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Research Article

Grüss-Type Inequalities Involving Bounds Constant via Analytic Kernel Fractional Integral

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Abstract: The primary aim of this study is to derive a generalization of certain Grüss-type inequalities under constant constraints via a generalized Analytical kernel Riemann-Liouville fractional integral. We derived new Grüss-type inequalities with constant limits using generalized fractional integral operators with identical and varying parameters. The results acquired are more broadly applicable.

***Keywords*:** Grüss type inequalities, Riemann-Liouville fractional integral, Analytic kernel fractional integral

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1. Introduction

Calculus has seen several phases of evolution because it can be hard to understand in some situations, mathematicians have employed inequalities to derive solutions for differential and integral equations by establishing upper bounds for particular parameters. This reasoning led to the concept of differential and integral inequality. In the field of integration, the inequality (1) that Grüss established in 1935 [1] also (see [2, 3]) is one of the most important.

$$\left|\frac{1}{c_2-c_1}\int_{c_1}^{c_2}\gamma(z)\varphi(z)\,dz-\frac{1}{(c_2-c_1)^2}\int_{c_1}^{c_2}\gamma(z)\,dz\int_{c_1}^{c_2}\varphi(z)\,dz\right|$$

$$\leq\frac{1}{4}(P-p)(Q-q), \tag{1}$$

where γ, φ are two integrable functions on $[c_1, c_2]$, satisfying the conditions

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$$p \leq \gamma(z) \leq P, \quad q \leq \varphi(z) \leq Q, \quad z \in [c_1, c_2], \quad p, P, q, Q \in \mathbb{R}. \tag{2}$$

This inequality has garnered substantial importance due to its extensive applications across various domains within mathematical sciences, such as difference equations, integral arithmetic mean, and h-integral arithmetic mean, which are associated with numerous current and applied sciences (see [4, 5]). Dahmani [6] introduced the fractional variant of inequality (1) utilising the Riemann-Liouville fractional integral as follows:

$$\left| \frac{z^{\tau}}{\Gamma(\tau+1)} \mathscr{I}_{0^+}^{\tau} \{\gamma\varphi\}(z) - \mathscr{I}_{0^+}^{\tau} \varphi(z)\ \mathscr{I}_{0^+}^{\tau} \gamma(z) \right|^2$$

$$\leq \frac{1}{4} \left(\frac{z^{\tau}}{\Gamma(\tau+1)} \right)^2 (Q-q)(P-p), \tag{3}$$

where $\gamma, \varphi : [0, +\infty) \longrightarrow \mathbb{R}$ be an integrable functions on $[0, +\infty)$ satisfying (2), and $\tau > 0$, $z > 0$.

The identical author, inside the identical paper, proposed the subsequent inequality:

Theorem 1.1 Let $\gamma, \varphi : [0, +\infty) \longrightarrow \mathbb{R}$ be an integrable functions on $[0, +\infty)$ satisfying (2). Then, $\forall\ \tau, \sigma > 0$, $z > 0$, the following inequality is holds

$$\left[\frac{z^{\sigma}}{\Gamma(\sigma+1)} \left(\mathscr{I}_{0^+}^{\tau} \gamma\varphi \right)(z) - \left(\mathscr{I}_{0^+}^{\sigma} \gamma \right)(z) \left(\mathscr{I}_{0^+}^{\tau} \varphi \right)(z) \right.$$

$$\left. + \frac{z^{\tau}}{\Gamma(\tau+1)} \left(\mathscr{I}_{0^+}^{\sigma} \gamma\varphi \right)(z) - \left(\mathscr{I}_{0^+}^{\sigma} \varphi \right)(z) \left(\mathscr{I}_{0^+}^{\tau} \gamma \right)(z) \right]^2$$

$$\leq \left(\frac{Pz^{\sigma}}{\Gamma(\sigma+1)} - \left(\mathscr{I}_{0^+}^{\sigma} \gamma \right)(z) \right) \left(\left(\mathscr{I}_{0^+}^{\tau} \gamma \right)(z) - \frac{pz^{\tau}}{\Gamma(\tau+1)} \right)$$

$$+ \left(\frac{Pz^{\tau}}{\Gamma(\tau+1)} - \left(\mathscr{I}_{0^+}^{\tau} \gamma \right)(z) \right) \left(\left(\mathscr{I}_{0^+}^{\sigma} \gamma \right)(z) - \frac{pz^{\sigma}}{\Gamma(\sigma+1)} \right)$$

$$\times \left(\frac{Qz^{\sigma}}{\Gamma(\sigma+1)} - \left(\mathscr{I}_{0^+}^{\sigma} \varphi \right)(z) \right) \left(\left(\mathscr{I}_{0^+}^{\tau} \varphi \right)(z) - \frac{qz^{\tau}}{\Gamma(\tau+1)} \right)$$

$$+ \left(\frac{Qz^{\tau}}{\Gamma(\tau+1)} - \left(\mathscr{I}_{0^+}^{\tau} \varphi \right)(z) \right) \left(\left(\mathscr{I}_{0^+}^{\sigma} \varphi \right)(z) - \frac{qz^{\sigma}}{\Gamma(\sigma+1)} \right). \tag{4}$$

In 2014, Tariboon et al. [7] introduced a novel fractional integral formulation of inequality (1). This was accomplished by substituting the constants p, P, q, Q with four positive integrable functions, thereby extending the scope and applicability of the original inequality. Their contribution represents a significant advancement in the field of fractional integral inequalities.

Theorem 1.2 Let $\gamma, \varphi : [0, +\infty) \longrightarrow \mathbb{R}$ be an integrable functions on $[0, +\infty)$ satisfying

$$\psi_1(z) \leq \gamma(z) \leq \psi_2(z), \quad u_1(z) \leq \varphi(z) \leq u_2(z), \quad z > 0, \tag{5}$$

for the integrable functions ψ_1, ψ_2, u_1, u_2 on $[0, +\infty)$. Then, for all $\tau > 0$ the following inequality holds,

$$\left[\frac{z^\tau}{\Gamma(\tau+1)}\left(\mathscr{I}_{0^+}^\tau \varphi\gamma\right)(z) - \left(\mathscr{I}_{0^+}^\tau \varphi\right)(z)\left(\mathscr{I}_{0^+}^\tau \gamma\right)(z)\right]^2 \leq T(\varphi, \psi_1, \psi_2)\, T(\gamma, u_1, u_2), \tag{6}$$

where $T(g, \phi, \eta)$ is defined by

$$\begin{aligned}
T(g, \phi, \eta) = & \left[\left(\mathscr{I}_{0^+}^\tau \eta\right)(z) - \left(\mathscr{I}_{0^+}^\tau g\right)(z)\right]\left[\left(\mathscr{I}_{0^+}^\tau g\right)(z) - \left(\mathscr{I}_{0^+}^\tau \phi\right)(z)\right] \\
& + \frac{z^\tau}{\Gamma(\tau+1)}\left(\mathscr{I}_{0^+}^\tau g\phi\right)(z) - \left(\mathscr{I}_{0^+}^\tau g\right)(z)\left(\mathscr{I}_{0^+}^\tau \phi\right)(z) \\
& + \frac{z^\tau}{\Gamma(\tau+1)}\left(\mathscr{I}_{0^+}^\tau g\eta\right)(z) - \left(\mathscr{I}_{0^+}^\tau g\right)(z)\left(\mathscr{I}_{0^+}^\tau \eta\right)(z) \\
& - \frac{z^\tau}{\Gamma(\tau+1)}\left(\mathscr{I}_{0^+}^\tau \phi\eta\right)(z) + \left(\mathscr{I}_{0^+}^\tau \phi\right)(z)\left(\mathscr{I}_{0^+}^\tau \eta\right)(z).
\end{aligned}$$

In 2012, Dragomir [8] introduced novel Grüss-type inequalities applicable to functions of bounded variation, exploring their applications within Hilbert spaces equipped with self-adjoint operators. Subsequently, Alomari [9] developed advanced Grüss inequalities incorporating double integrals, establishing precise bounds for these formulations. Concurrently, Chinchane and Pachpatte [10] proposed innovative Grüss inequalities by employing the Hadamard fractional integral operator. In 2015, Liu and Tuna [11] conducted a comprehensive investigation on time scales, deriving numerous weighted Grüss and Ostrowski-type inequalities through the framework of combined dynamic derivatives. Further advancements were made by Sousa et al. [12], who utilized the Katugampola fractional integral to derive a generalized form of the Grüss inequality. Rashid et al. [13] extended this work by establishing Grüss-type inequalities using generalized proportional fractional integrals. In the same year, Zhou et al. [14] provided a detailed exposition of the Grüss inequality and introduced several related inequalities based on the generalized proportional Hadamard fractional integral. In 2021, Naz et al. [15] conducted a study employing the generalized Hilfer-Katugampola k-fractional derivative to address various Grüss-type problems. Simultaneously, Al Qurashi et al. [16] investigated h-discrete frameworks, uncovering discrete dynamical Grüss inequalities associated with the Atangana-Baleanu fractional operator. For further significant contributions to the Grüss inequality, refer to [17, 18]. Most recently, in 2024, Radwan et al. [19] (see also [20]) introduced new Grüss-type inequalities utilizing φ-fractional integrals further expand these inequalities' theoretical and applicative scope.

