess $N_0 \in \mathscr{C}^p(Y)$, and consider the family $\{N_k\}$ defined by $N_{k+1} = \Psi(N_k)$ for any integer $k \geq 1$ we obtain

$$\lim_{k \to +\infty} H_p\left(\Psi(N_k), U_1\right) = 0. \tag{45}$$

Since $H_p(L, \Psi(L)) \leq \varepsilon$ for any $L \in \mathscr{C}^p(Y)$, we easily obtain

$$\begin{aligned} H_p(L, U_1) &\leq H_p(L, \Psi(L)) + H_p(\Psi(L), \Psi(U_1)) - \inf_{m \in \Psi(L} p(l, l) \\ &\leq \varepsilon + \Theta H_p(L, U_1), \end{aligned} \tag{46}$$

from which we deduce the inequality

$$H_p(L, U_1) \leq \frac{\varepsilon}{1 - \Theta}. \tag{47}$$

This completes the intended proof.

Example 1 Let $Y = [0, 1] \times [0, 1]$ and the partial metric $p \colon Y \times Y \to \mathbb{R}^+$ defined as:

$$p(\mathbf{x}, \mathbf{y}) = \max\{x_1, x_2\} + \max\{y_1, y_2\} \text{ for all } \mathbf{x} = (x_1, y_1), \mathbf{y} = (x_2, y_2) \in Y. \tag{48}$$

Define $T_1, T_2 \colon Y \to CB^p(Y)$ as

$$\begin{aligned} T_1(\mathbf{x}) &= \left[0, \frac{y_1}{3}\right] \times \left[0, \frac{x_1}{3}\right] \text{ for } \mathbf{x} = (x_1, y_1) \in Y, \\ T_2(\mathbf{x}) &= \left[0, \frac{x_1}{3}\right] \times \left[0, \frac{y_1 + 0.5}{3}\right] \text{ for } \mathbf{x} = (x_1, y_1) \in Y, \\ T_3(\mathbf{x}) &= \left[0, \frac{x_1 + 0.5}{3}\right] \times \left[0, \frac{y_1 + 0.5}{3}\right] \text{ for } \mathbf{x} = (x_1, y_1) \in Y, \\ T_4(\mathbf{x}) &= \left[0, \frac{y_1 + 0.5}{3}\right] \times \left[0, \frac{x_1}{3}\right] \text{ for } \mathbf{x} = (x_1, y_1) \in Y. \end{aligned} \tag{49}$$

First we are to show that

$$H_p\left(T_i\left(\mathbf{x}\right), T_i\left(\mathbf{y}\right)\right) \leq \lambda_i z_{T_i}(\mathbf{x}, \mathbf{y}), \tag{50}$$

holds for each $i = 1, 2, 3, 4,$ where

$$z_{T_i}(\mathbf{x}, \mathbf{y}) = \max\{p(\mathbf{x}, \mathbf{y}), p(\mathbf{x}, T_i(\mathbf{x})), p(\mathbf{y}, T_i(\mathbf{y})), \frac{p(\mathbf{x}, T_i(\mathbf{y})) + p(\mathbf{y}, T_i(\mathbf{x}))}{2}\}. \tag{51}$$

Note that for $\mathbf{x}, \mathbf{y} \in Y,$ we have

$$\begin{aligned} H_p\left(T_1\left(\mathbf{x}\right), T_1\left(\mathbf{y}\right)\right) &= \max\left\{\frac{y_1}{3}, \frac{y_2}{3}\right\} + \max\left\{\frac{x_1}{3}, \frac{x_2}{3}\right\} \\ &= \frac{1}{3}\max\{x_1, x_2\} + \frac{1}{3}\max\{y_1, y_2\} \\ &= \frac{1}{3}p(\mathbf{x}, \mathbf{y}) \\ &\leq \lambda_1 z_{T_1}(\mathbf{x}, \mathbf{y}), \end{aligned} \tag{52}$$

where $\lambda_1 = \frac{1}{3}$.

