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Research Article

On Estimation of Stress-Strength Model for Exponential Flexible Weibull Extension Distribution Based on K-Records

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Abstract: In this paper, we consider an extension of the record model, which is well known as the K-record. For the Random Variables X and Y, the stress-strength parameter $R = P(Y < X)$, and the EFWE distribution is applied, when the random variables X and Y approach the Exponential Flexible Weibull Extension (EFWE) distribution, which is an extension of the exponential flexible Weibull distribution. The behavior of stress-strength parameters and reliability have been estimated using the k-record data by using maximum likelihood and Bayesian estimators through the Monte Carlo simulation study, which showed satisfactory performance of the estimators. Finally, the stress-strength model of the EFWE distribution is utilized to analyze the actual data that shows the failure times of two distinct types of electrical insulators.

Keywords: exponential flexible weibull extension, stress-strength reliability model, maximum likelihood estimators, bayesian estimators, monte carlo simulation

1. Introduction

Stress-strength models have attracted the attention of many statisticians. In recent years, statistics has been extensively applied in various fields, including economics, quality control, and medical and engineering problems, demonstrating its versatility and relevance. Over the past ten years, numerous authors have focused on examining the use of simple stress-strength reliability models that are theoretically more manageable, easier, and more practical to implement. This reliability model involves the variables X and Y, with X representing the intensity and Y representing stress. The system functions effectively when X is greater than Y. Satisfying with $R = P(Y < X)$, which represents the reliability of the system. Let $\{X_i,\ i > 1\}$ be a sequence of continuous independent and identically distributed random variables (iid) with the cumulative distribution function (cdf) $F(x)$ and the probability density function (pdf) $f(x)$, respectively. A value X_j is considered an upper record if it is greater than all previous observations, meaning $X_j > X_i$, for every $i < j$. Similarly, a lower record value can be defined analogously. Many papers and several books on record-breaking data have

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been published today (see, *e.g.*, Chandler [1], Resnick [2], Alsadat et al. [3], Abd El-Monsef et al. [4], Shorrock [5], Glick [6], Samaniego and Whitaker [7], Arnold et al. [8] and Nevzorov [9]). The k-record process above is defined in terms of the kth largest X that is still seen. For a formal definition, the definition is described by Arnold et al. [8], in the continuous case, let $T_{1(k)} = k,\ R_{1(k)} = X_{1\,:\,k}$ and for $n \geq 2$, let $T_{1(k)} = \min\left\{ j\ :\ j > T_{n-1(k)},\ X_j > X_{T_{n-1(k)-k+1\,:\,T_{n-1(k)}}} \right\}$, where $X_{i\,:\,m}$ denotes the i-th order statistic in a sample of size m. The sequence of upper k-records is then defined by $S_{n(k)} = X_{T_{n(k)-k+1\,:\,T_{n(k)}}}$ for $n \geq 1$. Arnold et al. [8]. This is a type2 k-record sequence. Note that if $K = 1$, the normal record is recovered. Lower k-records can also be defined similarly. This series of k-records was first introduced by Dziubdziela and Kopocinski [10] and has been widely accepted in literature. Some progress has been made in statistical inference using k-records. References include Deheuvels and Nevzorov [11], Barred [12], Ali Mousa et al. [13], Malinowska and Szynal [14], Danielak and Raqab [15, 16], Ahmadi et al. [17], Fasandi and Ahmadi [18], and others. The Weibull distribution is a cornerstone of reliability modeling due to its adaptability to increasing, decreasing, or constant failure rates. The EFWE further enhances this flexibility by introducing an exponential parameterization, enabling superior fit to bathtub-shaped or multimodal hazard functions prevalent in modern engineering systems. Despite its potential, the integration of EFWE with K-record-based stress-strength models remains unexplored, leaving a gap in methods capable of balancing parametric flexibility with data efficiency. In the present paper, we focus on the estimation of $R = P(Y < X)$ for the EFWE distribution, which was introduced by Bieh et al. [19], when the data followed K-record values by using different methods. The paper is structured as follows: Section 2 presents Some properties of the EFEW. Section 3 presents the stress strength model of EFWE. Estimation studies of EFWE with various methods are derived in section 4. The results of a Monte Carlo simulation study, demonstrating the estimators' satisfactory performance in section 5. Section 6 utilized the Stress-Strength model to examine areal data regarding the failure times of two electrical insulators. The study's conclusion is presented in section 7.

