Contemporary Mathematics
https://ojs.wiserpub.com/index.php/CM/

UNIVERSAL WISER PUBLISHER

Research Article

$(k,\ g)$-Fractional Integral Hermite-Hadamard Type Inequalities Involving Convex and Symmetric Functions

Majid K. Neamah[1,2]*, Alawiah Ibrahim[2], Tariq A. Aljaaidi[3], Mohammed S. Abdo[4]

[1] Department of Mathematics, College of Sciences, University of Baghdad, Baghdad, Iraq
[2] Department of Mathematical Sciences, Faculty of Science and Technology, University of Kebangsaan Malaysia, 43600, Bangi, Selangor, Malaysia
[3] Department of Information Technology, Faculty of Engineering and Smart Computing, Modern Specialized University, Sana'a, Yemen
[4] Department of Mathematics, Hodeidah University, Al-Hodeidah, Yemen
E-mail: mmathemtic@yahoo.com

Received: 11 January 2025; **Revised:** 6 February 2025; **Accepted:** 10 February 2025

Abstract: This paper examines the importance of generalised $(k,\ g)$ fractional integral operators within the framework of mathematical inequalities. We introduce innovative generalised fractional integral Hermite-Hadamard inequalities, which, to our knowledge, constitute a new advancement in the area. These inequalities are formulated using contemporary generalised fractional integral operators and underscore the complex interconnections among convexity, symmetry, and fractional calculus. align, we present fractional integral Hermite-Hadamard-type inequalities that utilise these generalised operators, offering an expanded framework for comprehending the characteristics of convex and symmetric functions. Our discoveries enhance theoretical understanding and possess prospective applications in optimisation, numerical analysis, and diverse areas of applied mathematics. Furthermore, we enhance this work by discussing several special cases pertinent to this paper.

***Keywords*:** Hermite-Hadamard-type inequalities, Riemann-Liouville fractional integral, fractional integral

MSC: 26D10, 26A33

1. Introduction

Throughout the last two centuries, several mathematicians have proposed important and helpful mathematical inequalities. Among mathematical inequalities, one inequality holds a significant position within the theory of inequalities. This is the esteemed Hermite-Hadamard inequality. This disparity was originally proposed by Hermite in 1881. Nevertheless, before 1893, this result was unrecognised in the literature and not widely accepted as Hermite's inequality [1]. He wrote that the inequality:

$$\gamma\left(\frac{c_1+c_2}{2}\right) \leq \frac{1}{(c_2-c_1)}\int_{c_1}^{c_2}\gamma(z)\,dz \leq \frac{\gamma(c_1)+\gamma(c_2)}{2},\ c_1,\ c_2 \in \mathbb{R},\ c_1 < c_2, \tag{1}$$

DOI: https://doi.org/10.37256/cm.6220256396

The property applicable to convex functions γ was established by Hadamard in 1893 and is known as the Hermite-Hadamard inequality. Since then, it has undergone numerous generalisations and expansions for univariable, bivariable, and multivariable convex functions [2]. This inequality has attracted substantial interest from academics and mathematicians because of its significant applicability in several fields. Accordingly, a lot of research has appeared to contain generalisations and extensions of this inequality (see [3–5]). Sarikaya et al. [6] were able, by using the fractional operator Riemann-Liouville, to present the fractional the integral formula of this inequality for the convex functions as follows:

$$\frac{\gamma(c_1)+\gamma(c_2)}{2} \leq \frac{\Gamma(\tau+1)}{2(c_1+c_2)^{\tau}}\left[\mathscr{I}^{\tau}_{c_1^+}\gamma(c_2)+\mathscr{I}^{\tau}_{c_2^-}\gamma(c_1)\right] \leq \gamma\left(\frac{c_1+c_2}{2}\right), \tag{2}$$

where $\gamma: [c_1,\, c_2] \longrightarrow \mathbb{R}$ be a positive convex function on $[c_1,\, c_2]$ with $0 \leq c_1 < c_2$, and $\tau > 0$, and the $\mathscr{I}^{\tau}$ represents the Riemann-Liouville, which has been subsequently extended and referred to as Hermite-Hadamard-Mercer type inequalities. Chen [7], established the following Riemann-Liouville fractional integral Hermite-Hadamard type inequalities. Numerous researchers have extensively employed various operators on convex functions (see [8–12]), resulting in the derivation of different types of convex functions. The matter will not stop at this point, as different types of convex functions have appeared, and sometimes the convexity condition has been dispensed with. In an application of the Hermite-Hadamard type, Nowicka and Witkowski [13], showed how certain improvements may be employed for certain planar figures and three-dimensional bodies that meet specified regularity constraints, regardless of convexity. Dahmani [14], employed concave functions to establish Hermite-Hadamard-type inequalities using the Riemann-Liouville fractional integral. Set [15], examined the Hermite-Hadamard type for the second kind of s-convex functions, as established by Dragomir [16], and m-convex functions utilising fractional integrals. Noor [17], utilised q-differentiable convex and quasi-convex functions to formulate quantum estimates of the Hermite-Hadamard variety. Garwal et al. [18], employed the $(k,\, s)$-R-L fractional integrals to establish certain Hermite-Hadamard-type inequalities for convex functions. New Hermite-Hadamard-type inequalities were created using convex functions, s-convex functions, and coordinate convex functions through the conformable fractional integral [19]. Mohammed and Brevik [20] proposed a novel variant of the Hermite-Hadamard type for R-L fractional integrals. Awan et al. [21] established novel Hermite-Hadamard type inequalities for n-polynomial harmonically convex functions in the same year. Furthermore, Chudziak and Zołdak [22] proposed a concept for a coordinated $(F,\, G)$-convex function defined on an interval in $\mathbb{R}^2$. Recently, Khan et al. [23] examined the Hermite-Hadamard inequality concerning the coordinates of convex fuzzy interval-valued functions. Numerous writers have examined the Hermite-Hadamard inequalities across various forms of convexity in functions, utilising diverse fractional integral and derivative operators, for further information (see [24–28]).

This study aims to resolve notable gaps in the literature on fractional calculus, specifically on Hermite-Hadamard inequalities. While much of the existing research has focused on traditional fractional operators, like Riemann-Liouville and Caputo, there has been limited exploration of generalised fractional operators that could enhance both applicability and theoretical insights. This lack of generalisation limits the potential for broader applications in various fields. Moreover, most studies concentrate on standard convex functions, overlooking the rich landscape of generalised convexity concepts, such as S-convexity and coordinate convexity. This oversight limits the depth of understanding and the potential for new mathematical developments. The current body of work often repeats classical results without extending them to new contexts, resulting in a scarcity of novel inequalities derived from generalised fractional integral operators. Additionally, while theoretical advancements in fractional calculus are noteworthy, they frequently do not translate into practical applications. This disconnect highlights the need for research that connects mathematical developments with real-world problems, ensuring that the findings are relevant and useful in various domains. Furthermore, the field tends to remain isolated within specific mathematical domains, with limited interdisciplinary collaboration. This lack of connection between different fields restricts the potential for innovative applications of fractional calculus. In response to these gaps, this study aims to introduce new generalised fractional integral Hermite-Hadamard inequalities and demonstrate their applicability across various contexts.

Fractional calculus is a vital domain of mathematical analysis, offering crucial instruments for modelling intricate systems. The generalisation to (k, g)-fractional integral operators introduces a complexity that requires careful consideration of the conditions for validity. The derived inequalities typically depend on properties like continuity, convexity, and bounding. If these conditions are not met-such as with discontinuous or non-convex functions, the inequalities may fail or become less effective. Additionally, the choice of parameters k and g can affect the results, potentially leading to overly broad or inapplicable outcomes. Talking about these conditions and restrictions would help people understand better and direct future research towards finding valid situations and creating modified inequalities for a wider range of functions.

Discontinuities may result in undefined or divergent integrals, thus rendering the inequalities invalid. Likewise, nonconvex functions may contravene the requisite criteria for the inequalities to be valid, leading to erroneous limits. By looking into these cases, we might learn more about how strong the established inequalities are. This could lead to the creation of new mathematical tools or different inequalities that cover more complex function classes, which would make fractional calculus more useful.

The impetus for generalising classical operators originates from the necessity for more adaptable mathematical instruments capable of tackling complex systems and phenomena. Generalised operators expand application by including a broader spectrum of functions and behaviours, especially in fractional calculus, where systems frequently display non-integer dynamics. This generalisation yields fresh insights, including the formulation of innovative inequalities and the expansion of classical conclusions, thus augmenting our comprehension of mathematical properties and their interrelations. Moreover, generalised operators enhance numerical techniques for resolving differential equations, allowing more precise modelling across many domains. They offer adaptability in customising strategies for particular issues and stimulate more study, promoting multidisciplinary cooperation. These breakthroughs empower mathematicians to address intricate difficulties more efficiently and foster discoveries in scientific and technical fields.

The organisation of this paper is, as follows: In the first subsection, we present the reverse Hermite-Hadamard inequality for convex functions. The second paragraph presents supplementary results relating to Hermite-Hadamard-type inequalities that incorporate the (k, g)-fractional integral operator. The second part delineates our discoveries about Hermite-Hadamard inequalities and Hermite-Hadamard-type inequalities by utilising the generalised k-fractional integral operator, which encompasses symmetric functions.

2. Essential preliminaries

The definitions provided in this part are essential for building a robust foundation for the next debates and outcomes related to fractional calculus and its applications. Comprehending these notions is crucial for understanding the intricacies of generalised fractional integral operators and their ramifications in mathematical inequalities, especially the Hermite-Hadamard inequalities. The Riemann-Liouville fractional integral operator generalises conventional integration to non-integer orders, whereas the gamma function offers crucial resources for managing these fractionals. The g-Riemann-Liouville fractional integrals are flexible because they can integrate an increasing function g, which is important for studying a lot of different kinds of convexity and symmetry. Also, the idea of function composition is important for looking at how fractional integrals and derivatives affect each other. These concepts collectively facilitate the advancement of generalised proportional fractional integrals, essential for formulating new inequalities, connecting fractional calculus with classical inequality theory, and enhancing the mathematical framework.

Definition 1 [29] Let $\gamma \in L_1[c_1, c_2]$. The Riemann-Liouville fractional integral operator of order τ ($\tau > 0$) is defined as

$$\mathscr{I}^{\tau}\gamma(z)=\frac{1}{\Gamma(\tau)}\int_{c_1}^{z}(z-\mu)^{\tau-1}\gamma(\mu)\,d\mu,\ c_1<\mu<c_2, \tag{3}$$

$$\mathscr{I}^{\tau}\gamma(z)=\gamma(z),$$

where $\Gamma(\tau)$ is the gamma function (4).