$$\begin{aligned} H_p\left(T_2\left(\mathbf{x}\right), T_2\left(\mathbf{y}\right)\right) &= \max\left\{\frac{x_1}{3}, \frac{x_2}{3}\right\} + \max\left\{\frac{y_1+0.5}{3}, \frac{y_2+0.5}{3}\right\} \\ &= \frac{1}{3}[\max\{x_1, x_2\} + \max\{y_1+0.5, y_2+0.5\}] \\ &\leq \frac{4}{9}[\max\{x_1, \frac{x_2}{3}\} + \max\{y_1, \frac{y_2+0.5}{3}\} + \max\{x_2, \frac{x_1}{3}\} + \max\{y_2, \frac{y_1+0.5}{3}\}] \\ &= \frac{8}{9}[\frac{p(\mathbf{x}, T_2(\mathbf{y})) + p(\mathbf{y}, T_2(\mathbf{x}))}{2}] \\ &\leq \lambda_2 z_{T_2}(\mathbf{x}, \mathbf{y}), \end{aligned} \tag{53}$$

where $\lambda_2 = \frac{8}{9}$.

$$
\begin{aligned}
H_p\left(T_3(\mathbf{x}), T_3(\mathbf{y})\right) &= \max\left\{\frac{x_1+0.5}{3}, \frac{x_2+0.5}{3}\right\} + \max\left\{\frac{y_1+0.5}{3}, \frac{y_2+0.5}{3}\right\} \\
&= \frac{1}{3}[\max\{x_1+0.5, x_2+0.5\} + \max\{y_1+0.5, y_2+0.5\}] \\
&\leq \frac{4}{9}[\max\{x_1, \frac{x_2+0.5}{3}\} + \max\{y_1, \frac{y_2+0.5}{3}\} + \max\{x_2, \frac{x_1+0.5}{3}\} + \max\{y_2, \frac{y_1+0.5}{3}\}] \\
&= \frac{8}{9}[\frac{p(\mathbf{x}, T_3(\mathbf{y})) + p(\mathbf{y}, T_3(\mathbf{x}))}{2}] \\
&\leq \lambda_3 z_{T_3}(\mathbf{x}, \mathbf{y}),
\end{aligned} \tag{54}
$$

where $\lambda_3 = \frac{8}{9}$.

$$
\begin{aligned}
H_p\left(T_4(\mathbf{x}), T_4(\mathbf{y})\right) &= \max\left\{\frac{y_1+0.5}{3}, \frac{y_2+0.5}{3}\right\} + \max\left\{\frac{x_1}{3}, \frac{x_2}{3}\right\} \\
&= \frac{1}{3}[\max\{x_1, x_2\} + \max\{y_1+0.5, y_2+0.5\}] \\
&\leq \frac{4}{9}[\max\{y_1, \frac{y_2+0.5}{3}\} + \max\{x_1, \frac{x_2}{3}\} + \max\{y_2, \frac{y_1+0.5}{3}\} + \max\{x_2, \frac{x_1}{3}\}] \\
&= \frac{8}{9}[\frac{p(\mathbf{x}, T_4(\mathbf{y})) + p(\mathbf{y}, T_4(\mathbf{x}))}{2}] \\
&\leq \lambda_4 z_{T_4}(\mathbf{x}, \mathbf{y}),
\end{aligned} \tag{55}
$$

where $\lambda_4 = \frac{8}{9}$.

We now take the generalized multivalued IFS $\{Y; T_1, T_2, T_3, T4\}$ associated with the corresponding mapping Ψ given as

$$
\Psi(U) = T_1(U) \cup T_2(U) \cup T_3(U) \cup T_4(U) \text{ for all } U \in \mathscr{C}^p(Y). \tag{56}
$$

Owing to the results in Proposition 2, taking the pair $L, M \in \mathscr{C}^p(Y)$, we can infer that

$$
H_p(\Psi(L), \Psi(M)) \leq \Theta H_p(L, M), \tag{57}
$$

where

$$Z_{\Psi}(L, M) = \max\{H_p(L, M), H_p(L, \Psi(L)), H_p(M, \Psi(M)), \frac{H_p(L, \Psi(M)) + H_p(M, \Psi(L))}{2}\} \tag{58}$$

and $\Theta = \max\{\lambda_i, 1 \leq i \leq 4\} = \max\{\frac{1}{3}, \frac{8}{9}\} = \frac{8}{9}$.