The cumulative distribution function cdf of the EFWE is given by

$$F(x;\ \alpha,\ \beta,\ \lambda) = 1 - \exp\left\{ -\lambda \exp\left\{ \exp\left\{ \alpha x - \frac{\beta}{x} \right\} \right\} \right\}. \tag{1}$$

The probability density function pdf corresponding to Eq. (1) is given by

$$f(x;\ \alpha,\ \beta,\ \lambda) = \lambda \left(\alpha + \frac{\beta}{x^2} \right) \exp\left\{ \alpha x - \frac{\beta}{x} \right\} \exp\left\{ \exp\left\{ \alpha x - \frac{\beta}{x} \right\} \right\} \exp\left\{ -\lambda \exp\left\{ \exp\left\{ \alpha x - \frac{\beta}{x} \right\} \right\} \right\},$$

$$0 < \alpha < \beta < \infty,\ \lambda > 0,\ x > 0. \tag{2}$$

Equation (2) represents a probability density function that depends on three parameters: α, β, and λ. The function combines exponential and double-exponential terms, incorporating a shape component $\left(\alpha + \frac{\beta}{x^2} \right)$ and an exponent that includes nested exponential. The term $\exp\left(\alpha x - \frac{\beta}{x} \right)$ plays a crucial role in determining the distribution's behavior, and the final exponential term involving λ ensures proper scaling.

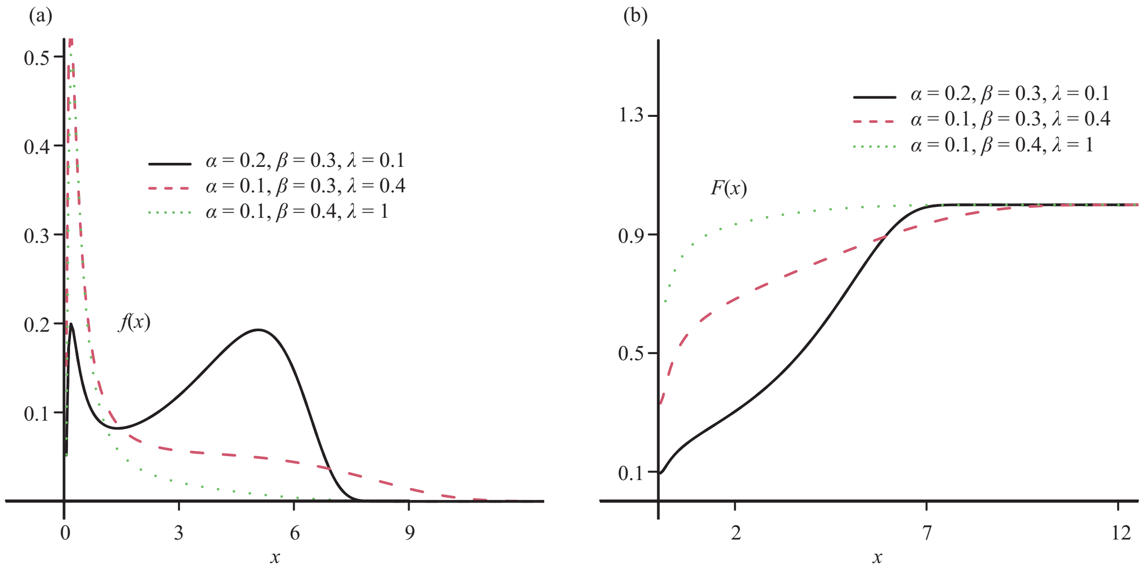

Figure 1. Plots of the probability density and the cumulative functions

Figure 1 shows the plots of the probability density and the cumulative functions with different values of the parameters.