Definition 2 [30] The gamma function, represented by $\Gamma(z)$, is a function defined by the Euler's integral [31], i.e.,

$$\Gamma(z)=\int_0^{\infty}z^{z-1}e^{-t}dt, \tag{4}$$

for any $(z\in\mathbb{C})$ such that $(\Re(z)>0)$. It is clear that from (4) we have, $\Gamma(1)=1$ and it satisfies the following recurrent formulas:

$$\Gamma(1+z)=z\Gamma(z),$$

$$\Gamma(1-z)=-z\Gamma(-z),$$

for all $z\in\mathbb{C}$ with $\Re(z)>1$. The gamma function (4) reduces to the factorial function when the argument is a positive real integer n, that is,

$$\Gamma(n)=(n-1)!.$$

The generalization of (4) is called k-gamma function. This interesting special function was introduced by [32] and has been given as,

$$\Gamma_k(z)=\int_0^{\infty}t^{z-1}exp\left(-\frac{t^k}{k}\right)dt,\ \Re(z)>0,\ k>0,$$

is referred to as the k-Gamma function when $z,\ k>0$. The k-Gamma function exhibits the following relationships: We observe that $\Gamma_k(z)$ converges to $\Gamma(z)$ as k approaches to 1, $\Gamma_k(z)=(K)^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right)$ and $\Gamma_k(z+k)=z\Gamma_k(z)$. Also, obviously, we have $\Gamma_k(k)=1$.

Definition 3 [29] For the integrable function γ on the interval $\digamma$ and for the increasing function g, where $g(z)\in C^1(\digamma,\ \mathbb{R})$ such that $g'(z)\neq 0,\ z\in\digamma$. We have for all $\tau>0$

$$\left({}^{g}\mathscr{I}_{c_1^+}^{\tau}\gamma\right)(z)=\frac{1}{\Gamma(\tau)}\int_{c_1}^{z}g'(\mu)\left[g(z)-g(\mu)\right]^{\tau-1}\gamma(\mu)\,d\mu \tag{5}$$

and

$$\left({}^{g}\mathscr{I}^{\tau}_{c_2^-}\gamma\right)(z) = \frac{1}{\Gamma(\tau)}\int_{z}^{c_2} g'(\mu)\left[g(\mu)-g(z)\right]^{\tau-1}\gamma(\mu)\,d\mu, \tag{6}$$

where $\left({}^{g}\mathscr{I}^{\tau}_{c_1^+}\gamma\right)(z)$ and $\left({}^{g}\mathscr{I}^{\tau}_{c_2^-}\gamma\right)(z)$ denotes the left and right-sided g Riemann-Liouville fractional integrals of the function γ, respectively.

Definition 4 [33] For the integrable function γ on the interval F and for the increasing function g, where $g(z) \in C^1(F, \mathbb{R})$ such that $g'(z) \neq 0, z \in F$. We have for all $\tau > 0$

$$\left({}^{(k,\,g)}\mathscr{I}^{\tau}_{c_1^+}\gamma\right)(z) = \frac{1}{\Gamma_k(\tau)}\int_{c_1}^{z} g'(\mu)\left[g(z)-g(\mu)\right]^{\frac{\tau}{k}-1}\gamma(\mu)\,d\mu \tag{7}$$

and

$$\left({}^{(k,\,g)}\mathscr{I}^{\tau}_{c_2^-}\gamma\right)(z) = \frac{1}{\Gamma_k(\tau)}\int_{z}^{c_2} g'(\mu)\left[g(\mu)-g(z)\right]^{\frac{\tau}{k}-1}\gamma(\mu)\,d\mu, \tag{8}$$

where $\left({}^{(k,\,g)}\mathscr{I}^{\tau}_{c_1^+}\gamma\right)(z)$ and $\left({}^{(k,\,g)}\mathscr{I}^{\tau}_{c_2^-}\gamma\right)(z)$ denotes the left and right-sided $(k,\ g)$ Riemann-Liouville fractional integrals of the function γ, respectively.

Definition 5 [34] If $\gamma: X \to Y$ and $\varphi: Y \to Z$ are mapping , we denote by $\varphi\circ\gamma$ their composition

$$\varphi\circ\gamma: X \to Z,\ \varphi\circ\gamma = \varphi(\gamma(x)),\ \forall x \in X. \tag{9}$$

2.1 *Some preliminary result*

Lemma 1 Let $\gamma: [c_1,\ c_2] \longrightarrow \mathbb{R}$, with $0 \le c_1 < c_2$ be a positive and twice differentiable function on $(c_1,\ c_2)$ satisfying γ be an integrable function on $[c_1,\ c_2]$. If γ'' is bounded in $[c_1,\ c_2]$. Then, the following inequalities are holds,

$$\begin{aligned}
&\frac{\tau Q_1}{2(c_2-c_1)^{\tau}}\int_{c_1}^{\left(\frac{c_1+c_2}{2}\right)}\left(\frac{c_1+c_2}{2}-\mu\right)^2\left\{(c_2-\mu)^{\tau-1}+(\mu-c_1)^{\tau-1}\right\}d\mu\\
&\le \frac{\Gamma(\tau+1)}{2(c_2-c_1)^{\tau}}\left\{\mathscr{I}^{\tau}_{c_1^+}\gamma(c_2)+\mathscr{I}^{\tau}_{c_2^-}\gamma(c_1)\right\}-\gamma\left\{\frac{c_1+c_2}{2}\right\}\\
&\le \frac{\tau Q_2}{2(c_2-c_1)^{\tau}}\int_{c_1}^{\left(\frac{c_1+c_2}{2}\right)}\left(\frac{c_1+c_2}{2}-\mu\right)^2\left\{(c_2-\mu)^{\tau-1}+(\mu-c_1)^{\tau-1}\right\}d\mu,
\end{aligned} \tag{10}$$

where the $\mathscr{I}^{\tau}$ is the Riemann-Liouville fractional integral .

Lemma 2 Let $\gamma: [c_1,\ c_2] \longrightarrow \mathbb{R}$, with $0 \le c_1 < c_2$ be a positive and twice differentiable function on $(c_1,\ c_2)$ satisfying γ be an integrable function on $[c_1,\ c_2]$. If γ'' is bounded in $[c_1,\ c_2]$. Then, the following inequalities are holds,

$$\frac{-\tau Q_2}{2(c_2-c_1)^\tau}\int_{c_1}^{\left(\frac{c_1+c_2}{2}\right)}[(c_2-\mu)(\mu-c_1)]\left\{(c_2-\mu)^{\tau-1}+(\mu-c_1)^{\tau-1}\right\}d\mu$$

$$\leq \frac{\Gamma(\tau+1)}{2(c_2-c_1)^\tau}\left\{\mathscr{I}^\tau_{c_1^+}\gamma(c_2)+\mathscr{I}^\tau_{c_2^-}\gamma(c_1)\right\}-\frac{\gamma(c_1)+\gamma(c_2)}{2}$$

$$\leq \frac{-\tau Q_1}{2(c_2-c_1)^\tau}\int_{c_1}^{\left(\frac{c_1+c_2}{2}\right)}[(c_2-\mu)(\mu-c_1)]\left\{(c_2-\mu)^{\tau-1}+(\mu-c_1)^{\tau-1}\right\}d\mu, \tag{11}$$

where the $\mathscr{I}^\tau$ is the Riemann-Liouville fractional integral.

In the same work [7] also demonstrated the fractional integral Hermite-Hadamard inequality (2) with relaxation of the convexity property of the function γ.

Liu et al. [35], presented the inequality (2) with respect to another positive increasing monotone function as follows:

Lemma 3 Suppose that $\gamma : [c_1,\ c_2] \longrightarrow \mathbb{R}$ with $0 \leq c_1 < c_2$ be a positive and convex function on $[c_1,\ c_2]$, and $g(\mu)$, be an increasing monotone and positive function having a continuous derivative on $(c_1,\ c_2)$. Then, the following inequalities are holds,

$$\gamma\left(\frac{c_1+c_2}{2}\right)\leq \frac{\Gamma(\tau+1)}{2(c_2-c_1)^\tau}$$

$$\left\{{}^g\mathscr{I}^\tau_{\{g^{-1}(c_1)\}^+}(\gamma\circ g)\left(g^{-1}(c_2)\right)+{}^g\mathscr{I}^\tau_{\{g^{-1}(c_2)\}^-}(\gamma\circ g)\left(g^{-1}(c_1)\right)\right\}$$

$$\leq \frac{\gamma(c_1)+\gamma(c_2)}{2}. \tag{12}$$

3. Main results

3.1 *(k, g)-fractional integral Hermite-Hadamard type inequalities involving convex functions*

The Hermite-Hadamard inequalities are very important in mathematical analysis because they give limits for integrals of convex functions and make fractional calculus a lot easier to understand. They help us understand non-integer order integrals and derivatives better, especially through the properties of convex functions, which proves that these differences are real. This section presents the Hermite-Hadamard inequality related to convex functions for $(k,\ g)$-fractional integral operators, obtained via function composition. This methodology seeks to extend classical findings and enhance our comprehension of the connection between fractional calculus and convexity.

Theorem 1 Let $g : \digamma \longrightarrow [c_1,\ c_2] \subseteq \mathbb{R}$, be a continuous and strictly increasing function satisfies $0 \leq c_1 < c_2$ and the differentiable convex function $\gamma : [c_1,\ c_2] \longrightarrow \mathbb{R}$ on $(c_1,\ c_2)$ such that the composition $(\gamma\circ g) : \digamma \longrightarrow \mathbb{R}$ be an integrable mapping on $\digamma$, then the following inequalities hold for all $k \in \mathbb{R}^+$

$$\gamma\left(\frac{c_1+c_2}{2}\right) \leq \frac{\delta^{\frac{\tau}{k}}\Gamma_k(\tau+k)}{2(c_2-c_1)^{\frac{\tau}{k}}}$$

$$\left({}^{(k,\, g)}\mathscr{I}^{\tau,\, \delta}_{\{g^{-1}(c_1)\}^+}(\gamma\circ g)\left(g^{-1}(c_2)\right) + {}^{(k,\, g)}\mathscr{I}^{\tau,\, \delta}_{\{g^{-1}(c_2)\}^-}(\gamma\circ g)\left(g^{-1}(c_1)\right)\right)$$

$$\leq \frac{\gamma(c_1)+\gamma(c_2)}{2}, \tag{13}$$

where $\left({}^{(k,\, g)}\mathscr{I}^{\tau,\, \delta}_{c_1^+}\gamma\right)(z)$ and $\left({}^{(k,\, g)}\mathscr{I}^{\tau,\, \delta}_{c_2^-}\gamma\right)(z)$ are the left and right-sided proportional k-fractional integrals, respectively.

