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Research Article

Quiescent Optical Solitons for the Concatenation Model Having Nonlinear Chromatic Dispersion and Kerr Law of Self-Phase Modulation

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Abstract: This paper focuses on the retrieval of quiescent optical solitons within the framework of the concatenation model, incorporating nonlinear chromatic dispersion and the Kerr law of self-phase modulation. These solitons, which remain stable and maintain their shape over time, are crucial for understanding the behavior of light in nonlinear optical media. The retrieval of these solitons is achieved through two distinct techniques. Each of these integration schemes offers a systematic way to derive analytical solutions, ensuring that the underlying dynamics of the optical solitons are accurately captured. In addition to the analytical solutions, this study presents numerical simulations to validate the theoretical findings. These simulations illustrate the behavior of the recovered quiescent solitons, confirming their stability and showcasing their dynamics under the influence of self-phase modulation and nonlinear chromatic dispersion. By bridging analytical methods with computational validation, the paper offers a thorough examination of these soliton structures and their real-world relevance, particularly in the design of advanced optical fiber networks and nonlinear optical devices.

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1. Introduction

It was exactly a decade ago when the concatenation model was conceived [1–5]. This model is formed by the conjunction of the familiar nonlinear Schrödinger's equation (NLSE), the Lakshmanan-Porsezian-Daniel (LPD) equation and the Sasa-Satsuma equation (SSE). This proposed model is projected to implement performance enhancement in the soliton transmission dynamics across transcontinental and transoceanic distances for pulse propagation [6–10]. There are several features that were subsequently addressed for this model. A full spectrum of soliton solutions were retrieved for the model by the aid of undetermined coefficients [11–15]. The conservation laws were derived [16–20]. This was followed by the numerical study of optical solitons for the model by the Laplace-Adomian decomposition method (LADM) [21–25]. The Painleve analysis for the model was carried out [26–30]. Later, the model was applied to recover gap solitons with fiber Bragg gratings [31–35].

After an exhaustive study of the concatenation model with Kerr law of self-phase modulation (SPM) in one-dimensional case, the model was subsequently considered with polarization-mode dispersion that gave way to an additional set of results being reported such as the implementation of the method of undetermined coefficients to retrieve the soliton solutions. The LADM scheme was applied here as well to obtain the numerical simulation of the solitons with impressive accuracy. Subsequently, the dispersive version of the concatenation model emerged later during the same year where the fundamental models that were conjoined to formulate this newer version of the concatenation model were the Schrödinger-Hirota equation, LPD model and the fifth-order NLSE. This too was later studied to recover its soliton solutions along with several additional interesting results.

While both the models were addressed with linear chromatic dispersion (CD), it is now time to turn the page. The current paper addresses the concatenation model with nonlinear CD and with Kerr law of SPM. It is quite well-known that if CD is rendered to be nonlinear the solitons are stalled during its propagation and thus the concept of quiescent optical solitons ensued. The current paper addresses just that. Two integration algorithms that recover the quiescent solitons for the model are projective Riccati equation approach as well as the enhanced Kudryashov's algorithm. The used methods are special cases of the generalized Riccati Equation Mapping Method and the new extended direct algebraic method. They yield a full spectrum of quiescent optical solitons to the concatenation model having nonlinear CD and Kerr-law of SPM. The results are derived after a quick and succinct re-visitation of the integration algorithms. The derived results are exhibited along with the respective parameter constraints, which are also enumerated, for the existence of such solitons.

1.1 *Governing model*

The concatenation model considered in this study follows a specific mathematical formulation, which has been explored in prior works [1–5, 21–27]:

$$iq_t + a\left(|q|^n q\right)_{xx} + b|q|^2 q + c_1\left[\delta_1 q_{xxxx} + \delta_2 (q_x)^2 q^* + \delta_3 |q_x|^2 q + \delta_4 |q|^2 q_{xx} + \delta_5 q^2 q^*_{xx} + \delta_6 |q|^4 q\right]$$

$$+ ic_2\left[\delta_7 q_{xxx} + \delta_8 |q|^2 q_x + \delta_9 q^2 q^*_x\right] = 0. \tag{1}$$

This model serves as a framework for studying complex interactions in nonlinear optical systems by coupling different physical effects. It incorporates nonlinear CD, which accounts for wavelength-dependent variations in refractive index, and Kerr nonlinearity, representing the intensity-dependent refractive index that leads to self-phase modulation.

The concatenation of these effects provides a more comprehensive representation of the dynamics in nonlinear optical media. By using this model, researchers can describe the evolution and propagation of solitons-localized wave packets that preserve their shape due to a balance between nonlinearity and dispersion. The mathematical structure of the concatenation model offers a versatile platform for deriving both analytical and numerical solutions, capturing a wide range of physical behaviors observed in experimental setups. This formulation, detailed across the cited literature, forms the backbone of the present study, enabling the retrieval of quiescent solitons through advanced integration techniques.

To provide a detailed description of equation (1), let's examine the effects of setting specific values for the parameters c_1 and c_2, and the resulting simplified models. Setting $c_1 = 0$ transforms equation (1) into the SSE, a well-known model that accounts for the higher-order nonlinear effects, such as third-order dispersion, that are significant in nonlinear optics. This equation offers insights into the behavior of ultrashort pulses in optical fibers, where high-order dispersion terms become relevant. When we further assume $c_1 = c_2 = 0$, equation (1) reduces to the NLSE, one of the fundamental models in nonlinear wave propagation that describes the evolution of wave envelopes in dispersive and nonlinear media. The NLSE characterizes optical solitons, stable pulse solutions that maintain their shape due to a balance between dispersion and nonlinearity. Additionally, setting only $c_2 = 0$ modifies equation (1) to resemble the LPD model. This model captures intermediate effects between the SSE and the NLSE by incorporating second-order dispersion along with additional nonlinearities, thus extending its applicability in systems with non-trivial dispersion management. In equation (1), the parameters c_1 and c_2 correspond to the dispersion terms and directly affect the propagation characteristics of the wave. The wave profile is represented by the function $q = q(x,\ t)$, describing the amplitude of the pulse as it evolves in both space x and time t. The parameter a is associated with nonlinear CD, which significantly influences the pulse dynamics in systems where the refractive index varies with wavelength. This effect leads to different spectral components of the pulse traveling at different speeds, altering the pulse shape over time. The first term in the equation represents the linear temporal evolution of the pulse and accounts for the rate of change of the pulse envelope with respect to the spatial variable x. It effectively models the change in amplitude as the pulse progresses through the medium. The nonlinear CD is described by the second term, which influences pulse broadening and temporal asymmetry, resulting from the complex interplay of nonlinearities and dispersion. When light waves of varying frequencies propagate at different velocities, they spread out over time, modifying the overall pulse structure. Finally, the third term embodies the Kerr nonlinearity through the SPM effect, driven by changes in the refractive index with intensity. This nonlinearity leads to a frequency shift within the pulse, allowing the pulse to maintain its shape and is a critical factor in soliton formation. The Kerr effect, quantified by parameter b, ensures that high-intensity parts of the pulse experience more rapid phase changes, thereby influencing the spectral and temporal characteristics of the pulse.

