Contemporary Mathematics
https://ojs.wiserpub.com/index.php/CM/

Research Article

A Novel Scheme for Numerical Analysis of Fractional Order Coupled System of Partial Differential Equations

Wasim Sajjad Hussain, Sajjad Ali*

Department of Mathematics, Shaheed Benazir Bhutto University, Sheringal, Dir Upper, KPK, Pakistan
E-mail: charsadamath@yahoo.com

Received: 17 January 2025; **Revised:** 6 March 2025; **Accepted:** 7 March 2025

Abstract: This work focused on developing and analyzing semi analytical scheme that is a computer-based approach for solving complex evolutionary partial differential equations (PDEs) of the Burger type. A novel scheme named Asymptotic Homotopy Perturbation Transform Method (AHPTM) is introduced for solving fractional order coupled nonlinear Burgers PDEs. The Caputo fractional form has been considered for derivatives. Nonlinear Burgers PDEs have various applications in fluid dynamics, traffic flow, nonlinear acoustics and signal processing. The algorithm of AHPTM is a fast convergent approach that has been developed by combining the Laplace transformation and the asymptotic homotopy perturbation method. Using the technique of AHPTM, three test problems of one-dimensional coupled nonlinear Burgers PDEs have been solved. This work demonstrates the smooth and easy implementation of solving problems by using MATLAB programming. The numerical results demonstrate that this novel approach is simple, easy and computationally capable for the problems. An error estimate is also required to solve the problem. To demonstrate the effectiveness and accuracy of the said solver, the solutions to the problems are tabulated and graphically displayed via using MATLAB software.

***Keywords*:** asymptotic homotopy method, numerical analysis, caputo derivative, burgers equation

MSC: 35A25, 35C10, 41A99, 92B05

1. Introduction

Computation mathematics with respect to computational fluid has many applications. The governing partial differential equations PDEs for fluid flow has much importance. But, it addresses computational challenges from complex mathematical structures and multiscale behavior. The showcase advancements in mathematical models and numerical algorithms for solving these complex PDEs is the need of modern science.

Recently, the governing PDEs of Burger models has various applications in fluid dynamics (turbulence, shock waves, and boundary layers), traffic flow (vehicle density and velocity in traffic systems), nonlinear acoustics (sound wave propagation in media with nonlinear and dissipative effects) and signal processing (used in denoising and smoothing signals/images due to its diffusion properties). In this context, various studies have been conducted. Esipov [1] developed the one-dimensional coupled viscous Burgers equation to investigate the polydispersive sedimentation model. Sedimentation, or the change in scaled volume concentrations of two kinds of particles under gravity, is simply described

DOI: https://doi.org/10.37256/cm.6420256447

by the set of coupled equations in fluid suspensions or colloids. This system of equations, according to Burgers [2] and Cole [3], describes a variety of phenomena, containing the approximate theory of flow through a shock wave flowing in a viscous fluid, as well as a mathematical model of turbulence. The coupled Burgers equation is interesting from a numerical perspective because there are typically no analytical solutions available. The coupled Burgers equations of one dimension were solved exactly by Kaya [4] using the Adomian decomposition technique and Soliman [5] using a modified extended tanh function method. Many researchers and scholars have worked out the numerical solution to one-dimensional coupled Burgers equations. Esipov [1] presented numerical calculations and compared them with experimental data. The one-dimensional Burgers equations and coupled Burgers equations were solved by Abdou and Soliman [6] using the variational iteration approach. Wei and Gu [7] employed the conjugate filter methodology, while Khater et al. [8], applied the Chebyshev spectral collocation technique. Dehghan et al. [9] used the Adomian-pade methodology to get the numerical results of coupled viscous Burgers equations. Rashid and Ismail [10] used the Fourier pseudo-spectral approach. To solve the coupled viscous Burgers equation, by linearizing the non-linear variables, Mittal and Arrora [11] employed a cubic B-Spline collocation approach based on the Crank-Nicolsion version for time integration and Cubic B-Spline functions for space integration, whereas a generalized differential quadrature method was employed by Mokhtari et al. [12]. Certain finite-difference approaches are studied for solving single one-dimensional Burgers equations as well as two- and three-dimensional Burgers equations, see references [13–22]. The work focused on analyzing numerical methods, algorithms, and computer-based approaches for solving complex PDEs of the Burgers type, see references [9, 23–25]. This study bridges mathematics, computer science, and applied fields to tackle problems that may be difficult or impossible to solve analytically. In the field of computational mathematics, It is known fact that the development of advanced methods for the solution of fractional order equations and their coupled systems is essential [26–28]. In this research, we study a novel approach of the homotopy methods to the analytical solution to the one-dimensional coupled non-linear Burgers equations. For the basic literature of this specific research work, we refer to the recent work [29–40]. The purpose of this research study is to develop a new scheme of AHPTM, which is a very simple, rapidly convergent technique. This approach combines the Laplace transform with the Asymptotic Homotopy Perturbation Method (AHPM) [41]. Three significant problems of one-dimensional coupled non-linear Burgers equations are used to illustrate this approach. This work demonstrates the rapid convergence to problem solutions.

This work is organized as follows: Section 2 provides the basic idea of the suggested strategy. Section 3 illustrates how the proposed method can be applied to fractional problems. The final section of the work includes a conclusion.

2. Basic structure of the proposed method

This study presents a novel asymptotic homotopy perturbation transform method for one-dimensional Fractional Coupled Nonlinear Burgers Equations.

Consider

$$\frac{\partial^{\eta} w}{\partial \tau^{\eta}} + \eta w \frac{\partial w}{\partial \hbar} + \alpha \left[w \frac{\partial y}{\partial \hbar} + y \frac{\partial w}{\partial \hbar} \right] + \sigma \frac{\partial^2 w}{\partial \hbar^2} = 0,\ 0 < \eta \leq 1, \tag{1}$$

$$\frac{\partial^{\eta} y}{\partial \tau^{\eta}} + \sigma y \frac{\partial y}{\partial \hbar} + \beta \left[w \frac{\partial y}{\partial \hbar} + y \frac{\partial w}{\partial \hbar} \right] + \mu \frac{\partial^2 y}{\partial \hbar^2} = 0,\ 0 < \eta \leq 1. \tag{2}$$

Initial conditions:

$$w(\hbar,\ 0) = a_1(\hbar),$$

$$y(\hbar,\ 0) = a_2(\hbar).$$

Consider

$$j = \frac{\partial^\eta w}{\partial \tau^\eta}$$

and

$$\mathbb{k} = \eta w \frac{\partial w}{\partial \hbar} + \alpha \left[w \frac{\partial y}{\partial \hbar} + y \frac{\partial w}{\partial \hbar} \right] + \sigma \frac{\partial^2 w}{\partial \hbar^2},$$

$$\mathbf{M} = \frac{\partial^\eta y}{\partial \tau^\eta}$$

and

$$\mathbf{N} = \sigma y \frac{\partial y}{\partial \hbar} + \beta \left[w \frac{\partial y}{\partial \hbar} + y \frac{\partial w}{\partial \hbar} \right] + \mu \frac{\partial^2 y}{\partial \hbar^2} = 0.$$

Where $\mathbb{k}$ and $\mathbf{N}$ denote nonlinear terms in original problem given in Equations (1) and (2). Substituting these into equation (1) and (2), we obtain

$$jw(\hbar,\ \tau) + \mathbb{k}w(\hbar,\ \tau) = 0, \tag{3}$$

$$\mathbf{M}y(\hbar,\ \tau) + \mathbf{N}y(\hbar,\ \tau) = 0. \tag{4}$$

By using the homotopy rule on the equations (3) and (4), as in AHPTM, we get

$$jw(\hbar,\ \tau) - p\mathbb{k}w(\hbar,\ \tau) = 0, \tag{5}$$

$$\mathbf{M}y(\hbar,\ \tau) - q\mathbf{N}y(\hbar,\ \tau) = 0, \tag{6}$$

$$\frac{\partial^\eta}{\partial \tau^\eta} w(\hbar,\ \tau) - p\mathbb{k}w(\hbar,\ \tau) = 0, \tag{7}$$

$$\frac{\partial^\eta}{\partial \tau^\eta} y(\hbar,\ \tau) - q\mathbf{N}y(\hbar,\ \tau) = 0. \tag{8}$$

Applying Laplace transform to equation (7) and (8), we get

$$L\left[\frac{\partial^\eta}{\partial \tau^\eta} w(\hbar,\ \tau)\right] = L\left[p\mathbb{k}w(\hbar,\ \tau)\right], \tag{9}$$

$$L\left[\frac{\partial^\eta}{\partial \tau^\eta} y(\hbar,\ \tau)\right] = L\left[q\mathbf{N}y(\hbar,\ \tau)\right]. \tag{10}$$

The Laplace transform of the Caputo fractional derivative is given by:

$$L\left[D_\tau^{m\eta} w(\hbar,\ \tau)\right] = S^{m\eta} w(\hbar,\ \tau) - \sum_{k=0}^{m-1} S^{m\eta-k-1} w^k(\hbar,\ 0),\ m-1 < m\eta \leq m. \tag{11}$$

$$L\left[D_\tau^{m\eta} y(\hbar,\ \tau)\right] = S^{m\eta} y(\hbar,\ \tau) - \sum_{k=0}^{m-1} S^{m\eta-k-1} y^k(\hbar,\ 0),\ m-1 < m\eta \leq m. \tag{12}$$

By substituting the expression of (11) in equation (9), we have

$$s^\eta w(s,\ \tau) - s^{\eta-1} w(\hbar,\ 0) - s^{\eta-2} w_\tau(\hbar,\ 0) \ldots = pL\left[\mathbb{k}w(\hbar,\ \tau)\right].$$

By substituting the expression of (12) in equation (10), we have

$$s^\eta y(s,\ \tau) - s^{\eta-1} y(\hbar,\ 0) - s^{\eta-2} y_\tau(\hbar,\ 0) \ldots = qL\left[\mathbf{N}y(\hbar,\ \tau)\right].$$

Using initial conditions and solving

$$s^\eta w(s,\ \tau) - \frac{1}{s} a_1(\hbar) = \frac{1}{s^\eta} pL\left[\mathbb{k}w(\hbar,\ \tau)\right], \tag{13}$$

$$s^\eta y(s,\ \tau) - \frac{1}{s} a_2(\hbar) = \frac{1}{s^\eta} qL\left[\mathbf{N}y(\hbar,\ \tau)\right]. \tag{14}$$

By taking the inverse of the Laplace transform operator, we have

$$L^{-1}\left[w(s,\ \tau)\right] = L^{-1}\left[\frac{a_1(\hbar)}{s}\right] + L^{-1}\left[\frac{pL\left[\mathbb{k}w(\hbar,\ \tau)\right]}{s^\eta}\right], \tag{15}$$

$$L^{-1}[y(s,\ \tau)] = L^{-1}\left[\frac{a_2(\hbar)}{s}\right] + L^{-1}\left[\frac{qL[\mathbf{N}y(\hbar,\ \tau)]}{s^{\eta}}\right]. \tag{16}$$

Let

$$L^{-1}[w(s,\ \tau)] = w(\hbar,\ \tau), \tag{17}$$

$$L^{-1}\left[\frac{1}{s}a_1(\hbar)\right] = O(\hbar,\ \tau), \tag{18}$$

$$L^{-1}[y(s,\ \tau)] = y(\hbar,\ \tau), \tag{19}$$

$$L^{-1}\left[\frac{1}{s}a_2(\hbar)\right] = O^{'}(\hbar,\ \tau). \tag{20}$$

By substituting the expression of (17) and (18) in equation (15), we get

$$w(\hbar,\ \tau) = O(\hbar,\ \tau) + pL^{-1}\left(\frac{1}{s^{\eta}}L[\Bbbk w(\hbar,\ \tau)]\right). \tag{21}$$

By substituting the expression of (19) and (20) in equation (16), we have

$$y(\hbar,\ \tau) = O^{'}(\hbar,\ \tau) + qL^{-1}\left(\frac{L[\mathbf{N}y(\hbar,\ \tau)]}{s^{\eta}}\right). \tag{22}$$

$$w(\hbar,\ \tau) = \sum_{m=0}^{\infty} p^m w_m(\hbar,\ \tau), \tag{23}$$

$$y(\hbar,\ \tau) = \sum_{m=0}^{\infty} q^m y_m(\hbar,\ \tau). \tag{24}$$

$$N(w(\hbar,\ \tau)) = B_1\Bbbk_0 + \sum_{m=1}^{\infty}\left(\sum_{i=0}^{m} B_{m+1-i}\Bbbk_i\right)p^m, \tag{25}$$

$$\mathbf{N}(y(\hbar,\ \tau)) = B_1\mathbf{N}_0 + \sum_{m=1}^{\infty}\left(\sum_{i=0}^{m} B_{m+1-i}\mathbf{N}_i\right)q^m. \tag{26}$$

By substituting equation (23), and (25) in (21), we have

$$\sum_{m=0}^{\infty} p^m w_m(\hbar,\ \tau) = O(\hbar,\ \tau) + pL^{-1}\left[\frac{1}{s^\eta} L\left(B_1 \mathbb{k}_0 + \sum_{m=1}^{\infty}\left(\sum_{i=0}^{m} B_{m+1-i}\mathbb{k}_i\right)p^m\right)\right].$$

By substituting equation (24), and (26) in (22), we have

$$\sum_{m=0}^{\infty} q^m y_m(\hbar,\ \tau)$$

$$= O^{'}(\hbar,\ \tau) + qL^{-1}\left[\frac{1}{s^\eta} L\left(B_1 \mathbf{N}_0 + \sum_{m=1}^{\infty}\left(\sum_{i=0}^{m} B_{m+1-i}\mathbf{N}_i\right)q^m\right)\right].$$

$$w_0(\hbar,\ \tau)p^0 + w_1(\hbar,\ \tau)p^1 + w_2(\hbar,\ \tau)p^2 + \ldots$$

$$= O(\hbar,\ \tau) + pL^{-1}\left[\frac{1}{S^\eta} L\left((B_1\mathbb{k}_0) + (B_2\mathbb{k}_0 + B_1\mathbb{k}_1)\,p^1 + (B_3\mathbb{k}_0 + B_2\mathbb{k}_1 + B_1\mathbb{k}_2)\,p^2\right)\right], \tag{27}$$

$$y_0(\hbar,\ \tau)q^0 + y_1(\hbar,\ \tau)q^1 + y_2(\hbar,\ \tau)q^2 + \ldots$$

$$= O^{'}(\hbar,\ \tau) + qL^{-1}\left[\frac{1}{S^\eta} L\left((B_1\mathbf{N}_0) + (B_2\mathbf{N}_0 + B_1\mathbf{N}_1)\,q^1 + (B_3\mathbf{N}_0 + B_2\mathbf{N}_1 + B_1\mathbf{N}_2)\,q^2\right)\right]. \tag{28}$$

In equation (27), we compare the coefficient of like power of p to obtain

$$p^0 : w_0(\hbar,\ \tau) = O(\hbar,\ \tau), \tag{29}$$

$$p^1 : w_1(\hbar,\ \tau) = L^{-1}\frac{1}{S^\eta} L\left[B_1\mathbb{k}_0\right], \tag{30}$$

$$p^2 : w_2(\hbar,\ \tau) = L^{-1}\frac{1}{S^\eta}\left[B_2\mathbb{k}_0 + B_1\mathbb{k}_1\right], \tag{31}$$

and the k^{th} order iteration is

$$p^k : w_k(\hbar,\ \tau) = L^{-1}\left[\frac{1}{S^\eta} L\left(\sum_{i=0}^{k-1} B_{k-1}\mathbb{k}_i\right)\right]. \tag{32}$$

In equation (28), we compare the coefficient of like power of p to obtain

$$q^0: y_0(\hbar,\ \tau) = O^{'}(\hbar,\ \tau), \tag{33}$$

$$q^1: y_1(\hbar,\ \tau) = L^{-1}\frac{1}{S^\eta}L\left[B_1\mathbf{N}_0\right], \tag{34}$$

$$q^2: y_2(\hbar,\ \tau) = L^{-1}\frac{1}{S^\eta}\left[B_2\mathbf{N}_0 + B_1\mathbf{N}_1\right], \tag{35}$$

and the k^{th} order iteration is

$$q^k: y_k(\hbar,\ \tau) = L^{-1}\left[\frac{L\left(\sum_{i=0}^{k-1}B_{k-1}\mathbf{N}_i\right)}{S^\eta}\right]. \tag{36}$$

Using the Laplace and inverse Laplace transform properties, the aforementioned equations are solved, and the approximate solution is provided as

$$\widetilde{w}(\hbar,\ \tau) = w_0(\hbar,\ \tau) + w_1(\hbar,\ \tau) + w_2(\hbar,\ \tau) + w_3(\hbar,\ \tau)\ldots, \tag{37}$$

$$\widetilde{y}(\hbar,\ \tau) = y_0(\hbar,\ \tau) + y_1(\hbar,\ \tau) + y_2(\hbar,\ \tau) + y_3(\hbar,\ \tau)\ldots \tag{38}$$

We will tabulate the results of the relevant problems after determining the analytical solution to the newly suggested technique. We will also apply the developed technique by using MATLAB code to show the results obtained by AHPTM and other methods via different surface and line graphs. To compare the approximate solution with exact solution through new suggested scheme by plotting errors.