Proof. Using the convexity of γ, and for each $\varsigma_1,\ \varsigma_2 \in [c_1,\ c_2]$, we can have

$$\gamma(\lambda\varsigma_1+(1-\lambda)\varsigma_2) \leq \lambda\gamma(\varsigma_1)+(1-\lambda)\gamma(\varsigma_2). \tag{14}$$

By putting $\lambda = \frac{1}{2}$, we get

$$\gamma\left(\frac{\varsigma_1+\varsigma_2}{2}\right) \leq \frac{\gamma(\varsigma_1)+\gamma(\varsigma_2)}{2}. \tag{15}$$

Taking

$$\varsigma_1 = \psi c_1+(1-\psi)c_2 \tag{16}$$

and

$$\varsigma_2 = (1-\psi)c_1+\psi c_2. \tag{17}$$

Substituting (16) and (17) in (15), we have

$$2\gamma\left(\frac{c_1+c_2}{2}\right) \leq \gamma\{\psi c_1+(1-\psi)c_2\}+\gamma\{(1-\psi)c_1+\psi c_2\}. \tag{18}$$

Taking product by $\exp\left[\frac{\delta-1}{\delta}\psi(c_2-c_1)\right]\psi^{\frac{\tau}{k}-1}$ on both sides of the inequality (18), then integrate the resulting inequality with respect to ψ from 0 to 1, we can have

$$2\gamma\left(\frac{c_1+c_2}{2}\right)\int_0^1 \exp\left[\frac{\delta-1}{\delta}\psi(c_2-c_1)\right]\psi^{\frac{\tau}{k}-1}d\psi$$

$$\leq \int_0^1 \exp\left[\frac{\delta-1}{\delta}\psi(c_2-c_1)\right]\psi^{\frac{\tau}{k}-1}\gamma\{\psi c_1+(1-\psi)c_2\}d\psi$$

$$+\int_0^1 \exp\left[\frac{\delta-1}{\delta}\psi(c_2-c_1)\right]\psi^{\frac{\tau}{k}-1}\gamma\{(1-\psi)c_1+\psi c_2\}d\psi. \tag{19}$$

To the left-hand side of the inequality (19), applying the substitution $\psi=\frac{c_2-g(\mu)}{c_2-c_1}$ and employing the fact that $\left({}^{(k,\,g)}\mathscr{I}_{c_1}^{\tau,\,\delta}[1]\right)(z)=\frac{(g(z)-g(c_1))^{\frac{\tau}{k}}}{\delta^{\frac{\tau}{k}}\Gamma_k(\tau+k)}$, we get

$$\frac{2k}{\tau}\gamma\left(\frac{c_1+c_2}{2}\right)$$

$$\leq \int_0^1 \exp\left[\frac{\delta-1}{\delta}\psi(c_2-c_1)\right]\psi^{\frac{\tau}{k}-1}\gamma\{\psi c_1+(1-\psi)c_2\}d\psi$$

$$+\int_0^1 \exp\left[\frac{\delta-1}{\delta}\psi(c_2-c_1)\right]\psi^{\frac{\tau}{k}-1}\gamma\{(1-\psi)c_1+\psi c_2\}d\psi. \tag{20}$$

Next,

$$\frac{\delta^{\frac{\tau}{k}}\Gamma_k(\tau+k)}{2(c_2-c_1)^{\frac{\tau}{k}}}\left({}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{\{g^{-1}(c_1)\}^+}(\gamma\circ g)\left(g^{-1}(c_2)\right)+{}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{\{g^{-1}(c_2)\}^-}(\gamma\circ g)\left(g^{-1}(c_1)\right)\right)$$

$$=\frac{\tau}{2k(c_2-c_1)^{\frac{\tau}{k}}}\left(\int_{g^{-1}(c_1)}^{g^{-1}(c_2)}\exp\left[\frac{\delta-1}{\delta}(c_2-g(\mu))\right](c_2-g(\mu))^{\frac{\tau}{k}-1}\gamma\{g(\mu)\}g'(\mu)d\mu\right.$$

$$\left.+\int_{g^{-1}(c_1)}^{g^{-1}(c_2)}\exp\left[\frac{\delta-1}{\delta}(g(u)-c_1)\right](g(u)-c_1)^{\frac{\tau}{k}-1}\gamma\{g(u)\}g'(u)du\right)$$

$$=\frac{\tau}{2k}\left(\int_{g^{-1}(c_1)}^{g^{-1}(c_2)}\exp\left[\frac{\delta-1}{\delta}(c_2-g(\mu))\right]\left(\frac{c_2-g(\mu)}{c_2-c_1}\right)^{\frac{\tau}{k}-1}\gamma\{g(\mu)\}\frac{g'(\mu)}{c_2-c_1}d\mu\right.$$

$$\left.+\int_{g^{-1}(c_1)}^{g^{-1}(c_2)}\exp\left[\frac{\delta-1}{\delta}(g(u)-c_1)\right]\left(\frac{g(u)-c_1}{c_2-c_1}\right)^{\frac{\tau}{k}-1}\gamma\{g(u)\}\frac{g'(u)}{c_2-c_1}du\right).$$

Putting $g(\mu) = \psi c_1 + (1-\psi) c_2$ and $g(u) = (1-\psi) c_1 + \psi c_2$, we obtain

$$\frac{\delta^{\frac{\tau}{k}} \Gamma_k(\tau+k)}{2(c_2-c_1)^{\frac{\tau}{k}}} \left\{ {}^{(k,\, g)}\mathscr{I}^{\tau,\, \delta}_{\{g^{-1}(c_1)\}^+} (\gamma \circ g)\left(g^{-1}(c_2)\right) + {}^{(k,\, g)}\mathscr{I}^{\tau,\, \delta}_{\{g^{-1}(c_2)\}^-} (\gamma \circ g)\left(g^{-1}(c_1)\right) \right\}$$

$$= \frac{\tau}{2k} \left\{ \int_0^1 \exp\left[\frac{\delta-1}{\delta} \psi(c_2-c_1)\right] \psi^{\frac{\tau}{k}-1} \gamma\{\psi c_1 + (1-\psi) c_2\} d\psi \right.$$

$$\left. + \int_0^1 \exp\left[\frac{\delta-1}{\delta} \psi(c_2-c_1)\right] \psi^{\frac{\tau}{k}-1} \gamma\{(1-\psi) c_1 + \psi c_2\} d\psi \right\}$$

$$\geq \gamma\left(\frac{c_1+c_2}{2}\right). \tag{21}$$

Which proves the first inequality in (13). To prove the second inequality and by using the convexity of γ, we can certain

$$\gamma(\psi c_1 + (1-\psi) c_2) \leq \psi\gamma(c_1) + (1-\psi)\gamma(c_2) \tag{22}$$

and

$$\gamma((1-\psi) c_1 + \psi c_2) \leq (1-\psi)\gamma(c_1) + \psi\gamma(c_2). \tag{23}$$

Adding both inequalities (22) and (23), we have

$$\gamma(\psi c_1 + (1-\psi) c_2) + \gamma((1-\psi) c_1 + \psi c_2)$$

$$\leq \gamma(c_1) + \gamma(c_2). \tag{24}$$

Taking product by $\exp\left[\frac{\delta-1}{\delta} \psi(c_2-c_1)\right] \psi^{\frac{\tau}{k}-1}$ on both sides of the inequality (24), then integrate the resulting inequality with respect to ψ from 0 to 1, we can get

$$\int_0^1 \exp\left[\frac{\delta-1}{\delta} \psi(c_2-c_1)\right] \psi^{\frac{\tau}{k}-1} \gamma(\psi c_1 + (1-\psi) c_2) d\psi$$

$$+ \int_0^1 \exp\left[\frac{\delta-1}{\delta} \psi(c_2-c_1)\right] \psi^{\frac{\tau}{k}-1} \gamma((1-\psi) c_1 + \psi c_2) d\psi$$

$$\leq \frac{[\gamma(c_1) + \gamma(c_2)]k}{\tau}.$$

So, it follows that

$$\frac{\delta^{\frac{\tau}{k}} k\Gamma_k(\tau)}{(c_2-c_1)^{\frac{\tau}{k}}}\left({}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{\{g^{-1}(c_1)\}^+}(\gamma\circ g)\left(g^{-1}(c_2)\right)+{}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{\{g^{-1}(c_2)\}^-}(\gamma\circ g)\left(g^{-1}(c_1)\right)\right)$$

$$\leq \frac{[\gamma(c_1)+\gamma(c_2)]k}{\tau}.$$

Which yields

$$\frac{\delta^{\frac{\tau}{k}}\Gamma_k(\tau+k)}{2(c_2-c_1)^{\frac{\tau}{k}}}\left({}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{\{g^{-1}(c_1)\}^+}(\gamma\circ g)\left(g^{-1}(c_2)\right)+{}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{\{g^{-1}(c_2)\}^-}(\gamma\circ g)\left(g^{-1}(c_1)\right)\right)$$

$$\leq \frac{\gamma(c_1)+\gamma(c_2)}{2}. \tag{25}$$

Therefore, based on the inequalities (21) and (25), we may deduce the desired inequality (13). □

Remark 1

(i): If we put $\delta=1,\ k=1$, then Theorem 1 will reduce to the Lemma 3 which is the result proved by Liu et al. [35] for g-R-L fractional integral.

(ii): Taking $\delta=1, k=1, g(\varsigma_1)=\varsigma_1, \forall\varsigma_1\in[c_1,\ c_2]$, so that Theorem 1 will reduce to Theorem 2 which is the result proved by Sarikaya et al. [6] for classical R-L fractional integral.

(iii): If we apply the inequalities (13) for $g(\varsigma_1)=\varsigma_1, \forall\varsigma_1\in[c_1,\ c_2]$ and $\delta=1,\ k=1,\ \tau=1$, we get the classical Hermite-Hadamard inequalities for a classical integral (1).

Theorem 2 Let $g:\digamma\longrightarrow[c_1,\ c_2]\subseteq\mathbb{R}$, be a continuous and strictly increasing function satisfies $0\leq c_1<c_2$ and the differentiable convex function $\gamma:[c_1,\ c_2]\longrightarrow\mathbb{R}$ on $(c_1,\ c_2)$ such that the composition $(\gamma\circ g):\digamma\longrightarrow\mathbb{R}$ be an integrable mapping on $\digamma$, then the following inequalities hold for all $k\in\mathbb{R}^+$

$$\gamma\left(\frac{c_1+c_2}{2}\right)\leq\frac{2^{\frac{\tau}{k}-1}\delta^{\frac{\tau}{k}}\Gamma_k(\tau+k)}{(c_2-c_1)^{\frac{\tau}{k}}}$$

$$\times\left({}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{\left\{g^{-1}\left(\frac{c_1+c_2}{2}\right)\right\}^-}(\gamma\circ g)\left(g^{-1}(c_1)\right)+{}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{\left\{g^{-1}\left(\frac{c_1+c_2}{2}\right)\right\}^+}(\gamma\circ g)\left(g^{-1}(c_2)\right)\right)$$

$$\leq\frac{\gamma(c_1)+\gamma(c_2)}{2}, \tag{26}$$

where $\left({}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{c_1^+}\gamma\right)(z)$ and $\left({}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{c_2^-}\gamma\right)(z)$ are the left and right-sided proportional k-fractional integrals, respectively.