2. Analytic kernel Riemann-Liouville fractional integral

Here, we drop the definition and basic concepts of the fractional integral operators uesd to present our new generalized results.

Definition 2.1 [21] For the function $\gamma \in L^1[c_1, c_2]$, and for $\tau, \delta \in (0, +\infty)$, the left fractional integral involving analytic kernel function A with parameters τ, δ of $\gamma(z)$ can be defined as

$$\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\right)(z) = \int_{c_1}^{z} (z-\kappa)^{\tau-1} A\left((z-\kappa)^{\delta}\right)\gamma(\kappa)\,d\kappa, \tag{7}$$

which provided that $A: D(0, K) \to \mathbb{C}$ be a complex analytic function with power series

$$A = \sum_{n=0}^{\infty} (c_1)_n z^n, \tag{8}$$

where the coefficients $(c_1)_n = (c_1)_n(\tau, \delta)$ are permitted to depend on the parameters τ, δ if desired, and $K > (c_2 - c_1)^{\delta}$.

Remark 2.1

(1) As originally written in [21], the parameters τ and δ may be complex, but according to the purposes of this work we restrict them to be real, since we cannot do inequalities in the complex plane.

(2) As in [21], the generalized fractional integral operator (7) can be written as an infinite series of Riemann-Liouville fractional integrals

$$\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\right)(z) = \sum_{n=0}^{\infty} (c_1)_n \Gamma(\tau+n\delta)\left({}_{c_1}^{RL}\mathscr{I}_z^{\tau+n\delta}\gamma\right)(z)$$

$$= \sum_{n=0}^{\infty} \frac{(c_1)_n z^{\tau+n\delta}}{\tau+n\delta} = z^{\tau}\beta\left(z^{\delta}\right), \tag{9}$$

where

$$\beta(z) = \sum_{n=0}^{\infty} \frac{(c_1)_n z^n}{\tau+n\delta}.$$

Now, we are ready to introduce our main results in this paper. The Grüss-type inequalities in the case of constant bounds are given and discussed throughout the following sections.

Theorem 2.1 (Cauchy-Schwartz Inequality) [22] Let $\gamma(\mu, \kappa)$ and $\varphi(\mu, \kappa)$ be measurable functions defined on a product measure space $D = [c_1, c_2] \times [b_1, b_2]$. Then, the Cauchy-Schwartz inequality states that:

$$\left(\iint_D \gamma(z_1, z_2)\varphi(z_1, z_2)dz_1dz_2\right)^2 \le \left(\iint_D \gamma(z_1, z_2)^2 dz_1dz_2\right)\left(\iint_D \varphi(z_1, z_2)^2 dz_1dz_2\right), \tag{10}$$

where c_1, c_2, b_1, b_2 are real numbers and measurable set $D \subseteq \mathbb{R}^2$.

Theorem 2.2 (Young's Inequality) [23, 24] Let γ, φ be continuous, strictly increasing, and mutually inverse for non-negative argument, with $\gamma(0) = \varphi(0) = 0$. Then

$$cd \leq \int_0^c \gamma(z)dz + \int_0^d \varphi(z)dz, \tag{11}$$

equality is satisfied if and only if d equals $\gamma(c)$. In from (11), if c, d are non-negative and p, $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ is satisfied, then the holds,

$$cd \leq \frac{c^p}{p} + \frac{d^q}{q}. \tag{12}$$

We are now prepared to present our primary findings in this paper. The Grüss-type inequalities in the case of constant bounds are given and discussed throughout the following sections.

3. Analytic kernel fractional integral Grüss inequalities involving constant bounds

The following result is valid for another strictly growing and continuous function. It also gives us a new way to look at the Grüss-type inequality using the newly created proportional fractional integral operator. Inspired by the work of Dahmani et al. [6].

To support our findings, we need the following lemmas.

Lemma 3.1 Let $A(z)$ be an analytic function and $\gamma : [0, +\infty) \longrightarrow \mathbb{R}$ be a positive integrable function, which satisfy $p \leq \gamma(z) \leq P$, $z > 0$, p, $P \in \mathbb{R}$. Then, for all τ, $\delta \in \mathbb{R}^+$, the following identity holds,

$$z^{\tau}\beta\left(z^{\delta}\right)\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma^2\right)(z) - 2\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\right)^2(z)$$

$$= \left[\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\right)(z) - pz^{\tau}\beta\left(z^{\delta}\right)\right]\left[Pz^{\tau}\beta\left(z^{\delta}\right) - \left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\right)(z)\right]$$

$$- z^{\tau}\beta\left(z^{\delta}\right)_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\left(P - \gamma(z)\right)\left(\gamma(z) - p\right), \tag{13}$$

where $\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\right)$ is the Analytic kernel Riemann-Liouville fractional integral (AKR-L) (7).

Proof. Consider μ, $\kappa \in [0, \infty)$, we have

$$(P - \gamma(\kappa))(\gamma(\mu) - p) + (P - \gamma(\mu))(\gamma(\kappa) - p)$$

$$-(P - \gamma(\mu))(\gamma(\mu) - p) - (P - \gamma(\kappa))(\gamma(\kappa) - p)$$

$$= \gamma^2(\mu) + \gamma^2(\kappa) - 2\gamma(\mu)\gamma(\kappa). \tag{14}$$

Taking the product by the positive factor $(z - \mu)^{\tau-1}A\left((z-\mu)^{\delta}\right)$, $\mu \in (c_1, z)$ on the both sides of (14), then integrating the estimating identity concerning the variable μ over (c_1, z), we get

$$
\begin{aligned}
&(P-\gamma(\kappa))\left[\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\right)(z)-pz^{\tau}\beta\left(z^{\delta}\right)\right]\\
&+\left[Pz^{\tau}\beta\left(z^{\delta}\right)-\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\right)(z)\right](\gamma(\kappa)-p)\\
&-{}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\,(P-\gamma(z))(\gamma(z)-p)\\
&-z^{\tau}\beta\left(z^{\delta}\right)(P-\gamma(\kappa))(\gamma(\kappa)-p)\\
=&\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma^2\right)(z)+z^{\tau}\beta\left(z^{\delta}\right)\gamma^2(\kappa)-2\gamma(\kappa)\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\right)(z),
\end{aligned}
\tag{15}
$$

where $\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\right)$ is the Analytic kernel Riemann-Liouville fractional integral (AKR-L) (7).

Again, multiplying with the positive factor $(z-\kappa)^{\tau-1}A\left((z-\kappa)^{\delta}\right)$, $\kappa\in(c_1,\,z)$ on the both sides of (15), then integrating the estimating identity concerning the variable κ over $(c_1,\,z)$, we have

$$
\begin{aligned}
&\left[\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\right)(z)-pz^{\tau}\beta\left(z^{\delta}\right)\right]\int_{c_1}^{z}(z-\kappa)^{\tau-1}A\left((z-\kappa)^{\delta}\right)(P-\gamma(\kappa))\,d\kappa\\
&+\left[Pz^{\tau}\beta\left(z^{\delta}\right)-\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\right)(z)\right]\int_{c_1}^{z}(z-\kappa)^{\tau-1}A\left((z-\kappa)^{\delta}\right)(\gamma(\kappa)-p)\,d\kappa\\
&-{}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\left\{(P-\gamma(z))(\gamma(z)-p)\right\}\int_{c_1}^{z}(z-\kappa)^{\tau-1}A\left((z-\kappa)^{\delta}\right)d\kappa\\
&-z^{\tau}\beta\left(z^{\delta}\right)\int_{c_1}^{z}(z-\kappa)^{\tau-1}A\left((z-\kappa)^{\delta}\right)(P-\gamma(\kappa))(\gamma(\kappa)-p)\,d\kappa\\
=&\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma^2\right)(z)\int_{c_1}^{z}(z-\kappa)^{\tau-1}A\left((z-\kappa)^{\delta}\right)d\kappa\\
&+z^{\tau}\beta\left(z^{\delta}\right)\int_{c_1}^{z}(z-\kappa)^{\tau-1}A\left((z-\kappa)^{\delta}\right)\gamma^2(\kappa)\,d\kappa\\
&-2\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\right)(z)\int_{c_1}^{z}(z-\kappa)^{\tau-1}A\left((z-\kappa)^{\delta}\right)\gamma(\kappa)\,d\kappa,
\end{aligned}
\tag{16}
$$