3. Proposed method's application

3.1 *Problem 01*

The first specific problem of fractional coupled non-linear Burgers equations is considered in the form of

$$\frac{\partial^\eta w}{\partial \tau^\eta} = \frac{\partial^2 w}{\partial \hbar^2} + 2w\frac{\partial w}{\partial \hbar} - \frac{\partial}{\partial \hbar}(wy), \tag{39}$$

$$\frac{\partial^\eta y}{\partial \tau^\eta} = \frac{\partial^2 y}{\partial \hbar^2} + 2y\frac{\partial y}{\partial \hbar} - \frac{\partial}{\partial \hbar}(wy),\ 0 < \eta \leq 1. \tag{40}$$

Under the given initial conditions

$$w(\hbar,\ 0) = sin(\hbar),\ y(\hbar,\ 0) = sin(\hbar).$$

Consider

$$j=\frac{\partial^{\eta} w}{\partial \tau^{\eta}},\ \mathbb{k}=-\frac{\partial^2 w}{\partial \hbar^2}-2w\frac{\partial w}{\partial \hbar}+\frac{\partial}{\partial \hbar}(wy),$$

and

$$\mathbf{M}=\frac{\partial^{\eta} y}{\partial \tau^{\eta}},\ \mathbf{N}=-\frac{\partial^2 y}{\partial \hbar^2}-2y\frac{\partial y}{\partial \hbar}+\frac{\partial}{\partial \hbar}(wy).$$

As deformation equation is

$$\mathrm{L}(j)-\mathrm{Lp}\mathbb{k}=0,$$

and

$$L\mathbf{M}-Lp\mathbf{N}=0,$$

where L is Laplace operator.

$$L(j)=Lp\mathbb{k},$$

$$L\mathbf{M}=Lp\mathbf{N},$$

$$\mathrm{L}\left(\frac{\partial^{\eta} w}{\partial \tau^{\eta}}\right)=L(P\mathbb{k}),$$

$$\mathrm{L}\left(\frac{\partial^{\eta} y}{\partial \tau^{\eta}}\right)=L(P\mathbf{N}).$$

$$L\left(\frac{\partial^{\eta} w}{\partial \tau^{\eta}}\right)=\mathrm{Lp}\left[\mathrm{B}_1\mathbb{k}_0+(\mathrm{B}_2\mathbb{k}_0+\mathrm{B}_1\mathbb{k}_1)\,\mathrm{p}+(\mathrm{B}_3\mathbb{k}_0+\mathrm{B}_2\mathbb{k}_1+\mathrm{B}_1\mathbb{k}_2)\,\mathrm{p}^2+\ldots\right],$$

$$L\left(\frac{\partial^{\eta} y}{\partial \tau^{\eta}}\right)=\mathrm{Lq}\left[\mathrm{B}_1\mathbf{N}_0+(\mathrm{B}_2\mathbf{N}_0+\mathrm{B}_1\mathbf{N}_1)\,\mathrm{q}+(\mathrm{B}_3\mathbf{N}_0+\mathrm{B}_2\mathbf{N}_1+\mathrm{B}_1\mathbf{N}_2)\,\mathrm{q}^2+\ldots\right],$$

$$s^{\eta} w(s,\ \tau) - s^{\eta-1} w(\hbar,\ 0) - s^{\eta-2} w_{\tau}(\hbar,\ 0)...$$

$$= \mathrm{L}\left[(\mathrm{B}_1 \Bbbk_0)\,\mathrm{p} + (\mathrm{B}_2 \Bbbk_0 + \mathrm{B}_1 \Bbbk_1)\,\mathrm{p}^2 + (\mathrm{B}_3 \Bbbk_0 + \mathrm{B}_2 \Bbbk_1 + \mathrm{B}_1 \Bbbk_2)\,\mathrm{p}^3 + ...,\ \right]$$

$$s^{\eta} y(s,\ \tau) - s^{\eta-1} y(\hbar,\ 0) - s^{\eta-2} y_{\tau}(\hbar,\ 0)...$$

$$= \mathrm{L}\left[(\mathrm{B}_1 \mathbf{N_0})\,\mathrm{q} + (\mathrm{B}_2 \mathbf{N_0} + \mathrm{B}_1 \mathbf{N_1})\,\mathrm{q}^2 + (\mathrm{B}_3 \mathbf{N_0} + \mathrm{B}_2 \mathbf{N_1} + \mathrm{B}_1 \mathbf{N}_2)\,\mathrm{q}^3 + ...\right],\ w(\hbar,\ \tau)$$

$$= sin(\hbar) + L^{-1} \frac{1}{S^{\eta}} L[B_1 \Bbbk_0] p + L^{-1} \frac{1}{S^{\eta}} L[B_2 \Bbbk_0 + B_1 \Bbbk_1] p^2$$

$$+ L^{-1} \frac{1}{S^{\eta}} L[B_3 \Bbbk_0 + B_2 \Bbbk_1 + B_1 \Bbbk_2] p^3 + ...,\ y(\hbar,\ \tau)$$

$$= sin(\hbar) + L^{-1} \frac{1}{S^{\eta}} L[B_1 \mathbf{N}_0] q + L^{-1} \frac{1}{S^{\eta}} L[B_2 \mathbf{N}_0 + B_1 \mathbf{N}_1] q^2$$

$$+ L^{-1} \frac{1}{S^{\eta}} L[B_3 \mathbf{N}_0 + B_2 \mathbf{N}_1 + B_1 \mathbf{N}_2] q^3 + ...,$$

$$w_0(\hbar,\ \tau) + w_1(\hbar,\ \tau) p + w_2(\hbar,\ \tau) p^2 + w_3(\hbar,\ \tau) p^3 + ...$$

$$= sin(\hbar) + L^{-1} \frac{1}{S^{\eta}} L[B_1 \Bbbk_0] p$$

$$+ L^{-1} \frac{1}{S^{\eta}} L[B_2 \Bbbk_0 + B_1 \Bbbk_1] p^2 + L^{-1} \frac{1}{S^{\eta}} L[B_3 \Bbbk_0 + B_2 \Bbbk_1 + B_1 \Bbbk_2] p^3 + ..., \tag{41}$$

$$y_0(\hbar,\ \tau) + y_1(\hbar,\ \tau) q + y_2(\hbar,\ \tau) q^2 + y_3(\hbar,\ \tau) q^3 + ...$$

$$= sin(\hbar) + L^{-1} \frac{1}{S^{\eta}} L[B_1 \mathbf{N}_0] q$$

$$+ L^{-1} \frac{1}{S^{\eta}} L[B_2 \mathbf{N}_0 + B_1 \mathbf{N}_1] q^2 + L^{-1} \frac{1}{S^{\eta}} L[B_3 \mathbf{N}_0 + B_2 \mathbf{N}_1 + B_1 \mathbf{N}_2] q^3 + ..., \tag{42}$$

Equating the coefficients of like powers of p in equation (39), we get

$$p^0 : w_0(\hbar,\ \tau) = sin(\hbar), \tag{43}$$

$$p^1 : w_1(\hbar,\ \tau) = L^{-1}\frac{1}{S^\eta}L\left[B_1\mathbb{k}_0\right], \tag{44}$$

$$p^2 : w_2(\hbar,\ \tau) = L^{-1}\frac{1}{S^\eta}\left[B_2\mathbb{k}_0 + B_1\mathbb{k}_1\right], \tag{45}$$

$$p^3 : w_3(\hbar,\ \tau) = L^{-1}\frac{1}{S^\eta}L\left[B_3\mathbb{k}_0 + B_2\mathbb{k}_1 + B_1\mathbb{k}_2\right]. \tag{46}$$

Equating the coefficients of like powers of q in equation (40), we get

$$q^0 : y_0(\hbar,\ \tau) = sin(\hbar), \tag{47}$$

$$q^1 : y_1(\hbar,\ \tau) = L^{-1}\frac{1}{S^\eta}L\left[B_1\mathbf{N_0}\right], \tag{48}$$

$$q^2 : y_2(\hbar,\ \tau) = L^{-1}\frac{1}{S^\eta}\left[B_2\mathbf{N_0} + B_1\mathbf{N_1}\right], \tag{49}$$

$$q^3 : y_3(\hbar,\ \tau) = L^{-1}\frac{1}{S^\eta}L\left[B_3\mathbf{N_0} + B_2\mathbf{N_1} + B_1\mathbf{N_2}\right]. \tag{50}$$

Consider equation (43) and equation (47), we have

$$w_0(\hbar,\ \tau) = sin(\hbar),\ y_0(\hbar,\ \tau) = sin(\hbar).$$

$$\mathbb{k}_0 = -\frac{\partial^2 w_0}{\partial \hbar^2} - 2w_0\frac{\partial w_0}{\partial \hbar} + \frac{\partial}{\partial \hbar}(w_0 y_0).$$

Hence $\mathbb{k}_0 = sin(\hbar)$.
Consider equation (44)

$$w_1(\hbar,\ \tau) = L^{-1}\frac{1}{S^\eta}L\left[B_1\mathbb{k}_0\right],$$

$$w_1(\hbar,\ \tau) = L^{-1}\frac{1}{S^\eta}L\left[B_1 sin(\hbar)\right],$$

$$w_1(\hbar,\ \tau) = B_1 sin(\hbar)L^{-1}\left(\frac{1}{S^{\eta+1}}\right),$$

On taking inverse L^{-1} of transformation as:

$$w_1(\hbar,\ \tau) = B_1 sin(\hbar)\left(\frac{\tau^\eta}{\Gamma(\eta+1)}\right). \tag{51}$$

$$\mathbf{N_0} = -\frac{\partial^2 y_0}{\partial \hbar^2} - 2y_0\frac{\partial y_0}{\partial \hbar} + \frac{\partial}{\partial \hbar}(w_0 y_0).$$

Hence $\mathbf{N_0} = sin(\hbar)$.
Consider equation (48)

$$y_1(\hbar,\ \tau) = L^{-1}\frac{1}{S^\eta}L\left[B_1\mathbf{N}_0\right],$$

$$y_1(\hbar,\ \tau) = L^{-1}\frac{1}{S^\eta}L\left[B_1 sin(\hbar)\right],$$

$$y_1(\hbar,\ \tau) = B_1 sin(\hbar)L^{-1}\left(\frac{1}{S^{\eta+1}}\right),$$

On taking inverse L^{-1} of transformation as:

$$y_1(\hbar,\ \tau) = B_1 sin(\hbar)\left(\frac{\tau^\eta}{\Gamma(\eta+1)}\right). \tag{52}$$

$$\mathbb{k}_1 = -\frac{\partial^2 w_1}{\partial \hbar^2} - 2w_1\frac{\partial w_1}{\partial \hbar} + \frac{\partial}{\partial \hbar}(w_1 y_1).$$

Hence

$$\mathbb{k}_1 = B_1 sin(\hbar)\left(\frac{\tau^\eta}{\Gamma(\eta+1)}\right).$$

Consider equation (45), we have

$$w_2(\hbar,\ \tau) = L^{-1}\frac{1}{S^\eta}L\left[B_2\mathbb{k}_0 + B_1\mathbb{k}_1\right],$$

$$w_2(\hbar,\ \tau) = L^{-1}\frac{1}{S^\eta}L\left[B_2 sin(\hbar) + B_1\left(B_1 sin(\hbar)\left(\frac{\tau^\eta}{\Gamma(\eta+1)}\right)\right)\right],$$

$$w_2(\hbar,\ \tau) = B_2 sin(\hbar)L^{-1}\left(\frac{1}{S^{\eta+1}}\right) + B_1^2 sin(\hbar)L^{-1}\left(\frac{1}{S^{2\eta+1}}\right),$$

On taking inverse transformation L^{-1}, we get

$$w_2(\hbar,\ \tau) = B_2 sin(\hbar)\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right) + B_1^2 sin(\hbar)\left(\frac{\tau^{2\eta}}{\Gamma(2\eta+1)}\right). \tag{53}$$

$$\mathbf{N}_1 = -\frac{\partial^2 y_1}{\partial \hbar^2} - 2y_1\frac{\partial y_1}{\partial \hbar} + \frac{\partial}{\partial \hbar}(w_1 y_1).$$

Hence

$$\mathbf{N}_1 = B_1 sin(\hbar)\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right).$$

Consider equation (49), we have

$$y_2(\hbar,\ \tau) = L^{-1}\frac{1}{S^{\eta}}L\left[B_2\mathbf{N}_0 + B_1\mathbf{N}_1\right],$$

$$y_2(\hbar,\ \tau) = B_2(sin(\hbar))L^{-1}\left(\frac{1}{S^{\eta+1}}\right) + B_1^2 sin(\hbar)L^{-1}\left(\frac{1}{S^{2\eta+1}}\right).$$

On taking inverse transformation L^{-1}, we get

$$y_2(\hbar,\ \tau) = B_2 sin(\hbar)\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right) + B_1^2 sin(\hbar)\left(\frac{\tau^{2\eta}}{\Gamma(2\eta+1)}\right). \tag{54}$$

$$\mathbb{k}_2 = -\frac{\partial^2 w_2}{\partial \hbar^2} - 2w_2\frac{\partial w_2}{\partial \hbar} + \frac{\partial}{\partial \hbar}(w_2 y_2).$$

Hence

$$\mathbb{k}_2 = B_2 sin(\hbar)\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right) + B_1^2 sin(\hbar)\left(\frac{\tau^{2\eta}}{\Gamma(2\eta+1)}\right).$$

Consider equation (46), we have

$$w_3(\hbar,\ \tau) = L^{-1}\frac{1}{S^{\eta}}L\left[B_3\mathbb{k}_0 + B_2\mathbb{k}_1 + B_1\mathbb{k}_2\right],$$

$$w_3(\hbar,\ \tau) = L^{-1}\frac{1}{S^{\eta}}L\left[B_3 sin(\hbar) + B_2\left(B_1 sin(\hbar)\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right)\right)\right]$$

$$+L^{-1}\frac{1}{S^{\eta}}L\left[B_1\left(B_2 sin(\hbar)(\frac{\tau^{\eta}}{\Gamma(\eta+1)})+B_1^2 sin(\hbar)\left(\frac{\tau^{2\eta}}{\Gamma(2\eta+1)}\right)\right)\right],$$

$$w_3(\hbar,\ \tau)=B_3 sin(\hbar)L^{-1}\left(\frac{1}{s^{\eta+1}}\right)+2B_1B_2 sin(\hbar)L^{-1}\left(\frac{1}{s^{2\eta+1}}\right)+B_1^3 sin(\hbar)L^{-1}\left(\frac{1}{s^{3\eta+1}}\right).$$

On taking inverse transformation L^{-1}, we get

$$w_3(\hbar,\ \tau)=B_3 sin(\hbar)\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right)+2B_1B_2 sin(\hbar)\left(\frac{\tau^{2\eta}}{\Gamma(2\eta+1)}\right)+B_1^3 sin(\hbar)\left(\frac{\tau^{3\eta}}{\Gamma(3\eta+1)}\right). \tag{55}$$

$$\mathbf{N}_2=-\frac{\partial^2 y_2}{\partial\hbar^2}-2y_2\frac{\partial y_2}{\partial\hbar}+\frac{\partial}{\partial\hbar}(w_2y_2).$$

Hence

$$\mathbf{N}_2=B_2 sin(\hbar)\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right)+B_1^2 sin(\hbar)\left(\frac{\tau^{2\eta}}{\Gamma(2\eta+1)}\right).$$

Consider equation (50), we have

$$y_3(\hbar,\ \tau)=L^{-1}\frac{1}{S^{\eta}}L\left[B_3\mathbf{N}_0+B_2\mathbf{N}_1+B_1\mathbf{N}_2\right]$$

$$y_3(\hbar,\ \tau)=B_3 sin(\hbar)L^{-1}\left(\frac{1}{s^{\eta+1}}\right)+2B_1B_2 sin(\hbar)L^{-1}\left(\frac{1}{s^{2\eta+1}}\right)+B_1^3 sin(\hbar)L^{-1}\left(\frac{1}{s^{3\eta+1}}\right).$$

On taking inverse transformation L^{-1}, we get

$$y_3(\hbar,\ \tau)=B_3 sin(\hbar)\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right)+2B_1B_2 sin(\hbar)\left(\frac{\tau^{2\eta}}{\Gamma(2\eta+1)}\right)+B_1^3 sin(\hbar)\left(\frac{\tau^{3\eta}}{\Gamma(3\eta+1)}\right). \tag{56}$$

By adding equations (43), (51), (53), and (55).

$$\widetilde{w}(\hbar,\ \tau)=w_0(\hbar,\ \tau)+w_1(\hbar,\ \tau)+w_2(\hbar,\ \tau)+w_3(\hbar,\ \tau),$$

$$\widetilde{w}(\hbar,\ \tau)=sin(\hbar)+B_1 sin(\hbar)\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right)+B_2 sin(\hbar)\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right)+B_1^2 sin(\hbar)\left(\frac{\tau^{2\eta}}{\Gamma(2\eta+1)}\right)$$

$$+B_3 sin(\hbar)\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right)+2B_1B_2 sin(\hbar)\left(\frac{\tau^{2\eta}}{2\eta+1}\right)+B_1^3 sin(\hbar)\left(\frac{\tau^{3\eta}}{3\eta+1}\right).$$

$$\widetilde{w}(\hbar,\ \tau)=sin(\hbar)+(B_1+B_2+B_3)\,sin(\hbar)\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right)+\left(B_1^2+2B_1B_2\right)sin(\hbar)\left(\frac{\tau^{2\eta}}{\Gamma(2\eta+1)}\right) \tag{57}$$

$$+B_1^3 sin(\hbar)\left(\frac{\tau^{3\eta}}{\Gamma(3\eta+1)}\right).$$

By adding equations (47), (52), (54), and (56).