This article is structured as follows: Section 2 provides the theoretical foundation for the study, introducing the key equations and assumptions of the concatenation model, nonlinear chromatic dispersion, and the Kerr law of self-phase modulation. Section 3 describes the two main techniques used to retrieve quiescent optical solitons. It explains the methodologies behind the projective Riccati equation method and the enhanced Kudryashov's algorithm, detailing how each approach leads to analytical soliton solutions. Section 4 focuses on the characteristics and behavior of quiescent optical solitons within the given model. It examines their stability and dynamics, specifically in the context of nonlinear chromatic dispersion and self-phase modulation. Section 5 presents the results from both the analytical solutions and numerical simulations. This section analyzes the soliton behavior, confirming their stability and dynamic properties under various conditions. It also discusses the implications of these findings for practical applications. Section 6 summarizes the key findings of the study, emphasizing the relevance of quiescent optical solitons in nonlinear optical media. It also offers recommendations for future research and potential applications in optical communication and nonlinear optics.

2. Mathematical analysis

The projected form of the soliton is outlined as

$$q(x,\ t) = U(\xi) e^{i(\omega t + \theta)},\ \xi = kx. \tag{2}$$

The amplitude $U(\xi)$, wave width k, wave number ω, and phase shift θ characterize the quiescent soliton. Transformation in Eq. (2) allows Eq. (1) to be split into its imaginary and real parts:

$$\begin{aligned} & ak^2 n(n+1) U^n \left(U'\right)^2 + ak^2 (n+1) U^{1+n} U'' + c_1 k^2 \left(\delta_2 + \delta_3\right) \left(U'\right)^2 U^2 + c_1 k^2 \left(\delta_4 + \delta_5\right) U^3 U'' \\ & + c_1 \delta_1 k^4 U^{(iv)} U - \omega U^2 + bU^4 + c_1 \delta_6 U^6 = 0, \end{aligned} \tag{3}$$

and

$$c_2 \delta_7 U''' U k^3 + c_2 k \left(\delta_8 + \delta_9\right) U^3 U' = 0. \tag{4}$$

Based on the imaginary part Eq. (4), we deduce:

$$\delta_7 = 0,\ \delta_8 + \delta_9 = 0. \tag{5}$$

To comply with the integrability restriction, we take $n = 2$. As a result, Equation (1) can be modified to read

$$\begin{aligned} & iq_t + a\left(|q|^2 q\right)_{xx} + b|q|^2 q + c_1 \left[\delta_1 q_{xxxx} + \delta_2 \left(q_x\right)^2 q^* + \delta_3 |q_x|^2 q + \delta_4 |q|^2 q_{xx} + \delta_5 q^2 q_{xx}^* + \delta_6 |q|^4 q\right] \\ & + ic_2 \left[\delta_8 |q|^2 q_x + \delta_9 q^2 q_x^*\right] = 0, \end{aligned} \tag{6}$$

and Eq. (3) can be rewritten as:

$$c_1 \delta_1 k^4 U^{(iv)} + \left[3ak^2 + c_1 k^2 \left(\delta_4 + \delta_5\right)\right] U^2 U'' + \left[6ak^2 + c_1 k^2 \left(\delta_2 + \delta_3\right)\right] U \left(U'\right)^2 - \omega U + bU^3 + c_1 \delta_6 U^5 = 0. \tag{7}$$

Eq. (7) can be simplified to the following simple ordinary differential equation (ODE).

$$k^2 U'''' + \Omega_5 U^2 U'' + \Omega_4 U \left(U'\right)^2 + \Omega_1 U + \Omega_2 U^3 + \Omega_3 U^5 = 0, \tag{8}$$

with

$$
\begin{cases}
\Omega_5 = \dfrac{[3a + c_1(\delta_4 + \delta_5)]}{c_1\delta_1}, \\
\Omega_4 = \dfrac{[6a + c_1(\delta_2 + \delta_3)]}{c_1\delta_1}, \\
\Omega_3 = \dfrac{\delta_6}{\delta_1 k^2}, \\
\Omega_2 = \dfrac{b}{c_1\delta_1 k^2}, \\
\Omega_1 = \dfrac{-\omega}{c_1\delta_1 k^2}.
\end{cases} \tag{9}
$$

3. Methodology

Take into account the model structured as [11–20]:

$$
F(u,\ u_x,\ u_t,\ u_{xt},\ u_{xx},\ ....) = 0. \tag{10}
$$

In this context, $u = u(x,\ t)$ serves as the wave profile, in which x and t correspond to the space and time domains. The substitutions:

$$
\xi = k(x - vt),\ u(x,\ t) = U(\xi), \tag{11}
$$

reduce Eq. (10) to:

$$
P(U,\ -kvU',\ kU',\ k^2U'',\ ...) = 0. \tag{12}
$$

In this formulation, v represents the wave's velocity, k denotes the wave width, and ξ specifies the wave variable. Next, the basic procedures of the techniques will be outlined in the following subsections.