which leads to

$$\left[\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\right)(z)-z^{\tau}\beta\left(z^{\delta}\right)p\right]\left[Pz^{\tau}\beta\left(z^{\delta}\right)-\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\right)(z)\right]$$

$$+\left[Pz^{\tau}\beta\left(z^{\delta}\right)-\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\right)(z)\right]\left[\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\right)(z)-pz^{\tau}\beta\left(z^{\delta}\right)\right]$$

$$-z^{\tau}\beta\left(z^{\delta}\right){}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\left\{(P-\gamma(z))(\gamma(z)-p)\right\}$$

$$-z^{\tau}\beta\left(z^{\delta}\right){}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\left\{(P-\gamma(z))(\gamma(z)-p)\right\}$$

$$=z^{\tau}\beta\left(z^{\delta}\right)\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma^2\right)(z)+z^{\tau}\beta\left(z^{\delta}\right)\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma^2\right)(z)$$

$$-2\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\right)^2(z), \tag{17}$$

where $\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\right)$ is the Analytic kernel Riemann-Liouville fractional integral (AKR-L) (7).

Clearly, by (17), it yields the desired result (13). This concludes the proof. □

The subsequent lemma is necessary to establish the forthcoming result:

Lemma 3.2 Let $A(z)$, be a positive analytic function and $\gamma,\ \varphi:[0,\,+\infty)\longrightarrow\mathbb{R}$, be positive integrable functions that satisfy (22). Then, for all $z>0$, $\tau,\ \sigma,\ \delta\in\mathbb{R}^+$, the following inequality holds,

$$\Big[z^{\sigma}\beta\left(z^{\delta}\right)\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\varphi\right)(z)-\left({}_{c_1}^{A}\mathscr{I}_z^{\sigma,\,\delta}\gamma\right)(z)\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\varphi\right)(z)$$

$$+z^{\tau}\beta\left(z^{\delta}\right)\left({}_{c_1}^{A}\mathscr{I}_z^{\sigma,\,\delta}\gamma\varphi\right)(z)-\left({}_{c_1}^{A}\mathscr{I}_z^{\sigma,\,\delta}\varphi\right)(z)\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\right)(z)\Big]^2$$

$$\leq\Big[z^{\sigma}\beta\left(z^{\delta}\right)\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma^2\right)(z)+z^{\tau}\beta\left(z^{\delta}\right)\left({}_{c_1}^{A}\mathscr{I}_z^{\sigma,\,\delta}\gamma^2\right)(z)$$

$$-2\left({}_{c_1}^{A}\mathscr{I}_z^{\sigma,\,\delta}\gamma\right)(z)\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\right)(z)\Big]$$

$$\times\Big[z^{\sigma}\beta\left(z^{\delta}\right)\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\varphi^2\right)(z)+z^{\tau}\beta\left(z^{\delta}\right)\left({}_{c_1}^{A}\mathscr{I}_z^{\sigma,\,\delta}\varphi^2\right)(z)$$

$$-2\left({}_{c_1}^{A}\mathscr{I}_z^{\sigma,\,\delta}\varphi\right)(z)\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\varphi\right)(z)\Big]. \tag{18}$$

where $\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\right)$ is the Analytic kernel Riemann-Liouville fractional integral (AKR-L) (7).

Proof. Taking multiplication in both sides of the (26) by $(z-\kappa)^{\sigma-1}A\left((z-\kappa)^{\delta}\right)$, $\kappa\in(c_1,\,z)$, then integrating the estimating identity concerning the variable κ over $(c_1,\,z)$, we get

$$\int_{c_1}^{z}\int_{c_1}^{z}(z-\mu)^{\tau-1}A\left((z-\mu)^{\delta}\right)(z-\kappa)^{\sigma-1}A\left((z-\kappa)^{\delta}\right)H(\mu,\,\kappa)\,d\mu d\kappa$$

$$=z^{\sigma}\beta\left(z^{\delta}\right)\left({}_{c_1}^{A}\mathscr{I}_{z}^{\tau,\,\delta}\gamma\varphi\right)(z)+z^{\tau}\beta\left(z^{\delta}\right)\left({}_{c_1}^{A}\mathscr{I}_{z}^{\sigma,\,\delta}\gamma\varphi\right)(z)$$

$$-\left({}_{c_1}^{A}\mathscr{I}_{z}^{\sigma,\,\delta}\gamma\right)(z)\left({}_{c_1}^{A}\mathscr{I}_{z}^{\tau,\,\delta}\varphi\right)(z)-\left({}_{c_1}^{A}\mathscr{I}_{z}^{\sigma,\,\delta}\varphi\right)(z)\left({}_{c_1}^{A}\mathscr{I}_{z}^{\tau,\,\delta}\gamma\right)(z). \tag{19}$$

where $\left({}_{c_1}^{A}\mathscr{I}_{z}^{\tau,\,\delta}\right)$ is the Analytic kernel Riemann-Liouville fractional integral (AKR-L) (7).

Applying the Cauchy-Schwarz (10) on the right-hand-side of the equation (27), we obtain the desired (18). □

Lemma 3.3 Let $A(z)$, $z>0$, be an analytic function and $\gamma:[0,\,+\infty)\longrightarrow\mathbb{R}$, be a positive integrable function, which satisfies the condition (22). Then, for all $\tau,\,\sigma,\,\delta\in\mathbb{R}^{+}$, the following equations holds,

$$\left({}_{c_1}^{A}\mathscr{I}_{z}^{\tau,\,\delta}\gamma^{2}\right)(z)z^{\sigma}\beta\left(z^{\delta}\right)+z^{\tau}\beta\left(z^{\delta}\right)\left({}_{c_1}^{A}\mathscr{I}_{z}^{\sigma,\,\delta}\gamma^{2}\right)(z)$$

$$-2\left({}_{c_1}^{A}\mathscr{I}_{z}^{\tau,\,\delta}\gamma\right)(z)\left({}_{c_1}^{A}\mathscr{I}_{z}^{\sigma,\,\delta}\gamma\right)(z)$$

$$=\left[Pz^{\sigma}\beta\left(z^{\delta}\right)-\left({}_{c_1}^{A}\mathscr{I}_{z}^{\sigma,\,\delta}\gamma\right)(z)\right]\left[\left({}_{c_1}^{A}\mathscr{I}_{z}^{\tau,\,\delta}\gamma\right)(z)-pz^{\tau}\beta\left(z^{\delta}\right)\right]$$

$$+\left[Pz^{\tau}\beta\left(z^{\delta}\right)-\left({}_{c_1}^{A}\mathscr{I}_{z}^{\tau,\,\delta}\gamma\right)(z)\right]\left[\left({}_{c_1}^{A}\mathscr{I}_{z}^{\sigma,\,\delta}\gamma\right)(z)-pz^{\sigma}\beta\left(z^{\delta}\right)\right]$$

$$-z^{\sigma}\beta\left(z^{\delta}\right){}_{c_1}^{A}\mathscr{I}_{z}^{\tau,\,\delta}\left\{(P-\gamma(z))(\gamma(z)-p)\right\}$$

$$-z^{\tau}\beta\left(z^{\delta}\right){}_{c_1}^{A}\mathscr{I}_{z}^{\sigma,\,\delta}\left\{(P-\gamma(z))(\gamma(z)-p)\right\}, \tag{20}$$

where $\left({}_{c_1}^{A}\mathscr{I}_{z}^{\tau,\,\delta}\right)$ is the Analytic kernel Riemann-Liouville fractional integral (AKR-L) (7).