We thus deduce that the results in Theorem 2 are satisfied. Furthermore, if we consider an initial guess $L_0 \in \mathscr{C}^p(Y)$, we thus assert that the iterate family

$$\{L_0, \Psi(L_0), \Psi^2(L_0), \cdots\} \tag{59}$$

of compact subsets converges to the attractor of Ψ as its limit point. Figures 1, 2, 3, and 4 show the convergence process of sequence steps at $n = 1, 3, 5,$ and 7, respectively.

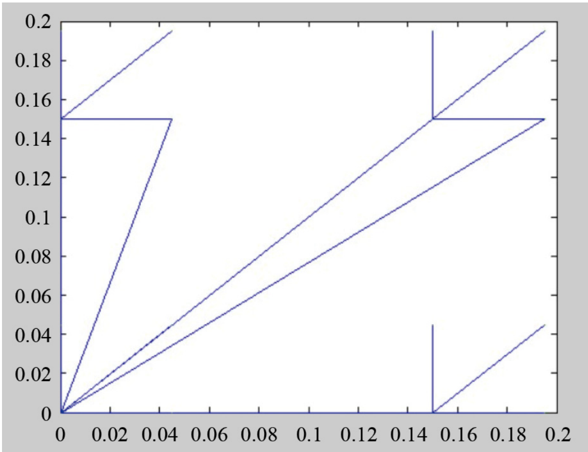

Figure 1. Iteration steps for $n = 1$

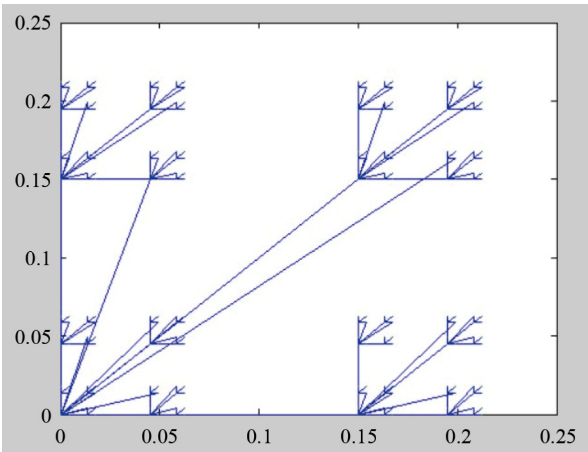

Figure 2. Iteration steps for $n = 3$

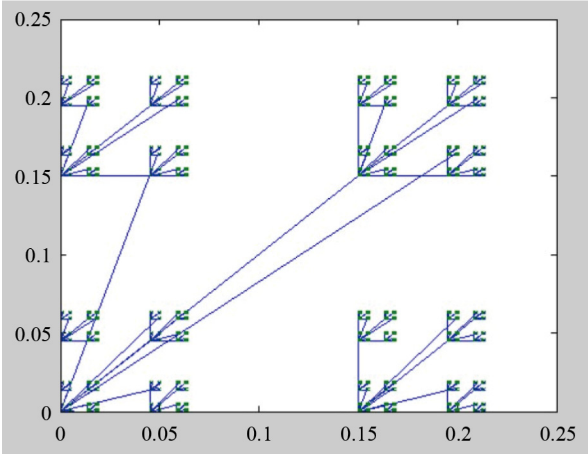

Figure 3. Iteration steps for $n = 5$

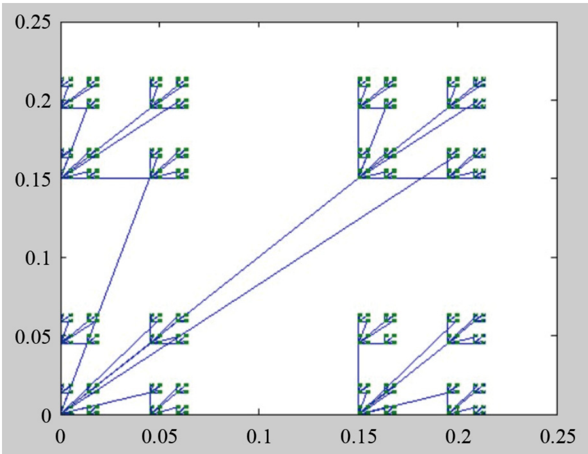

Figure 4. Iteration steps for $n = 7$

In particular, applying the results of Theorem 3 we have the following Remark. Note that whenever we refer to the pair (Y, d) is to denote the complete partial metric space in the following couple of results.

Remark 1 Let $\mathscr{S}^p(Y)$ denotes the family of all unit set emanating from Y. We infer that $\mathscr{S}^p(Y) \subseteq \mathscr{C}^p(Y)$. Note that for $T_k = T$ for any k, where $T = T_1$, then the operator Ψ satisfies

$$\Psi(y_1) = T(y_1). \tag{60}$$

This observation of Remark 1 leads us to the fixed point result below.

Corollary 1 We first assume that $\{Y; T_k, 1 \leq k \leq r\}$ is a generalized multivalued IFS system acting on the pair (Y, p) (complete) and with $T: Y \to Y$ the map as defined in Remark 1. Secondly, we suppose that there is some $\lambda \in (0, 1)$ satisfying the condition:

$$p\left(Ty_1,\, Ty_2\right) \le \lambda Z_T\left(y_1,\, y_2\right),\ \text{ for any } y_1,\, y_2 \in Y, \tag{61}$$

where

$$Z_T\left(y_1,\, y_2\right) = \max\{p(y_1,\, y_2),\, p(y_1,\, Ty_1),\, p(y,\, Ty_2),\, \frac{p(y_1,\, Ty_2) + p(y,\, Ty_1)}{2}\}. \tag{62}$$

T admits a unique fixed point u in Y. Beside this convergences, if we consider any initial guest $u_0 \in Y$, we can assert that the sequence of iterate $\{u_0,\, Tu_0,\, T^2u_0,\, \cdots\}$ is convergent and its limit is the fixed point of T.

Corollary 2 Given $\{Y;\, T_k,\, 1 \le k \le r\}$ a generalized multivalued IFS acting on the pair $(Y,\, p)$ and we consider T_k for $\le k \le r$ to denotes the generalized multivalued contractive self-map defined on Y. We then assert that the mapping $\Psi\colon \mathscr{C}^p(Y) \to \mathscr{C}^p(Y)$ as considered in Theorem 3 above admits at most one attractor which belongs to $\mathscr{C}^p\left(Y\right)$. Furthermore, considering a chosen starting set $L_0 \in \mathscr{C}^p\left(Y\right),$ the sequence of iterates $\{L_0,\, \Phi\left(L_0\right),\, \Psi^2\left(L_0\right),\, \cdots\}$ of compact subsets admits a limit point that is an attractor of Ψ as described above.

In order for us to verify the results of Corollary 2 we intend to introduce an example as below.

Example 2 We set in particular $Y = [0,\, 10]$ and we endow the space Y with the function p given by

$$p(y_1,\, y_2) = \frac{1}{2}\max\{y_1,\, y_2\} + \frac{1}{4}|y_1 - y_2| \text{ for all } y_1,\, y_2 \in Y. \tag{63}$$

Indeed p is a partial metric.