3.1 *Projective Riccati equation algorithm*

The basic procedures of the technique is outlined in the steps [11–20]:

Step-1: Eq. (10) admits:

$$
U(\xi) = \alpha_0 + \sum_{i=1}^{N} f^{i-1}(\xi)\Big(\alpha_i f(\xi) + \beta_i g(\xi)\Big), \tag{13}
$$

where

$$
g'(\xi) = 1 - g^2(\xi) - rf(\xi),
$$

$$
f'(\xi) = -f(\xi)g(\xi), \tag{14}
$$

with

$$g(\xi)^2 = 1 - 2rf(\xi) + \varphi(r)f(\xi)^2. \tag{15}$$

Here, r, α_0, α_i, and β_i denote constants, with N based on the balancing scheme in Eq. (12).
Step-2: Eq. (14) holds the solutions:
Case-1: $\varphi(r) = r^2 + 1$

$$g(\xi) = \frac{\coth[\xi]}{r\ \mathrm{csch}[\xi] + 1}, \text{ and } f(\xi) = \frac{\mathrm{csch}[\xi]}{r\ \mathrm{csch}[\xi] + 1}. \tag{16}$$

Case-2: $\varphi(r) = r^2 - 1$

$$g(\xi) = \frac{5\ \tanh[\xi] + 3}{3\ \tanh[\xi] + 4r\ \mathrm{sech}[\xi] + 5}, \text{ and } f(\xi) = \frac{4\ \mathrm{sech}[\xi]}{3\ \tanh[\xi] + 4r\ \mathrm{sech}[\xi] + 5}, \tag{17}$$

or

$$g(\xi) = \frac{\tanh[\xi]}{r\ \mathrm{sech}[\xi] + 1}, \text{ and } f(\xi) = \frac{\mathrm{sech}[\xi]}{r\ \mathrm{sech}[\xi] + 1}. \tag{18}$$

Case-3: $\varphi(r) = \frac{5}{9}r^2$

$$g(\xi) = \frac{2}{2\ \coth[\xi] \pm 3\ \mathrm{csch}[\xi]}, \text{ and } f(\xi) = \frac{1}{r}\frac{3\ \mathrm{sech}[\xi]}{3\ \mathrm{sech}[\xi] \pm 2}. \tag{19}$$

Case-4: $\varphi(r) = \frac{24}{25}r^2$

$$g(\xi) = \frac{\tanh[\xi]}{1 \pm 5\ \mathrm{sech}[\xi]}, \text{ and } f(\xi) = \frac{1}{r}\frac{5\ \mathrm{sech}[\xi]}{5\ \mathrm{sech}[\xi] \pm 1}. \tag{20}$$

Case-5: $\varphi(r) = 0$

$$g(\xi) = \tanh\left[\frac{\xi}{2}\right], \text{ and } f(\xi) = \frac{1}{2r}\mathrm{sech}^2\left[\frac{\xi}{2}\right], \tag{21}$$

or

$$g(\xi) = \coth\left[\frac{\xi}{2}\right], \text{ and } f(\xi) = -\frac{1}{2r}\mathrm{csch}^2\left[\frac{\xi}{2}\right]. \tag{22}$$

Step-3: Upon inserting Eq. (13) together with (14) and (15) into Eq. (12), a polynomial is obtained. The resulting coefficients yield the required parameters in (11) and (13).

3.2 *The enhanced Kudryashov's algorithm*

The basic procedures of the technique is outlined in the steps [11–20]:

Step-1: Eq. (12) admits:

$$U(\xi)=\alpha_0+\sum_{i=1}^{N}\left\{\alpha_i R(\xi)^i+\beta_i\left(\frac{R'(\xi)}{R(\xi)^i}\right)\right\}, \tag{23}$$

where

$$R'(\xi)^2=R(\xi)^2\left(1-\chi R(\xi)^2\right). \tag{24}$$

In this formulation, α_0, χ, and β_i are constants, where N is based on the balancing procedure outlined in Eq. (12).

Step-2: Eq. (22) satisfies:

$$R(\xi)=\frac{4d}{4d^2e^{\xi}+\chi e^{-\xi}}, \tag{25}$$

where d is nonzero constant.

Step-3: Plugging Eq. (23) and Eq. (24) into Eq. (12) yields the findings necessary for Eq. (11) and Eq. (23). These restrictions, when reinserted into Eq. (23) with Eq. (25), result in straddled solitons, reducible to bright, singular, or dark solitons.

4. Quiescent optical solitons

4.1 *The projective Riccati equations algorithm*

With the homogeneous balance principle applied U^5 and $U^{(iv)}$ in Eq. (7), $N=1$ is determined, leading to:

$$U(\xi)=\alpha_0+\alpha_1 f(\xi)+\beta_1 g(\xi). \tag{26}$$

When Eq. (26), along with Eqs. (14) and (15), is substituted into Eq. (7), the resulting expressions are:

Case-1: Assuming $\delta[r]=0$, the result is:

$$\alpha_0=0,\ \alpha_1=0,\ \beta_1=\sqrt{\frac{4\Omega_1}{4\Omega_5-\Omega_4}},\ k=\sqrt{\frac{4\Omega_1\Omega_5}{\Omega_4-4\Omega_5}},\ \Omega_3=\frac{(\Omega_4-4\Omega_5)^2}{16\Omega_1},\ \Omega_2=\frac{1}{2}(\Omega_4-4\Omega_5). \tag{27}$$

Accordingly, Eq. (1) admits the findings:

Dark soliton:

$$q(x,\ t)=\sqrt{\frac{4\Omega_1}{4\Omega_5-\Omega_4}}\tanh\left[\sqrt{\frac{\Omega_1\Omega_5}{\Omega_4-4\Omega_5}}x\right]e^{i(\omega t+\theta)}. \tag{28}$$

Singular soliton:

$$q(x,\ t)=\sqrt{\frac{4\Omega_1}{4\Omega_5-\Omega_4}}\coth\left[\sqrt{\frac{\Omega_1\Omega_5}{\Omega_4-4\Omega_5}}x\right]e^{i(\omega t+\theta)}. \tag{29}$$