Proof. Considering Lemma 3.1, multiplying both sides of (15) by $(z-\kappa)^{\sigma-1}A\left((z-\kappa)^{\delta}\right)$, κ over $(c_1,\,z)$ then integrating the estimating identity concerning the variable k, we get

$$\left[\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\right)(z)-pz^{\tau}\beta\left(z^{\delta}\right)\right]\int_{c_1}^{z}(z-\kappa)^{\sigma-1}A\left((z-\kappa)^{\delta}\right)(P-\gamma(\kappa))\,d\kappa$$

$$+\left(Pz^{\tau}\beta\left(z^{\delta}\right)-\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\right)(z)\right)\int_{c_1}^{z}(z-\kappa)^{\sigma-1}A\left((z-\kappa)^{\delta}\right)(\gamma(\kappa)-p)\,d\kappa$$

$$-{}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\left\{(P-\gamma(z))(\gamma(z)-p)\right\}\int_{c_1}^{z}(z-\kappa)^{\sigma-1}A\left((z-\kappa)^{\delta}\right)d\kappa$$

$$-z^{\tau}\beta\left(z^{\delta}\right)\int_{c_1}^{z}(z-\kappa)^{\sigma-1}A\left((z-\kappa)^{\delta}\right)(P-\gamma(\kappa))(\gamma(\kappa)-p)\,d\kappa$$

$$=\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma^2\right)(z)\int_{c_1}^{z}(z-\kappa)^{\sigma-1}A\left((z-\kappa)^{\delta}\right)d\kappa+z^{\tau}\beta\left(z^{\delta}\right)$$

$$\times\int_{c_1}^{z}(z-\kappa)^{\sigma-1}A\left((z-\kappa)^{\delta}\right)\gamma^2(\kappa)\,d\kappa$$

$$-2\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\right)(z)\int_{c_1}^{z}(z-\kappa)^{\sigma-1}A\left((z-\kappa)^{\delta}\right)\gamma(\kappa)\,d\kappa, \tag{21}$$

where $\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\right)$ is the Analytic kernel Riemann-Liouville fractional integral (AKR-L) (7). This yields the required result (20). □

The following results are devoted to the *R-L* fractional integral Grüss-type inequalities, which concern the constant bounds in a single and distinct order. These results provide some generalizations of the *R-L* fractional integral Grüss-type inequalities proposed by Dahmani et al. [6].

First, we establish the following result on utilizing Lemma 3.1.

Theorem 3.1 Let $A(z)$, $z>0$ be an analytic function and γ, $\varphi:[0,+\infty)\longrightarrow\mathbb{R}$, be positive and integrable functions which satisfy the condition

$$p\le\gamma(z)\le P,\quad q\le\varphi(z)\le Q,\quad p,\,P,\,q,\,Q\in\mathbb{R}. \tag{22}$$

Then, for all $\tau,\,\delta\in\mathbb{R}^{+}$, the following inequality holds,

$$\left|z^{\tau}\beta\left(z^{\delta}\right)\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\varphi\right)(z)-\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\varphi\right)(z)\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\right)(z)\right|$$

$$\le z^{2\tau}\beta^2\left(z^{\delta}\right)(Q-q)(P-p), \tag{23}$$

where $\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\right)$ is the Analytic kernel Riemann-Liouville fractional integral (AKR-L) (7). This yields the required result (20).

Proof. Define the function $H(\mu,\,\kappa)$ as follows

$$H(\mu,\,\kappa) = (\gamma(\mu) - \gamma(\kappa))\,(\varphi(\mu) - \varphi(\kappa)), \quad \mu,\,\kappa \in (c_1,\,z), \quad z > 0. \tag{24}$$

So, we have

$$H(\mu,\,\kappa) = \gamma(\mu)\,\varphi(\mu) - \gamma(\mu)\,\varphi(\kappa) - \gamma(\kappa)\,\varphi(\mu) + \gamma(\kappa)\,\varphi(\kappa), \tag{25}$$

multiplying (25) by $(z-\mu)^{\tau-1} A\left((z-\mu)^{\delta}\right)$, where $\mu \in (c_1,\,z)$ then integrating the estimating identity concerning μ over, $(c_1,\,z)$, we obtain

$$\begin{aligned}
&\int_{c_1}^{z} (z-\mu)^{\tau-1} A\left((z-\mu)^{\delta}\right) H(\mu,\,\kappa)\,d\mu \\
&= \left({}^{A}_{c_1}\mathscr{I}_z^{\tau,\,\delta}\gamma\varphi\right)(z) - \varphi(\kappa)\left({}^{A}_{c_1}\mathscr{I}_z^{\tau,\,\delta}\gamma\right)(z) \\
&\quad - \gamma(\kappa)\left({}^{A}_{c_1}\mathscr{I}_z^{\tau,\,\delta}\varphi\right)(z) + z^{\tau}\beta\left(z^{\delta}\right)\gamma(\kappa)\,\varphi(\kappa).
\end{aligned} \tag{26}$$

Then, multiplying (26) by $(z-\kappa)^{\tau-1} A\left((z-\kappa)^{\delta}\right)$, (26), where $\kappa \in (c_1,\,z)$ then integrating the estimating identity with regard to κ yields

$$\begin{aligned}
&\int_{c_1}^{z}\int_{c_1}^{z} (z-\mu)^{\tau-1} A\left((z-\mu)^{\delta}\right)(z-\kappa)^{\tau-1} A\left((z-\kappa)^{\delta}\right) H(\mu,\,\kappa)\,d\mu d\kappa \\
&= 2\left[z^{\tau}\beta\left(z^{\delta}\right)\left({}^{A}_{c_1}\mathscr{I}_z^{\tau,\,\delta}\gamma\varphi\right)(z) - \left({}^{A}_{c_1}\mathscr{I}_z^{\tau,\,\delta}\varphi\right)(z)\left({}^{A}_{c_1}\mathscr{I}_z^{\tau,\,\delta}\gamma\right)(z)\right].
\end{aligned} \tag{27}$$

where $\left({}^{A}_{c_1}\mathscr{I}_z^{\tau,\,\delta}\right)$ is the Analytic kernel Riemann-Liouville fractional integral (AKR-L) (7). This yields the required result (20).

Now, applying the Cauchy-Schwarz (10) on the right-hand-side of the (27), we have

$$\begin{aligned}
&\left[z^{\tau}\beta\left(z^{\delta}\right)\left({}^{A}_{c_1}\mathscr{I}_z^{\tau,\,\delta}\gamma\varphi\right)(z) - \left({}^{A}_{c_1}\mathscr{I}_z^{\tau,\,\delta}\varphi\right)(z)\left({}^{A}_{c_1}\mathscr{I}_z^{\tau,\,\delta}\gamma\right)(z)\right]^2 \\
&\leq \left[z^{\tau}\beta\left(z^{\delta}\right)\left({}^{A}_{c_1}\mathscr{I}_z^{\tau,\,\delta}\gamma^2\right)(z) - \left({}^{A}_{c_1}\mathscr{I}_z^{\tau,\,\delta}\gamma\right)^2(z)\right] \\
&\quad \times \left[z^{\tau}\beta\left(z^{\delta}\right)\left({}^{A}_{c_1}\mathscr{I}_z^{\tau,\,\delta}\varphi^2\right)(z) - \left({}^{A}_{c_1}\mathscr{I}_z^{\tau,\,\delta}\varphi\right)^2(z)\right].
\end{aligned} \tag{28}$$

Also, according to the hypothesis condition, we have that

$$(P-\gamma(z))(\gamma(z)-p)\geq 0,$$

and

$$(Q-\varphi(z))(\varphi(z)-q)\geq 0,$$

for all $z\in[0,+\infty)$, which leads to

$$z^{\tau}\beta\left(z^{\delta}\right){}_{c_1}^{A}\mathscr{I}_{z}^{\tau,\,\delta}(P-\gamma(z))(\gamma(z)-p)\geq 0,$$

and

$$z^{\tau}\beta\left(z^{\delta}\right){}_{c_1}^{A}\mathscr{I}_{z}^{\tau,\,\delta}(Q-\varphi(z))(\varphi(z)-q)\geq 0.$$

Therefore, by Lemma 3.1, we obtain from (28) that

$$\begin{aligned}&z^{\tau}\beta\left(z^{\delta}\right)\left({}_{c_1}^{A}\mathscr{I}_{z}^{\tau,\,\delta}\gamma^{2}\right)(z)-\left({}_{c_1}^{A}\mathscr{I}_{z}^{\tau,\,\delta}\gamma\right)^{2}(z)\\ \leq&\left[Pz^{\tau}\beta\left(z^{\delta}\right)-\left({}_{c_1}^{A}\mathscr{I}_{z}^{\tau,\,\delta}\gamma\right)(z)\right]\\ &\times\left[\left({}_{c_1}^{A}\mathscr{I}_{z}^{\tau,\,\delta}\gamma\right)(z)-pz^{\tau}\beta\left(z^{\delta}\right)\right],\end{aligned}\tag{29}$$

and

$$\begin{aligned}&z^{\tau}\beta\left(z^{\delta}\right)\left({}_{c_1}^{A}\mathscr{I}_{z}^{\tau,\,\delta}\varphi^{2}\right)(z)-\left({}_{c_1}^{A}\mathscr{I}_{z}^{\tau,\,\delta}\varphi\right)^{2}(z)\\ \leq&\left[Qz^{\tau}\beta\left(z^{\delta}\right)-\left({}_{c_1}^{A}\mathscr{I}_{z}^{\tau,\,\delta}\varphi\right)(z)\right]\\ &\times\left[\left({}_{c_1}^{A}\mathscr{I}_{z}^{\tau,\,\delta}\varphi\right)(z)-qz^{\tau}\beta\left(z^{\delta}\right)\right].\end{aligned}\tag{30}$$