Define $T_1,\, T_2\colon Y \to CB^p\left(Y\right)$ as

$$\begin{aligned} T_1\left(y\right) &= \left[0,\, \frac{10-y}{3}\right] \quad \text{for any } y \in Y \text{and} \\ T_2\left(y\right) &= \left[0,\, \frac{16-y}{4}\right] \quad \text{for each } y \in Y. \end{aligned} \tag{64}$$

Now, for $y_1,\, y_2 \in Y,$ we have

$$\begin{aligned} H_p\left(T_1\left(y_1\right),\, T_1\left(y_2\right)\right) &= \frac{1}{2}\max\left\{\frac{10-y_1}{3},\, \frac{10-y_2}{3}\right\} + \frac{1}{4}\left|\frac{10-y_1}{3} - \frac{10-y_2}{3}\right| \\ &= \frac{1}{3}\left[\frac{1}{2}\max\{10-y_1,\, 10-y_2\} + \frac{1}{4}\left|(10-y_1) - (10+y_2)\right|\right] \\ &\le \frac{1}{3}[\frac{1}{2}\max\{y_1,\, y_2\} + \frac{1}{4}\left|y_1 - y_2\right|] \\ &= \lambda_1 p\left(y_1,\, y_2\right), \end{aligned} \tag{65}$$

where $\lambda_1 = \frac{1}{3}$.

Also, for $y_1, y_2 \in Y$, we have

$$\begin{aligned} H_p\left(T_2\left(y_1\right), T_2\left(y_2\right)\right) &= \frac{1}{2}\max\left\{\frac{16-y_1}{4}, \frac{16-y_2}{4}\right\} + \frac{1}{4}\left|\frac{16-y_1}{4} - \frac{16-y_2}{4}\right| \\ &= \frac{1}{4}\left[\frac{1}{2}\max\{16-y_1, 16-y_2\} + \frac{1}{4}\left|(16-y_1)-(16-y_2)\right|\right] \\ &\leq \frac{1}{4}[\frac{1}{2}\max\{y_1, y_2\} + \frac{1}{4}\left|y_1 - y_2\right|] \\ &= \lambda_2 p\left(y_1, y_2\right), \end{aligned} \tag{66}$$

where $\lambda_2 = \frac{1}{4}$.

We now take the generalized multivalued IFS $\{Y; T_1, T_2\}$ associated with the corresponding mapping Ψ defined by

$$\Psi(U) = T_1(U) \cup T_2(U) \text{ for all } U \in \mathscr{C}^p\left(Y\right). \tag{67}$$

Owing to the results in Proposition 2, taking the pair $L, M \in \mathscr{C}^p\left(Y\right)$, we can infer that

$$H_p\left(\Psi\left(L\right), \Psi\left(M\right)\right) \leq \Theta H_p\left(L, M\right), \tag{68}$$

where $\Theta = \max\left\{\frac{1}{3}, \frac{1}{4}\right\} = \frac{1}{3}$.

We thus deduce that the results in Corollary 2 are satisfied. Furthermore, if we consider an initial guess $L_0 \in \mathscr{C}^p(Y)$, we thus assert that the iterate family

$$\left\{L_0, \Psi\left(L_0\right), \Psi^2\left(L_0\right), \cdots\right\} \tag{69}$$

of compact subsets converges to the attractor of Ψ as its limit point.

3. Conclusions

In this paper, we proved the existence of attractor of generalized IFS based on generalized multivalued mapping in partial metric spaces. In addition, we constructed the above example to illustrate the results presented therein. Further, the generalized collage theorem for these maps in the setup of partial metric space is demonstrated. As partial metric space is linked with the natural way having applications in computer science and mathematics problems (see [20, 21, 23, 25]), allowing the results in this paper to be explored in relation to partial b-metric space and more general metric spaces with much more applications and other areas such as nonlinear analysis or differential equations.

The study of IFS of generalized multivalued mappings in partial metric spaces could be extended in the future to explore their applications for example in fuzzy partial metric spaces, b-fuzzy partial metric spaces and modular metric spaces in connection with the theory of neutrosophic with various engineering applications. These systems should be studied in relation to fractal generation, dynamical systems, and optimization problems. The theory could be extended to stochastic or hybrid mappings which would create new research opportunities for modeling uncertainty and complex systems in applied sciences.

Conflict of interest

The authors declare no competing financial interest.

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