2. Some properties of the exponential flexible Weibull extension

There are several properties can be written for understand the proposed distribution.

2.1 *The relationship between the pdf and the cdf*

The relationship between Eqs. (1) and (2), is defined as

$$F(x;\ \alpha,\ \beta,\ \lambda) = 1 - \frac{f(x;\ \alpha,\ \beta,\ \lambda)}{\lambda\left(\alpha + \frac{\beta}{x^2}\right)\exp\left\{\alpha x - \frac{\beta}{x}\right\}\exp\left\{\exp\left\{\alpha x - \frac{\beta}{x}\right\}\right\}}. \tag{3}$$

the function captures the cumulative probability up to a given x ensuring that $F(x)$ increases monotonically from 0 to 1 as x grows. The formulation allows flexibility in modeling different types of distributions depending on the parameter values.

2.2 *The moment function*

The moment function around zero can be computed using the form

$$\mu_k' = \int_0^\infty x^k f(x)dx$$

$$= \int_0^\infty \lambda x^k \left(\alpha + \frac{\beta}{x^2}\right)\exp\left\{\alpha x - \frac{\beta}{x}\right\}\exp\left\{\exp\left\{\alpha x - \frac{\beta}{x}\right\}\right\}\exp\left\{-\lambda \exp\left\{\exp\left\{\alpha x - \frac{\beta}{x}\right\}\right\}\right\}, \tag{4}$$

The moment function around μ can be computed using the form

$$\mu_k = \int_0^\infty (x-\mu)^k f(x)dx$$

$$= \int_0^\infty \lambda(x-\mu)^k \left(\alpha + \frac{\beta}{x^2}\right) \exp\left\{\alpha x - \frac{\beta}{x}\right\} \exp\left\{\exp\left\{\alpha x - \frac{\beta}{x}\right\}\right\} \exp\left\{-\lambda \exp\left\{\exp\left\{\alpha x - \frac{\beta}{x}\right\}\right\}\right\}, \tag{5}$$

where $\mu = \mu_1'$. Since the distribution is more complicated, the sum of Riemann can be used ro compute the expatiation. And the Numerical method can be used to compute the integration. The Table 1 shows the results. Note that

$$\text{Skewness} = \frac{\mu_3^2}{\mu_2^3}. \tag{6}$$

$$\text{Kurtosis} = \frac{\mu_4}{\mu_2^2}. \tag{7}$$

Table 1. The results of the computing of $E(X^K)$ for different values of parameters

Parameters	μ	Var(X)	Skewness	Kurtosis
$\alpha = 0.2, \beta = 0.3, \lambda = 0.1$	0.2571	0.3606	11.8297	14.2861
$\alpha = 0.1, \beta = 0.3, \lambda = 0.4$	0.3669	0.9785	17.1332	20.6206
$\alpha = 0.1, \beta = 0.4, \lambda = 1$	0.4194	1.1330	14.7513	17.8258

The Table 1 shows the numerical results, the probability density function is positive Skewness and it is Kurtosis, since the results of Kurtosis are larger than 3. This table is important for the parameters estimation.

3. Stress-strength reliability model

If X and Y are independent random variables distributed as $X \sim (\alpha, \beta_1, \lambda)$ and $Y \sim (\alpha, \beta_2, \lambda)$, then the reliability of the system with stress variable (Y) and strength variable (X), represented as $P(Y < X)$, can be calculated as By utilizing Eqs. (1) and (2), we obtain

$$R(x) = P(Y < X) = \int_0^\infty F_Y(x) f_X(x)dx$$

$$= \int_0^\infty 1 - \exp\left\{-\lambda_2 \exp\left\{\exp\left\{\alpha_2 x - \frac{\beta_2}{x}\right\}\right\}\right\} \lambda_1 \left(\alpha_1 + \frac{\beta_1}{x^2}\right)$$

$$\times \exp\left\{\alpha_1 x - \frac{\beta_1}{x}\right\} \exp\left\{\exp\left\{\alpha_1 x - \frac{\beta_1}{x}\right\}\right\} \exp\left\{-\lambda_1 \exp\left\{\alpha_1 x - \frac{\beta_1}{x}\right\}\right\} dx, \tag{8}$$