Proof. Using the convexity of the function γ, for each $\varsigma_1,\ \varsigma_2\in[c_1,\ c_2]$, we can have

$$\gamma\left(\frac{\varsigma_1+\varsigma_2}{2}\right)\leq\frac{\gamma(\varsigma_1)+\gamma(\varsigma_2)}{2}.$$

Putting $\varsigma_1 = \frac{\psi}{2}c_2 + \frac{2-\psi}{2}c_1$ and $\varsigma_2 = \frac{2-\psi}{2}c_2 + \frac{\psi}{2}c_1$, it follows that for all $\varsigma_1,\ \varsigma_2 \in [c_1,\ c_2]$, $\psi \in [0,\ 1]$,

$$\gamma\left(\frac{c_1+c_2}{2}\right) \leq \frac{1}{2}\left\{\gamma\left(\frac{\psi}{2}c_2 + \frac{2-\psi}{2}c_1\right) + \gamma\left(\frac{2-\psi}{2}c_2 + \frac{\psi}{2}c_1\right)\right\}. \tag{27}$$

Taking product by $\exp\left[\frac{\delta-1}{\delta}\frac{\psi}{2}(c_2-c_1)\right]\left(\frac{\psi}{2}\right)^{\frac{\tau}{k}-1}$, on both sides of the inequality (27), then integrate the resulting inequality with respect to ψ from $\psi = 0$ to $\psi = 1$, we obtain

$$\begin{aligned}
&\frac{2^{2-\frac{\tau}{k}}k}{\tau}\gamma\left(\frac{c_1+c_2}{2}\right) \\
&\leq \int_0^1 \exp\left[\frac{\delta-1}{\delta}\frac{\psi}{2}(c_2-c_1)\right]\left(\frac{\psi}{2}\right)^{\frac{\tau}{k}-1}\gamma\left(\frac{\psi}{2}c_2 + \frac{2-\psi}{2}c_1\right)d\psi \\
&\quad + \int_0^1 \exp\left[\frac{\delta-1}{\delta}\frac{\psi}{2}(c_2-c_1)\right]\left(\frac{\psi}{2}\right)^{\frac{\tau}{k}-1}\gamma\left(\frac{2-\psi}{2}c_2 + \frac{\psi}{2}c_1\right)d\psi
\end{aligned} \tag{28}$$

Next,

$$\begin{aligned}
&\frac{2^{\frac{\tau}{k}-1}\delta^{\frac{\tau}{k}}\Gamma_k(\tau+k)}{(c_2-c_1)^{\frac{\tau}{k}}}\left({}^{(k,\ g)}\mathscr{I}^{\tau,\ \delta}_{\left\{g^{-1}\left(\frac{c_1+c_2}{2}\right)\right\}^-}(\gamma\circ g)\left(g^{-1}(c_1)\right)\right. \\
&\left. + {}^{(k,\ g)}\mathscr{I}^{\tau,\ \delta}_{\left\{g^{-1}\left(\frac{c_1+c_2}{2}\right)\right\}^+}(\gamma\circ g)\left(g^{-1}(c_2)\right)\right) \\
&= \frac{2^{\frac{\tau}{k}-1}\tau}{k(c_2-c_1)^{\frac{\tau}{k}}} \\
&\times \left(\int_{g^{-1}(c_1)}^{g^{-1}\left(\frac{c_1+c_2}{2}\right)} \exp\left[\frac{\delta-1}{\delta}(g(\mu)-c_1)\right](g(\mu)-c_1)^{\frac{\tau}{k}-1}(\gamma\circ g)(\mu)g'(\mu)d\mu\right. \\
&\left. + \int_{g^{-1}\left(\frac{c_1+c_2}{2}\right)}^{g^{-1}(c_2)} \exp\left[\frac{\delta-1}{\delta}(c_2-g(u))\right](c_2-g(u))^{\frac{\tau}{k}-1}(\gamma\circ g)(u)g'(u)du\right) \\
&= \frac{2^{\frac{\tau}{k}-1}\tau}{k}\left(\int_{g^{-1}(c_1)}^{g^{-1}\left(\frac{c_1+c_2}{2}\right)} \exp\left[\frac{\delta-1}{\delta}(g(\mu)-c_1)\right]\left(\frac{g(\mu)-c_1}{c_2-c_1}\right)^{\frac{\tau}{k}-1}(\gamma\circ g)(\mu)\frac{g'(\mu)}{c_2-c_1}d\mu\right. \\
&\left. + \int_{g^{-1}\left(\frac{c_1+c_2}{2}\right)}^{g^{-1}(c_2)} \exp\left[\frac{\delta-1}{\delta}(c_2-g(u))\right]\left(\frac{c_2-g(u)}{c_2-c_1}\right)^{\frac{\tau}{k}-1}(\gamma\circ g)(u)\frac{g'(u)}{c_2-c_1}du\right).
\end{aligned}$$

Putting $g(\mu)=\frac{\psi}{2}c_2+\frac{2-\psi}{2}c_1$ and $g(u)=\frac{2-\psi}{2}c_2+\frac{\psi}{2}c_1$, we obtain

$$\frac{2^{\frac{\tau}{k}-1}\delta^{\frac{\tau}{k}}\Gamma_k(\tau+k)}{(c_2-c_1)^{\frac{\tau}{k}}}$$

$$\times\left({}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{\left\{g^{-1}\left(\frac{c_1+c_2}{2}\right)\right\}^-}(\gamma\circ g)\left(g^{-1}(c_1)\right)+{}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{\left\{g^{-1}\left(\frac{c_1+c_2}{2}\right)\right\}^+}(\gamma\circ g)\left(g^{-1}(c_2)\right)\right)$$

$$=\frac{2^{\frac{\tau}{k}-2}\tau}{k}\left(\int_0^1\exp\left[\frac{\delta-1}{\delta}\frac{\psi}{2}(c_2-c_1)\right]\left(\frac{\psi}{2}\right)^{\frac{\tau}{k}-1}\gamma\left(\frac{\psi}{2}c_2+\frac{2-\psi}{2}c_1\right)d\psi\right.$$

$$\left.+\int_0^1\exp\left[\frac{\delta-1}{\delta}\frac{\psi}{2}(c_2-c_1)\right]\left(\frac{\psi}{2}\right)^{\frac{\tau}{k}-1}\gamma\left(\frac{2-\psi}{2}c_2+\frac{\psi}{2}c_1\right)d\psi\right)$$

$$\geq\gamma\left(\frac{c_1+c_2}{2}\right),\tag{29}$$

This produces the initial inequality in (26). To demonstrate the alternative inequality and consider the convexity of the function γ, we ascertain:

$$\gamma\left(\frac{\psi}{2}c_2+\frac{2-\psi}{2}c_1\right)\leq\frac{\psi}{2}\gamma(c_2)+\frac{2-\psi}{2}\gamma(c_1),\tag{30}$$

$$\gamma\left(\frac{2-\psi}{2}c_2+\frac{\psi}{2}c_1\right)\leq\left(\frac{2-\psi}{2}\gamma(c_2)+\frac{\psi}{2}\gamma(c_1)\right).\tag{31}$$

Adding the inequalities (30), (31), we get

$$\gamma\left(\frac{\psi}{2}c_2+\frac{2-\psi}{2}c_1\right)+\gamma\left(\frac{2-\psi}{2}c_2+\frac{\psi}{2}c_1\right)\leq\gamma(c_1)+\gamma(c_2).\tag{32}$$

Taking product by $\exp\left[\frac{\delta-1}{\delta}\frac{\psi}{2}(c_2-c_1)\right]\left(\frac{\psi}{2}\right)^{\frac{\tau}{k}-1}$, on both sides of the inequality (32), then integrate the resulting inequality with respect to ψ from $\psi=0$ to $\psi=1$, we get

$$\frac{2^{\frac{\tau}{k}-1}\tau}{k}\left\{\int_0^1 \exp\left[\frac{\delta-1}{\delta}\frac{\psi}{2}(c_2-c_1)\right]\left(\frac{\psi}{2}\right)^{\tau-1}\gamma\left(\frac{\psi}{2}c_2+\frac{2-\psi}{2}c_1\right)d\psi\right.$$

$$\left.+\int_0^1 \exp\left[\frac{\delta-1}{\delta}\frac{\psi}{2}(c_2-c_1)\right]\left(\frac{\psi}{2}\right)^{\tau-1}\gamma\left(\frac{2-\psi}{2}c_2+\frac{\psi}{2}c_1\right)d\psi\right\}$$

$$\leq \gamma(c_1)+\gamma(c_2).$$

Which yields

$$\frac{2^{\frac{\tau}{k}-2}\tau}{k}\left\{\int_0^1 \exp\left[\frac{\delta-1}{\delta}\frac{\psi}{2}(c_2-c_1)\right]\left(\frac{\psi}{2}\right)^{\tau-1}\gamma\left(\frac{\psi}{2}c_2+\frac{2-\psi}{2}c_1\right)d\psi\right.$$

$$\left.+\int_0^1 \exp\left[\frac{\delta-1}{\delta}\frac{\psi}{2}(c_2-c_1)\right]\left(\frac{\psi}{2}\right)^{\tau-1}\gamma\left(\frac{2-\psi}{2}c_2+\frac{\psi}{2}c_1\right)d\psi\right\}$$

$$\leq \frac{\gamma(c_1)+\gamma(c_2)}{2}. \tag{33}$$

By comparing the left-hand side of inequality (29) with that of inequality (33), we may deduce that

$$\frac{2^{\frac{\tau}{k}-1}\delta^{\frac{\tau}{k}}\Gamma_k(\tau+k)}{(c_2-c_1)^{\frac{\tau}{k}}}$$

$$\times\left({}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{\left\{g^{-1}\left(\frac{c_1+c_2}{2}\right)\right\}^-}(\gamma\circ g)\left(g^{-1}(c_1)\right)+{}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{\left\{g^{-1}\left(\frac{c_1+c_2}{2}\right)\right\}^+}(\gamma\circ g)\left(g^{-1}(c_2)\right)\right)$$

$$\leq \frac{\gamma(c_1)+\gamma(c_2)}{2},$$

It represents the requisite second inequality in (26). The proof is, therefore, complete. □

3.2 *(k, g)-fractional integral Hermite-Hadamard type inequalities involving symmetric functions*

The current section involves a new generalisation for some types of Hermite-Hadamard inequalities, which were introduced using the R-L integral by Chinchane and Pachpatte [36]. To achieve our results here, we use the fractional integral established by Jarad et al. [37] , which was later generalized to $(k,\, g)$-fractional integral version by Aljaaidi et al. [33] .