Case-2: When $\delta[r]$ is taken as $\frac{24}{25}r^2$, this gives:

$$\alpha_0=0,\ \alpha_1=\sqrt{\frac{96\Omega_1 r^2}{25\,(4\Omega_5-\Omega_4)}},\ \beta_1=\sqrt{\frac{4\Omega_1}{4\Omega_5-\Omega_4}},\ k=\sqrt{\frac{4\Omega_1\Omega_5}{\Omega_4-4\Omega_5}},\ \Omega_3=\frac{(\Omega_4-4\Omega_5)^2}{16\Omega_1},$$

$$\Omega_2=\frac{1}{2}\left(\Omega_4-4\Omega_5\right). \tag{30}$$

In turn, this yields the straddled soliton:

$$q(x,\ t)=\left\{\sqrt{\frac{4\Omega_1}{4\Omega_5-\Omega_4}}\left(\frac{\sqrt{24}\operatorname{sech}\left[\sqrt{\frac{4\Omega_1\Omega_5}{\Omega_4-4\Omega_5}}x\right]}{5\operatorname{sech}\left[\sqrt{\frac{4\Omega_1\Omega_5}{\Omega_4-4\Omega_5}}x\right]\pm 1}+\frac{\tanh\left[\sqrt{\frac{4\Omega_1\Omega_5}{\Omega_4-4\Omega_5}}x\right]}{1\pm 5\operatorname{sech}\left[\sqrt{\frac{4\Omega_1\Omega_5}{\Omega_4-4\Omega_5}}x\right]}\right)\right\}e^{i(\omega t+\theta)}. \tag{31}$$

Case-3: Assuming $\delta[r]=\frac{5}{9}r^2$, the result is:

$$\alpha_0=0,\ \alpha_1=\sqrt{\frac{20\Omega_1 r^2}{9\,(4\Omega_5-\Omega_4)}},\ \beta_1=\sqrt{\frac{4\Omega_1}{4\Omega_5-\Omega_4}},\ k=\sqrt{\frac{4\Omega_1\Omega_5}{\Omega_4-4\Omega_5}},\ \Omega_3=\frac{(\Omega_4-4\Omega_5)^2}{16\Omega_1},$$

$$\Omega_2=\frac{1}{2}\left(\Omega_4-4\Omega_5\right). \tag{32}$$

Accordingly, Eq. (1) satisfies the straddled soliton:

$$q(x,\ t)=\left\{\sqrt{\frac{4\Omega_1}{4\Omega_5-\Omega_4}}\left(\frac{\sqrt{5}\operatorname{sech}\left[\sqrt{\frac{4\Omega_1\Omega_5}{\Omega_4-4\Omega_5}}x\right]}{3\operatorname{sech}\left[\sqrt{\frac{4\Omega_1\Omega_5}{\Omega_4-4\Omega_5}}x\right]\pm 2}+\frac{2}{2\coth\left[\sqrt{\frac{4\Omega_1\Omega_5}{\Omega_4-4\Omega_5}}x\right]\pm 3\operatorname{csch}\left[\sqrt{\frac{4\Omega_1\Omega_5}{\Omega_4-4\Omega_5}}x\right]}\right)\right\}$$

$$\times e^{i(\omega t+\theta)}. \tag{33}$$

Case-4: By defining $\delta[r] = r^2 - 1$, we arrive at:

$$\alpha_1 = 0,\ \beta_1 = \sqrt{\frac{10\Omega_1\left(r^2-1\right)}{13\Omega_5}},\ k = \sqrt{\frac{2\Omega_1}{13}},\ \Omega_3 = -\frac{13\Omega_5^2\left(5r^2-17\right)}{100\Omega_1\left(r^2-1\right)^2},\ \Omega_4 = -\frac{3\Omega_5\left(r^2+3\right)}{2\left(r^2-1\right)},$$

$$\Omega_2 = \frac{\Omega_5\left(r^2-6\right)}{2\left(r^2-1\right)}. \tag{34}$$

In turn, this yields the straddled solitons:

$$q(x,\ t) = \left\{\sqrt{\frac{10\Omega_1\left(r^2-1\right)}{13\Omega_5}}\left(\frac{5\tanh\left[\sqrt{\frac{2\Omega_1}{13}}x\right]+3}{3\tanh\left[\sqrt{\frac{2\Omega_1}{13}}x\right]+4r\operatorname{sech}\left[\sqrt{\frac{2\Omega_1}{13}}x\right]+5}\right)\right\}e^{i(\omega t+\theta)}, \tag{35}$$

and

$$q(x,\ t) = \left\{\sqrt{\frac{10\Omega_1\left(r^2-1\right)}{13\Omega_5}}\left(\frac{\tanh\left[\sqrt{\frac{2\Omega_1}{13}}x\right]}{r\operatorname{sech}\left[\sqrt{\frac{2\Omega_1}{13}}x\right]+1}\right)\right\}e^{i(\omega t+\theta)}. \tag{36}$$

Case-5: When $\delta[r]$ is taken as r^2+1, this gives:

$$\alpha_1 = 0,\ \beta_1 = \sqrt{-\frac{10\Omega_1\left(r^2+1\right)}{13\Omega_5}},\ k = \sqrt{\frac{2\Omega_1}{13}},\ \Omega_3 = \frac{13\Omega_5^2\left(5r^2+17\right)}{100\Omega_1\left(r^2+1\right)^2},\ \Omega_4 = -\frac{3\Omega_5\left(r^2-3\right)}{2\left(r^2+1\right)},$$

$$\Omega_2 = \frac{\Omega_5\left(r^2+6\right)}{2\left(r^2+1\right)}. \tag{37}$$

As a result, Eq. (1) admits the straddled soliton:

$$q(x,\ t) = \left\{\sqrt{-\frac{10\Omega_1\left(r^2+1\right)}{13\Omega_5}}\left(\frac{\coth\left[\sqrt{\frac{2\Omega_1}{13}}x\right]}{r\operatorname{csch}\left[\sqrt{\frac{2\Omega_1}{13}}x\right]+1}\right)\right\}e^{i(\omega t+\theta)}. \tag{38}$$

4.2 *The enhanced Kudryashov's algorithm*

With the homogeneous balance principle applied $U^{(iv)}$ and U^5 in Eq. (7), one finds $N=1$, which allows us to express Eq. (7) in the form:

$$U(\xi)=\alpha_0+\alpha_1R(\xi)+\beta_1\left(\frac{R'(\xi)}{R(\xi)}\right). \tag{39}$$