4. Estimation of Stress-strength reliability measure based on K records

In this section, the Stress-strength model $R = P(Y < X)$ will be estimated, when $X \sim EFWE(\alpha_1,\ \beta_1,\ \lambda_1)$ distribution and $Y \sim EFWE(\alpha_2,\ \beta_2,\ \lambda_2)$. The study focuses on MLE and non-informative Bayesian estimators. Additionally, it presents MLE and non-informative Bayesian estimator in the scenario where the parameters are all unknown.

4.1 *Maximum likelihood estimation of Stress-strength model*

In the general case Suppose further $(x_1,\ x_2,\ \ldots,\ x_m)$ is a random sample from EFWE distribution $(\alpha_1,\ \beta_1,\ \lambda_1)$ and $(y_1,\ y_2,\ \ldots,\ y_n)$ is another random sample from EFEW distribution $(\alpha_2,\ \beta_2,\ \lambda_2)$. We can find The likelihood density function has the form

$$f(x_1,\ x_2,\ \ldots,\ x_m) = z^m \prod_{i=1}^{m} \frac{f(x_i)}{1 - F(x_i)} [1 - F(x_m)]^z. \tag{9}$$

$$f(y_1,\ y_2,\ \ldots,\ y_n) = p^n \prod_{i=1}^{n} \frac{f(y_i)}{1 - F(y_i)} [1 - F(y_n)]^p. \tag{10}$$

Then, the log likelihood function of Stress-strength reliability function has the form

$$\begin{aligned}
\ell = {} & m \log(z\lambda_1) - z\lambda_1 \exp\left\{\exp\left\{\alpha_1 x_m - \frac{\beta_1}{x_m}\right\}\right\} + \sum_{i=1}^{m} \log\left(\alpha_1 + \frac{\beta_1}{x_i^2}\right) + \sum_{i=1}^{m}\left(\alpha_1 x_i - \frac{\beta_1}{x_i}\right) \\
& + \sum_{i=1}^{m} \exp\left\{\alpha_1 x_i - \frac{\beta_1}{x_i}\right\} + n\log(p\lambda_2) - p\lambda_2 \exp\left\{\exp\left\{\alpha_2 y_n - \frac{\beta_2}{y_n}\right\}\right\} + \sum_{j=1}^{n} \log\left(\alpha_2 + \frac{\beta_2}{y_j^2}\right) \\
& + \sum_{j=1}^{n}\left(\alpha_2 y_j - \frac{\beta_2}{y_j}\right) + \sum_{j=1}^{n} \exp\left\{\alpha_2 y_j - \frac{\beta_2}{y_j}\right\}.
\end{aligned} \tag{11}$$

The estimated values of $(\alpha_1,\ \beta_1,\ \lambda_1,\ \alpha_2,\ \beta_2,\ \lambda_2)$ denoted by $(\hat{\alpha}_1,\ \hat{\beta}_1,\ \hat{\lambda}_1,\ \hat{\alpha}_2,\ \hat{\beta}_2,\ \hat{\lambda}_2)$ can be derived as follows

$$\begin{aligned}
\frac{\partial \ell}{\partial \alpha_1} = {} & -z\lambda_1 x_m \exp\left\{\alpha_1 x_m - \frac{\beta_1}{x_m}\right\} \exp\left\{\exp\left\{\alpha_1 x_m - \frac{\beta_1}{x_m}\right\}\right\} + \sum_{i=1}^{m} \frac{1}{\alpha_1 + \frac{\beta_1}{x_i^2}} \\
& + \sum_{i=1}^{m} x_i + \sum_{i=1}^{m} x_i \exp\left\{\alpha_1 x_i - \frac{\beta_1}{x_i}\right\},
\end{aligned} \tag{12}$$