In the view of the (28), (29) and (30), we obtain

$$\left[z^{\tau}\beta\left(z^{\delta}\right)\left({}_{c_1}^{A}\mathscr{I}_{z}^{\tau,\,\delta}\gamma\varphi\right)(z)-\left({}_{c_1}^{A}\mathscr{I}_{z}^{\tau,\,\delta}\varphi\right)(z)\left({}_{c_1}^{A}\mathscr{I}_{z}^{\tau,\,\delta}\gamma\right)(z)\right]^{2}$$

$$\leq\left[Pz^{\tau}\beta\left(z^{\delta}\right)-\left({}_{c_1}^{A}\mathscr{I}_{z}^{\tau,\,\delta}\gamma\right)(z)\right]\left[\left({}_{c_1}^{A}\mathscr{I}_{z}^{\tau,\,\delta}\gamma\right)(z)-pz^{\tau}\beta\left(z^{\delta}\right)\right]$$

$$\times\left[Qz^{\tau}\beta\left(z^{\delta}\right)-\left({}_{c_1}^{A}\mathscr{I}_{z}^{\tau,\,\delta}\varphi\right)(z)\right]\left[\left({}_{c_1}^{A}\mathscr{I}_{z}^{\tau,\,\delta}\varphi\right)(z)-qz^{\tau}\beta\left(z^{\delta}\right)\right]. \tag{31}$$

Now, applying the inequality property that $4cd\leq(c+d)^{2}$, $c,\,d\in\mathbb{R}$, that is

$$4\left[Pz^{\tau}\beta\left(z^{\delta}\right)-\left({}_{c_1}^{A}\mathscr{I}_{z}^{\tau,\,\delta}\gamma\right)(z)\right]\left[\left({}_{c_1}^{A}\mathscr{I}_{z}^{\tau,\,\delta}\gamma\right)(z)-pz^{\tau}\beta\left(z^{\delta}\right)\right]$$

$$\leq\left[z^{\tau}\beta\left(z^{\delta}\right)(P-p)\right]^{2} \tag{32}$$

and

$$4\left[Qz^{\tau}\beta\left(z^{\delta}\right)-\left({}_{c_1}^{A}\mathscr{I}_{z}^{\tau,\,\delta}\varphi\right)(z)\right]\left[\left({}_{c_1}^{A}\mathscr{I}_{z}^{\tau,\,\delta}\varphi\right)(z)-qz^{\tau}\beta\left(z^{\delta}\right)\right]$$

$$\leq\left[z^{\tau}\beta\left(z^{\delta}\right)(Q-q)\right]^{2}. \tag{33}$$

Thus, via (31), (32), and (33), we promptly obtain the required result (23). We can, therefore, conclude the proof. □

Remark 3.1

1. If we choose $A(z)=\dfrac{1}{\Gamma(\tau)}$ and $\delta=0$, in Theorem 3.1, then we recapture the inequality involving *R-L* fractional integral version obtained proved by Dahmani et al. [6].

2. When we apply for $A(z)=\dfrac{1}{\Gamma(\tau)}$, $\delta=0$ and $\tau=1$, in Theorem 3.1, then we obtain the Grüss inequality (1).

Further, we present the inequality related to fractional integral Grüss-type for content bounds distinct order. This result is obtained by utilizing Lemma 3.3. The following lemma is required to prove the next result.

Theorem 3.2 Let $A(z)$, $z>0$ be an analytic function and $\gamma,\,\varphi:[0,\,+\infty)\longrightarrow\mathbb{R}$, be a positive and integrable function, which satisfyies the condition (22). Then, for all $\tau,\,\sigma,\,\delta\in\mathbb{R}^{+}$, the following inequality holds,

$$\Big[z^{\sigma}\beta\left(z^{\delta}\right)\left({}_{c_1}^{A}\mathscr{I}_{z}^{\tau,\,\delta}\gamma\varphi\right)(z)-\left({}_{c_1}^{A}\mathscr{I}_{z}^{\sigma,\,\delta}\gamma\right)(z)\left({}_{c_1}^{A}\mathscr{I}_{z}^{\tau,\,\delta}\varphi\right)(z)$$

$$+z^{\tau}\beta\left(z^{\delta}\right)\left({}_{c_1}^{A}\mathscr{I}_{z}^{\sigma,\,\delta}\gamma\varphi\right)(z)-\left({}_{c_1}^{A}\mathscr{I}_{z}^{\sigma,\,\delta}\varphi\right)(z)\left({}_{c_1}^{A}\mathscr{I}_{z}^{\tau,\,\delta}\gamma\right)(z)\Big]^{2}$$

$$\leq\left[Pz^{\sigma}\beta\left(z^{\delta}\right)-\left({}_{c_1}^{A}\mathscr{I}_{z}^{\sigma,\,\delta}\gamma\right)(z)\right]\left[\left({}_{c_1}^{A}\mathscr{I}_{z}^{\tau,\,\delta}\gamma\right)(z)-pz^{\tau}\beta\left(z^{\delta}\right)\right]$$

$$+\left[Pz^{\tau}\beta\left(z^{\delta}\right)-\left({}_{c_1}^{A}\mathscr{I}_{z}^{\tau,\,\delta}\gamma\right)(z)\right]\left[\left({}_{c_1}^{A}\mathscr{I}_{z}^{\sigma,\,\delta}\gamma\right)(z)-pz^{\sigma}\beta\left(z^{\delta}\right)\right]$$

$$\times\left[Qz^{\sigma}\beta\left(z^{\delta}\right)-\left({}_{c_1}^{A}\mathscr{I}_{z}^{\sigma,\,\delta}\varphi\right)(z)\right]\left[\left({}_{c_1}^{A}\mathscr{I}_{z}^{\tau,\,\delta}\varphi\right)(z)-qz^{\tau}\beta\left(z^{\delta}\right)\right]$$

$$+\left[Qz^{\tau}\beta\left(z^{\delta}\right)-\left({}_{c_1}^{A}\mathscr{I}_{z}^{\tau,\,\delta}\varphi\right)(z)\right]\left[\left({}_{c_1}^{A}\mathscr{I}_{z}^{\sigma,\,\delta}\varphi\right)(z)-qz^{\sigma}\beta\left(z^{\delta}\right)\right]. \tag{34}$$

where $\left({}_{c_1}^{A}\mathscr{I}_{z}^{\tau,\,\delta}\right)$ is the Analytic kernel Riemann-Liouville fractional integral (AKR-L) (7).

Proof. According to the condition (22), we have

$$(P-\gamma(z))(\gamma(z)-p)\geq 0,$$

and

$$(Q-\varphi(z))(\varphi(z)-q)\geq 0.$$

It follows that

$$-z^{\sigma}\beta\left(z^{\delta}\right){}_{c_1}^{A}\mathscr{I}_{z}^{\tau,\,\delta}\left\{(P-\gamma(z))(\gamma(z)-p)\right\}$$

$$-z^{\tau}\beta\left(z^{\delta}\right){}_{c_1}^{A}\mathscr{I}_{z}^{\sigma,\,\delta}\left\{(P-\gamma(z))(\gamma(z)-p)\right\}\leq 0, \tag{35}$$

and

$$-z^{\sigma}\beta\left(z^{\delta}\right){}_{c_1}^{A}\mathscr{I}_{z}^{\tau,\,\delta}\left\{(Q-\varphi(z))(\varphi(z)-q)\right\}$$

$$-z^{\tau}\beta\left(z^{\delta}\right){}_{c_1}^{A}\mathscr{I}_{z}^{\sigma,\,\delta}\left\{(Q-\varphi(z))(\varphi(z)-q)\right\}\leq 0. \tag{36}$$