Incorporating Eq. (39) along with Eq. (24) into Eq. (7) results in:
Result-1:

$$\alpha_0=0,\ \alpha_1=0,\ \beta_1=\pm\sqrt{\frac{5\Omega_1}{-2\Omega_2-\Omega_4+4\Omega_5}},\ k=\sqrt{\frac{\Omega_1\left(-\Omega_2+2\Omega_4+2\Omega_5\right)}{16\Omega_2+8\Omega_4-32\Omega_5}},$$

$$\Omega_3=\frac{6\Omega_2^2+\left(\Omega_2-\Omega_4+4\Omega_5\right)\left(\Omega_4-4\Omega_5\right)}{25\Omega_1}. \tag{40}$$

Thus, Eq. (1) satisfies:

$$q(x,\,t)=\left\{\mp\sqrt{\frac{5\Omega_1}{-2\Omega_2-\Omega_4+4\Omega_5}}\left(\frac{4d^2\exp\left[2\sqrt{\frac{\Omega_1\left(-\Omega_2+2\Omega_4+2\Omega_5\right)}{16\Omega_2+8\Omega_4-32\Omega_5}}x\right]-\chi}{4d^2\exp\left[2\sqrt{\frac{\Omega_1\left(-\Omega_2+2\Omega_4+2\Omega_5\right)}{16\Omega_2+8\Omega_4-32\Omega_5}}x\right]+\chi}\right)\right\}e^{i(\omega t+\theta)}. \tag{41}$$

Selecting $\chi=\pm4d^2$ to recover dark and singular solitons for $\frac{5\Omega_1}{-2\Omega_2-\Omega_4+4\Omega_5}>0$ and $\frac{\Omega_1\left(-\Omega_2+2\Omega_4+2\Omega_5\right)}{16\Omega_2+8\Omega_4-32\Omega_5}>0$:

$$q(x,\,t)=\left\{\mp\sqrt{\frac{5\Omega_1}{-2\Omega_2-\Omega_4+4\Omega_5}}\tanh\left[\sqrt{\frac{\Omega_1\left(-\Omega_2+2\Omega_4+2\Omega_5\right)}{16\Omega_2+8\Omega_4-32\Omega_5}}x\right]\right\}e^{i(\omega t+\theta)}, \tag{42}$$

and

$$q(x,\,t)=\left\{\mp\sqrt{\frac{5\Omega_1}{-2\Omega_2-\Omega_4+4\Omega_5}}\coth\left[\sqrt{\frac{\Omega_1\left(-\Omega_2+2\Omega_4+2\Omega_5\right)}{16\Omega_2+8\Omega_4-32\Omega_5}}x\right]\right\}e^{i(\omega t+\theta)}. \tag{43}$$

Result-2:

$$\alpha_0 = 0,\ \alpha_1 = \pm\sqrt{-\frac{20\Omega_1 \chi}{\Omega_2 + \Omega_4 + \Omega_5}},\ \beta_1 = 0,\ k = \sqrt{-\Omega_1},$$

$$\Omega_3 = \frac{6\Omega_2^2 + \Omega_2 (7\Omega_4 + 2\Omega_5) + \Omega_4^2 - 4\Omega_5^2 - 3\Omega_4\Omega_5}{100\Omega_1}. \tag{44}$$

Accordingly, Eq. (1) holds:

$$q(x,\ t) = \left\{ \pm\sqrt{-\frac{20\Omega_1 \chi}{\Omega_2 + \Omega_4 + \Omega_5}} \left(\frac{4de^{x\sqrt{-\Omega_1}}}{4d^2 e^{2x\sqrt{-\Omega_1}} + \chi} \right) \right\} e^{i(\omega t + \theta)}. \tag{45}$$

Selecting $\chi = \pm 4d^2$ to recover bright and singular solitons for $\frac{20\Omega_1}{\Omega_2 + \Omega_4 + \Omega_5} < 0$ and $\Omega_1 < 0$:

$$q(x,\ t) = \left\{ \pm\sqrt{-\frac{20\Omega_1}{\Omega_2 + \Omega_4 + \Omega_5}} \operatorname{sech}\left[\sqrt{-\Omega_1} x \right] \right\} e^{i(\omega t + \theta)}, \tag{46}$$

and

$$q(x,\ t) = \left\{ \pm\sqrt{\frac{20\Omega_1}{\Omega_2 + \Omega_4 + \Omega_5}} \operatorname{csch}\left[\sqrt{-\Omega_1} x \right] \right\} e^{i(\omega t + \theta)}. \tag{47}$$

5. Results and discussion

We provide an in-depth analysis of Figures 1-3, examining the modulus profiles of various solitons, namely dark, bright-dark, and bright solitons, as described by the corresponding equations along with $a = 1$, $c_1 = 1$, $\delta_5 = 1$, $\delta_4 = 1$, $\delta_3 = 1$, $\delta_2 = 1$, $\delta_1 = 1$, and $\omega = 1$. The analysis focuses on the influence of the wave width parameter k across a range from $k = 0.5$ to $k = 1.4$ on the soliton shape, amplitude, and width. Each figure uses 2D plots to explore how changing k modulates each soliton's structure and dynamics, shedding light on their distinctive characteristics and applications.

Figure 1 demonstrates the modulus of the dark soliton governed by equation (28) under various values of k. Dark solitons are known for their characteristic dips in amplitude, where a localized dark region occurs in the center of the soliton profile. As k increases from 0.5 to 1.4, the soliton's width broadens, and the depth of the dip in amplitude decreases. This broadening effect suggests that higher k values cause the dark soliton to become less localized, spreading out over a wider region and appearing flatter in profile. This behavior aligns with the known dynamics of dark solitons, where increasing the wave width variable k reduces the soliton's spatial confinement. For smaller k, the soliton's amplitude dip is more pronounced, indicating stronger localization and a steeper transition between the low and high amplitude regions. This response can be useful in applications where control over the soliton's confinement and the sharpness of amplitude changes is needed, such as in optical signal processing where the rapid transition between high and low states is advantageous.