$$\frac{\partial \ell}{\partial \alpha_2} = -p\lambda_2 y_n \exp\left\{\alpha_2 y_n - \frac{\beta_2}{x_n}\right\}\exp\left\{\exp\left\{\alpha_2 y_n - \frac{\beta_2}{y_n}\right\}\right\} + \sum_{j=1}^{n} \frac{1}{\alpha_2 + \frac{\beta_2}{y_j^2}}$$

$$+\sum_{j=1}^{n} y_j + \sum_{j=1}^{n} y_j \exp\left\{\alpha_2 y_j - \frac{\beta_2}{y_j}\right\}, \tag{13}$$

$$\frac{\partial \ell}{\partial \beta_1} = \frac{z\lambda_1 \exp\left\{\alpha_1 x_m - \frac{\beta_1}{x_m}\right\}\exp\left\{\exp\left\{\alpha_1 x_m - \frac{\beta_1}{x_m}\right\}\right\}}{x_m} + \sum_{i=1}^{m} \frac{\frac{1}{x_i^2}}{\alpha_1 + \frac{\beta_1}{x_i^2}}$$

$$-\sum_{i=1}^{m} \frac{1}{x_i} - \sum_{i=1}^{m} \frac{1}{x_i} \exp\left\{\alpha_1 x_i - \frac{\beta_1}{x_i}\right\}, \tag{14}$$

$$\frac{\partial \ell}{\partial \beta_2} = \frac{p\lambda_2 \exp\left\{\alpha_2 y_n - \frac{\beta_2}{y_n}\right\}\exp\left\{\exp\left\{\alpha_2 y_n - \frac{\beta_2}{y_n}\right\}\right\}}{y_n} + \sum_{j=1}^{n} \frac{\frac{1}{y_j^2}}{\alpha_2 + \frac{\beta_2}{y_j^2}}$$

$$-\sum_{j=1}^{n} \frac{1}{y_j} - \sum_{j=1}^{n} \frac{1}{y_j} \exp\left\{\alpha_2 y_j - \frac{\beta_2}{y_j}\right\}, \tag{15}$$

$$\frac{\partial \ell}{\partial \lambda_1} = \frac{m}{\lambda_1} - z\exp\left\{\exp\left\{\alpha_1 x_m - \frac{\beta_1}{x_m}\right\}\right\}, \tag{16}$$

and

$$\frac{\partial \ell}{\partial \lambda_2} = \frac{n}{\lambda_2} - p\exp\left\{\exp\left\{\alpha_2 y_n - \frac{\beta_2}{y_n}\right\}\right\} \tag{17}$$

The solution of equations (12)-(17) is not possible in closed form, so a numerical technique is needed to solve it and substitute $\alpha_1,\ \alpha_2,\ \beta_1,\ \beta_2,\ \lambda_1,\ \lambda_2$ in (8).

4.2 *Maximum likelihood estimation of Stress-strength model with common unknown parameters*

The Single Maximum Likelihood Perimeter (SML) will implement for estimating R. Suppose that $(x_1,\ x_2,\ \ldots,\ x_m)$ and $(y_1,\ y_2,\ \ldots,\ y_n)$ are random samples from EFWE distribution $(\alpha,\ \beta,\ \lambda)$. The likelihood density function has the form

$$\ell = m\log(z\lambda) - z\lambda \exp\left\{\exp\left\{\alpha x_m - \frac{\beta}{x_m}\right\}\right\} + \sum_{i=1}^{m}\log\left(\alpha + \frac{\beta}{x_i^2}\right) + \sum_{i=1}^{m}\left(\alpha x_i - \frac{\beta}{x_i}\right)$$

$$+ \sum_{i=1}^{m}\exp\left\{\alpha x_i - \frac{\beta}{x_i}\right\} + n\log(p\lambda) - p\lambda \exp\left\{\exp\left\{\alpha y_n - \frac{\beta}{y_n}\right\}\right\} + \sum_{j=1}^{n}\log\left(\alpha + \frac{\beta}{y_j^2}\right)$$

$$+ \sum_{j=1}^{n}\left(\alpha y_j - \frac{\beta}{y_j}\right) + \sum_{j=1}^{n}\exp\left\{\alpha y_j - \frac{\beta}{y_j}\right\}. \tag{18}$$