$$\widetilde{y}(\hbar,\ \tau)=y_0(\hbar,\ \tau)+y_1(\hbar,\ \tau)+y_2(\hbar,\ \tau)+y_3(\hbar,\ \tau),$$

$$\widetilde{y}(\hbar,\ \tau)=sin(\hbar)+B_1 sin(\hbar)\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right)+B_2 sin(\hbar)\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right)+B_1^2 sin(\hbar)\left(\frac{\tau^{2\eta}}{\Gamma(2\eta+1)}\right)$$

$$+B_3 sin(\hbar)\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right)+2B_1B_2 sin(\hbar)\left(\frac{\tau^{2\eta}}{2\eta+1}\right)+B_1^3 sin(\hbar)\left(\frac{\tau^{3\eta}}{3\eta+1}\right),$$

$$\widetilde{y}(\hbar,\ \tau)=sin(\hbar)+(B_1+B_2+B_3)\,sin(\hbar)\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right)+\left(B_1^2+2B_1B_2\right)sin(\hbar)\left(\frac{\tau^{2\eta}}{\Gamma(2\eta+1)}\right)$$

$$+B_1^3 sin(\hbar)\left(\frac{\tau^{3\eta}}{\Gamma(3\eta+1)}\right). \tag{58}$$

Residual for $w(\hbar,\ \tau)$

$$R=\frac{\partial^{\eta}\widetilde{w}}{\partial\tau^{\eta}}-\frac{\partial^2\widetilde{w}}{\partial\hbar^2}-2\widetilde{w}\frac{\partial\widetilde{w}}{\partial\hbar}+\frac{\partial}{\partial\hbar}(\widetilde{w}\widetilde{y}),$$

$$R=sin(\hbar)+(B_1+B_2+B_3)\,sin(\hbar)+\left(B_1^2+2B_1B_2+B_1+B_2+B_3\right)sin(\hbar)\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right)$$

$$+\left(B_1^3+B_1^2+2B_1B_2\right)sin(\hbar)\left(\frac{\tau^{2\eta}}{\Gamma(2\eta+1)}\right)+B_1^3 sin(\hbar)\left(\frac{\tau^{3\eta}}{\Gamma(3\eta+1)}\right).$$

$$R=\left((1+B_1+B_2+B_3)+\left(B_1^2+2B_1B_2+B_1+B_2+B_3\right)\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right)\right)sin(\hbar)$$

$$+\left(\left(B_1^3+B_1^2+2B_1B_2\right)\left(\frac{\tau^{2\eta}}{\Gamma(2\eta+1)}\right)+B_1^3\left(\frac{\tau^{3\eta}}{\Gamma(3\eta+1)}\right)\right)sin(\hbar).$$

Residual for $y(\hbar,\ \tau)$

$$R = \frac{\partial^{\eta}\widetilde{y}}{\partial\tau^{\eta}} - \frac{\partial^{2}\widetilde{y}}{\partial\hbar^{2}} - 2\widetilde{y}\frac{\partial\widetilde{y}}{\partial\hbar} + \frac{\partial}{\partial\hbar}(\widetilde{w}\widetilde{y}),$$

$$R = sin(\hbar) + (B_1 + B_2 + B_3)\,sin(\hbar) + \left(B_1^2 + 2B_1B_2 + B_1 + B_2 + B_3\right) sin(\hbar)\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right)$$

$$+ \left(B_1^3 + B_1^2 + 2B_1B_2\right) sin(\hbar)\left(\frac{\tau^{2\eta}}{\Gamma(2\eta+1)}\right) + B_1^3 sin(\hbar)\left(\frac{\tau^{3\eta}}{\Gamma(3\eta+1)}\right).$$

$$R = \left((1 + B_1 + B_2 + B_3) + \left(B_1^2 + 2B_1B_2 + B_1 + B_2 + B_3\right)\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right)\right) sin(\hbar)$$

$$+ \left(\left(B_1^3 + B_1^2 + 2B_1B_2\right)\left(\frac{\tau^{2\eta}}{\Gamma(2\eta+1)}\right) + B_1^3\left(\frac{\tau^{3\eta}}{\Gamma(3\eta+1)}\right)\right) sin(\hbar).$$

The constant values are determined using least square method $B_1 = -0.9999996,\ B_2 = -0.00000027502669,$ and $B_3 = 0.000000040182729.$

By substituting the values of the auxiliary constants, we get the analytical solution for the fractional-order problem 3.1 through AHPTM. When $\eta = 1,$ the AHPTM solution of classical integer-order problem corresponding to problem 3.1 is given by:

$$\widetilde{w}(\hbar,\ \tau) = sin(\hbar)(-0.1667\tau^3 + 0.5\tau^2 - \tau + 0.1),$$

$$\widetilde{y}(\hbar,\ \tau) = sin(\hbar)(-0.1667\tau^3 + 0.5\tau^2 - \tau + 0.1).$$

The numerical results of problem 3.1 , which were produced using the AHPTM algorithm for various values in range of $0 < \eta \leq 1$, are shown in Table 1. The concerned errors estimates of solutions obtained using the suggested method are shown in Tables 2 to 3. The suggested method is accurate and efficient, as shown by the minor error values in the last columns of Tables 2 and 3. The results of the proposed method are compared with other existing methods in Table 4, wherein high accuracy of proposed method is observed. Several graphs showing different fractional solutions within the interval $0 < \eta \leq 1$ are used to illustrate the dynamic behavior of the solution to the given problem. These details are described in the captions of Figures 1 to 5. Finally, the visual representations in Figures 1 to 5 and Tables 1 to 4 demonstrate the accuracy and convergence of the proposed technique.

Table 1. Numerical results of problem 3.1 by taking fixed value of τ and numerous values of $\hbar$ and η

$\hbar$	AHPTM	AHPTM	AHPTM	AHPTM
at $(\tau = 0.001)$	$(\eta = 0.2)$	$(\eta = 0.3)$	$(\eta = 0.4)$	$(\eta = 0.5)$
-3.1416	$-9.5502e-17$	$-1.0721e-16$	$-1.1425e-16$	$-1.1822e-16$
-2.8274	-0.24098	-0.27052	-0.28829	-0.2983
-2.5133	-0.45837	-0.51456	-0.54837	-0.56739
-2.1991	-0.6309	-0.70823	-0.75476	-0.78095
-1.885	-0.74166	-0.83257	-0.88728	-0.91806
-1.5708	-0.77983	-0.87542	-0.93294	-0.96531
-1.2566	-0.74166	-0.83257	-0.88728	-0.91806
-0.94248	-0.6309	-0.70823	-0.75476	-0.78095
-0.62832	-0.45837	-0.51456	-0.54837	-0.56739
-0.31416	-0.24098	-0.27052	-0.28829	-0.2983
0	0	0	0	0
0.31416	0.24098	0.27052	0.28829	0.2983
0.62832	0.45837	0.51456	0.54837	0.56739
0.94248	0.6309	0.70823	0.75476	0.78095
1.2566	0.74166	0.83257	0.88728	0.91806
1.5708	0.77983	0.87542	0.93294	0.96531
1.885	0.74166	0.83257	0.88728	0.91806
2.1991	0.6309	0.70823	0.75476	0.78095
2.5133	0.45837	0.51456	0.54837	0.56739
2.8274	0.24098	0.27052	0.28829	0.2983
3.1416	$9.5502e-17$	$1.0721e-16$	$1.1425e-16$	$1.1822e-16$

Table 2. The error estimate at $\tau = 0.001$ and $\eta = 1$ for $\hbar$ is provided for comparison purposes of the solution via AHPTM and exact solution to problem 3.1

$\hbar$	Exact	AHPTM	Error
$\tau = 0.001$	$(\eta = 1)$	$(\eta = 1)$	$(\eta = 1)$
-3.1416	$-1.2234e-16$	$-1.2234e-16$	$9.1952e-30$
-2.8274	-0.30871	-0.30871	$2.3148e-14$
-2.5133	-0.5872	-0.5872	$4.4076e-14$
-2.1991	-0.80821	-0.80821	$6.0729e-14$
-1.885	-0.95011	-0.95011	$7.1387e-14$
-1.5708	-0.999	-0.999	$7.5051e-14$
-1.2566	-0.95011	-0.95011	$7.1387e-14$
-0.94248	-0.80821	-0.80821	$6.0729e-14$
-0.62832	-0.5872	-0.5872	$4.4076e-14$
-0.31416	-0.30871	-0.30871	$2.3148e-14$
0	0	0	0
0.31416	0.30871	0.30871	$2.3148e-14$
0.62832	0.5872	0.5872	$4.4076e-14$
0.94248	0.80821	0.80821	$6.0729e-14$
1.2566	0.95011	0.95011	$7.1387e-14$
1.5708	0.999	0.999	$7.5051e-14$
1.885	0.95011	0.95011	$7.1387e-14$
2.1991	0.80821	0.80821	$6.0729e-14$
2.5133	0.5872	0.5872	$4.4076e-14$
2.8274	0.30871	0.30871	$2.3148e-14$
3.1416	$1.2234e-16$	$1.2234e-16$	$9.1952e-30$

Table 3. The absolute error at $\eta = 1$ for $(\hbar,\ \tau)$ is provided for comparison purposes of the solution via AHPTM and exact solution to problem 3.1

$\hbar$	τ	Exact	AHPTM	Error
		$\eta = 1$	$\eta = 1$	$\eta = 1$
-3.1416	0	$-1.2246e-16$	$-1.2246e-16$	0
-2.8274	0.0005	-0.30886	-0.30886	$2.1094e-15$
-2.5133	0.001	-0.5872	-0.5872	$4.4076e-14$
-2.1991	0.0015	-0.8078	-0.8078	$2.6157e-13$
-1.885	0.002	-0.94916	-0.94916	$8.8729e-13$
-1.5708	0.0025	-0.9975	-0.9975	$2.1476e-12$
-1.2566	0.003	-0.94821	-0.94821	$4.0639e-12$
-0.94248	0.0035	-0.80619	-0.80619	$6.2111e-12$
-0.62832	0.004	-0.58544	-0.58544	$7.5187e-12$
-0.31416	0.0045	-0.30763	-0.30763	$6.2138e-12$
0	0.005	0	0	0
0.31416	0.0055	0.30732	0.30732	$1.3483e-11$
0.62832	0.006	0.58427	0.58427	$3.5934e-11$
0.94248	0.0065	0.80378	0.80378	$6.75e-11$
1.2566	0.007	0.94442	0.94442	$1.0589e-10$
1.5708	0.0075	0.99253	0.99253	$1.457e-10$
1.885	0.008	0.94348	0.94348	$1.7829e-10$
2.1991	0.0085	0.80217	0.80217	$1.9223e-10$
2.5133	0.009	0.58252	0.58252	$1.7468e-10$
2.8274	0.0095	0.3061	0.3061	$1.1351e-10$
3.1416	0.01	$1.2125e-16$	$1.2125e-16$	$5.5007e-26$

Table 4. The absolute error at $\eta = 1$ for $(\hbar,\ \tau)$ is provided for comparison purposes of the solution to problem 3.1

$\hbar$	τ	Fvim [23]	Adm [9]	Gdtm [24]	Hpm [25]	AHPTM
		Error	Error	Error	Error	Error
-10	0.001	9.0610^{-11}	9.9910^{-4}	2.7110^{-7}	9.9910^{-4}	$4.0856e-14$
-10	0.002	7.2410^{-10}	1.9910^{-3}	1.0810^{-6}	1.9910^{-3}	$5.0759e-13$
-10	0.003	2.4410^{-9}	2.9910^{-3}	2.4410^{-6}	2.9910^{-3}	$2.3246e-12$
-10	0.004	5.7910^{-9}	3.9910^{-3}	4.3410^{-6}	3.9910^{-3}	$6.9588e-12$
-10	0.005	1.1310^{-8}	4.9910^{-3}	6.7810^{-6}	4.9910^{-3}	$1.642e-11$
00	0.001	0	1.0010^{-3}	0	1.0010^{-3}	0
00	0.002	0	2.0010^{-3}	0	2.0010^{-3}	0
00	0.003	0	3.0010^{-3}	0	3.0010^{-3}	0
00	0.004	0	4.0110^{-3}	0	4.0110^{-3}	0
00	0.005	0	5.0210^{-3}	0	5.0210^{-3}	0
00	0.001	9.0610^{-11}	9.9810^{-4}	2.7110^{-7}	9.9810^{-4}	$4.0856e-14$
00	0.002	7.2410^{-10}	1.9910^{-3}	1.0810^{-6}	1.9910^{-3}	$5.0759e-13$
00	0.003	2.4410^{-9}	2.9810^{-3}	2.4410^{-6}	2.9810^{-3}	$2.3246e-12$
00	0.004	5.7910^{-9}	3.9710^{-3}	4.3410^{-6}	3.9710^{-3}	$6.9588e-12$
00	0.005	1.1310^{-8}	4.9610^{-3}	6.7810^{-6}	4.9610^{-3}	$1.642e-11$

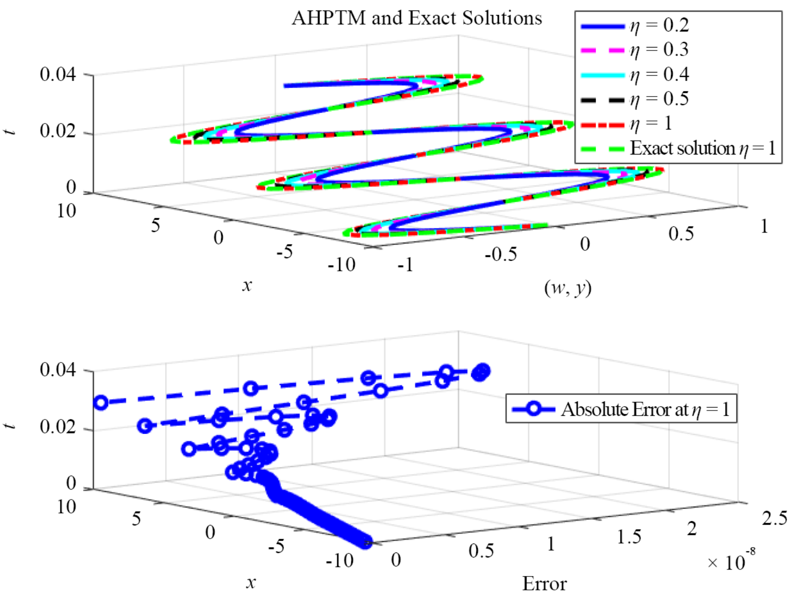

Figure 1. AHPTM and exact solutions to the problem 3.1 are compared graphically for various values of $(\hbar, \ \tau) = (x, \ t)$, and η

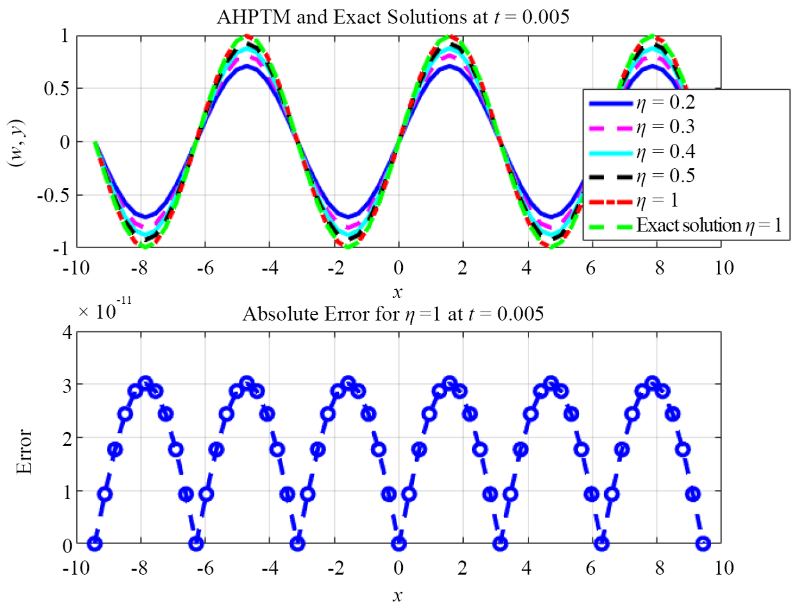

Figure 2. AHPTM and exact solutions to the problem 3.1 are compared graphically for various values of $(\hbar, \ \eta) = (x, \ \eta)$, and at $\tau = 0.005$ (as t denotes τ)

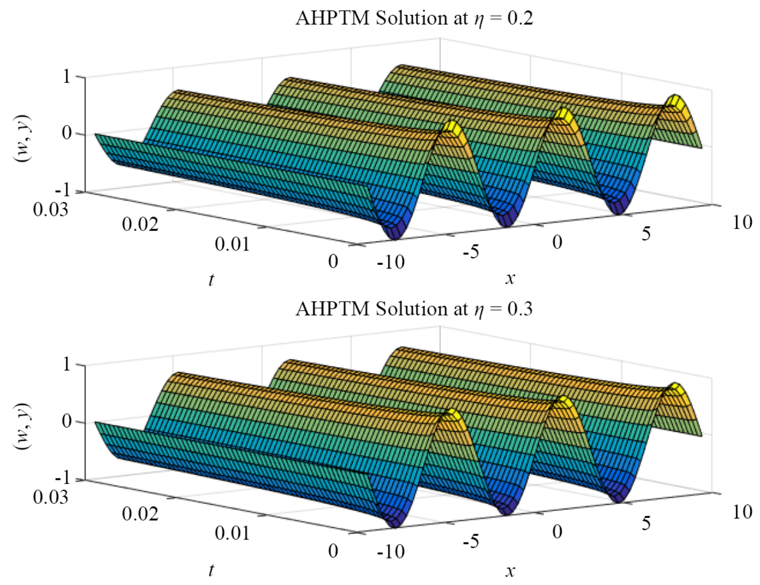

Figure 3. Surface plots at various $(\hbar,\ \tau) = (x,\ t)$, and $\eta = 0.3,\ 0.2$ of AHPTM's solution to the problem 3.1