In view of (35), (36), and Lemma 3.3, we write

$$z^{\sigma}\beta\left(z^{\delta}\right)\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma^2\right)(z)+z^{\tau}\beta\left(z^{\delta}\right)\left({}_{c_1}^{A}\mathscr{I}_z^{\sigma,\,\delta}\gamma^2\right)(z)$$

$$-2\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\right)(z)\left({}_{c_1}^{A}\mathscr{I}_z^{\sigma,\,\delta}\gamma\right)(z)$$

$$\leq\left[Pz^{\sigma}\beta\left(z^{\delta}\right)-\left({}_{c_1}^{A}\mathscr{I}_z^{\sigma,\,\delta}\gamma\right)(z)\right]\left[\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\right)(z)-pz^{\tau}\beta\left(z^{\delta}\right)\right]$$

$$+\left[Pz^{\tau}\beta\left(z^{\delta}\right)-\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\right)(z)\right]\left[\left({}_{c_1}^{A}\mathscr{I}_z^{\sigma,\,\delta}\gamma\right)(z)-pz^{\sigma}\beta\left(z^{\delta}\right)\right],\tag{37}$$

and

$$z^{\sigma}\beta\left(z^{\delta}\right)\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\varphi^2\right)(z)+z^{\tau}\beta\left(z^{\delta}\right)\left({}_{c_1}^{A}\mathscr{I}_z^{\sigma,\,\delta}\varphi^2\right)(z)$$

$$-2\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\varphi\right)(z)\left({}_{c_1}^{A}\mathscr{I}_z^{\sigma,\,\delta}\varphi\right)(z)$$

$$\leq\left[Qz^{\sigma}\beta\left(z^{\delta}\right)-\left({}_{c_1}^{A}\mathscr{I}_z^{\sigma,\,\delta}\varphi\right)(z)\right]\left[\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\varphi\right)(z)-qz^{\tau}\beta\left(z^{\delta}\right)\right]$$

$$+\left[Qz^{\tau}\beta\left(z^{\delta}\right)-\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\varphi\right)(z)\right]\left[\left({}_{c_1}^{A}\mathscr{I}_z^{\sigma,\,\delta}\varphi\right)(z)-qz^{\sigma}\beta\left(z^{\delta}\right)\right].\tag{38}$$

again, by the inequalities (37), (38), and Lemma 3.2, we achieve the desired outcome (34). Consequently, the proof is concluded. □

Remark 3.2

1. If we consider $A(z)=\dfrac{1}{\Gamma(\tau)}$ and $\tau=\sigma$, in Theorem 3.2, then we get Theorem 3.1.

2. If we apply Theorem 3.2 for $A(z)=\dfrac{1}{\Gamma(\tau)}$, $\delta=0$ and $\tau=1$, $\sigma=1$, then we get the classical Grüss (1).

More inequalities have been investigated which are related to analytic kernel fractional integral Grüss-type involving constant bounds.

Theorem 3.3 Let $A(z)$, $z>0$ be an analytic function and γ, $\varphi:[0,+\infty)\longrightarrow\mathbb{R}$, be a positive and integrable function, which is satisfying $\dfrac{1}{P}+\dfrac{1}{Q}=1$, $P, Q>1$. Then, for all τ, $\delta\in\mathbb{R}^+$, the following inequalities holds,

$$1)\qquad \frac{\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma^P\right)(z)}{P}+\frac{\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\varphi^Q\right)(z)}{Q}\geq\frac{1}{z^{\tau}\beta\left(z^{\delta}\right)}\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\right)(z)\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\varphi\right)(z),\tag{39}$$

2) $$\frac{\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma^P\right)(z)\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\varphi^P\right)(z)}{P}+\frac{\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma^Q\right)(z)\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\varphi^Q\right)(z)}{Q}$$

$$\geq\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\varphi\right)^2(z), \tag{40}$$

3) $$\frac{\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma^P\right)(z)\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\varphi^Q\right)(z)}{P}+\frac{\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma^Q\right)(z)\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\varphi^P\right)(z)}{Q}$$

$$\geq\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\varphi^{P-1}\right)(z)\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\varphi^{Q-1}\right)(z), \tag{41}$$

4) $$\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma^P\right)(z)\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\varphi^Q\right)(z)\geq\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\varphi\right)(z)\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma^{P-1}\varphi^{Q-1}\right)(z). \tag{42}$$

Proof. Utilising Young's inequality (12), we obtain:

$$\frac{\gamma^P(\mu)}{P}+\frac{\varphi^Q(\kappa)}{Q}\geq\gamma(\mu)\,\varphi(\kappa),\quad \mu,\,\kappa\in(c_1,\,z). \tag{43}$$

Taking the product in both sides of (43) by $(z-\mu)^{\tau-1}A\left((z-\mu)^{\delta}\right)$, $\mu\in(c_1,\,z)$, then integrating the resulting inequality concerning μ over $(c_1,\,z)$, we get

$$\frac{1}{P}\int_{c_1}^{z}(z-\mu)^{\tau-1}A\left((z-\mu)^{\delta}\right)\gamma^P(\mu)\,d\mu+\frac{\varphi^Q(\kappa)}{Q}\int_{c_1}^{z}(z-\mu)^{\tau-1}A\left((z-\mu)^{\delta}\right)d\mu$$

$$\geq\varphi(\kappa)\int_{c_1}^{z}(z-\mu)^{\tau-1}A\left((z-\mu)^{\delta}\right)\gamma(\mu)\,d\mu.$$

It follows that

$$\frac{\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma^P\right)(z)}{P}+\frac{\varphi^Q(\kappa)}{Q}z^{\tau}\beta\left(z^{\delta}\right)\geq\varphi(\kappa)\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\right)(z). \tag{44}$$

Now, multiplying both sides of the (44) by $(z-\kappa)^{\tau-1}A\left((z-\kappa)^{\delta}\right)$, $\kappa\in(c_1,\,z)$, then integrate the estimating inequality concerning κ over $(c_1,\,z)$, we get

$$\frac{\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma^P\right)(z)}{P}+\frac{\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\varphi^Q\right)(z)}{Q}\geq\frac{1}{z^{\tau}\beta\left(z^{\delta}\right)}\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\varphi\right)(z)\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\right)(z),$$

The first (39) is thus proved. We once more utilise Young's inequality (12), by taking $C = \gamma(\mu)\,\varphi(\kappa)$ and $D = \gamma(\kappa)\,\varphi(\mu)$, where $\mu,\ \kappa \in (c_1, z)$, we get

$$\frac{\gamma^P(\mu)\,\varphi^P(\kappa)}{P} + \frac{\gamma^Q(\kappa)\,\varphi^Q(\mu)}{Q} \geq \gamma(\mu)\,\varphi(\kappa)\,\gamma(\kappa)\,\varphi(\mu). \tag{45}$$

Clearly, we get the inequality (40) by employing the same rationale demonstrated in the proof of (39). Now, utilising Young's inequality (12) taking $C = \dfrac{\gamma(\mu)}{\varphi(\mu)}$, and $D = \dfrac{\gamma(\kappa)}{\varphi(\kappa)}$, $\mu,\ \kappa \in (c_1, z)$, where, $\varphi(\mu),\ \varphi(\kappa) \neq 0$, hence we have

$$\frac{\gamma^P(\mu)\,\varphi^Q(\kappa)}{P} + \frac{\gamma^Q(\kappa)\,\varphi^P(\mu)}{Q} \geq \gamma(\mu)\,\varphi^{P-1}(\mu)\,\gamma(\kappa)\,\varphi^{Q-1}(\kappa). \tag{46}$$

We get (41). Finally, putting $C = \dfrac{\gamma(\mu)}{\gamma(\kappa)}$, and $D = \dfrac{\varphi(\mu)}{\varphi(\kappa)}$, where, $\mu,\ \kappa \in (c_1, z)$ in (12), such that $\gamma(\kappa) \neq 0$ and $\varphi(\kappa) \neq 0$, we have

$$\frac{\gamma^P(\mu)\,\varphi^Q(\kappa)}{P} + \frac{\gamma^P(\kappa)\,\varphi^Q(\mu)}{Q} \geq \gamma(\mu)\,\varphi(\mu)\,\gamma^{P-1}(\kappa)\,\varphi^{Q-1}(\kappa). \tag{47}$$