Theorem 3 For the continuous and strictly increasing function $g:[c_1,\, c_2]\longrightarrow \digamma\subseteq\mathbb{R}$ on $[c_1,\, c_2]$, satisfying $0\leq c_1<c_2$ and the twice differentiable positive and symmetric function $\gamma:\digamma\longrightarrow\mathbb{R}$ on $\digamma^\circ$ such that the composition $(\gamma\circ g):[c_1,\, c_2]\longrightarrow\mathbb{R}$ be an integrable mapping on $[c_1,\, c_2]$, provided that γ'' is bounded in $[c_1,\, c_2]$. Then, the following inequalities hold for all $k\in\mathbb{R}^+$

$$
\frac{\tau Q_1}{2k(c_2-c_1)^{\frac{\tau}{k}}}\int_{g^{-1}(c_1)}^{g^{-1}\left(\frac{c_1+c_2}{2}\right)}\left(\frac{c_1+c_2}{2}-g(\mu)\right)^2
$$

$$
\times\left(\exp\left[\frac{\delta-1}{\delta}(c_2-g(\mu))\right](c_2-g(\mu))^{\frac{\tau}{k}-1}\right.
$$

$$
\left.+\exp\left[\frac{\delta-1}{\delta}(g(\mu)-c_1)\right](g(\mu)-c_1)^{\frac{\tau}{k}-1}\right)g'(\mu)d\mu
$$

$$
\leq\frac{\delta^{\frac{\tau}{k}}\Gamma(\tau+k)}{2(c_2-c_1)^{\frac{\tau}{k}}}\left({}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{\{g^{-1}(c_1)\}^+}(\gamma\circ g)\left(g^{-1}(c_2)\right)\right.
$$

$$
\left.+{}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{\{g^{-1}(c_2)\}^-}(\gamma\circ g)\left(g^{-1}(c_1)\right)\right)-\gamma\left\{\frac{c_1+c_2}{2}\right\}
$$

$$
\leq\frac{\tau Q_2}{2k(c_2-c_1)^{\frac{\tau}{k}}}\int_{g^{-1}(c_1)}^{g^{-1}\left(\frac{c_1+c_2}{2}\right)}\left(\frac{c_1+c_2}{2}-g(\mu)\right)^2
$$

$$
\times\left(\exp\left[\frac{\delta-1}{\delta}(c_2-g(\mu))\right](c_2-g(\mu))^{\frac{\tau}{k}-1}\right.
$$

$$
\left.+\exp\left[\frac{\delta-1}{\delta}(g(\mu)-c_1)\right](g(\mu)-c_1)^{\frac{\tau}{k}-1}\right)g'(\mu)d\mu, \tag{34}
$$

with $\tau>0$, where $Q_1=\inf_{t\in[c_1,\,c_2]}f''(t),\ Q_2=\sup_{t\in[c_1,\,c_2]}f''(t)$ and $\left({}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{c_1^+}\gamma\right)(z)$ and $\left({}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{c_2^-}\gamma\right)(z)$ are the left and right-sided proportional k-fractional integrals, respectively.

Proof. Using the definition 4, we can write

$$
\frac{\delta^{\frac{\tau}{k}}\Gamma(\tau+k)}{2(c_2-c_1)^{\frac{\tau}{k}}}
$$

$$
\times\left({}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{\{g^{-1}(c_1)\}^+}(\gamma\circ g)\left(g^{-1}(c_2)\right)+{}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{\{g^{-1}(c_2)\}^-}(\gamma\circ g)\left(g^{-1}(c_1)\right)\right)
$$

$$
=\frac{\tau}{2k(c_2-c_1)^{\frac{\tau}{k}}}\left(\int_{g^{-1}(c_1)}^{g^{-1}(c_2)}\exp\left[\frac{\delta-1}{\delta}(c_2-g(\mu))\right](c_2-g(\mu))^{\frac{\tau}{k}-1}\gamma\{g(\mu)\}g'(\mu)d\mu\right.
$$

$$
\left.+\int_{g^{-1}(c_1)}^{g^{-1}(c_2)}\exp\left[\frac{\delta-1}{\delta}(g(\mu)-c_1)\right](g(\mu)-c_1)^{\frac{\tau}{k}-1}\gamma\{g(\mu)\}g'(\mu)d\mu\right)
$$

$$= \frac{\tau}{2k(c_2-c_1)^{\frac{\tau}{k}}} \int_{g^{-1}(c_1)}^{g^{-1}(c_2)} \left(\exp\left[\frac{\delta-1}{\delta}(c_2-g(\mu))\right](c_2-g(\mu))^{\frac{\tau}{k}-1} \right.$$

$$\left. + \exp\left[\frac{\delta-1}{\delta}(g(\mu)-c_1)\right](g(\mu)-c_1)^{\frac{\tau}{k}-1} \right) \gamma\{g(\mu)\} g'(\mu)\, d\mu$$

$$= \frac{\tau}{2k(c_2-c_1)^{\frac{\tau}{k}}} \int_{g^{-1}(c_1)}^{g^{-1}(c_2)} \left(\exp\left[\frac{\delta-1}{\delta}(c_2-g(\mu))\right](c_2-g(\mu))^{\frac{\tau}{k}-1} \right.$$

$$\left. + \exp\left[\frac{\delta-1}{\delta}(g(\mu)-c_1)\right](g(\mu)-c_1)^{\frac{\tau}{k}-1} \right) \gamma\{c_1+c_2-g(\mu)\} g'(\mu)\, d\mu.$$

So, we have

$$\frac{\delta^{\frac{\tau}{k}}\Gamma(\tau+k)}{2(c_2-c_1)^{\frac{\tau}{k}}}$$

$$\times \left({}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{\{g^{-1}(c_1)\}^+} (\gamma\circ g)\left(g^{-1}(c_2)\right) + {}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{\{g^{-1}(c_2)\}^-} (\gamma\circ g)\left(g^{-1}(c_1)\right) \right)$$

$$= \frac{\tau}{4k(c_2-c_1)^{\frac{\tau}{k}}} \int_{g^{-1}(c_1)}^{g^{-1}(c_2)} \left(\exp\left[\frac{\delta-1}{\delta}(c_2-g(\mu))\right](c_2-g(\mu))^{\frac{\tau}{k}-1} \right. \tag{35}$$

$$\left. + \exp\left[\frac{\delta-1}{\delta}(g(\mu)-c_1)\right](g(\mu)-c_1)^{\frac{\tau}{k}-1} \right)$$

$$\times \left(\gamma\{g(\mu)\} + \gamma\{c_1+c_2-g(\mu)\}\right) g'(\mu)\, d\mu,$$

which we can rewrite as

$$\frac{\delta^{\frac{\tau}{k}}\Gamma(\tau+k)}{2(c_2-c_1)^{\frac{\tau}{k}}} \left({}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{\{g^{-1}(c_1)\}^+} (\gamma\circ g)\left(g^{-1}(c_2)\right) \right.$$

$$\left. + {}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{\{g^{-1}(c_2)\}^-} (\gamma\circ g)\left(g^{-1}(c_1)\right) \right) - \gamma\left\{\frac{c_1+c_2}{2}\right\}$$

$$= \frac{\tau}{4k(c_2-c_1)^{\frac{\tau}{k}}} \int_{g^{-1}(c_1)}^{g^{-1}(c_2)} \left(\exp\left[\frac{\delta-1}{\delta}(c_2-g(\mu))\right](c_2-g(\mu))^{\frac{\tau}{k}-1} \right.$$

$$+\exp\left[\frac{\delta-1}{\delta}(g(\mu)-c_1)\right](g(\mu)-c_1)^{\frac{\tau}{k}-1}\Bigg)$$

$$\times\left(\gamma\{c_1+c_2-g(\mu)\}+\gamma\{g(\mu)\}-2\gamma\left\{\frac{c_1+c_2}{2}\right\}\right)g'(\mu)\,d\mu.$$

Since

$$\Bigg(\exp\left[\frac{\delta-1}{\delta}(c_2-g(\mu))\right](c_2-g(\mu))^{\frac{\tau}{k}-1}$$

$$+\exp\left[\frac{\delta-1}{\delta}(g(\mu)-c_1)\right](g(\mu)-c_1)^{\frac{\tau}{k}-1}\Bigg)$$

$$\times\left(\gamma\{c_1+c_2-g(\mu)\}+\gamma\{g(\mu)\}-2\gamma\left\{\frac{c_1+c_2}{2}\right\}\right), \tag{36}$$

is symmetric about $\mu=g^{-1}\left(\frac{c_1+c_2}{2}\right)$, we can deduce

$$\frac{\tau}{4k(c_2-c_1)^{\frac{\tau}{k}}}\int_{g^{-1}(c_1)}^{g^{-1}(c_2)}\Bigg(\exp\left[\frac{\delta-1}{\delta}(c_2-g(\mu))\right](c_2-g(\mu))^{\frac{\tau}{k}-1}$$

$$+\exp\left[\frac{\delta-1}{\delta}(g(\mu)-c_1)\right](g(\mu)-c_1)^{\frac{\tau}{k}-1}\Bigg)$$

$$\times\left(\gamma\{c_1+c_2-g(\mu)\}+\gamma\{g(\mu)\}-2\gamma\left\{\frac{c_1+c_2}{2}\right\}\right)g'(\mu)\,d\mu$$

$$=\frac{\tau}{2k(c_2-c_1)^{\frac{\tau}{k}}}\int_{g^{-1}(c_1)}^{g^{-1}\left(\frac{c_1+c_2}{2}\right)}\Bigg(\exp\left[\frac{\delta-1}{\delta}(c_2-g(\mu))\right](c_2-g(\mu))^{\frac{\tau}{k}-1}$$

$$+\exp\left[\frac{\delta-1}{\delta}(g(\mu)-c_1)\right](g(\mu)-c_1)^{\frac{\tau}{k}-1}\Bigg)$$

$$\times\left(\gamma\{c_1+c_2-g(\mu)\}+\gamma\{g(\mu)\}-2\gamma\left\{\frac{c_1+c_2}{2}\right\}\right)g'(\mu)\,d\mu,$$

which yields

$$\frac{\delta^{\frac{\tau}{k}}\Gamma(\tau+k)}{2(c_2-c_1)^{\frac{\tau}{k}}}\left({}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{\{g^{-1}(c_1)\}^+}(\gamma\circ g)\left(g^{-1}(c_2)\right)\right.$$

$$\left.+\,{}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{\{g^{-1}(c_2)\}^-}(\gamma\circ g)\left(g^{-1}(c_1)\right)\right)-\gamma\left\{\frac{c_1+c_2}{2}\right\}$$

$$=\frac{\tau}{2k(c_2-c_1)^{\frac{\tau}{k}}}\int_{g^{-1}(c_1)}^{g^{-1}\left(\frac{c_1+c_2}{2}\right)}\left(\exp\left[\frac{\delta-1}{\delta}(c_2-g(\mu))\right](c_2-g(\mu))^{\frac{\tau}{k}-1}\right.$$

$$\left.+\exp\left[\frac{\delta-1}{\delta}(g(\mu)-c_1)\right](g(\mu)-c_1)^{\frac{\tau}{k}-1}\right) \tag{37}$$

$$\times\left(\gamma\{c_1+c_2-g(\mu)\}+\gamma\{g(\mu)\}-2\gamma\left\{\frac{c_1+c_2}{2}\right\}\right)g'(\mu)\,d\mu.$$