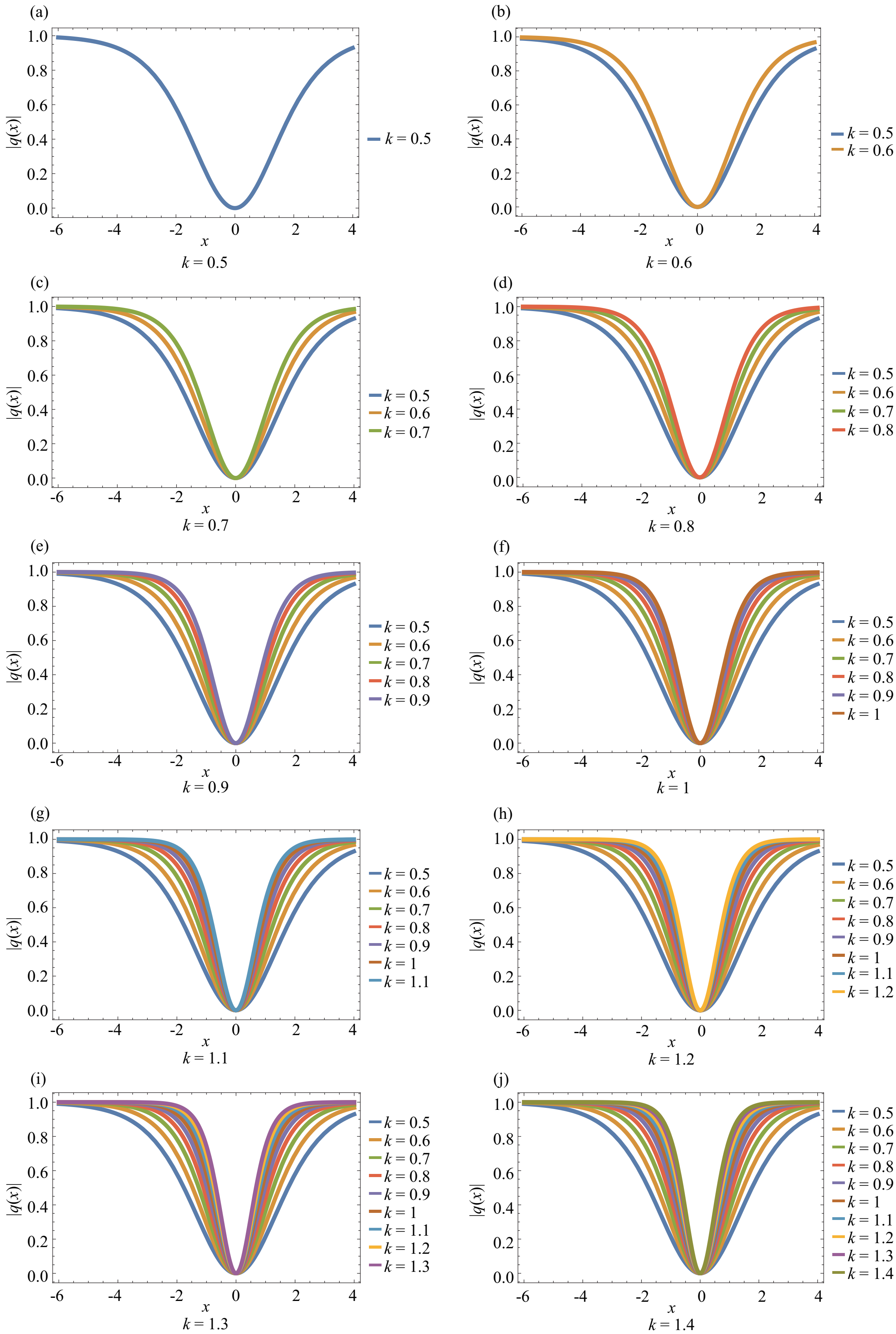


Figure 1. Investigating the properties of a dark soliton, with particular attention to its amplitude