The estimated values of $(\alpha,\ \beta,\ \lambda)$ denoted by $(\hat{\alpha},\ \hat{\beta},\ \hat{\lambda})$ can be derived as follows

$$\frac{\partial \ell}{\partial \alpha} = -z\lambda x_m \exp\left\{\alpha x_m - \frac{\beta}{x_m}\right\}\exp\left\{\exp\left\{\alpha x_m - \frac{\beta}{x_m}\right\}\right\} + \sum_{i=1}^{m}\frac{1}{\alpha + \frac{\beta}{x_i^2}} + \sum_{i=1}^{m} x_i$$

$$+ \sum_{i=1}^{m} x_i \exp\left\{\alpha x_i - \frac{\beta}{x_i}\right\} - p\lambda y_n \exp\left\{\alpha y_n - \frac{\beta}{y_n}\right\}\exp\left\{\exp\left\{\alpha y_n - \frac{\beta}{y_n}\right\}\right\}$$

$$+ \sum_{j=1}^{n}\frac{1}{\alpha + \frac{\beta}{y_j^2}} + \sum_{j=1}^{n} y_j + \sum_{j=1}^{n} y_j \exp\left\{\alpha y_j - \frac{\beta}{y_j}\right\}, \tag{19}$$

$$\frac{\partial \ell}{\partial \beta} = \frac{z\lambda x_m \exp\left\{\alpha x_m - \frac{\beta}{x_m}\right\}\exp\left\{\exp\left\{\alpha x_m - \frac{\beta}{x_m}\right\}\right\}}{x_m} + \sum_{i=1}^{m}\frac{\frac{1}{x_i^2}}{\alpha + \frac{\beta}{x_i^2}} - \sum_{i=1}^{m}\frac{1}{x_i}$$

$$- \sum_{i=1}^{m}\frac{1}{x_i}\exp\left\{\alpha x_i - \frac{\beta}{x_i}\right\} + \frac{p\lambda y_n \exp\left\{\alpha y_n - \frac{\beta}{y_n}\right\}\exp\left\{\exp\left\{\alpha y_n - \frac{\beta}{y_n}\right\}\right\}}{y_n}$$

$$+ \sum_{j=1}^{n}\frac{\frac{1}{y_j^2}}{\alpha + \frac{\beta}{y_j^2}} - \sum_{j=1}^{n}\frac{1}{y_j} - \sum_{j=1}^{n}\frac{1}{y_j}\exp\left\{\alpha y_j - \frac{\beta}{y_j}\right\}, \tag{20}$$

and

$$\frac{\partial \ell}{\partial \lambda} = \frac{m}{\lambda} - z\exp\left\{\exp\left\{\alpha x_m - \frac{\beta}{x_m}\right\}\right\} + \frac{n}{\lambda} - p\exp\left\{\exp\left\{\alpha y_n - \frac{\beta}{y_n}\right\}\right\}. \tag{21}$$

The solution of equations (19)-(21) is not possible in closed form, so numerical technique is needed to solve it and substitute $\alpha,\ \beta,\ \lambda$ in (8).

4.3 *Bayesian estimation of Stress-strength model in the general case*

The Double Bayesian Estimation Perimeter (DBE) will implement for estimating R. The non-informative prior information of $\mathbf{\Omega} = (\alpha_1,\ \beta_1,\ \lambda_1,\ \alpha_2,\ \beta_2,\ \lambda_2)$, is given by

$$f(\mathbf{\Omega}) \propto \frac{1}{\prod_{i=1}^{6} \Omega_i},\ \Omega_i > 0,\ i = 1,\ 2,\ 3,\ 4,\ 5,\ 6, \tag{22}$$