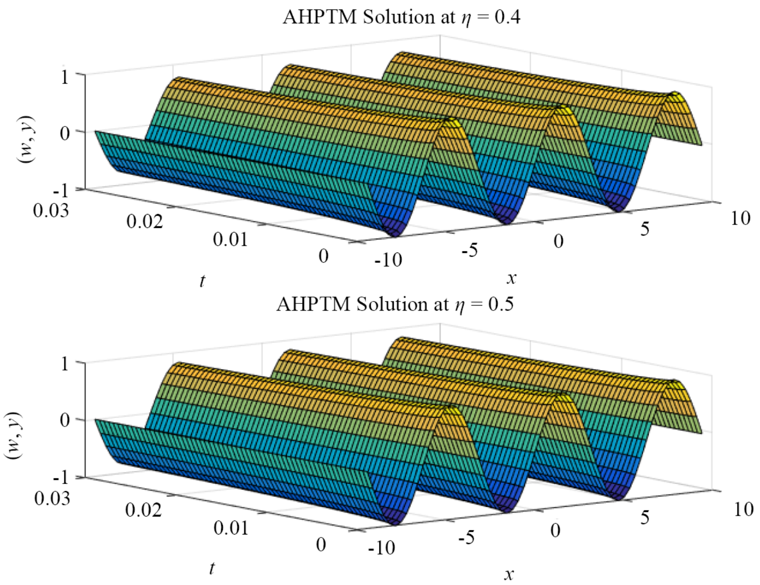

Figure 4. Surface plots at various $(\hbar,\ \tau) = (x,\ t)$, and $\eta = 0.5,\ 0.4$ of AHPTM's solution to the problem 3.1

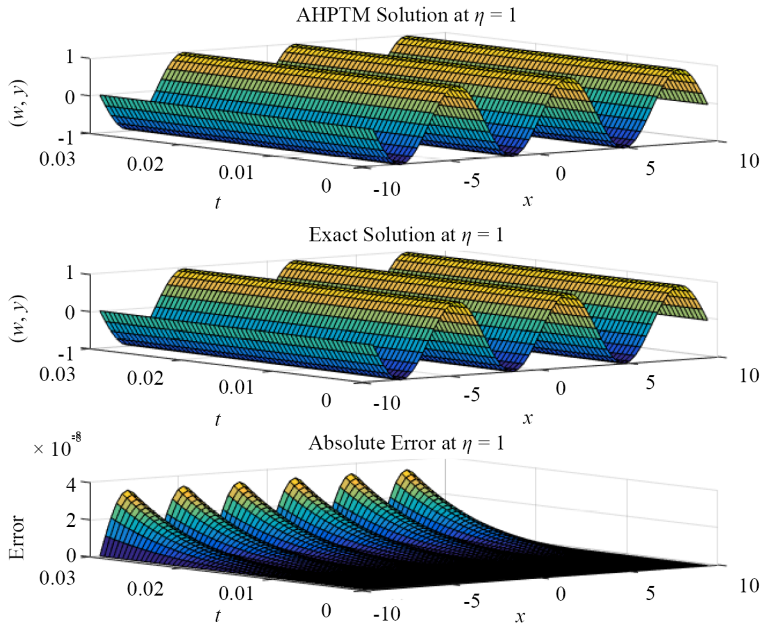

Figure 5. A comparison of solution via AHPTM with the exact solution of problem 3.1, for all $(\hbar,\ \tau) = (x,\ t)$ at fixed value $\eta = 1$

3.2 *Problem 02*

The second specific problem of fractional coupled non-linear Burgers equations is considered in the form of

$$\frac{\partial^{\eta} w}{\partial \tau^{\eta}} + 2w\frac{\partial w}{\partial \hbar} - \frac{\partial}{\partial \hbar}(wy) = \frac{\partial^2 w}{\partial \hbar^2}, \tag{59}$$

$$\frac{\partial^{\eta} y}{\partial \tau^{\eta}} + 2y\frac{\partial y}{\partial \hbar} - \frac{\partial}{\partial \hbar}(wy) = \frac{\partial^2 y}{\partial \hbar^2},\ 0 < \eta \leq 1. \tag{60}$$

Under the given initial conditions

$$w(\hbar,\ 0) = e^{\hbar},\ y(\hbar,\ 0) = e^{\hbar}.$$

Consider

$$j = \frac{\partial^{\eta} w}{\partial \tau^{\eta}},\ \mathbb{k} = 2w\frac{\partial w}{\partial \hbar} - \frac{\partial}{\partial \hbar}(wy) - \frac{\partial^2 w}{\partial \hbar^2},$$

and

$$\mathbf{M} = \frac{\partial^{\eta} y}{\partial \tau^{\eta}},\ \mathbf{N} = 2y\frac{\partial y}{\partial \hbar} - \frac{\partial}{\partial \hbar}(wy) - \frac{\partial^2 y}{\partial \hbar^2}.$$

As deformation equation is

$$\mathrm{L}(j) - \mathrm{Lp}\Bbbk = 0,$$

and

$$\mathrm{L}(\mathbf{M}) - \mathrm{Lp}\mathbf{N} = 0,$$

where L denotes the Laplace transform operator.

$$\mathrm{L}(j) = \mathrm{Lp}\Bbbk,$$

$$\mathrm{L}(\mathbf{M}) = \mathrm{Lp}\mathbf{N},$$

$$\mathrm{L}\left(\frac{\partial^{\eta} w}{\partial \tau^{\eta}}\right) = L(P\Bbbk),$$

$$\mathrm{L}\left(\frac{\partial^{\eta} y}{\partial \tau^{\eta}}\right) = L(P\mathbf{N}).$$

$$L\left(\frac{\partial^{\eta} w}{\partial \tau^{\eta}}\right) = \mathrm{Lp}\left[\mathrm{B}_1\Bbbk_0 + (\mathrm{B}_2\Bbbk_0 + \mathrm{B}_1\Bbbk_1)\,\mathrm{p} + (\mathrm{B}_3\Bbbk_0 + \mathrm{B}_2\Bbbk_1 + \mathrm{B}_1\Bbbk_2)\,\mathrm{p}^2 + ...\right],$$

$$L\left(\frac{\partial^{\eta} y}{\partial \tau^{\eta}}\right) = \mathrm{Lq}\left[\mathrm{B}_1\mathbf{N}_0 + (\mathrm{B}_2\mathbf{N}_0 + \mathrm{B}_1\mathbf{N}_1)\,\mathrm{q} + (\mathrm{B}_3\mathbf{N}_0 + \mathrm{B}_2\mathbf{N}_1 + \mathrm{B}_1\mathbf{N}_2)\,\mathrm{q}^2 + ...\right],$$

$$s^{\eta} w(s,\ \tau) - s^{\eta-1} w(\hbar,\ 0) - s^{\eta-2} w_{\tau}(\hbar,\ 0)...$$

$$= \mathrm{L}\left[(\mathrm{B}_1\Bbbk_0)\,\mathrm{p} + (\mathrm{B}_2\Bbbk_0 + \mathrm{B}_1\Bbbk_1)\,\mathrm{p}^2 + (\mathrm{B}_3\Bbbk_0 + \mathrm{B}_2\Bbbk_1 + \mathrm{B}_1\Bbbk_2)\,\mathrm{p}^3 + ...,\ \right]$$

$$s^{\eta} y(s,\ \tau) - s^{\eta-1} y(\hbar,\ 0) - s^{\eta-2} y_{\tau}(\hbar,\ 0)...$$

$$= \mathrm{L}\left[(\mathrm{B}_1\mathbf{N}_0)\,\mathrm{q} + (\mathrm{B}_2\mathbf{N}_0 + \mathrm{B}_1\mathbf{N}_1)\,\mathrm{q}^2 + (\mathrm{B}_3\mathbf{N}_0 + \mathrm{B}_2\mathbf{N}_1 + \mathrm{B}_1\mathbf{N}_2)\,\mathrm{q}^3 + ...,\ \right]$$

$$w(\hbar,\ \tau) = e^{\hbar} + L^{-1}\frac{1}{S^{\eta}}L\left[B_1\Bbbk_0\right]p + L^{-1}\frac{1}{S^{\eta}}L\left[B_2\Bbbk_0 + B_1\Bbbk_1\right]p^2$$

$$+ L^{-1}\frac{1}{S^{\eta}}L\left[B_3\Bbbk_0 + B_2\Bbbk_1 + B_1\Bbbk_2\right]p^3 + ...,$$

$$y(\hbar,\ \tau) = e^{\hbar} + L^{-1}\frac{1}{S^{\eta}}L[B_1\mathbf{N}_0]q + L^{-1}\frac{1}{S^{\eta}}L[B_2\mathbf{N}_0 + B_1\mathbf{N}_1]q^2$$

$$+ L^{-1}\frac{1}{S^{\eta}}L[B_3\mathbf{N}_0 + B_2\mathbf{N}_1 + B_1\mathbf{N}_2]q^3 + \ldots,$$

$$w_0(\hbar,\ \tau) + w_1(\hbar,\ \tau)p + w_2(\hbar,\ \tau)p^2 + w_3(\hbar,\ \tau)p^3 + \ldots$$

$$= e^{\hbar} + L^{-1}\frac{1}{S^{\eta}}L[B_1\mathbb{k}_0]p$$

$$+ L^{-1}\frac{1}{S^{\eta}}L[B_2\mathbb{k}_0 + B_1\mathbb{k}_1]p^2 + L^{-1}\frac{1}{S^{\eta}}L[B_3\mathbb{k}_0 + B_2\mathbb{k}_1 + B_1\mathbb{k}_2]p^3 + \ldots, \tag{61}$$

$$y_0(\hbar,\ \tau) + y_1(\hbar,\ \tau)q + y_2(\hbar,\ \tau)q^2 + y_3(\hbar,\ \tau)q^3 + \ldots$$

$$= e^{\hbar} + L^{-1}\frac{1}{S^{\eta}}L[B_1\mathbf{N}_0]q$$

$$+ L^{-1}\frac{1}{S^{\eta}}L[B_2\mathbf{N}_0 + B_1\mathbf{N}_1]q^2 + L^{-1}\frac{1}{S^{\eta}}L[B_3\mathbf{N}_0 + B_2\mathbf{N}_1 + B_1\mathbf{N}_2]q^3 + \ldots, \tag{62}$$

Equating the coefficients of like powers of p in equation (59), we get

$$p^0 : w_0(\hbar,\ \tau) = e^{\hbar}, \tag{63}$$

$$p^1 : w_1(\hbar,\ \tau) = L^{-1}\frac{1}{S^{\eta}}L\left[B_1\mathbb{k}_0\right], \tag{64}$$

$$p^2 : w_2(\hbar,\ \tau) = L^{-1}\frac{1}{S^{\eta}}\left[B_2\mathbb{k}_0 + B_1\mathbb{k}_1\right], \tag{65}$$

$$p^3 : w_3(\hbar,\ \tau) = L^{-1}\frac{1}{S^{\eta}}L\left[B_3\mathbb{k}_0 + B_2\mathbb{k}_1 + B_1\mathbb{k}_2\right]. \tag{66}$$

Equating the coefficients of like powers of q in equation (60), we get

$$q^0 : y_0(\hbar,\ \tau) = e^{\hbar}, \tag{67}$$

$$q^1 : y_1(\hbar,\ \tau) = L^{-1}\frac{1}{S^{\eta}}L\left[B_1\mathbf{N}_0\right], \tag{68}$$

$$q^2 : y_2(\hbar,\ \tau) = L^{-1}\frac{1}{S^\eta}\left[B_2\mathbf{N}_0 + B_1\mathbf{N}_1\right], \tag{69}$$

$$q^3 : y_3(\hbar,\ \tau) = L^{-1}\frac{1}{S^\eta}L\left[B_3\mathbf{N}_0 + B_2\mathbf{N}_1 + B_1\mathbf{N}_2\right]. \tag{70}$$

Consider equation (63) and equation (67), we have

$$w_0(\hbar,\ \tau) = e^{\hbar},\ y_0(\hbar,\ \tau) = e^{\hbar}.$$

$$\mathbb{k}_0 = 2w_0\frac{\partial w_0}{\partial\hbar} - \frac{\partial}{\partial\hbar}(w_0y_0) - \frac{\partial^2 w_0}{\partial\hbar^2}.$$

Hence $\mathbb{k}_0 = -e^{\hbar}$.
Consider equation (64), we have

$$w_1(\hbar,\ \tau) = L^{-1}\frac{1}{S^\eta}L\left[B_1\mathbb{k}_0\right],$$

$$w_1(\hbar,\ \tau) = L^{-1}\frac{1}{S^\eta}L\left[B_1(-e^{\hbar})\right],$$

$$w_1(\hbar,\ \tau) = -B_1e^{\hbar}L^{-1}\left(\frac{1}{S^{\eta+1}}\right),$$

On taking inverse L^{-1} of transformation as:

$$w_1(\hbar,\ \tau) = -B_1e^{\hbar}\left(\frac{\tau^\eta}{\Gamma(\eta+1)}\right). \tag{71}$$

$$\mathbf{N}_0 = 2y_0\frac{\partial y_0}{\partial\hbar} - \frac{\partial}{\partial\hbar}(w_0y_0) - \frac{\partial^2 y_0}{\partial\hbar^2}.$$

Hence $\mathbf{N}_0 = -e^{\hbar}$.
Consider equation (68), we have

$$y_1(\hbar,\ \tau) = L^{-1}\frac{1}{S^\eta}L\left[B_1\mathbf{N}_0\right],$$

$$y_1(\hbar,\ \tau) = L^{-1}\frac{1}{S^\eta}L\left[B_1(-e^\hbar)\right],$$

$$y_1(\hbar,\ \tau) = -B_1e^\hbar L^{-1}\left(\frac{1}{S^{\eta+1}}\right),$$

On taking inverse L^{-1} of transformation, we get

$$y_1(\hbar,\ \tau) = -B_1e^\hbar\left(\frac{\tau^\eta}{\Gamma(\eta+1)}\right). \tag{72}$$

$$\mathbb{k}_1 = 2w_1\frac{\partial w_1}{\partial\hbar} - \frac{\partial}{\partial\hbar}(w_1y_1) - \frac{\partial^2 w_1}{\partial\hbar^2}.$$

Hence $\mathbb{k}_1 = B_1e^\hbar\left(\frac{\tau^\eta}{\Gamma(\eta+1)}\right)$.
Consider equation (65), we have

$$w_2(\hbar,\ \tau) = L^{-1}\frac{1}{S^\eta}L\left[B_2\mathbb{k}_0 + B_1\mathbb{k}_1\right],$$

$$w_2(\hbar,\ \tau) = L^{-1}\frac{1}{S^\eta}L\left[B_2(-e^\hbar) + B_1\left(B_1e^\hbar\left(\frac{\tau^\eta}{\Gamma(\eta+1)}\right)\right)\right],$$

$$w_2(\hbar,\ \tau) = -B_2e^\hbar L^{-1}\left(\frac{1}{S^{\eta+1}}\right) + B_1^2e^\hbar L^{-1}\left(\frac{1}{S^{2\eta+1}}\right).$$

On taking inverse L^{-1} of transformation as:

$$w_2(\hbar,\ \tau) = -B_2e^\hbar\left(\frac{\tau^\eta}{\Gamma(\eta+1)}\right) + B_1^2e^\hbar\left(\frac{\tau^{2\eta}}{\Gamma(2\eta+1)}\right). \tag{73}$$

$$\mathbf{N}_1 = 2y_1\frac{\partial y_1}{\partial\hbar} - \frac{\partial}{\partial\hbar}(w_1y_1) - \frac{\partial^2 y_1}{\partial\hbar^2}.$$

Hence $\mathbf{N}_1 = B_1e^\hbar\left(\frac{\tau^\eta}{\Gamma(\eta+1)}\right)$.
Consider equation (69), we have

$$y_2(\hbar,\ \tau) = L^{-1}\frac{1}{S^{\eta}}L\left[B_2\mathbf{N}_0 + B_1\mathbf{N}_1\right],$$

$$y_2(\hbar,\ \tau) = L^{-1}\frac{1}{S^{\eta}}\left[B_2(-e^{\hbar}) + B_1\left(B_1\left(e^{\hbar}\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right)\right)\right],$$

$$y_2(\hbar,\ \tau) = -B_2e^{\hbar}L^{-1}\left(\frac{1}{S^{\eta+1}}\right) + B_1^2e^{\hbar}L^{-1}\left(\frac{1}{S^{2\eta+1}}\right).$$

On taking inverse L^{-1} of transformation as:

$$y_2(\hbar,\ \tau) = -B_2e^{\hbar}\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right) + B_1^2e^{\hbar}\left(\frac{\tau^{2\eta}}{\Gamma(2\eta+1)}\right). \tag{74}$$

$$\mathbb{k}_2 = 2w_2\frac{\partial w_2}{\partial \hbar} - \frac{\partial}{\partial \hbar}(w_2y_2) - \frac{\partial^2 w_2}{\partial \hbar^2}.$$