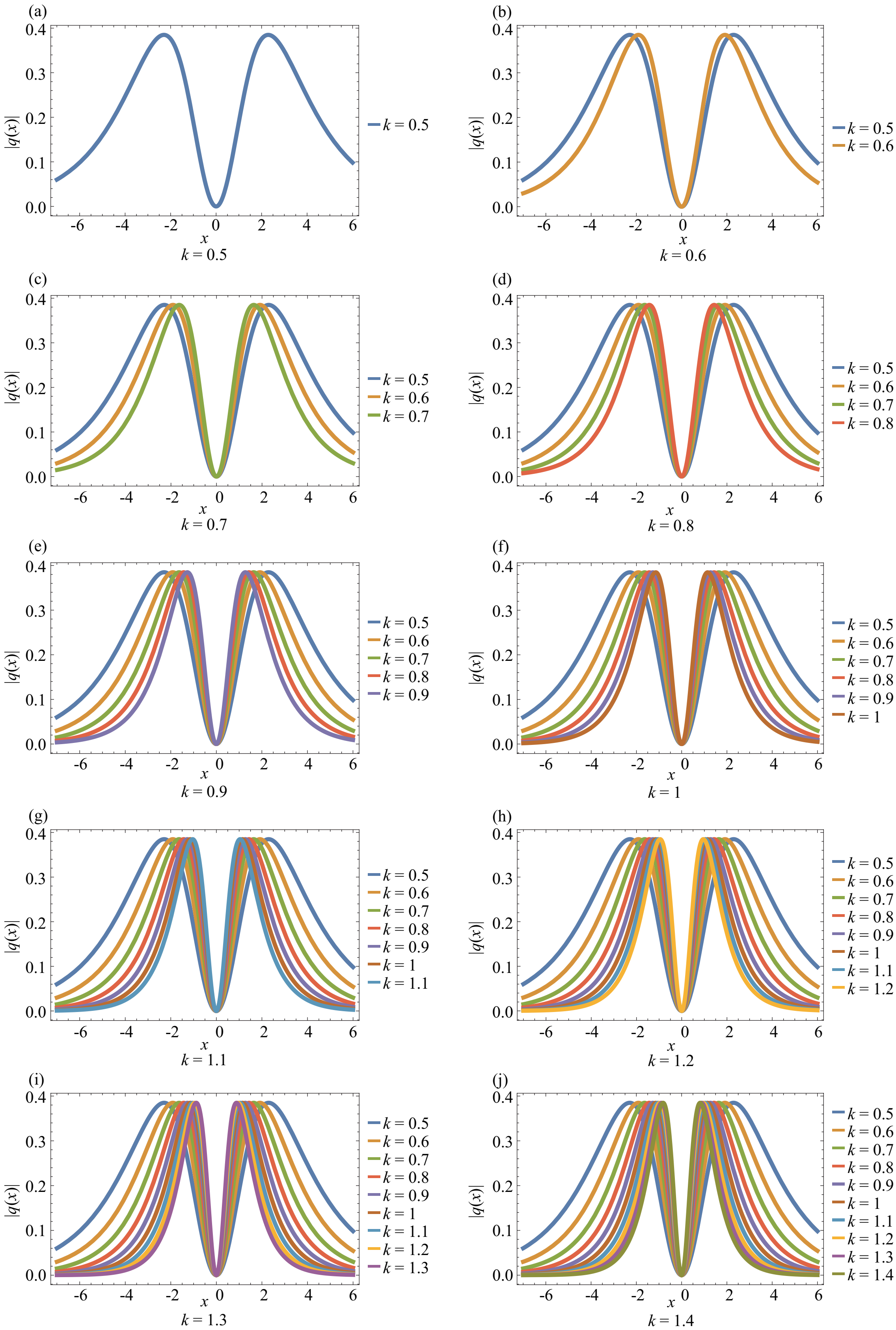
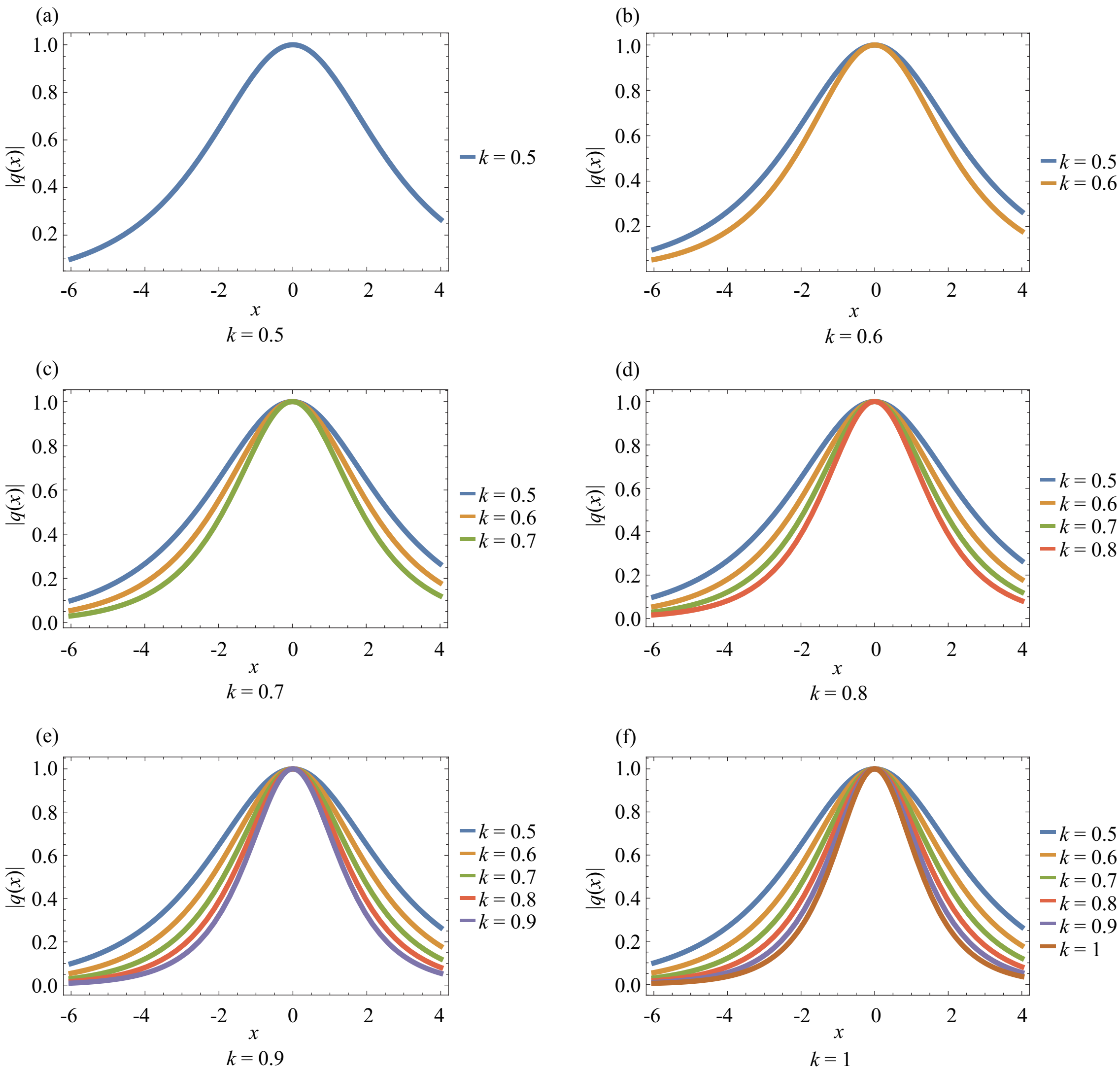


Figure 2. Investigating the properties of a bright-dark soliton, with particular attention to its amplitude

Figure 2 examines the modulus of the dark-bright soliton, a composite soliton described by equation (31). Bright-dark solitons display a dual characteristic, with both a peak (bright component) and a dip (dark component) within the same profile. As k increases, the amplitude of the bright component slightly rises, while the width of the dark dip broadens. This dual modulation effect highlights the unique nature of bright-dark solitons, where each component responds differently to variations in k. The results suggest that larger k values cause the bright portion to become more pronounced while the dark component spreads out. This behavior can be advantageous in applications requiring complex wave structures with controllable bright and dark regions. For instance, bright-dark solitons could be leveraged in optical lattices or in the modulation of light in photonic circuits, where tailored intensity profiles are desirable for different functional roles in transmission channels.