Since

$$\gamma\{c_1+c_2-g(\mu)\}-\gamma\left\{\frac{c_1+c_2}{2}\right\}=\int_{\frac{c_1+c_2}{2}}^{c_1+c_2-g(\mu)}\gamma'\{g(\theta)\}\,d\{g(\theta)\},$$

and

$$\gamma\left\{\frac{c_1+c_2}{2}\right\}-\gamma\{g(\mu)\}=\int_{g(\mu)}^{\frac{c_1+c_2}{2}}\gamma'\{g(\theta)\}\,d\{g(\theta)\}.$$

Therefore, we can have

$$\gamma\{g(\mu)\}+\gamma\{c_1+c_2-g(\mu)\}-2\gamma\left\{\frac{c_1+c_2}{2}\right\}$$

$$=\int_{\frac{c_1+c_2}{2}}^{c_1+c_2-g(\mu)}\gamma'\{g(\theta)\}\,d\{g(\theta)\}-\int_{g(\mu)}^{\frac{c_1+c_2}{2}}\gamma'\{g(\theta)\}\,d\{g(\theta)\}$$

$$=\int_{g(\mu)}^{\frac{c_1+c_2}{2}}\gamma'\{c_1+c_2-g(\theta)\}\,d\{g(\theta)\}-\int_{g(\mu)}^{\frac{c_1+c_2}{2}}\gamma'\{g(\theta)\}\,d\{g(\theta)\}$$

$$=\int_{g(\mu)}^{\frac{c_1+c_2}{2}}\left[\gamma'\{c_1+c_2-g(\theta)\}-\gamma'\{g(\theta)\}\right]d\{g(\theta)\}. \tag{38}$$

Since

$$\gamma'\{c_1+c_2-g(\theta)\}-\gamma'\{g(\theta)\}=\int_{g(\theta)}^{c_1+c_2-g(\theta)}\gamma''\{g(z)\}\,d\{g(z)\},$$

therefore, for $g(\theta)\in\left[c_1,\ \frac{c_1+c_2}{2}\right]$, where $\theta\in\digamma$, we have

$$Q_1(c_1+c_2-2g(\theta))\leq\gamma'\{c_1+c_2-g(\theta)\}-\gamma'\{g(\theta)\}\leq Q_2(c_1+c_2-2g(\theta)),$$

which leads to

$$Q_1\left(\frac{c_1+c_2}{2}-g(\mu)\right)^2\leq\int_{g(\mu)}^{\frac{c_1+c_2}{2}}\left[\gamma'\{c_1+c_2-g(\theta)\}-\gamma'\{g(\theta)\}\right]d\{g(\theta)\}$$

$$\leq Q_2\left(\frac{c_1+c_2}{2}-g(\mu)\right)^2.$$

That is,

$$Q_1\left(\frac{c_1+c_2}{2}-g(\mu)\right)^2\leq\gamma\{g(\mu)\}+\gamma\{c_1+c_2-g(\mu)\}-2\gamma\left\{\frac{c_1+c_2}{2}\right\}$$

$$\leq Q_2\left(\frac{c_1+c_2}{2}-g(\mu)\right)^2. \tag{39}$$

In the view of both inequalities (39) and (37), we can conclude the required (34), which completes our proof. □

Theorem 4 For the continuous and strictly increasing function $g:[c_1,\ c_2]\longrightarrow\digamma\subseteq\mathbb{R}$ on $[c_1,\ c_2]$, satisfies $0\leq c_1<c_2$ and the twice differentiable positive and symmetric function $\gamma:\digamma\longrightarrow\mathbb{R}$ on $\digamma^\circ$ such that the composition $(\gamma\circ g):[c_1,\ c_2]\longrightarrow\mathbb{R}$ be an integrable mapping on $[c_1,\ c_2]$, provided that γ'' is bounded in $[c_1,\ c_2]$. Then, for all $k\in\mathbb{R}^+$ the following inequalities hold,

$$\frac{-\tau Q_2}{2k(c_2-c_1)^{\frac{\tau}{k}}}\int_{g^{-1}(c_1)}^{g^{-1}\left(\frac{c_1+c_2}{2}\right)}\left[(c_2-g(\mu))(g(\mu)-c_1)\right]$$

$$\times\left(\exp\left[\frac{\delta-1}{\delta}(c_2-g(\mu))\right](c_2-g(\mu))^{\frac{\tau}{k}-1}\right.$$

$$\left.+\exp\left[\frac{\delta-1}{\delta}(g(\mu)-c_1)\right](g(\mu)-c_1)^{\frac{\tau}{k}-1}\right)g'(\mu)\,d\mu$$

$$\leq \frac{\delta^{\frac{\tau}{k}}\Gamma(\tau+k)}{2(c_2-c_1)^{\frac{\tau}{k}}}\left({}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{\{g^{-1}(c_1)\}^+}(\gamma\circ g)\left(g^{-1}(c_2)\right)\right.$$

$$\left. + {}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{\{g^{-1}(c_2)\}^-}(\gamma\circ g)\left(g^{-1}(c_1)\right)\right) - \frac{\gamma(c_1)+\gamma(c_2)}{2}$$

$$\leq \frac{-\tau Q_1}{2k(c_2-c_1)^{\frac{\tau}{k}}}\int_{g^{-1}(c_1)}^{g^{-1}\left(\frac{c_1+c_2}{2}\right)}[(c_2-g(\mu))(g(\mu)-c_1)]$$

$$\times\left(\exp\left[\frac{\delta-1}{\delta}(c_2-g(\mu))\right](c_2-g(\mu))^{\frac{\tau}{k}-1}\right.$$

$$+\exp\left[\frac{\delta-1}{\delta}(g(\mu)-c_1)\right](g(\mu)-c_1)^{\frac{\tau}{k}-1}g'(\mu)\,d\mu, \tag{40}$$

where $\left({}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{c_1^+}\gamma\right)(z)$ and $\left({}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{c_2^-}\gamma\right)(z)$ are the left and right-sided proportional k-fractional integrals, respectively.

Proof. In view of Theorem 3, using the inequality (35), we have

$$\frac{\delta^{\frac{\tau}{k}}\Gamma(\tau+k)}{2(c_2-c_1)^{\frac{\tau}{k}}}\left({}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{\{g^{-1}(c_1)\}^+}(\gamma\circ g)\left(g^{-1}(c_2)\right)\right.$$

$$\left. + {}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{\{g^{-1}(c_2)\}^-}(\gamma\circ g)\left(g^{-1}(c_1)\right)\right) - \frac{\gamma(c_1)+\gamma(c_2)}{2}$$

$$= \frac{\tau}{4k(c_2-c_1)^{\frac{\tau}{k}}}\int_{g^{-1}(c_1)}^{g^{-1}(c_2)}\left(\exp\left[\frac{\delta-1}{\delta}(c_2-g(\mu))\right](c_2-g(\mu))^{\frac{\tau}{k}-1}\right.$$

$$\left.+\exp\left[\frac{\delta-1}{\delta}(g(\mu)-c_1)\right](g(\mu)-c_1)^{\frac{\tau}{k}-1}\right)$$

$$\times\left(\gamma\{c_1+c_2-g(\mu)\}+\gamma\{g(\mu)\}-[\gamma(c_1)+\gamma(c_1)]\right)g'(\mu)\,d\mu.$$

Now, the form

$$\left(\exp\left[\frac{\delta-1}{\delta}(c_2-g(\mu))\right](c_2-g(\mu))^{\frac{\tau}{k}-1}+\exp\left[\frac{\delta-1}{\delta}(g(\mu)-c_1)\right](g(\mu)-c_1)^{\frac{\tau}{k}-1}\right)$$

$$\times\left(\gamma\{c_1+c_2-g(\mu)\}+\gamma\{g(\mu)\}-[\gamma(c_1)+\gamma(c_1)]\right),$$

is symmetric about $\mu = g^{-1}\left(\frac{c_1+c_2}{2}\right)$, so we write

$$\frac{\delta^{\frac{\tau}{k}}\Gamma(\tau+k)}{2(c_2-c_1)^{\frac{\tau}{k}}}\left({}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{\{g^{-1}(c_1)\}^+}(\gamma\circ g)\left(g^{-1}(c_2)\right)\right.$$

$$\left.+{}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{\{g^{-1}(c_2)\}^-}(\gamma\circ g)\left(g^{-1}(c_1)\right)\right)-\frac{\gamma(c_1)+\gamma(c_2)}{2}$$

$$=\frac{\tau}{2k(c_2-c_1)^{\frac{\tau}{k}}}\int_{g^{-1}(c_1)}^{g^{-1}\left(\frac{c_1+c_2}{2}\right)}\left(\exp\left[\frac{\delta-1}{\delta}(c_2-g(\mu))\right](c_2-g(\mu))^{\frac{\tau}{k}-1}\right.$$

$$\left.+\exp\left[\frac{\delta-1}{\delta}(g(\mu)-c_1)\right](g(\mu)-c_1)^{\frac{\tau}{k}-1}\right) \tag{41}$$

$$\times\left(\gamma\{c_1+c_2-g(\mu)\}+\gamma\{g(\mu)\}-[\gamma(c_1)+\gamma(c_2)]\right)g'(\mu)\,d\mu.$$

Since

$$\gamma\{c_2\}-\gamma\{c_1+c_2-g(\mu)\}=\int_{c_1+c_2-g(\mu)}^{c_2}\gamma'\{g(\theta)\}\,d\{g(\theta)\}$$

and

$$\gamma\{g(\mu)\}-\gamma\{c_1\}=\int_{c_1}^{g(\mu)}\gamma'\{g(\theta)\}\,d\{g(\theta)\}.$$

It follows that

$$\gamma\{g(\mu)\}+\gamma\{c_1+c_2-g(\mu)\}-[\gamma\{c_1\}+\gamma\{c_2\}]$$

$$=\int_{c_1}^{g(\mu)}\gamma'\{g(\theta)\}\,d\{g(\theta)\}-\int_{c_1+c_2-g(\mu)}^{c_2}\gamma'\{g(\theta)\}\,d\{g(\theta)\}$$

$$=\int_{c_1}^{g(\mu)}\gamma'\{g(\theta)\}\,d\{g(\theta)\}-\int_{c_1}^{g(\mu)}\gamma'\{c_1+c_2-g(\theta)\}\,d\{g(\theta)\}$$

$$=-\int_{c_1}^{g(\mu)}\left[\gamma'\{c_1+c_2-g(\theta)\}-\gamma'\{g(\theta)\}\right]d\{g(\theta)\}. \tag{42}$$