Hence

$$\mathbb{k}_2 = B_2e^{\hbar}\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right) - B_1^2e^{\hbar}\left(\frac{\tau^{2\eta}}{\Gamma(2\eta+1)}\right).$$

Consider equation (66), we have

$$w_3(\hbar,\ \tau) = L^{-1}\frac{1}{S^{\eta}}L\left[B_3\mathbb{k}_0 + B_2\mathbb{k}_1 + B_1\mathbb{k}_2\right],$$

$$w_3(\hbar,\ \tau) = -B_3e^{\hbar}L^{-1}\left(\frac{1}{s^{\eta+1}}\right) + 2B_1B_2e^{\hbar}L^{-1}\left(\frac{1}{s^{2\eta+1}}\right) - B_1^3e^{\hbar}L^{-1}\left(\frac{1}{s^{3\eta+1}}\right).$$

On taking inverse L^{-1} of transformation, we get

$$w_3(\hbar,\ \tau) = -B_3e^{\hbar}\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right) + 2B_1B_2e^{\hbar}\left(\frac{\tau^{2\eta}}{\Gamma(2\eta+1)}\right) - B_1^3e^{\hbar}\left(\frac{\tau^{3\eta}}{\Gamma(3\eta+1)}\right). \tag{75}$$

$$\mathbf{N}_2 = 2y_2\frac{\partial y_2}{\partial \hbar} - \frac{\partial}{\partial \hbar}(w_2y_2) - \frac{\partial^2 y_2}{\partial \hbar^2}.$$

Hence

$$\mathbf{N}_2 = B_2 e^{\hbar}\left(\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}\right) - B_1^2 e^{\hbar}\left(\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}\right).$$

Consider equation (70), we have

$$y_3(\hbar,\ \tau) = L^{-1}\frac{1}{S^{\eta}}L\left[B_3\mathbf{N}_0 + B_2\mathbf{N}_1 + B_1\mathbf{N}_2\right],$$

$$y_3(\hbar,\ \tau) = -B_3 e^{\hbar}L^{-1}\left(\frac{1}{s^{\eta+1}}\right) + 2B_1B_2e^{\hbar}L^{-1}\left(\frac{1}{s^{2\eta+1}}\right) - B_1^3 e^{\hbar}L^{-1}\left(\frac{1}{s^{3\eta+1}}\right).$$

On taking inverse L^{-1} of transformation, we get

$$y_3(\hbar,\ \tau) = -B_3 e^{\hbar}\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right) + 2B_1B_2e^{\hbar}\left(\frac{\tau^{2\eta}}{\Gamma(2\eta+1)}\right) - B_1^3 e^{\hbar}\left(\frac{\tau^{3\eta}}{\Gamma(3\eta+1)}\right). \tag{76}$$

By adding equations (63), (71), (73), and (75).

$$\widetilde{w}(\hbar,\ \tau) = w_0(\hbar,\ \tau) + w_1(\hbar,\ \tau) + w_2(\hbar,\ \tau) + w_3(\hbar,\ \tau),$$

$$\widetilde{w}(\hbar,\ \tau) = e^{\hbar} - (B_1+B_2+B_3)e^{\hbar}\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right) + \left(B_1^2 + 2B_1B_2\right)e^{\hbar}\left(\frac{\tau^{2\eta}}{\Gamma(2\eta+1)}\right) - B_1^3 e^{\hbar}\left(\frac{\tau^{3\eta}}{\Gamma(3\eta+1)}\right). \tag{77}$$

By adding equations (67), (72), (74), and (76).

$$\widetilde{y}(\hbar,\ \tau) = y_0(\hbar,\ \tau) + y_1(\hbar,\ \tau) + y_2(\hbar,\ \tau) + y_3(\hbar,\ \tau),$$

$$\widetilde{y}(\hbar,\ \tau) = e^{\hbar} - (B_1+B_2+B_3)e^{\hbar}\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right) + \left(B_1^2 + 2B_1B_2\right)e^{\hbar}\left(\frac{\tau^{2\eta}}{\Gamma(2\eta+1)}\right) - B_1^3 e^{\hbar}\left(\frac{\tau^{3\eta}}{\Gamma(3\eta+1)}\right). \tag{78}$$

Residual for $w(\hbar,\ \tau)$

$$R = \frac{\partial^{\eta}\widetilde{w}}{\partial\tau^{\eta}} + 2\widetilde{w}\frac{\partial\widetilde{w}}{\partial\hbar} - \frac{\partial}{\partial\hbar}(\widetilde{w}\widetilde{y}) - \frac{\partial^2\widetilde{w}}{\partial\tau^2},$$

$$R = -e^{\hbar} - (B_1+B_2+B_3)e^{\hbar} + \left(B_1^2 + 2B_1B_2 + B_1 + B_2 + B_3\right)e^{\hbar}\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right)$$

$$-\left(B_1^3 + B_1^2 + 2B_1B_2\right)e^{\hbar}\left(\frac{\tau^{2\eta}}{\Gamma(2\eta+1)}\right) + B_1^3 e^{\hbar}\left(\frac{\tau^{3\eta}}{\Gamma(3\eta+1)}\right).$$

Residual for $y(\hbar,\ \tau)$

$$R = \frac{\partial^{\eta}\widetilde{y}}{\partial\tau^{\eta}} + 2\widetilde{y}\frac{\partial\widetilde{y}}{\partial\hbar} - \frac{\partial}{\partial\hbar}(\widetilde{w}\widetilde{y}) - \frac{\partial^2\widetilde{y}}{\partial\tau^2},$$

$$R = -e^{\hbar} - (B_1 + B_2 + B_3)\, e^{\hbar} + \left(B_1^2 + 2B_1B_2 + B_1 + B_2 + B_3\right) e^{\hbar}\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right)$$

$$-\left(B_1^3 + B_1^2 + 2B_1B_2\right) e^{\hbar}\left(\frac{\tau^{2\eta}}{\Gamma(2\eta+1)}\right) + B_1^3 e^{\hbar}\left(\frac{\tau^{3\eta}}{\Gamma(3\eta+1)}\right).$$

The constant values are determined using the least squares method $B_1 = -1.0000053,\ \ B_2 = -0.000036227039$, and $B_3 = 6.7016015e - 10$.

Table 5. Numerical results of AHPTM solutions for the problem 3.2 by taking fixed value of τ and numerous values of $\hbar$ and η

$\hbar$ at $(\tau = 0.001)$	AHPTM $(\eta = 0.2)$	AHPTM $(\eta = 0.3)$	AHPTM $(\eta = 0.4)$	AHPTM $(\eta = 0.5)$
-3.1416	0.058877	0.050132	0.046482	0.0448
-2.8274	0.080609	0.068637	0.063638	0.061336
-2.5133	0.11036	0.093971	0.087128	0.083976
-2.1991	0.1511	0.12866	0.11929	0.11497
-1.885	0.20687	0.17614	0.16332	0.15741
-1.5708	0.28323	0.24116	0.2236	0.21551
-1.2566	0.38777	0.33017	0.30613	0.29506
-0.94248	0.5309	0.45204	0.41913	0.40396
-0.62832	0.72685	0.6189	0.57383	0.55307
-0.31416	0.99514	0.84734	0.78563	0.75721
0	1.3625	1.1601	1.0756	1.0367
0.31416	1.8653	1.5883	1.4726	1.4194
0.62832	2.5539	2.1745	2.0162	1.9433
0.94248	3.4965	2.9772	2.7604	2.6605
1.2566	4.7871	4.0761	3.7793	3.6426
1.5708	6.5541	5.5806	5.1742	4.9871
1.885	8.9732	7.6405	7.0841	6.8278
2.1991	12.285	10.461	9.6989	9.348
2.5133	16.82	14.322	13.279	12.798
2.8274	23.028	19.608	18.18	17.522
3.1416	31.528	26.845	24.891	23.99

By substituting the values of the auxiliary constants, we get the analytical solution for the fractional-order problem 3.2 through AHPTM. When $\eta = 1$, the AHPTM solution of classical integer-order problem corresponding to problem 3.2 is given by:

$$\widetilde{w}(\hbar,\ \tau) = e^{\hbar}(0.16667\tau^3 + 0.50004\tau^2 + \tau + 1.0),$$

$$\widetilde{y}(\hbar,\ \tau) = e^{\hbar}(0.16667\tau^3 + 0.50004\tau^2 + \tau + 1.0).$$

The numerical results of the problem 3.2 with their error estimates, determined by the AHPTM technique, are recorded in Tables 5 to 7. The minimal error estimates in final columns are observed in Tables 6 and 7, indicating that the suggested method is accurate and efficient. Different graphs representing various fractional solutions within the interval $0 < \eta \leq 1$ are used to illustrate the dynamic behavior of the solution to the given problem 3.2 . These details are described in the captions of Figures 6 to 10. Finally, the visual representations in Figures 6 to 10 and Tables 5 to 7 demonstrate the accuracy and convergence of the proposed technique.

Table 6. The error estimate at $\tau = 0.001$ and $\eta = 1$ for $\hbar$ is provided for comparison purposes of the solution via AHPTM and exact solution to problem 3.2

$\hbar$	Exact	AHPTM	Error
$\tau = 0.001$	$(\eta = 1)$	$(\eta = 1)$	$(\eta = 1)$
-3.1416	0.043257	0.043257	$1.7269e-12$
-2.8274	0.059224	0.059224	$2.3643e-12$
-2.5133	0.081084	0.081084	$3.237e-12$
-2.1991	0.11101	0.11101	$4.4318e-12$
-1.885	0.15199	0.15199	$6.0676e-12$
-1.5708	0.20809	0.20809	$8.3072e-12$
-1.2566	0.28489	0.28489	$1.1373e-11$
-0.94248	0.39005	0.39005	$1.5571e-11$
-0.62832	0.53402	0.53402	$2.1319e-11$
-0.31416	0.73113	0.73113	$2.9188e-11$
0	1.001	1.001	$3.9962e-11$
0.31416	1.3705	1.3705	$5.4712e-11$
0.62832	1.8763	1.8763	$7.4906e-11$
0.94248	2.5689	2.5689	$1.0255e-10$
1.2566	3.5171	3.5171	$1.4041e-10$
1.5708	4.8153	4.8153	$1.9223e-10$
1.885	6.5927	6.5927	$2.6319e-10$
2.1991	9.0261	9.0261	$3.6034e-10$
2.5133	12.358	12.358	$4.9334e-10$
2.8274	16.919	16.919	$6.7543e-10$
3.1416	23.164	23.164	$9.2474e-10$

Table 7. The absolute error at $\eta = 1$ for $(\hbar, \tau)$ is provided for comparison purposes of the solution via AHPTM and exact solution to problem 3.2

$\hbar$	τ	Exact	AHPTM	Error
		$\eta = 1$	$\eta = 1$	$\eta = 1$
-3.1416	0	0.043214	0.043241	0
-2.8274	0.0005	0.059194	0.059194	$5.9153e-13$
-2.5133	0.001	0.081084	0.081084	$3.237e-12$
-2.1991	0.0015	0.11107	0.11107	$9.959e-12$
-1.885	0.002	0.15214	0.15214	$2.4197e-11$
-1.5708	0.0025	0.2084	0.2084	$5.1642e-11$
-1.2566	0.003	0.28546	0.28546	$1.0152e-10$
-0.94248	0.0035	0.39103	0.39103	$1.8855e-10$
-0.62832	0.004	0.53563	0.53563	$3.3585e-10$
-0.31416	0.0045	0.7337	0.7337	$5.7936e-10$
0	0.005	1.005	1.005	$9.7435e-10$
0.31416	0.0055	1.3767	1.3767	$1.6051e-09$
0.62832	0.006	1.8857	1.8857	$2.5992e-09$
0.94248	0.0065	2.5831	2.5831	$4.1483e-09$
1.2566	0.007	3.5383	3.5383	$6.5386e-09$
1.5708	0.0075	4.8467	4.8467	$1.0195e-08$
1.885	0.008	6.639	6.639	$1.5746e-08$
2.1991	0.0085	9.094	9.094	$2.4113e-08$
2.5133	0.009	12.457	12.457	$3.6648e-08$
2.8274	0.0095	17.063	17.063	$5.5318e-08$
3.1416	0.01	23.373	23.373	$8.2979e-08$

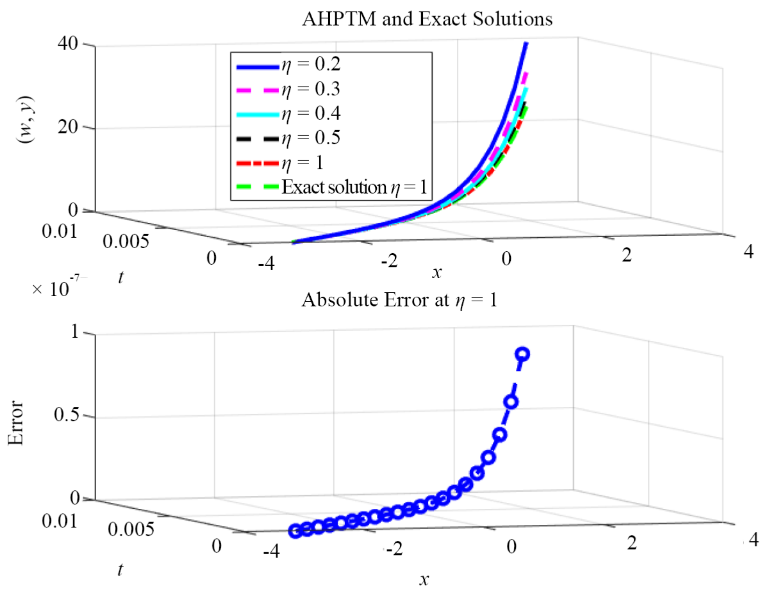

Figure 6. AHPTM and exact solutions to the problem 3.2 are compared graphically for various values of $(\hbar, \tau) = (x, t)$, and η

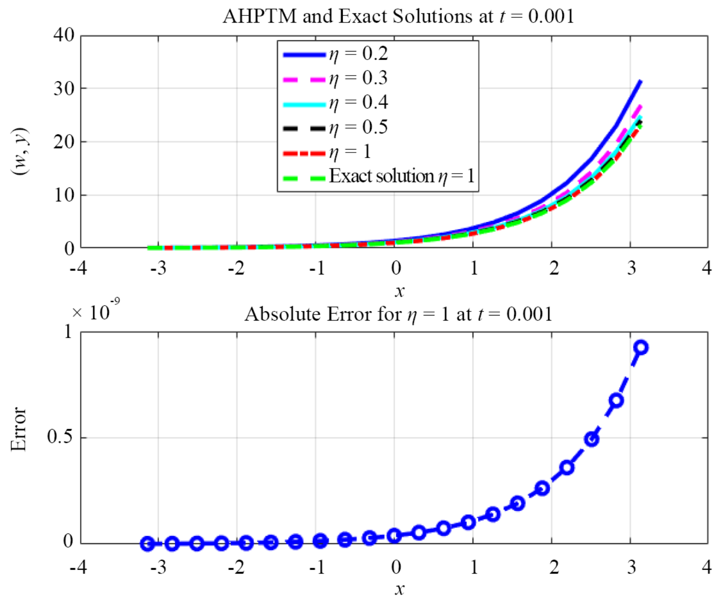

Figure 7. AHPTM and exact solutions to the problem 3.2 are compared graphically for various values of $(\hbar,\ \eta) = (x,\ \eta)$, and at $\tau = 0.001$ (as t denotes τ)

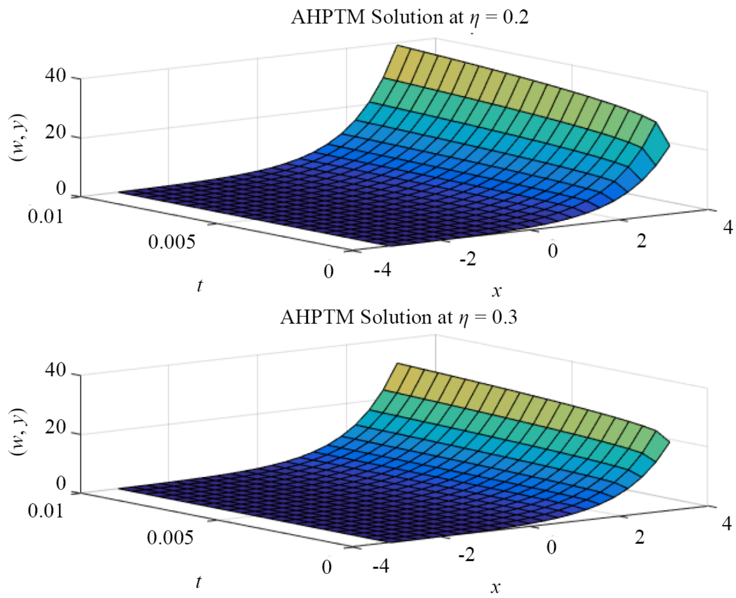

Figure 8. Surface plots at various $(\hbar,\ \tau) = (x,\ t)$, and $\eta = 0.3,\ 0.2$ of AHPTM's solution to the problem 3.2