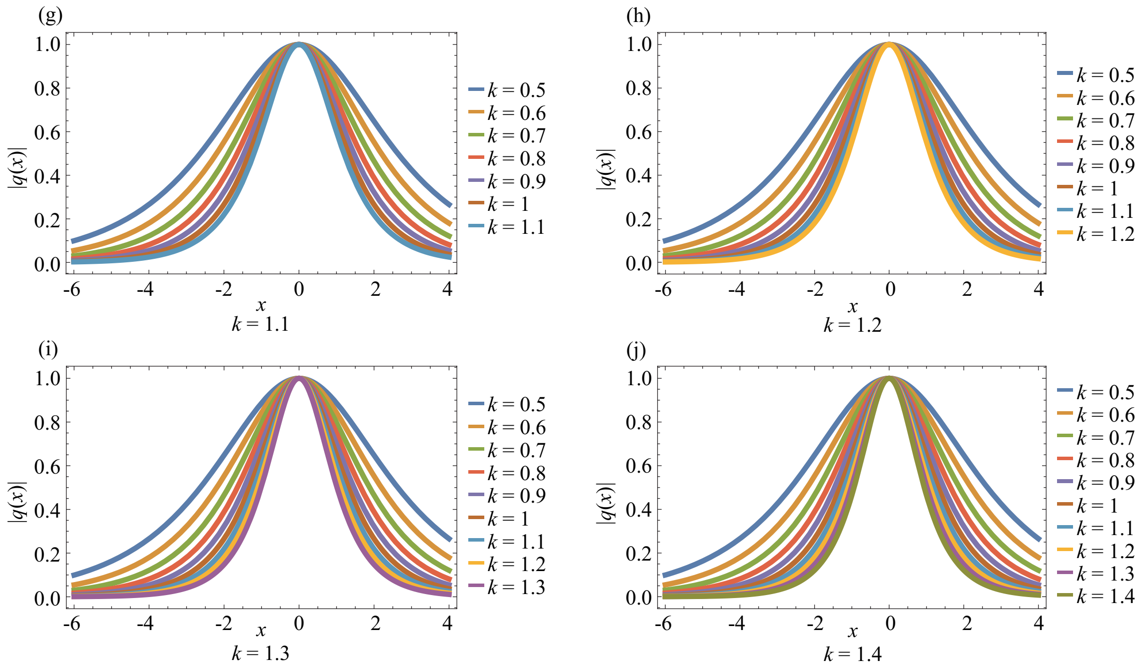


Figure 3. Investigating the properties of a bright soliton, with particular attention to its amplitude

Figure 3 examines the modulus of the bright soliton, as described by equation (46). Unlike other soliton types, the bright soliton primarily shows an increase in amplitude as k rises, with its width remaining relatively stable. This behavior indicates that bright solitons retain their shape while experiencing amplitude enhancement with higher k values, underscoring their resilience to changes in the wave width parameter. The bright soliton's consistent structure and amplitude stability make it suitable for applications requiring robust wave profiles, such as data transmission in optical fibers where the preservation of signal shape over long distances is critical. The ability to control the amplitude without affecting the soliton width also suggests its potential in amplitude-sensitive applications where modulation without altering spatial confinement is required.

In summary, Figures 1-3 provide comprehensive insights into the influence of the wave width parameter k on various soliton types. Dark and bright solitons exhibit primarily amplitude and width adjustments. Hybrid solitons, such as dark-bright solitons, display combined behaviors where each component responds distinctly to k variations. This analysis highlights the tunability of solitons for applications across optical communication, photonic systems, and nonlinear optics, where tailored soliton profiles and precise control over intensity and localization are essential.

6. Conclusions

The current study revisits and systematically derives quiescent optical solitons for the concatenation model, which is formulated here with a Kerr law SPM framework combined with nonlinear CD. This setup is essential to understand as it governs the propagation dynamics of optical pulses, especially under conditions where the nonlinear effects in the medium influence the stability of the transmitted pulses. The quiescent solitons retrieved in this study are derived through the application of two distinct integration algorithms: the projective Riccati equation approach and the enhanced Kudryashov's method. Each of these algorithms contributes uniquely to the retrieval process, collectively enabling a comprehensive recovery of quiescent solitons in the concatenation model. The full range of soliton solutions, along with the necessary parameter constraints that ensure their existence, are presented and thoroughly analyzed. These constraints are crucial as they outline the specific conditions under which stable quiescent solitons can be sustained in this nonlinear model.

The implications of the results presented in this paper are particularly significant for the telecommunication industry, as well as for engineers working with ground-based transmission systems. One of the critical findings highlighted here is the adverse effect that nonlinear CD has on pulse propagation. Specifically, the study serves as a warning: it is essential to prevent CD from becoming nonlinear during data transmission, whether intentionally or inadvertently. Any scenario in which nonlinear CD occurs could lead to potentially catastrophic consequences, where optical pulses might stall or freeze mid-transmission, particularly in scenarios such as transoceanic data transfer. This stalling effect would be highly detrimental, especially given the extensive distances involved in intercontinental communication, where stable pulse propagation is paramount for reliable data transfer. Therefore, ensuring that CD remains linear is fundamental to maintaining optimal pulse dynamics and avoiding transmission failures over long distances.

Furthermore, the study suggests that future research will extend this model by incorporating additional forms of SPM structures, which will provide a more versatile framework for exploring other nonlinear effects in optical fibers. Additionally, the concatenation model will be investigated under conditions involving polarization-mode dispersion and with the application of dispersion-flattened fibers. These fibers are particularly relevant in optical communications as they help maintain consistent dispersion properties across a broader wavelength range. Once these advanced models and the recovered soliton solutions align with pre-existing results in the literature, the findings will be broadly disseminated and shared across academic and engineering communities, thereby contributing valuable insights to both theoretical research and practical applications in fiber optics and telecommunication technologies [36–40].

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Conflict of interest

The authors claim that there is no conflict of interest.

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