Again, according to

$$\gamma'\left\{c_1+c_2-g(\theta)\right\}-\gamma'\left\{g(\theta)\right\}=\int_{g(\theta)}^{c_1+c_2-g(\theta)}\gamma''\left\{g(z)\right\}d\left\{g(z)\right\},$$

therefore, for $g(\theta)\in\left[c_1,\ \frac{c_1+c_2}{2}\right]$, where $\theta\in F$, we have

$$Q_1\left(c_1+c_2-2g(\theta)\right)\leq\gamma'\left\{c_1+c_2-g(\theta)\right\}-\gamma'\left\{g(\theta)\right\}\leq Q_2\left(c_1+c_2-2g(\theta)\right),$$

which leads to

$$\begin{aligned}&-Q_2\left[\left(c_2-g(\mu)\right)\left(g(\mu)-c_1\right)\right]\\ \leq\ &\int_{c_1}^{g(\mu)}\left[\gamma'\left\{c_1+c_2-g(\theta)\right\}-\gamma'\left\{g(\theta)\right\}\right]d\left\{g(\theta)\right\}\\ \leq\ &-Q_1\left[\left(c_2-g(\mu)\right)\left(g(\mu)-c_1\right)\right],\end{aligned}$$

that is

$$\begin{aligned}&-Q_2\left[\left(c_2-g(\mu)\right)\left(g(\mu)-c_1\right)\right]\\ \leq\ &\gamma\{g(\mu)\}+\gamma\{c_1+c_2-g(\mu)\}-\left[\gamma\{c_1\}+\gamma\{c_2\}\right]\\ \leq\ &-Q_1\left[\left(c_2-g(\mu)\right)\left(g(\mu)-c_1\right)\right].\end{aligned}\tag{43}$$

In view of (43) and (41), we can conclude (40), which completes our proof. □

Remark 2 Putting $\delta=1$, $k=1$, $g(\varsigma_1)=\varsigma_1$, $\forall\varsigma_1\in[c_1,\ c_2]$, then both Theorems (3) and (4) reduce to the Lemma 1 and the Lemma 2 respectively, which was proved by Chen [7] for classical R-L fractional integral.

The next result re-introduces the inequality (15) without the function γ being convex on its defined interval employing the view of Theorem 3 and Theorem 4.

Theorem 5 For the continuous and strictly increasing function $g:[c_1,\ c_2]\longrightarrow F\subseteq\mathbb{R}$ on $[c_1,\ c_2]$, satisfies $0\leq c_1<c_2$ and the differentiable positive and symmetric function $\gamma:F\longrightarrow\mathbb{R}$ on F° such that the composition $(\gamma\circ g):[c_1,\ c_2]\longrightarrow\mathbb{R}$ be an integrable mapping on $[c_1,\ c_2]$. If $\gamma'\left\{c_1+c_2-g(\mu)\right\}\geq\gamma'\left\{g(\mu)\right\}$ for all $g(\mu)\in\left[c_1,\ \frac{c_1+c_2}{2}\right]$. Then, for all $k\in\mathbb{R}^+$ the following inequalities hold,

$$\gamma\left(\frac{c_1+c_2}{2}\right) \leq \frac{\delta^{\frac{\tau}{k}}\Gamma(\tau+k)}{2(c_2-c_1)^{\frac{\tau}{k}}}$$

$$\left({}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{\{g^{-1}(c_1)\}^+}(\gamma\circ g)\left(g^{-1}(c_2)\right) + {}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{\{g^{-1}(c_2)\}^-}(\gamma\circ g)\left(g^{-1}(c_1)\right)\right)$$

$$\leq \frac{\gamma(c_1)+\gamma(c_2)}{2},$$

where $\left({}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{c_1^+}\gamma\right)(z)$ and $\left({}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{c_2^-}\gamma\right)(z)$ are the left and right-sided proportional k-fractional integrals, respectively.

Proof. Applying the assumptions of the Theorem 3, (37) and (38), we have

$$\frac{\delta^{\frac{\tau}{k}}\Gamma(\tau+k)}{2(c_2-c_1)^{\frac{\tau}{k}}}\left({}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{\{g^{-1}(c_1)\}^+}(\gamma\circ g)\left(g^{-1}(c_2)\right)\right.$$

$$\left.+\,{}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{\{g^{-1}(c_2)\}^-}(\gamma\circ g)\left(g^{-1}(c_1)\right)\right) - \gamma\left\{\frac{c_1+c_2}{2}\right\}$$

$$= \frac{\tau}{2k(c_2-c_1)^{\frac{\tau}{k}}}\int_{g^{-1}(c_1)}^{g^{-1}\left(\frac{c_1+c_2}{2}\right)}\left(\int_{g(\mu)}^{\frac{c_1+c_2}{2}}\left[\gamma'\{c_1+c_2-g(\theta)\}-\gamma'\{g(\theta)\}\right]d\{g(\theta)\}\right)$$

$$\times\left(\exp\left[\frac{\delta-1}{\delta}(c_2-g(\mu))\right](c_2-g(\mu))^{\frac{\tau}{k}-1}\right. \tag{44}$$

$$\left.+\exp\left[\frac{\delta-1}{\delta}(g(\mu)-c_1)\right](g(\mu)-c_1)^{\frac{\tau}{k}-1}\right)g'(\mu)\,d\mu \geq 0.$$

Similarly, employing the assumption of the Theorem 4, then from (41) and (42), we obtain

$$\frac{\delta^{\frac{\tau}{k}}\Gamma(\tau+k)}{2(c_2-c_1)^{\frac{\tau}{k}}}\left({}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{\{g^{-1}(c_1)\}^+}(\gamma\circ g)\left(g^{-1}(c_2)\right)\right.$$

$$\left.+\,{}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{\{g^{-1}(c_2)\}^-}(\gamma\circ g)\left(g^{-1}(c_1)\right)\right) - \frac{\gamma(c_1)+\gamma(c_1)}{2}$$

$$= \frac{\tau}{2k(c_2-c_1)^{\frac{\tau}{k}}}\int_{g^{-1}(c_1)}^{g^{-1}\left(\frac{c_1+c_2}{2}\right)}\left(-\int_{c_1}^{g(\mu)}\left[\gamma'\{c_1+c_2-g(\theta)\}-\gamma'\{g(\theta)\}\right]d\{g(\theta)\}\right)$$

$$\times \left(\exp\left[\frac{\delta-1}{\delta}(c_2-g(\mu))\right](c_2-g(\mu))^{\frac{\tau}{k}-1} \right. \tag{45}$$

$$\left. +\exp\left[\frac{\delta-1}{\delta}(g(\mu)-c_1)\right](g(\mu)-c_1)^{\frac{\tau}{k}-1} g'(\mu)\, d\mu \right.$$

$$\leq 0.$$

This completes the proof. □

4. Application

This section provides an example to confirm the validity and conditions of Theorem 1.

Example 1 Let $g(z)=z,\ \gamma(z)=z^2$ and $z\in[c_1,\ c_2]=[1,\ 2]$. Clearly, g is strictly increasing and continuous function on [1, 2], and γ is a differentiable convex function. Thus $(\gamma\circ g)(z)=\gamma(g(z))=\gamma(z)=z^2$ is an integrable function over the interval [1, 2] .

Now, we compute the first inequality in (13) for $\delta=0.6\in(0,\ 1]$, $k=2\in R^+$ and $\tau=5>0$, we obtain

$$\gamma\left(\frac{c_1+c_2}{2}\right)=\gamma\left(\frac{1+2}{2}\right)=\gamma(1.5)=(1.5)^2=2.25,$$

and

$$\frac{\delta^{\frac{\tau}{k}}\Gamma_k(\tau+k)}{2(c_2-c_1)^{\frac{\tau}{k}}}=\frac{(0.6)^{\frac{5}{2}}\Gamma_2(5)}{2(2-1)^{\frac{1}{2}}}=\frac{(0.9)^{\frac{5}{2}}(2)^{\frac{5}{2}}\Gamma(\frac{7}{2})}{2}=2.62,$$

where $\Gamma_k(w)=k^{\frac{w}{k}-1}\Gamma\left(\frac{w}{k}\right)$.

Since 2.25 < 2.62. Then the first inequality of Theorem 1 is verified.

Next, we compute the fractional integral with $\delta=\frac{1}{2}$, $\tau=1$ and $k=2$ for the left-hand side of inequality (13), we have

$$\left({}^{(k,\ g)}\mathscr{I}^{\tau,\ \delta}_{\{g^{-1}(c_1)\}^+}(\gamma\circ g)(g^{-1}(c_2))\right)+\left({}^{(k,\ g)}\mathscr{I}^{\tau,\ \delta}_{\{g^{-1}(c_2)\}^-}(\gamma\circ g)(g^{-1}(c_1))\right).$$

In this case: where $g^{-1}(c_1)=g^{-1}(1)=1,\ \ g^{-1}(c_2)=g^{-1}(2)=2,$ because $g(z)=z$ is its own inverse. This is true for any $z\in[1,\ 2]$.

Also, $(\gamma\circ g)(g^{-1}(c_2))=(\gamma\circ g)(2)=\gamma(g(2))=\gamma(2)=4$, and $(\gamma\circ g)(g^{-1}(c_1))=(\gamma\circ g)(1)=\gamma(g(1))=\gamma(1)=1$. Thus

$$\left({}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{\{g^{-1}(c_1)\}^+}(\gamma\circ g)(g^{-1}(c_2))\right)+\left({}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{\{g^{-1}(c_2)\}^-}(\gamma\circ g)(g^{-1}(c_1))\right)$$

$$=\left({}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{\{g^{-1}(c_1)\}^+}(4)\right)+\left({}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{\{g^{-1}(c_2)\}^-}(1)\right).$$

From (7) and (8) we have

$$\left({}^{(k,\,g)}\mathscr{I}^{\tau,\,\delta}_{c_1^+}\gamma\right)(z)=\frac{1}{\delta^{\frac{\tau}{k}}k\Gamma_k(\tau)}\int_{c_1}^{z}\exp\left[\frac{\delta-1}{\delta}(g(z)-g(\mu))\right](g(z)-g(\mu))^{\frac{\tau}{k}-1}g'(\mu)\gamma(\mu)d\mu. \tag{46}$$

Hence, for $\delta=\dfrac{1}{2}$, $k=2$ and $\tau=1$, we obtain

$$\left({}^{(k,\,g)}\mathscr{I}^{1,\,\frac{1}{2}}_{\{g^{-1}(c_1)\}^+}(4)\right)(z)=\frac{1}{\sqrt{\frac{1}{2}}2\Gamma_2(1)}\int_{1}^{z}\exp\left[-(z-\mu)\right](z-\mu)^{-\frac{1}{2}}(4)d\mu.$$