Multiplying in each side of (47) by both factors $(z-\mu)^{\tau-1} A\left((z-\mu)^{\delta}\right)$, $(z-\kappa)^{\tau-1} A\left((z-\kappa)^{\delta}\right)$, $\mu,\ \kappa \in (c_1, z)$, then taking the double integrating of the estimating inequality concerning μ, κ over (c_1, z), we get

$$\begin{aligned}
&\int_{c_1}^{z}\int_{c_1}^{z} \frac{\gamma^P(\mu)\,\varphi^Q(\kappa)}{P}(z-\mu)^{\tau-1} A\left((z-\mu)^{\delta}\right)(z-\kappa)^{\tau-1} A\left((z-\kappa)^{\delta}\right) d\mu d\kappa \\
&+\int_{c_1}^{z}\int_{c_1}^{z} \frac{\gamma^P(\kappa)\,\varphi^Q(\mu)}{Q}(z-\mu)^{\tau-1} A\left((z-\mu)^{\delta}\right)(z-\kappa)^{\tau-1} A\left((z-\kappa)^{\delta}\right) d\mu d\kappa \\
\geq &\int_{c_1}^{z}\int_{c_1}^{z} \gamma(\mu)\,\varphi(\mu)\,\gamma^{P-1}(\kappa)\,\varphi^{Q-1}(\kappa)(z-\mu)^{\tau-1} A\left((z-\mu)^{\delta}\right) \\
&\times (z-\kappa)^{\tau-1} A\left((z-\kappa)^{\delta}\right) d\mu d\kappa,
\end{aligned} \tag{48}$$

which is yields the required (42). □

The next result is as follows:

Theorem 3.4 Let $A(z)$, $z > 0$ be an analytic function and $\gamma,\ \varphi : [0, +\infty) \longrightarrow \mathbb{R}$, be a non-negative integrable functions, satisfy $\dfrac{1}{P} + \dfrac{1}{Q} = 1$, $P, Q > 1$. Then, $\forall\ \tau,\ \delta \in \mathbb{R}^+$, the following inequalities holds,

$$1)\quad \frac{\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma^P\right)(z)\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\varphi^2\right)(z)}{P}+\frac{\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma^2\right)(z)\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\varphi^Q\right)(z)}{Q}$$

$$\geq\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\varphi\right)(z)\,{}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\left\{\varphi^{\frac{2}{P}}(z)\,\gamma^{\frac{2}{Q}}(z)\right\}. \tag{49}$$

$$2)\quad \frac{\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma^2\right)(z)\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\varphi^Q\right)(z)}{P}+\frac{\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\varphi^2\right)(z)\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma^P\right)(z)}{Q}$$

$$\geq\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\left\{\gamma^{\frac{2}{P}}(z)\,\varphi^{\frac{2}{Q}}(z)\right\}\right)\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\left\{\gamma^{P-1}(z)\,\varphi^{Q-1}(z)\right\}\right). \tag{50}$$

Proof. Putting $C=\gamma(\mu)\,\varphi^{\frac{2}{P}}(\kappa)$ and $D=\gamma^{\frac{2}{Q}}(\kappa)\,\varphi(\mu)$, when $\mu,\,\kappa\in(c_1,\,z)$ in Young's inequality (12), we obtain

$$\frac{\gamma^P(\mu)\,\varphi^2(\kappa)}{P}+\frac{\gamma^2(\kappa)\,\varphi^Q(\mu)}{Q}\geq\gamma(\mu)\,\varphi(\mu)\,\varphi^{\frac{2}{P}}(\kappa)\,\gamma^{\frac{2}{Q}}(\kappa). \tag{51}$$

Multiplying each side of (51) by both factors $(z-\mu)^{\tau-1}A\left((z-\mu)^{\delta}\right)$ and $(z-\kappa)^{\tau-1}A\left((z-\kappa)^{\delta}\right)$, then taking the double integrating of the estimating inequality concerning $\mu,\,\kappa$ over $(c_1,\,z)$, we get

$$\int_{c_1}^{z}\int_{c_1}^{z}\frac{\gamma^P(\mu)\,\varphi^2(\kappa)}{P}(z-\mu)^{\tau-1}A\left((z-\mu)^{\delta}\right)(z-\kappa)^{\tau-1}A\left((z-\kappa)^{\delta}\right)d\mu d\kappa$$

$$+\int_{c_1}^{z}\int_{c_1}^{z}\frac{\gamma^2(\kappa)\,\varphi^Q(\mu)}{Q}(z-\mu)^{\tau-1}A\left((z-\mu)^{\delta}\right)(z-\kappa)^{\tau-1}A\left((z-\kappa)^{\delta}\right)d\mu d\kappa$$

$$\geq\int_{c_1}^{z}\int_{c_1}^{z}\gamma(\mu)\,\varphi(\mu)\,\varphi^{\frac{2}{P}}(\kappa)\,\gamma^{\frac{2}{Q}}(\kappa)\,(z-\mu)^{\tau-1}A\left((z-\mu)^{\delta}\right)$$

$$\times(z-\kappa)^{\tau-1}A\left((z-\kappa)^{\delta}\right)d\mu d\kappa. \tag{52}$$

It follows that inequality (49).

Next, by substituting $C=\dfrac{\gamma^{\frac{2}{P}}(\mu)}{\gamma(\kappa)}$ and $D=\dfrac{\varphi^{\frac{2}{Q}}(\mu)}{\varphi(\kappa)}$, where $\mu,\,\kappa\in(c_1,\,z)$ and $\gamma(\kappa)\neq 0$, $\varphi(\kappa)\neq 0$ in (12), we have

$$\frac{\gamma^2(\mu)}{P\gamma^P(\kappa)}+\frac{\varphi^2(\mu)}{Q\varphi^Q(\kappa)}\geq\frac{\gamma^{\frac{2}{P}}(\mu)}{\gamma(\kappa)}\frac{\varphi^{\frac{2}{Q}}(\mu)}{\varphi(\kappa)},$$

which can be rewritten as

$$\frac{\gamma^2(\mu)\varphi^Q(\kappa)}{P}+\frac{\varphi^2(\mu)\gamma^P(\kappa)}{Q}\geq\left(\gamma^{\frac{2}{P}}(\mu)\varphi^{\frac{2}{Q}}(\mu)\right)\left(\gamma^{P-1}(\kappa)\varphi^{Q-1}(\kappa)\right). \tag{53}$$

Multiplying sides of the (53) by both factors $(z-\mu)^{\tau-1}A\left((z-\mu)^\delta\right)$ and $(z-\kappa)^{\tau-1}A\left((z-\kappa)^\delta\right)$, $\mu,\ \kappa\in(c_1,\ z)$, then taking t double integrating of the estimating inequality concerning μ, κ over $(c_1,\ z)$, we obtain

$$\begin{aligned}&\int_{c_1}^{z}\int_{c_1}^{z}\frac{\gamma^2(\mu)\varphi^Q(\kappa)}{P}(z-\mu)^{\tau-1}A\left((z-\mu)^\delta\right)(z-\kappa)^{\tau-1}A\left((z-\kappa)^\delta\right)d\mu d\kappa\\&+\int_{c_1}^{z}\int_{c_1}^{z}\frac{\varphi^2(\mu)\gamma^P(\kappa)}{Q}(z-\mu)^{\tau-1}A\left((z-\mu)^\delta\right)(z-\kappa)^{\tau-1}A\left((z-\kappa)^\delta\right)d\mu d\kappa\\\geq&\int_{c_1}^{z}\int_{c_1}^{z}\left(\gamma^{\frac{2}{P}}(\mu)\varphi^{\frac{2}{Q}}(\mu)\right)\left(\gamma^{P-1}(\kappa)\varphi^{Q-1}(\kappa)\right)(z-\mu)^{\tau-1}A\left((z-\mu)^\delta\right)\\&\times(z-\kappa)^{\tau-1}A\left((z-\kappa)^\delta\right)d\mu d\kappa.\end{aligned} \tag{54}$$

Clearly, the (54) leads to the desired (50). □

Theorem 3.5 Let $A(z)$, $z>0$ be an analytic function and γ, $\varphi:[0,+\infty)\longrightarrow\mathbb{R}$, be a positive and integrable functions. Suppose that

$$P:=\min_{0\leq\mu\leq z}\frac{\gamma(\mu)}{\varphi(\mu)}\quad\text{and}\quad Q:=\max_{0\leq\mu\leq z}\frac{\gamma(\mu)}{\varphi(\mu)}. \tag{55}$$

Then, for all τ, $\delta\in\mathbb{R}^+$, the following inequalities holds,

$$\begin{aligned}1)\qquad&0\leq\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma^2\right)(z)\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\varphi^2\right)(z)\\&\leq\frac{(Q+P)^2}{4PQ}\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\varphi\right)^2(z),\end{aligned} \tag{56}$$

$$\begin{aligned}2)\qquad&0\leq\sqrt{\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma^2\right)(z)\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\varphi^2\right)(z)}-\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\varphi\right)(z)\\&\leq\frac{\left(\sqrt{Q}-\sqrt{P}\right)^2}{2\sqrt{PQ}}\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\varphi\right)(z),\end{aligned} \tag{57}$$