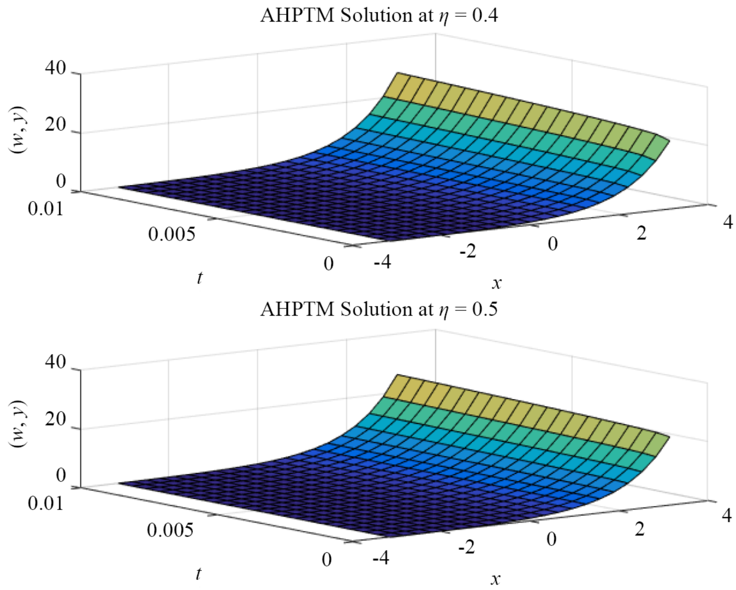

Figure 9. Surface plots at various $(\hbar,\ \tau) = (x,\ t)$, and $\eta = 0.5,\ 0.4$ of AHPTM's solution to the problem 3.2

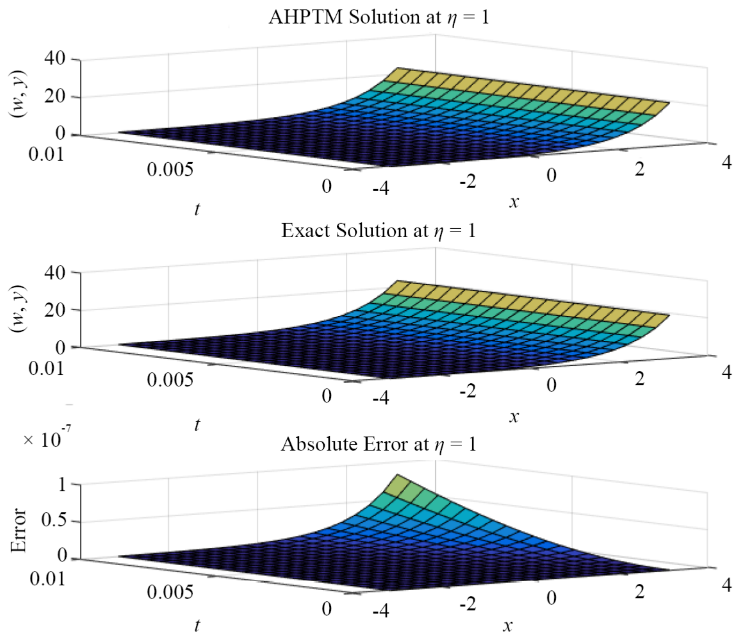

Figure 10. A comparison of solution via AHPTM with the exact solution of problem 3.2 , for all $(\hbar,\ \tau) = (x,\ t)$ at fixed value $\eta = 1$

3.3 *Problem 03*

The third specific problem of fractional coupled non-linear Burgers equations is considered in the form of

$$\frac{\partial^\eta w}{\partial \tau^\eta} = \frac{\partial^2 w}{\partial \hbar^2} + 2w\frac{\partial w}{\partial \hbar} - \frac{\partial}{\partial \hbar}(wy), \tag{79}$$

$$\frac{\partial^\eta y}{\partial \tau^\eta} = \frac{\partial^2 y}{\partial \hbar^2} + 2y\frac{\partial y}{\partial \hbar} - \frac{\partial}{\partial \hbar}(wy),\ 0 < \eta \leq 1. \tag{80}$$

Under the given initial conditions

$$w(\hbar,\ 0) = cos(\hbar),\ y(\hbar,\ 0) = cos(\hbar).$$

Consider

$$j = \frac{\partial^\eta w}{\partial \tau^\eta},\ \Bbbk = -\frac{\partial^2 w}{\partial \hbar^2} - 2w\frac{\partial w}{\partial \hbar} + \frac{\partial}{\partial \hbar}(wy),$$

and

$$\mathbf{M} = \frac{\partial^\eta y}{\partial \tau^\eta},\ \mathbf{N} = -\frac{\partial^2 y}{\partial \hbar^2} - 2y\frac{\partial y}{\partial \hbar} + \frac{\partial}{\partial \hbar}(wy).$$

As deformation equation is

$$\mathrm{L}(j) - \mathrm{Lp}\Bbbk = 0,$$

and

$$L\mathbf{M} - Lp\mathbf{N} = 0,$$

where L is Laplace operator.

$$\mathrm{L}(j) = \mathrm{Lp}\Bbbk,$$

$$L\mathbf{M} = Lp\mathbf{N},$$

$$\mathrm{L}\left(\frac{\partial^\eta w}{\partial \tau^\eta}\right) = L(P\Bbbk),$$

$$\mathrm{L}\left(\frac{\partial^\eta y}{\partial \tau^\eta}\right) = L(P\mathbf{N}).$$

$$L\left(\frac{\partial^{\eta} w}{\partial \tau^{\eta}}\right) = \mathrm{Lp}\left[\mathrm{B}_1\mathbb{k}_0 + (\mathrm{B}_2\mathbb{k}_0 + \mathrm{B}_1\mathbb{k}_1)\,\mathrm{p} + (\mathrm{B}_3\mathbb{k}_0 + \mathrm{B}_2\mathbb{k}_1 + \mathrm{B}_1\mathbb{k}_2)\,\mathrm{p}^2 + ...\right],$$

$$L\left(\frac{\partial^{\eta} y}{\partial \tau^{\eta}}\right) = \mathrm{Lq}\left[\mathrm{B}_1\mathbf{N}_0 + (\mathrm{B}_2\mathbf{N}_0 + \mathrm{B}_1\mathbf{N}_1)\,\mathrm{q} + (\mathrm{B}_3\mathbf{N}_0 + \mathrm{B}_2\mathbf{N}_1 + \mathrm{B}_1\mathbf{N}_2)\,\mathrm{q}^2 + ...\right],$$

$$s^{\eta} w(s,\ \tau) - s^{\eta-1} w(\hbar,\ 0) - s^{\eta-2} w_{\tau}(\hbar,\ 0)...$$

$$= \mathrm{L}\left[(\mathrm{B}_1\mathbb{k}_0)\,\mathrm{p} + (\mathrm{B}_2\mathbb{k}_0 + \mathrm{B}_1\mathbb{k}_1)\,\mathrm{p}^2 + (\mathrm{B}_3\mathbb{k}_0 + \mathrm{B}_2\mathbb{k}_1 + \mathrm{B}_1\mathbb{k}_2)\,\mathrm{p}^3 + ...\right],$$

$$s^{\eta} y(s,\ \tau) - s^{\eta-1} y(\hbar,\ 0) - s^{\eta-2} y_{\tau}(\hbar,\ 0)...$$

$$= \mathrm{L}\left[(\mathrm{B}_1\mathbf{N_0})\,\mathrm{q} + (\mathrm{B}_2\mathbf{N_0} + \mathrm{B}_1\mathbf{N_1})\,\mathrm{q}^2 + (\mathrm{B}_3\mathbf{N_0} + \mathrm{B}_2\mathbf{N_1} + \mathrm{B}_1\mathbf{N}_2)\,\mathrm{q}^3 + ...\right],$$

$$w(\hbar,\ \tau) = cos(\hbar) + L^{-1}\frac{1}{S^{\eta}}L[B_1\mathbb{k}_0]p + L^{-1}\frac{1}{S^{\eta}}L[B_2\mathbb{k}_0 + B_1\mathbb{k}_1]p^2$$

$$+ L^{-1}\frac{1}{S^{\eta}}L[B_3\mathbb{k}_0 + B_2\mathbb{k}_1 + B_1\mathbb{k}_2]p^3 + ...,$$

$$y(\hbar,\ \tau) = cos(\hbar) + L^{-1}\frac{1}{S^{\eta}}L[B_1\mathbf{N}_0]p + L^{-1}\frac{1}{S^{\eta}}L[B_2\mathbf{N}_0 + B_1\mathbf{N}_1]p^2$$

$$+ L^{-1}\frac{1}{S^{\eta}}L[B_3\mathbf{N}_0 + B_2\mathbf{N}_1 + B_1\mathbf{N}_2]p^3 + ...,$$

$$w_0(\hbar,\ \tau) + w_1(\hbar,\ \tau)p + w_2(\hbar,\ \tau)p^2 + w_3(\hbar,\ \tau)p^3 + ...$$

$$= cos(\hbar) + L^{-1}\frac{1}{S^{\eta}}L[B_1\mathbb{k}_0]p$$

$$+ L^{-1}\frac{1}{S^{\eta}}L[B_2\mathbb{k}_0 + B_1\mathbb{k}_1]p^2 + L^{-1}\frac{1}{S^{\eta}}L[B_3\mathbb{k}_0 + B_2\mathbb{k}_1 + B_1\mathbb{k}_2]p^3 + ..., \tag{81}$$

$$y_0(\hbar,\ \tau) + y_1(\hbar,\ \tau)q + y_2(\hbar,\ \tau)q^2 + y_3(\hbar,\ \tau)q^3 + ...$$

$$= cos(\hbar) + L^{-1}\frac{1}{S^{\eta}}L[B_1\mathbf{N}_0]q$$

$$+ L^{-1}\frac{1}{S^{\eta}}L[B_2\mathbf{N}_0 + B_1\mathbf{N}_1]q^2 + L^{-1}\frac{1}{S^{\eta}}L[B_3\mathbf{N}_0 + B_2\mathbf{N}_1 + B_1\mathbf{N}_2]q^3 + ..., \tag{82}$$

Equating the coefficients of like powers of p in equation (81), we get

$$p^0 : w_0(\hbar,\ \tau) = cos(\hbar), \tag{83}$$

$$p^1 : w_1(\hbar,\ \tau) = L^{-1}\frac{1}{S^\eta}L\left[B_1\mathbb{k}_0\right], \tag{84}$$

$$p^2 : w_2(\hbar,\ \tau) = L^{-1}\frac{1}{S^\eta}\left[B_2\mathbb{k}_0 + B_1\mathbb{k}_1\right], \tag{85}$$

$$p^3 : w_3(\hbar,\ \tau) = L^{-1}\frac{1}{S^\eta}L\left[B_3\mathbb{k}_0 + B_2\mathbb{k}_1 + B_1\mathbb{k}_2\right]. \tag{86}$$

Equating the coefficients of like powers of q in equation (82), we get

$$q^0 : y_0(\hbar,\ \tau) = cos(\hbar), \tag{87}$$

$$q^1 : y_1(\hbar,\ \tau) = L^{-1}\frac{1}{S^\eta}L\left[B_1\mathbf{N_0}\right], \tag{88}$$

$$q^2 : y_2(\hbar,\ \tau) = L^{-1}\frac{1}{S^\eta}\left[B_2\mathbf{N_0} + B_1\mathbf{N_1}\right], \tag{89}$$

$$q^3 : y_3(\hbar,\ \tau) = L^{-1}\frac{1}{S^\eta}L\left[B_3\mathbf{N_0} + B_2\mathbf{N_1} + B_1\mathbf{N_2}\right]. \tag{90}$$

Consider equation (83) and equation (87), we have

$$w_0(\hbar,\ \tau) = cos(\hbar),\ y_0(\hbar,\ \tau) = cos(\hbar).$$

$$\mathbb{k}_0 = -\frac{\partial^2 w_0}{\partial \hbar^2} - 2w_0\frac{\partial w_0}{\partial \hbar} + \frac{\partial}{\partial \hbar}(w_0 y_0).$$

Hence $\mathbb{k}_0 = cos(\hbar)$.
Consider equation (84)

$$w_1(\hbar,\ \tau) = L^{-1}\frac{1}{S^\eta}L\left[B_1\mathbb{k}_0\right],$$

$$w_1(\hbar,\ \tau) = L^{-1}\frac{1}{S^\eta}L\left[B_1 cos(\hbar)\right],$$

$$w_1(\hbar,\ \tau) = B_1 cos(\hbar)L^{-1}\left(\frac{1}{S^{\eta+1}}\right),$$

On taking inverse transformation L^{-1}, we get

$$w_1(\hbar,\ \tau) = B_1 cos(\hbar)\left(\frac{\tau^\eta}{\Gamma(\eta+1)}\right). \tag{91}$$

$$\mathbf{N_0} = -\frac{\partial^2 y_0}{\partial \hbar^2} - 2y_0\frac{\partial y_0}{\partial \hbar} + \frac{\partial}{\partial \hbar}(w_0 y_0).$$

Hence $\mathbf{N_0} = cos(\hbar)$.
Consider equation (88)

$$y_1(\hbar,\ \tau) = L^{-1}\frac{1}{S^\eta}L\left[B_1\mathbf{N}_0\right],$$

$$y_1(\hbar,\ \tau) = L^{-1}\frac{1}{S^\eta}L\left[B_1 cos(\hbar)\right],$$

$$y_1(\hbar,\ \tau) = B_1 cos(\hbar)L^{-1}\left(\frac{1}{S^{\eta+1}}\right),$$

On taking inverse transformation L^{-1}, we get

$$y_1(\hbar,\ \tau) = B_1 cos(\hbar)\left(\frac{\tau^\eta}{\Gamma(\eta+1)}\right). \tag{92}$$

$$\mathbb{k}_1 = -\frac{\partial^2 w_1}{\partial \hbar^2} - 2w_1\frac{\partial w_1}{\partial \hbar} + \frac{\partial}{\partial \hbar}(w_1 y_1).$$

Hence

$$\mathbb{k}_1 = B_1 cos(\hbar)\left(\frac{\tau^\eta}{\Gamma(\eta+1)}\right).$$

Consider equation (85), we have

$$w_2(\hbar,\ \tau) = L^{-1}\frac{1}{S^\eta}L\left[B_2\mathbb{k}_0 + B_1\mathbb{k}_1\right],$$

$$w_2(\hbar,\ \tau) = L^{-1}\frac{1}{S^\eta}L\left[B_2 cos(\hbar) + B_1\left(B_1 cos(\hbar)\left(\frac{\tau^\eta}{\Gamma(\eta+1)}\right)\right)\right],$$

$$w_2(\hbar,\ \tau) = B_2 cos(\hbar)L^{-1}\left(\frac{1}{S^{\eta+1}}\right) + B_1^2 cos(\hbar)L^{-1}\left(\frac{1}{S^{2\eta+1}}\right),$$

On taking inverse transformation L^{-1}, we get

$$w_2(\hbar,\ \tau) = B_2 cos(\hbar)\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right) + B_1^2 cos(\hbar)\left(\frac{\tau^{2\eta}}{\Gamma(2\eta+1)}\right). \tag{93}$$

$$\mathbf{N}_1 = -\frac{\partial^2 y_1}{\partial \hbar^2} - 2y_1\frac{\partial y_1}{\partial \hbar} + \frac{\partial}{\partial \hbar}(w_1 y_1).$$

Hence

$$\mathbf{N}_1 = B_1 cos(\hbar)\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right).$$

Consider equation (89), we have

$$y_2(\hbar,\ \tau) = L^{-1}\frac{1}{S^{\eta}}L\left[B_2\mathbf{N}_0 + B_1\mathbf{N}_1\right],$$

$$y_2(\hbar,\ \tau) = B_2(cos(\hbar))L^{-1}\left(\frac{1}{S^{\eta+1}}\right) + B_1^2 cos(\hbar)L^{-1}\left(\frac{1}{S^{2\eta+1}}\right).$$

On taking inverse transformation L^{-1}, we get

$$y_2(\hbar,\ \tau) = B_2 cos(\hbar)\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right) + B_1^2 cos(\hbar)\left(\frac{\tau^{2\eta}}{\Gamma(2\eta+1)}\right). \tag{94}$$

$$\Bbbk_2 = -\frac{\partial^2 w_2}{\partial \hbar^2} - 2w_2\frac{\partial w_2}{\partial \hbar} + \frac{\partial}{\partial \hbar}(w_2 y_2).$$

Hence

$$\Bbbk_2 = B_2 cos(\hbar)\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right) + B_1^2 cos(\hbar)\left(\frac{\tau^{2\eta}}{\Gamma(2\eta+1)}\right).$$

Consider equation (86), we have

$$w_3(\hbar,\ \tau) = L^{-1}\frac{1}{S^{\eta}}L\left[B_3\Bbbk_0 + B_2\Bbbk_1 + B_1\Bbbk_2\right],$$

$$w_3(\hbar,\ \tau) = L^{-1}\frac{1}{S^{\eta}}L\left[B_3 cos(\hbar) + B_2\left(B_1 cos(\hbar)\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right)\right)\right]$$

$$+ L^{-1}\frac{1}{S^{\eta}}L\left[B_1\left(B_2 cos(\hbar)(\frac{\tau^{\eta}}{\Gamma(\eta+1)}) + B_1^2 cos(\hbar)\left(\frac{\tau^{2\eta}}{\Gamma(2\eta+1)}\right)\right)\right],$$

$$w_3(\hbar,\ \tau) = B_3 cos(\hbar) L^{-1}\left(\frac{1}{s^{\eta+1}}\right) + 2B_1B_2 cos(\hbar) L^{-1}\left(\frac{1}{s^{2\eta+1}}\right) + B_1^3 cos(\hbar) L^{-1}\left(\frac{1}{s^{3\eta+1}}\right).$$