At $z=1$,

$$\left({}^{(k,\,g)}\mathscr{I}^{1,\,\frac{1}{2}}_{\{g^{-1}(c_1)\}^+}(4)\right)(1)=\frac{4}{\sqrt{2}\Gamma_2(1)}\int_{1}^{1}\exp\left[-(1-\mu))\right](1-\mu)^{-\frac{1}{2}}d\mu=0,$$

and

$$\left({}^{(k,\,g)}\mathscr{I}^{1,\,\frac{1}{2}}_{\{g^{-1}(c_2)\}^-}(1)\right)(z)=\frac{1}{\sqrt{\frac{1}{2}}2\Gamma_2(1)}\int_{z}^{2}\exp\left[-(\mu-z))\right](\mu-z)^{-\frac{1}{2}}(1)d\mu.$$

At $z=1$,

$$\left({}^{(k,\,g)}\mathscr{I}^{1,\,\frac{1}{2}}_{\{g^{-1}(c_2)\}^-}(1)\right)(1.5)=\frac{1}{\Gamma(\frac{1}{2})}\int_{1}^{2}\exp\left[-(\mu-1))\right](\mu-1)^{-\frac{1}{2}}d\mu$$

$$=\frac{1}{\sqrt{\pi}}\int_{0}^{1}e^{-u}u^{-\frac{1}{2}}du$$

$$=\frac{1}{\sqrt{\pi}}\sqrt{\pi}\,erf(1)\approx 0.843.$$

The right-hand side of inequality (13),

$$\frac{\gamma(c_1)+\gamma(c_2)}{2}=\frac{\gamma(1)+\gamma(2)}{2}=\frac{1+4}{2}=2.5.$$

So, the second inequality becomes:

$$\left({}^{(k,\ g)}\mathscr{I}^{\tau,\ \delta}_{\{g^{-1}(c_1)\}^+}(\gamma\circ g)(g^{-1}(c_2))\right)+\left({}^{(k,\ g)}\mathscr{I}^{\tau,\ \delta}_{\{g^{-1}(c_2)\}^-}(\gamma\circ g)(g^{-1}(c_1))\right)$$

$$=\left({}^{(k,\ g)}\mathscr{I}^{\tau,\ \delta}_{\{g^{-1}(c_1)\}^+}(4)\right)+\left({}^{(k,\ g)}\mathscr{I}^{\tau,\ \delta}_{\{g^{-1}(c_2)\}^-}(1)\right)$$

$$=0+0.843=0.843$$

$$<2.5=\frac{\gamma(1)+\gamma(2)}{2}=\frac{\gamma(c_1)+\gamma(c_2)}{2}.$$

Therefore, all postulates of Theorem 1 are satisfied.

5. Conclusion

A multitude of scholars have generalised several fractional operators employing classical approaches and operators inside fractional calculus. One of these operators is the generalised $(k,\ g)$-proportional fractional integral. This paper presents innovative approaches for fractional integral Hermite-Hadamard inequalities associated with the generalised proportional k-fractional integral concerning another strictly increasing continuous function g. We have employed the current fractional integral to establish several new fractional integral Hermite-Hadamard-type inequalities. We used convex and symmetric functions to articulate our generalisation of the Hermite-Hadamard inequality. Furthermore, our discussion of this study has clearly clarified numerous particular examples related to the main conclusions.

Future research could focus on exploring generalised fractional operators like Atangana-Baleanu and Caputo-Fabrizio to establish new inequalities and their applications in fractional differential equations. You can also find new things by using numbers to find close solutions to fractional integrals, applying results to functions that are not convex, and looking into multivariable extensions.

Conflict of interest

The authors declare no competing financial interest.

References

[1] Beckenbach EF. Generalized convex functions. *Bulletin of the American Mathematical Society*. 1937; 43(6): 363-371.

[2] Dragomir SS. On the Hadamard's inequality for convex functions on the coordinates in a rectangle from the plane. *Taiwanese Journal of Mathematics*. 2001; 5(4): 775-788.

[3] Dragomir SS. A generalization of Gruss inequality in inner product spaces and applications. *Journal of Mathematical Analysis and Applications*. 1999; 237(1): 74-82.

[4] Farissi AE, Latreuch Z, Belaïdi B. Hadamard-type inequalities for twice differentiable functions. *RGMIA Research Report Collection*. 2009; 12(1): 1-7.
[5] Florea A, Niculescu CP. A hermite-hadamard inequality for convex-concave symmetric functions. *Bulletin Mathématique de la Société des Sciences Mathé-matiques de Roumanie*. 2007; 50(2): 149-156.
[6] Sarikaya M, Set E, Yaldiz H, Basak N. Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities. *Mathematical and Computer Modelling*. 2013; 57(9-10): 2403-2407.
[7] Chen F. Extensions of the Hermite-Hadamard inequality for convex functions via fractional integrals. *Journal of Mathematical Inequalities*. 2016; 10(1): 75-81.
[8] Dragomir SS, Agarwal RP. Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formulas. *Applied Mathematics Letters*. 1998; 11(5): 91-95.
[9] Viloria JM, Cortez MV. Hermite-Hadamard type inequalities for harmonically convex functions on *n*-coordinates. *Applied Mathematics & Information Sciences Letters*. 2018; 6(2): 1-6.
[10] Neamah MK, Ibrahim A. Generalized proportional fractional integral inequalities for convex functions. *AIMS Mathematics*. 2021; 6(10): 10765-10777.
[11] Neamah MK, Ibrahim A, Mehdy HS, Redhwan SS, Abdo MS. Some new fractional inequalities involving convex functions and generalized fractional integral operator. *Journal of Function Spaces*. 2022; 2022: 2350193.
[12] Neamah MK, Ibrahim A, Aljaaidi TA, Abdo MS. Certain weighted fractional integral inequalities involving convex functions. *Contemporary Mathematics*. 2024; 5(4): 5863-5880.
[13] Nowicka M, Witkowski A. Applications of the Hermite-Hadamard inequality. *arXiv:160307170*. 2016. Available from: https://doi.org/10.48550/arXiv.1603.07170.
[14] Dahmani Z. On Minkowski and Hermite-Hadamard integral inequalities via fractional integration. *Annals of Functional Analysis*. 2010; 1(1): 51-58.
[15] Set E, Sarikaya MZ, Özdemir ME, Yildirim H. The Hermite-Hadamard's inequality for some convex functions via fractional integrals and related results. *Journal of Applied Mathematics, Statistics, and Informatics*. 2014; 10(2): 69-83.
[16] Dragomir SS. A generalization of Grüss inequality in inner product spaces and applications. *Journal of Mathematical Analysis and Applications*. 1999; 237(1): 74-82.
[17] Noor MA, Noor KI, Awan MU. Some quantum estimates for Hermite-Hadamard inequalities. *Applied Mathematics and Computation*. 2015; 251(1): 675-679.
[18] Agarwal P, Jleli M, Tomar M. Certain Hermite-Hadamard type inequalities via generalized *k*-fractional integrals. *Journal of Inequalities and Applications*. 2017; 2017(1): 55.
[19] Khan MA, Chu YM, Kashuri A, Liko R, Ali G. Conformable fractional integrals are versions of Hermite-Hadamard inequalities and their generalizations. *Journal of Function Spaces*. 2018; 2018(1): 6928130.
[20] Mohammed PO, Brevik I. A new version of the Hermite-Hadamard inequality for Riemann-Liouville fractional integrals. *Symmetry*. 2020; 12(4): 610.
[21] Awan MU, Akhtar N, Iftikhar S, Noor MA, Chu YM. New Hermite-Hadamard type inequalities for n-polynomial harmonically convex functions. *Journal of Inequalities and Applications*. 2020; 2020(1): 1-12.
[22] Chudziak M, Żołdak M. Hermite-Hadamard and Fejér inequalities for coordinated (f, g)-convex functions on a rectangle. *Symmetry*. 2020; 12(1): 13.
[23] Khan S, Khan MA, Chu YM. Converses of the Jensen inequality derived from the Green functions with applications in information theory. *Mathematical Methods in the Applied Sciences*. 2020; 43(5): 2577-2587.
[24] Delavar MR, Dragomir SS, Sen DL. A note on characterization of *h*-convex functions via Hermite-Hadamard type inequality. *Problemy Analiza*. 2019; 26(2): 28-36.
[25] Mohammed PO. Hermite-Hadamard inequalities for Riemann-Liouvillee fractional integrals of a convex function with respect to a monotone function. *Mathematical Methods in the Applied Sciences*. 2021; 44(3): 2314-2324.
[26] Abdeljawad T, Rashid S, Khan H, Chu YM. On new fractional integral inequalities for *p*-convexity within interval-valued functions. *Advances in Difference Equations*. 2020; 2020(1): 1-17.
[27] Yan P, Li Q, Chu YM, Mukhtar S, Waheed S. On some fractional integral inequalities for generalized strongly modified *h*-convex function. *AIMS Mathematics*. 2020; 5(6): 6620-6638.
[28] Zhou SS, Rashid S, Jarad F, Kalsoom H, Chu YM. New estimates considering the generalized proportional Hadamard fractional integral operators. *Advances in Difference Equations*. 2020; 2020(1): 1-15.

[29] Kilbas A, Srivastava HM, Trujillo JJ. *Theory and Applications of Fractional Differential Equations*. Amsterdam: Elsevier; 2006.
[30] Wrede R, Spiegel MR. *Theory and Problems of Advanced Calculus*. 2nd ed. New York: McGraw-Hill; 2002.
[31] Podlubny I. *Fractional Differential Equations*. San Diego: Academic Press; 1999.
[32] Diaz R, Pariguan E. On hypergeometric functions and Pochhammer k-symbol. *arXiv:math/0405596*. 2005. Available from: https://doi.org/10.48550/arXiv.math/0405596.
[33] Aljaaidi TA, Pachpatte DB, Abdo MS, Botmart T, Ahmad H, Almalahi MA, et al. (k, ψ)-proportional fractional integral Pólya-Szegö and Grüss-type inequalities. *Fractal Fract*. 2021; 5(4): 172.
[34] Folland GB. *Real Analysis: Modern Techniques and Their Applications*. 2nd ed. New York: John Wiley & Sons; 1999.
[35] Liu K, Wang J, O'Regan D. On the Hermite-Hadamard type inequality for ψ-Riemann-Liouville fractional integrals via convex functions. *Journal of Inequalities and Applications*. 2019; 2019(1): 1-10.
[36] Chinchane VL, Pachpatte DB. On new fractional integral inequalities involving convex functions using Hadamard fractional integral. *Bulletin of the Allahabad Mathematical Society*. 2016; 31(2): 183-192.
[37] Jarad F, Alqudah MA, Abdeljawad T. On more general forms of proportional fractional operators. *Open Mathematics*. 2020; 18(1): 167-176.