3) $$0 \leq \left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma^2\right)(z)\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\varphi^2\right)(z) - \left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\varphi\right)^2(z)$$

$$\leq \frac{(Q-P)^2}{4PQ}\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\varphi\right)^2(z). \tag{58}$$

Proof. Considering the assumption (55), we have

$$\left(\frac{\gamma(\mu)}{\varphi(\mu)} - P\right)\left(Q - \frac{\gamma(\mu)}{\varphi(\mu)}\right)\varphi^2(\mu) \geq 0, \quad 0 \leq \mu \leq z,$$

which is equivalent to

$$(\gamma(\mu) - P\varphi(\mu))(Q\varphi(\mu) - \gamma(\mu)) \geq 0.$$

It follows that

$$(Q+P)\gamma(\mu)\varphi(\mu) \geq \gamma^2(\mu) + PQ\varphi^2(\mu). \tag{59}$$

Where $\mu \in (c_1, z)$, multiplying in both sides of (59) by $(z-\mu)^{\tau-1} A\left((z-\mu)^{\delta}\right)$, then integrating the estimating inequality concerning μ over (c_1, z), we obtain

$$(Q+P)\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\varphi\right)(z) \geq \left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma^2\right)(z) + PQ\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\varphi^2\right)(z). \tag{60}$$

Since $PQ > 0$, therefore $\left(\sqrt{\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma^2\right)(z)} - \sqrt{PQ\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\varphi^2\right)(z)}\right)^2 \geq 0$. It follows that

$$\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma^2\right)(z) + PQ\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\varphi^2\right)(z) \geq 2\sqrt{\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma^2\right)(z)}\sqrt{PQ\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\varphi^2\right)(z)}. \tag{61}$$

Employing the both (60) and (61), we get

$$\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma^2\right)(z)\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\varphi^2\right)(z) \leq \frac{(Q+P)^2}{4PQ}\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\varphi\right)^2(z). \tag{62}$$

The required (56) is thus obtained. Now, from (62), we have

$$\sqrt{\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma^2\right)(z)\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\varphi^2\right)(z)} \leq \frac{Q+P}{2\sqrt{PQ}}\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\varphi\right)(z). \tag{63}$$

Adding $\left[-\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\varphi\right)(z)\right]$ to each side of (63), we get

$$\sqrt{\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma^2\right)(z)\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\varphi^2\right)(z)}-\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\varphi\right)(z)$$

$$\leq\frac{Q+P}{2\sqrt{PQ}}\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\varphi\right)(z)-\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\varphi\right)(z),$$

which yields

$$\sqrt{\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma^2\right)(z)\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\varphi^2\right)(z)}-\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\varphi\right)(z)$$

$$\leq\frac{\left(\sqrt{Q}-\sqrt{P}\right)^2}{2\sqrt{PQ}}\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\varphi\right)(z).$$

Finally, taking the square of each side of (63), then subtracting the factor $\left[\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\varphi\right)(z)\right]^2$ from each side of the estimating inequality, we obtain the required (58). Thus, the proof is completed. □

4. Application

In this part, we provide an example to confirm the validity and conditions of Theorem 3.1.

Example 4.1 Consider two integrable functions $\gamma(z)=z+1$, $\varphi(z)=z^2+1$ such that satisfying the conditions (2).

Case 1: For finite z range, it $z\in[0,Z]$ implies $p=1$, $P=Z+1$, $q=1$, $Q=Z^2+1$.

Case 2: For unbounded z requires $p=1$, $P\to\infty$, $q=1$, $Q\to\infty$.

These conditions (2) will ensure that the inequalities $\gamma(z)$, $\varphi(z)$ hold for all $z>0$.

Let $A(z)=e^z=\sum_{n=0}^{\infty}\dfrac{z^n}{n!}$. Clearly, $A(z)$ is an analytic function and positive for all $z>0$.

Using Definition 2.1, the fractional integral of a function $\gamma(z)$ involving the analytic kernel function $A(z)$ with parameters τ and δ is given by:

$$\left({}_{c_1}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\right)(z)=\int_{c_1}^{z}(z-\kappa)^{\tau-1}A\left((z-\kappa)^{\delta}\right)\gamma(\kappa)\,d\kappa.$$

Now, for $A(z)=e^z$, $\gamma(z)=z+1$, $\varphi(z)=z^2+1$, $\tau=2$, and $\delta=1$, the fractional integrals become:

$$\left({}_{0}^{A}\mathscr{I}_z^{\tau,\,\delta}\gamma\right)(z)=\int_{0}^{z}(z-\kappa)^{\tau-1}e^{(z-\kappa)}(1-\kappa)\,d\kappa.$$

At $z=1$, we get

$$\left({}_0^A\mathscr{I}_z^{2,\,1}\gamma\right)(1) = \int_0^1 e^{1-\kappa}(1-\kappa^2)\,d\kappa = 4-e \approx 1.2817,$$

$$\left({}_0^A\mathscr{I}_z^{\tau,\,\delta}\varphi\right)(z) = \int_0^z (z-\kappa)^{\tau-1} e^{(z-\kappa)}(\kappa^2+1)\,d\kappa.$$

At $z=1$, this becomes:

$$\left({}_0^A\mathscr{I}_1^{2,\,1}\varphi\right)(1) = \int_0^1 e^{1-\kappa}(1-\kappa+\kappa^2-\kappa^3)\,d\kappa = -4e+12 \approx 1.1296$$

$$\left({}_{c_1}^A\mathscr{I}_z^{\tau,\,\delta}\gamma\varphi\right)(z) = \int_{c_1}^z (z-\kappa)^{\tau-1} e^{z-\kappa}(\kappa+1)(\kappa^2+1)\,d\kappa.$$

At $z=1$, we obtain

$$\left({}_0^A\mathscr{I}_z^{2,\,1}\gamma\varphi\right)(z) = \int_0^1 (1-\kappa)e^{1-\kappa}(\kappa^3+\kappa^2+\kappa+1)\,d\kappa = 10e-24 \approx 3.1828.$$

Finally, we apply the inequality from Theorem 3.1:

$$\left| z^\tau \beta(z^\delta)\left({}_{c_1}^A\mathscr{I}_z^{\tau,\,\delta}\gamma\varphi\right)(z) - \left({}_{c_1}^A\mathscr{I}_z^{\tau,\,\delta}\varphi\right)(z)\left({}_{c_1}^A\mathscr{I}_z^{\tau,\,\delta}\gamma\right)(z)\right| \leq z^{2\tau}\beta^2(z^\delta)(Q-q)(P-p).$$

Case 1: If we restrict the domain of z to $z \in [1,\,2]$, i.e. $Z=2$, which implies that $p=2$, $P=3$, $q=2$, $Q=5$. Using $\beta(1) \approx 1.083$ and substituting the values we have computed:

$$\left| z^\tau \beta(z^\delta)\left({}_{c_1}^A\mathscr{I}_z^{\tau,\,\delta}\gamma\varphi\right)(z) - \left({}_{c_1}^A\mathscr{I}_z^{\tau,\,\delta}\varphi\right)(z)\left({}_{c_1}^A\mathscr{I}_z^{\tau,\,\delta}\gamma\right)(z)\right|$$

$$= |1.083\cdot 3.1828 - 1.1296\cdot 1.2817| = 1.9992$$

$$< 3.5187 \approx 1.083^2(5-2)(3-2) = z^{2\tau}\beta^2(z^\delta)(Q-q)(P-p).$$

This satisfies the Theorem 3.1.

Case 2: For unbounded z: $p=1$, $P\to\infty$, $q=1$, $Q\to\infty$. This case makes the right side of the inequality (23) tend to infinity. Thus, the conditions of the inequality (23) are satisfied, which leads to the Theorem 3.1 being true.

5. Conclusion

In the initial half of this study, we have re-examined and articulated Grüss inequality inside a novel framework using constant bounds. We employ the recently generalised fractional integral operator that incorporates an analytic kernel.

We derived novel Grüss-type inequalities with constant limits using generalised fractional integral operators of single and different orders. The results acquired are more generalised in character. Additionally, we established several novel associated inequalities utilising the contemporary fractional integral operator. Certain specific instances of the reported findings have been examined.

Conflict of interest

The authors declare no competing financial interest.

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