On taking inverse transformation L^{-1}, we get

$$w_3(\hbar,\ \tau) = B_3 cos(\hbar)\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right) + 2B_1B_2 cos(\hbar)\left(\frac{\tau^{2\eta}}{\Gamma(2\eta+1)}\right) + B_1^3 cos(\hbar)\left(\frac{\tau^{3\eta}}{\Gamma(3\eta+1)}\right). \tag{95}$$

$$\mathbf{N}_2 = -\frac{\partial^2 y_2}{\partial \hbar^2} - 2y_2\frac{\partial y_2}{\partial \hbar} + \frac{\partial}{\partial \hbar}(w_2 y_2).$$

Hence

$$\mathbf{N}_2 = B_2 cos(\hbar)\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right) + B_1^2 cos(\hbar)\left(\frac{\tau^{2\eta}}{\Gamma(2\eta+1)}\right).$$

Consider equation (90), we have

$$y_3(\hbar,\ \tau) = L^{-1}\frac{1}{S^{\eta}} L\left[B_3\mathbf{N}_0 + B_2\mathbf{N}_1 + B_1\mathbf{N}_2\right]$$

$$y_3(\hbar,\ \tau) = B_3 cos(\hbar) L^{-1}\left(\frac{1}{s^{\eta+1}}\right) + 2B_1B_2 cos(\hbar) L^{-1}\left(\frac{1}{s^{2\eta+1}}\right) + B_1^3 cos(\hbar) L^{-1}\left(\frac{1}{s^{3\eta+1}}\right).$$

On taking inverse transformation L^{-1}, we get

$$y_3(\hbar,\ \tau) = B_3 cos(\hbar)\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right) + 2B_1B_2 cos(\hbar)\left(\frac{\tau^{2\eta}}{\Gamma(2\eta+1)}\right) + B_1^3 cos(\hbar)\left(\frac{\tau^{3\eta}}{\Gamma(3\eta+1)}\right). \tag{96}$$

By adding equations (83), (91), (93), and (95).

$$\widetilde{w}(\hbar,\ \tau) = w_0(\hbar,\ \tau) + w_1(\hbar,\ \tau) + w_2(\hbar,\ \tau) + w_3(\hbar,\ \tau),$$

$$\widetilde{w}(\hbar,\ \tau) = cos(\hbar) + B_1 cos(\hbar)\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right) + B_2 cos(\hbar)\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right) + B_1^2 cos(\hbar)\left(\frac{\tau^{2\eta}}{\Gamma(2\eta+1)}\right)$$
$$+ B_3 cos(\hbar)\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right) + 2B_1B_2 cos(\hbar)\left(\frac{\tau^{2\eta}}{2\eta+1}\right) + B_1^3 cos(\hbar)\left(\frac{\tau^{3\eta}}{3\eta+1}\right).$$

$$\widetilde{w}(\hbar,\ \tau) = cos(\hbar) + (B_1 + B_2 + B_3)\,cos(\hbar)\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right) + \left(B_1^2 + 2B_1B_2\right)cos(\hbar)\left(\frac{\tau^{2\eta}}{\Gamma(2\eta+1)}\right)$$

$$+ B_1^3 cos(\hbar)\left(\frac{\tau^{3\eta}}{\Gamma(3\eta+1)}\right). \tag{97}$$

By adding equations (87), (92), (94), and (96).

$$\widetilde{y}(\hbar,\ \tau) = y_0(\hbar,\ \tau) + y_1(\hbar,\ \tau) + y_2(\hbar,\ \tau) + y_3(\hbar,\ \tau),$$

$$\widetilde{y}(\hbar,\ \tau) = cos(\hbar) + B_1 cos(\hbar)\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right) + B_2 cos(\hbar)\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right) + B_1^2 cos(\hbar)\left(\frac{\tau^{2\eta}}{\Gamma(2\eta+1)}\right)$$

$$+ B_3 cos(\hbar)\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right) + 2B_1B_2 cos(\hbar)\left(\frac{\tau^{2\eta}}{2\eta+1}\right) + B_1^3 cos(\hbar)\left(\frac{\tau^{3\eta}}{3\eta+1}\right),$$

$$\widetilde{y}(\hbar,\ \tau) = cos(\hbar) + (B_1 + B_2 + B_3)\,cos(\hbar)\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right) + \left(B_1^2 + 2B_1B_2\right)cos(\hbar)\left(\frac{\tau^{2\eta}}{\Gamma(2\eta+1)}\right)$$

$$+ B_1^3 cos(\hbar)\left(\frac{\tau^{3\eta}}{\Gamma(3\eta+1)}\right). \tag{98}$$

Residual for $w(\hbar,\ \tau)$

$$R = \frac{\partial^{\eta}\widetilde{w}}{\partial\tau^{\eta}} - \frac{\partial^2\widetilde{w}}{\partial\hbar^2} - 2\widetilde{w}\frac{\partial\widetilde{w}}{\partial\hbar} + \frac{\partial}{\partial\hbar}(\widetilde{w}\widetilde{y}),$$

$$R = cos(\hbar) + (B_1 + B_2 + B_3)\,cos(\hbar) + \left(B_1^2 + 2B_1B_2 + B_1 + B_2 + B_3\right)cos(\hbar)\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right)$$

$$+ \left(B_1^3 + B_1^2 + 2B_1B_2\right)cos(\hbar)\left(\frac{\tau^{2\eta}}{\Gamma(2\eta+1)}\right) + B_1^3 cos(\hbar)\left(\frac{\tau^{3\eta}}{\Gamma(3\eta+1)}\right).$$

$$R = \left((1 + B_1 + B_2 + B_3) + \left(B_1^2 + 2B_1B_2 + B_1 + B_2 + B_3\right)\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right)\right)cos(\hbar)$$

$$+ \left(\left(B_1^3 + B_1^2 + 2B_1B_2\right)\left(\frac{\tau^{2\eta}}{\Gamma(2\eta+1)}\right) + B_1^3\left(\frac{\tau^{3\eta}}{\Gamma(3\eta+1)}\right)\right)cos(\hbar).$$

Residual for $y(\hbar,\ \tau)$

$$R = \frac{\partial^{\eta}\widetilde{y}}{\partial\tau^{\eta}} - \frac{\partial^2\widetilde{y}}{\partial\hbar^2} - 2\widetilde{y}\frac{\partial\widetilde{y}}{\partial\hbar} + \frac{\partial}{\partial\hbar}(\widetilde{w}\widetilde{y}),$$

$$R = cos(\hbar) + (B_1 + B_2 + B_3)\,cos(\hbar) + \left(B_1^2 + 2B_1B_2 + B_1 + B_2 + B_3\right) cos(\hbar)\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right)$$

$$+ \left(B_1^3 + B_1^2 + 2B_1B_2\right) cos(\hbar)\left(\frac{\tau^{2\eta}}{\Gamma(2\eta+1)}\right) + B_1^3 cos(\hbar)\left(\frac{\tau^{3\eta}}{\Gamma(3\eta+1)}\right).$$

$$R = \left((1 + B_1 + B_2 + B_3) + \left(B_1^2 + 2B_1B_2 + B_1 + B_2 + B_3\right)\left(\frac{\tau^{\eta}}{\Gamma(\eta+1)}\right)\right) cos(\hbar)$$

$$+ \left(\left(B_1^3 + B_1^2 + 2B_1B_2\right)\left(\frac{\tau^{2\eta}}{\Gamma(2\eta+1)}\right) + B_1^3\left(\frac{\tau^{3\eta}}{\Gamma(3\eta+1)}\right)\right) cos(\hbar).$$

The constant values are determined using least square method $B_1 = -0.99999994,\ B_2 = 0.000000055359782$, and $B_3 = 0.000000056483867$.

By substituting the values of the auxiliary constants, we get the analytical solution for the fractional-order problem 3.3 through AHPTM. When $\eta = 1,$ the AHPTM solution of classical integer-order problem corresponding to problem 3.3 is given by:

$$\widetilde{w}(\hbar,\ \tau) = cos(\hbar)(-0.16667\tau^3 + 0.5\tau^2 - \tau + 1),$$

$$\widetilde{y}(\hbar,\ \tau) = cos(\hbar)(-0.16667\tau^3 + 0.5\tau^2 - \tau + 1).$$

The numerical results of the problem 3.3 with their errors estimates, determined by the AHPTM technique, are recorded in Tables 8 to 10. The suggested method is accurate and efficient, as shown by the minimal error values in the last columns of Tables 9 and 10. Several graphs representing different fractional solutions within the interval $0 < \eta \leq 1$ are used to illustrate the dynamic behavior of the solution to the given problem 3.3 . These details are described in the captions of Figures 11 to 15. Finally, the visual representations in Figures 11 to 15 and Tables 8 to 10 demonstrate the accuracy and convergence of the proposed technique.

Table 8. Numerical results of problem 3.3 by taking fixed value of τ and numerous values of $\hbar$ and η

$\hbar$	AHPTM	AHPTM	AHPTM	AHPTM
at $(\tau = 0.001)$	$(\eta = 0.2)$	$(\eta = 0.3)$	$(\eta = 0.4)$	$(\eta = 0.5)$
−3.1416	−0.77981	−0.87539	−0.93293	−0.96529
−2.8274	−0.74164	−0.83255	−0.88727	−0.91805
−2.5133	−0.63088	−0.70821	−0.75476	−0.78094
−2.1991	−0.45836	−0.51454	−0.54836	−0.56739
−1.885	−0.24097	−0.27051	−0.28829	−0.29829
−1.5708	$4.7749e-17$	$5.3602e-17$	$5.7126e-17$	$5.9107e-17$
−1.2566	0.24097	0.27051	0.28829	0.29829
−0.94248	0.45836	0.51454	0.54836	0.56739
−0.62832	0.63088	0.70821	0.75476	0.78094
−0.31416	0.74164	0.83255	0.88727	0.91805
0	0.77981	0.87539	0.93293	0.96529
0.31416	0.74164	0.83255	0.88727	0.91805
0.62832	0.63088	0.70821	0.75476	0.78094
0.94248	0.45836	0.51454	0.54836	0.56739
1.2566	0.24097	0.27051	0.28829	0.29829
1.5708	$4.7749e-17$	$5.3602e-17$	$5.7126e-17$	$5.9107e-17$
1.885	−0.24097	−0.27051	−0.28829	−0.29829
2.1991	−0.45836	−0.51454	−0.54836	−0.56739
2.5133	−0.63088	−0.70821	−0.75476	−0.78094
2.8274	−0.74164	−0.83255	−0.88727	−0.91805
3.1416	−0.77981	−0.87539	−0.93293	−0.96529

Table 9. The error estimate at $\tau = 0.001$ and $\eta = 1$ for $\hbar$ is provided for comparison purposes of the solution via AHPTM and exact solution to problem 3.3

$\hbar$	Exact	AHPTM	Error
$\tau = 0.001$	$(\eta = 1)$	$(\eta = 1)$	$(\eta = 1)$
−3.1416	−0.999	−0.999	$4.5075e-14$
−2.8274	−0.95011	−0.95011	$4.2855e-14$
−2.5133	−0.80821	−0.80821	$3.6415e-14$
−2.1991	−0.5872	−0.5872	$2.6423e-14$
−1.885	−0.30871	−0.30871	$1.3933e-14$
−1.5708	$6.1171e-17$	$6.1171e-17$	$2.761e-30$
−1.2566	0.30871	0.30871	$1.3933e-14$
−0.94248	0.5872	0.5872	$2.6423e-14$
−0.62832	0.80821	0.80821	$3.6415e-14$
−0.31416	0.95011	0.95011	$4.2855e-14$
0	0.999	0.999	$4.5075e-14$
0.31416	0.95011	0.95011	$4.2855e-14$
0.62832	0.80821	0.80821	$3.6415e-14$
0.94248	0.5872	0.5872	$2.6423e-14$
1.2566	0.30871	0.30871	$1.3933e-14$
1.5708	$6.1171e-17$	$6.1171e-17$	$2.761e-30$
1.885	−0.30871	−0.30871	$1.3933e-14$
2.1991	−0.5872	−0.5872	$2.6423e-14$
2.5133	−0.80821	−0.80821	$3.6415e-14$
2.8274	−0.95011	−0.95011	$4.2855e-14$
3.1416	−0.999	−0.999	$4.5075e-14$

Table 10. The absolute error at $\eta = 1$ for $(\hbar, \tau)$ is provided for comparison purposes of the solution via AHPTM and exact solution to problem 3.3

$\hbar$	τ	Exact	AHPTM	Error
		$\eta = 1$	$\eta = 1$	$\eta = 1$
-3.1416	0	-1	-1	0
-2.8274	0.0005	-0.95058	-0.95058	$2.8866e-15$
-2.5133	0.001	-0.80821	-0.80821	$3.6415e-14$
-2.1991	0.0015	-0.5869	-0.5869	$1.3056e-13$
-1.885	0.002	-0.3084	-0.3084	$2.1422e-13$
-1.5708	0.0025	$6.1079e-17$	$6.1079e-17$	$1.028e-28$
-1.2566	0.003	0.30809	0.30809	$1.0701e-12$
-0.94248	0.0035	0.58573	0.58573	$3.7567e-12$
-0.62832	0.004	0.80579	0.80579	$8.7952e-12$
-0.31416	0.0045	0.94679	0.94679	$1.6524e-11$
0	0.005	0.99501	0.99501	$2.6432e-11$
0.31416	0.0055	0.94584	0.94584	$3.6749e-11$
0.62832	0.006	0.80418	0.80418	$4.4217e-11$
0.94248	0.0065	0.58398	0.58398	$4.4199e-11$
1.2566	0.007	0.30686	0.30686	$3.1225e-11$
1.5708	0.0075	$6.0775e-17$	$6.0775e-17$	$8.1466e-27$
1.885	0.008	-0.30655	-0.30655	$5.3182e-11$
2.1991	0.0085	-0.58281	-0.58281	$1.2883e-10$
2.5133	0.009	-0.80177	-0.80177	$2.2273e-10$
2.8274	0.0095	-0.94206	-0.94206	$3.2487e-10$
3.1416	0.01	-0.99005	-0.99005	$4.1917e-10$

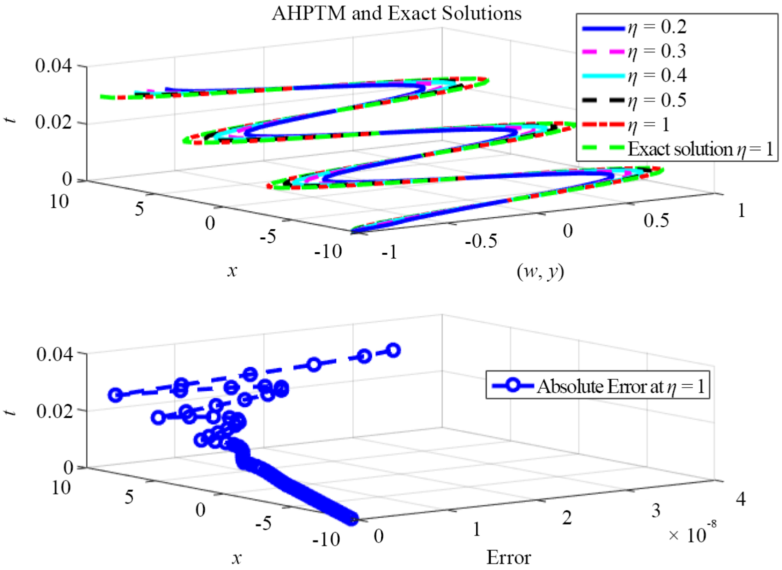

Figure 11. AHPTM and exact solutions to the problem 3.3 are compared graphically for various values of $(\hbar, \tau) = (x, t)$, and η

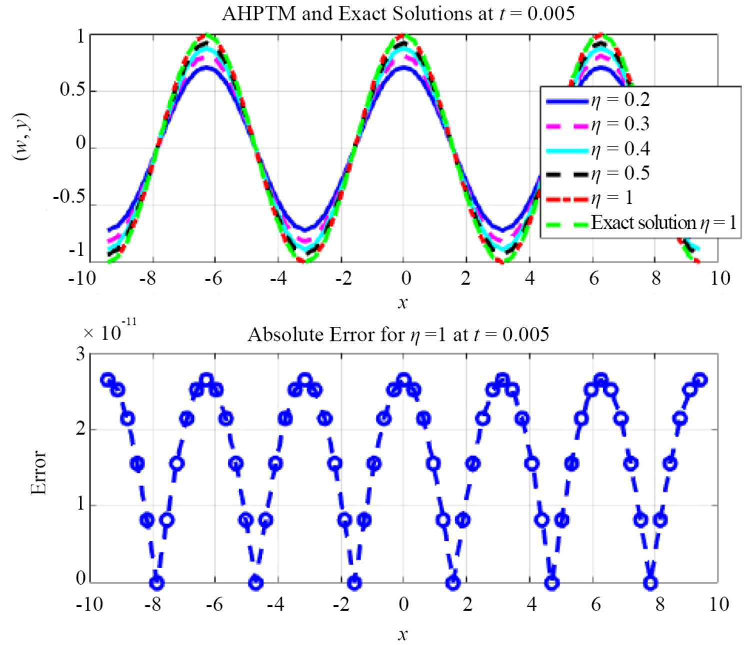

Figure 12. AHPTM and exact solutions to the problem 3.3 are compared graphically for various values of $(\hbar, \eta) = (x, \eta)$, and at $\tau = 0.005$ (as t denotes τ)

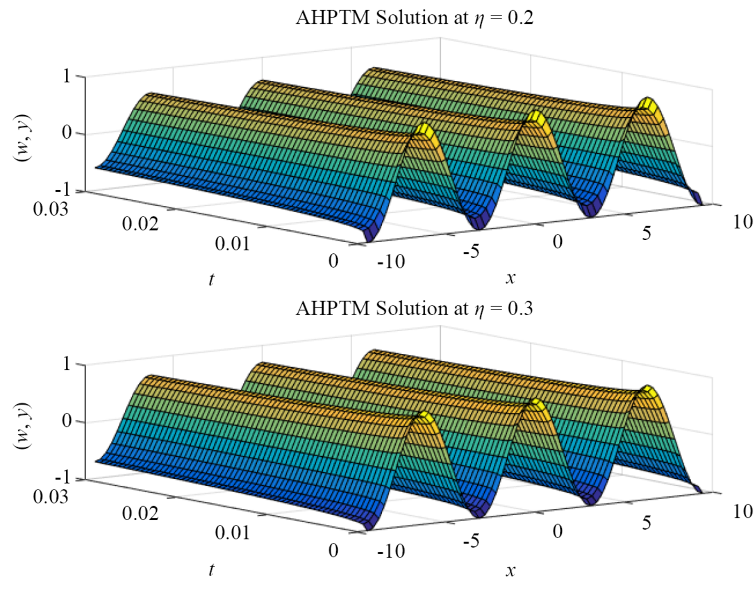

Figure 13. Surface plots at various $(\hbar, \tau) = (x, t)$, and $\eta = 0.3, 0.2$ of AHPTM's solution to the problem 3.3

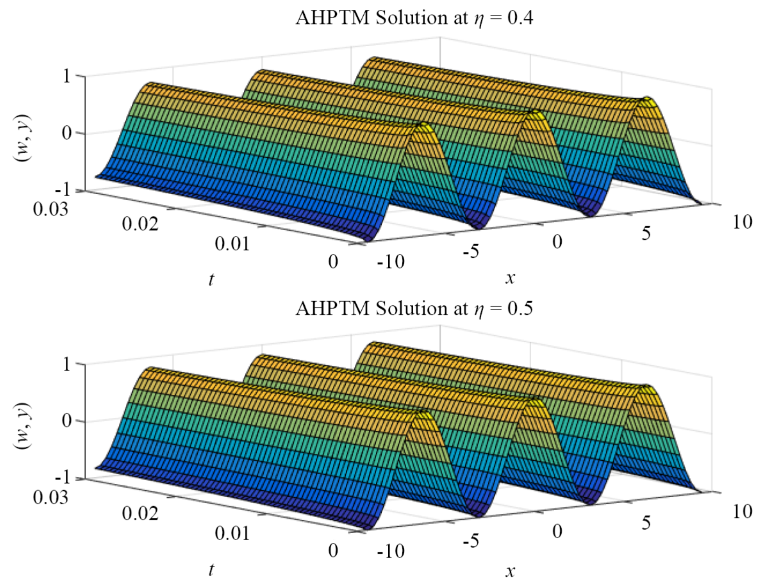

Figure 14. Surface plots at various $(\hbar,\ \tau) = (x,\ t)$, and $\eta = 0.5,\ 0.4$ of AHPTM's solution to the problem 3.3

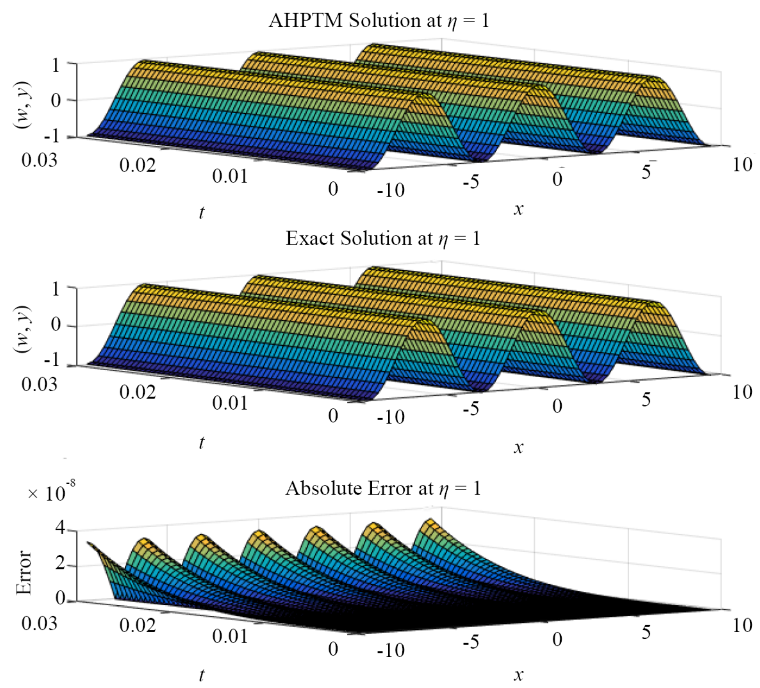

Figure 15. A comparison of solution via AHPTM with the exact solution of problem 3.3, for all $(\hbar,\ \tau) = (x,\ t)$ at fixed value $\eta = 1$.

4. Conclusion

In this research work, the asymptotic homotopy perturbation transform method is proposed to solve one-dimensional nonlinear coupled Burgers equations. Three numerical examples of one-dimensional coupled non-linear Burgers equations are examined. Focusing on the method's effectiveness and computational efficiency, AHPTM is compared with other existing methods wherein the proposed method is found to be a highly accurate scheme for solving coupled Burgers equations. No limitation of this method is observed in this work. It is also observed that AHPTM has simple construction, ease of use, and effectiveness in computation. The graphical and numerical results for the fractional-order problem serve as a testament to the accuracy and reliability of the AHPTM. Regarding AHPTM accuracy and simple implementation, the potential areas for future research are other practical and daily problems.

Regarding future research directions, AHPTM may be extended to efficiently solve fractional-order delay differential equations, time-fractional optimal control problems, stochastic differential equations, fractional stochastic processes, biomedical models, and fractional reaction-diffusion equations for simulations. The applications of AHPTM may also be investigated in quantum mechanics and statistical physics, nonlinear wave propagation, seismic wave modeling and plasma physics, etc.

Acknowledgement

Authors are thankful to Prince Sultan University for support through TAS research lab.

Conflict of interest

The authors declare no competing financial interest.

References

[1] Esipov SE. Coupled Burgers equations: A model of polydispersive sedimentation. *Physical Review E*. 1995; 52: 3711-3718. Available from: https://doi.org/10.1103/physreve.52.3711.

[2] Burgers JM. A mathematical model illustrating the theory of turbulence. *Advances in Applied Mechanics*. 1948; 1: 171-199. Available from: https://doi.org/10.1016/S0065-2156(08)70100-5.

[3] Cole JD. On a quasi-linear parabolic equations occurring in aerodynamics. *Quarterly of Applied Mathematics*. 1951; 9: 225-236. Available from: https://doi.org/10.1090/qam/42889.

[4] Kaya D. An explicit solution of coupled viscous Burgers' equations by the decomposition method. *International Journal of Mathematics and Mathematical Sciences*. 2001; 27(11): 675-680. Available from: https://doi.org/10.1155/S0161171201010249.

[5] Soliman A. The modified extended tanh-function method for solving Burgers-type equations. *Physica A: Statistical Mechanics and its Applications*. 2006; 361(2): 394-404. Available from: https://doi.org/10.1016/j.physa.2005.07.008.

[6] Abdou MA, Soliman AA. Variational iteration method for solving Burgers and coupled Burgers equations. *Journal of Computational and Applied Mathematics*. 2005; 181(2): 245-251. Available from: https://doi.org/10.1016/j.cam.2004.11.032.

[7] Wei GW, Gu Y. Conjugate filter approach for solving Burgers' equation. *Journal of Computational and Applied Mathematics*. 2002; 149(2): 439-456. Available from: https://doi.org/10.1016/S0377-0427(02)00488-0.

[8] Khater AH, Temsah RS, Hassan M. A Chebyshev spectral collocation method for solving Burgers'-type equations. *Journal of Computational and Applied Mathematics*. 2008; 222(2): 333-350. Available from: https://doi.org/10.1016/j.cam.2007.11.007.

[9] Dehghan M, Hamidi A, Shakourifar M. The solution of coupled Burgers's equations using Adomian-Pade technique. *Applied Mathematics and Computation*. 2007; 189(2): 1034-1047. Available from: https://doi.org/10.1016/j.amc.2006.11.179.

[10] Rashid A, Ismail AI. A fourier pseudo-spectral method for solving coupled viscous Burgers' equations. *Computational Methods in Applied Mathematics*. 2009; 9(4): 412-420. Available from: https://doi.org/10.2478/cmam-2009-0026.

[11] Mittal RC, Arora G. Numerical solution of the coupled viscous Burgers' equation. *Communications in Nonlinear Science and Numerical Simulation*. 2011; 16(3): 1304-1313. Available from: https://doi.org/10.1016/j.cnsns.2010.06.028.

[12] Mokhtari R, Toodar AS, Chegini NG. Application of the generalized differential quadrature method in solving Burgers' equations. *Communications in Theoretical Physics*. 2011; 56(6): 1009-1015. Available from: https://doi.org/10.1088/0253-6102/56/6/06.

[13] Kutley SE, Bahadir AR, Ozdes A. Numerical solution of one-dimensional Burgers equation: Explicit and exact-explicit finite difference methods. *Journal of Computational and Applied Mathematics*. 1999; 103(2): 251-261. Available from: https://doi.org/10.1016/S0377-0427(98)00261-1.

[14] Kadalbajoo MK, Awasthi A. A numerical method based on Crank-Nicolson scheme for Burgers' equation. *Applied Mathematics and Computation*. 2006; 182(2): 1430-1442. Available from: https://doi.org/10.1016/j.amc.2006.05.030.

[15] Jain PC, Holla DN. Numerical solution of coupled Burgers' equations. *International Journal of Non-Linear Mechanics*. 1978; 13(4): 213-222. Available from: https://doi.org/10.1016/0020-7462(78)90024-0.

[16] Fletcher CA. A comparison of finite element and finite difference of the one and two-dimensional Burgers' equations. *Journal of Computational Physics*. 1983; 51(1): 159-188. Available from: https://doi.org/10.1016/0021-9991(83)90085-2.

[17] Wubs FW, De Goede ED. An explicit-implicit method for a class of time-dependent partial differential equations. *Applied Numerical Mathematics*. 1992; 9(2): 157-181. Available from: https://doi.org/10.1016/0168-9274(92)90012-3.

[18] Goyon O. Multilevel schemes for solving unsteady equations. *International Journal for Numerical Methods in Fluids*. 1996; 22: 937-959. Available from: https://doi.org/10.1002/(SICI)1097-0363(19960530)22:10<937::AID-FLD387>3.0.CO;2-4.

[19] Bahadir AR. A fully implicit finite-difference scheme for two-dimensional Burgers' equation. *Applied Mathematics and Computation*. 2003; 137(1): 131-137. Available from: https://doi.org/10.1016/S0096-3003(02)00091-7.

[20] Srivastava VK, Awasthi MK, Singh S. An implicit logarithmic finite-difference technique for two dimensional coupled viscous Burgers' equation. *AIP Advances*. 2013; 3(12): 122105. Available from: https://doi.org/10.1063/1.4842595.

[21] Jima M, Shiferaw A, Tsegaye A. Numerical solution of the coupled viscous Burgers' equation using differential quadrature method based on fourier expansion basis. *Applied Mathematics*. 2018; 9(7): 821-835. Available from: https://doi.org/10.4236/am.2018.97057.

[22] Nazir T, Abbas M, Iqbal MK. New cubic B-spline approximation technique for numerical solutions of coupled viscous Burgers equations. *Engineering Computations*. 2020; 38(1): 83-106. Available from: https://doi.org/10.1108/EC-08-2019-0365.

[23] Prakash A, Kumar M, Sharma KK. Numerical method for solving fractional coupled Burgers equations. *Applied Mathematics and Computation*. 2015; 260(1): 314-320. Available from: https://doi.org/10.1016/j.amc.2015.03.037.

[24] Liu J, Hou G. Numerical solutions of the space- and time-fractional coupled Burgers equations by generalized differential transform method. *Applied Mathematics and Computation*. 2011; 217(16): 7001-7008. Available from: https://doi.org/10.1016/j.amc.2011.01.111.

[25] Yildirim A, Kelleci A. Homotopy perturbation method for numerical solutions of coupled Burgers equations with time- and space-fractional derivatives. *International Journal of Numerical Methods for Heat & Fluid Flow*. 2010; 20(8): 897-909. Available from: https://doi.org/10.1108/09615531011081423.

[26] Ali A, Ansari KJ, Alrabaiah H, Aloqaily A, Mlaiki N. Coupled system of fractional impulsive problem involving power-law kernel with piecewise order. *Fractal and Fractional*. 2023; 7(6): 436. Available from: https://doi.org/10.3390/fractalfract7060436.

[27] Hammad HA, Aydi H, Mlaiki N. Contributions of the fixed point technique to solve the 2D Volterra integral equations, Riemann-Liouville fractional integrals, and Atangana-Baleanu integral operators. *Advances in Difference Equations*. 2021; 2021(1): 97. Available from: https://doi.org/10.1186/s13662-021-03255-6.

[28] Shah FA, Kamran, Santina D, Mlaiki N, Aljawi S. Application of a hybrid pseudospectral method to a new two-dimensional multi-term mixed sub-diffusion and wave-diffusion equation of fractional order. *Networks and Heterogeneous Media*. 2024; 19(1): 44-85. Available from: https://doi.org/10.3934/nhm.2024003.

[29] Hussain WS, Ali S, Fatima N, Shah K, Abdeljawad T. Presentation of the efficient scheme for solving fractional order telegraph problems. *Partial Differential Equations in Applied Mathematics*. 2024; 12: 100976. Available from: https://doi.org/10.1016/j.padiff.2024.100976.

[30] Mirparizi M, Aski F. Interlayer shear effect on vibrational behavior of bilayer graphene using the molecular mechanics simulation. *Propulsion and Power Research*. 2016; 5(3): 250-260. Available from: https://doi.org/10.1016/j.jppr.2016.07.006.

[31] Almarashi A, Hussin AM, Mirparizi M, Zhang C, Saad HA. Influence of nanoparticles on freezing inside container equipped with fins. *Scientific Reports*. 2022; 12: 14792. Available from: https://doi.org/10.1038/s41598-022-18714-7.

[32] Mohammadpour M, Valipour P, Shambooli M, Ayani M, Mirparizi M. Nanofluid flow and forced convection heat transfer over a stretching surface considering heat source. *European Physical Journal Plus*. 2015; 130: 1-8. Available from: https://doi.org/10.1140/epjp/i2015-15155-8.

[33] Mirparizi M, Shariyat M, Fotuhi AR. Large deformation hermitian finite element coupled thermoelasticity analysis of wave propagation and reflection in a finite domain. *Journal of Solid Mechanics*. 2021; 13: 1-15. Available from: https://doi.org/10.22034/jsm.2020.1913430.1653.

[34] Hussain WS, Ali S, Shah K, Abdalla B, Abdeljawad T. Semi analytical scheme for the presentation of solution to Fractional Fokker-Planck Equation. *Partial Differential Equations in Applied Mathematics*. 2024; 10: 100740. Available from: https://doi.org/10.1016/j.padiff.2024.100740.

[35] Youssri YH, Atta AG. Adopted Chebyshev collocation algorithm for modeling human corneal shape via the Caputo fractional derivative. *Contemporary Mathematics*. 2025; 6(1): 1223-1238. Available from: https://doi.org/10.37256/cm.6120256387.

[36] Youssri YH, Atta AG. Chebyshev Petrov-Galerkin method for nonlinear time-fractional integro-differential equations with a mildly singular kernel. *Journal of Applied Mathematics and Computing*. 2025; 1: 1-21. Available from: https://doi.org/10.1007/s12190-025-02371-w.

[37] Sayed SM, Mohamed AS, Abo-Eldahab EM, Youssri YH. Spectral framework using modified shifted Chebyshev polynomials of the third-kind for numerical solutions of one- and two-dimensional hyperbolic telegraph equations. *Boundary Value Problems*. 2025; 2025(1): 7. Available from: https://doi.org/10.1186/s13661-024-01987-4.

[38] Youssri YH, Atta AG, Moustafa M, Abu Waar ZY. Explicit collocation algorithm for the nonlinear fractional duffing equation via third-kind Chebyshev polynomials. *Iranian Journal of Numerical Analysis and Optimization*. 2025; 1: 20. Available from: https://doi.org/10.22067/ijnao.2025.90483.1543.

[39] Taema MA, Youssri YH. Third-kind Chebyshev spectral collocation method for solving models of two interacting biological species. *Contemporary Mathematics*. 2024; 5(4): 6189-6207. Available from: https://doi.org/10.37256/cm.5420245950.

[40] Youssri YH, Atta AG. Radical Petrov-Galerkin approach for the time-fractional KdV-Burgers' equation. *Mathematical and Computational Applications*. 2024; 29(6): 107. Available from: https://doi.org/10.3390/mca29060107.

[41] Gul H, Ali S, Shah K, Muhammad S, Sitthiwirattham T, Chasreechai S. Application of asymptotic homotopy perturbation method to fractional order partial differential equation. *Symmetry*. 2021; 13(11): 2215. Available from: https://doi.org/10.3390/sym13112215.