Combining the joint non informative prior density and the log-likelihood function to obtain the joint posterior density of $\mathbf{\Omega}$ as the form

$$\pi(\mathbf{\Omega}|X,\ Y) = \frac{f(\mathbf{\Omega})\ell}{K} \tag{23}$$

where $K = \prod_{i=1}^{6} \int_{\Omega_i} f(\mathbf{\Omega})\ell d\Omega_i$. Then, the posterior density of $\mathbf{\Omega}$ can be written as

$$\begin{aligned}\pi(\mathbf{\Omega}|X,\ Y) = \frac{K^{-1}}{\prod_{i=1}^{6} \Omega_i} \Bigg[& m\log(z\lambda_1) - z\lambda_1 \exp\left\{\exp\left\{\alpha_1 x_m - \frac{\beta_1}{x_m}\right\}\right\} + \sum_{i=1}^{m} \log\left(\alpha_1 + \frac{\beta_1}{x_i^2}\right) \\ & + \sum_{i=1}^{m}\left(\alpha_1 x_i - \frac{\beta_1}{x_i}\right) + \sum_{i=1}^{m} \exp\left\{\alpha_1 x_i - \frac{\beta_1}{x_i}\right\} + n\log(p\lambda_2) - p\lambda_2 \exp\left\{\exp\left\{\alpha_2 y_n - \frac{\beta_2}{y_n}\right\}\right\} \\ & + \sum_{j=1}^{n} \log\left(\alpha_2 + \frac{\beta_2}{y_j^2}\right) + \sum_{j=1}^{n}\left(\alpha_2 y_j - \frac{\beta_2}{y_j}\right) + \sum_{j=1}^{n} \exp\left\{\alpha_2 y_j - \frac{\beta_2}{y_j}\right\}\Bigg].\end{aligned} \tag{24}$$

Then, the estimated parameters are given by

$$E(\hat{\Omega}_i) = \prod_{j\neq i} \int_{\Omega_j} \Omega_i \pi(\mathbf{\Omega}|X,\ Y) d\Omega_j,\ i \neq j,\ i = j = 1,\ 2,\ 3,\ 4,\ 5,\ 6. \tag{25}$$

4.4 *Bayesian estimation of Stress-strength model with common unknown parameters*

The Single Bayesian Estimation Perimeter (SBE) will implement for estimating R. The non-informative prior information of $\mathbf{\Theta} = (\alpha,\ \beta,\ \lambda)$ is given by

$$f(\mathbf{\Theta}) \propto \frac{1}{\prod_{i=1}^{3} \Theta_i},\ \Theta_i > 0. \tag{26}$$

By combining the joint non-informative prior density with the log-likelihood function, we can derive the joint posterior density of $\mathbf{\Theta}$ in the following form

$$\pi(\boldsymbol{\Theta}|Y,\ X) = \frac{f(\boldsymbol{\Theta})\ell}{\prod_{i=1}^{3}\Theta_i \int_{\Theta_i} f(\boldsymbol{\Theta})d\Theta_i}$$

$$= \frac{1}{\prod_{i=1}^{3}\Theta_i \int_{\Theta_i} f(\boldsymbol{\Theta})d\Theta_i}\left[m\log(z\lambda) - z\lambda\exp\left\{\exp\left\{\alpha x_m - \frac{\beta}{x_m}\right\}\right\}\right.$$

$$+\sum_{i=1}^{m}\log\left(\alpha + \frac{\beta}{x_i^2}\right) + \sum_{i=1}^{m}\left(\alpha x_i - \frac{\beta}{x_i}\right) + \sum_{i=1}^{m}\exp\left\{\alpha x_i - \frac{\beta}{x_i}\right\}$$

$$+n\log(p\lambda) - p\lambda\exp\left\{\exp\left\{\alpha y_n - \frac{\beta}{y_n}\right\}\right\} + \sum_{j=1}^{n}\log\left(\alpha + \frac{\beta}{y_j^2}\right)$$

$$\left.+\sum_{j=1}^{n}\left(\alpha y_j - \frac{\beta}{y_j}\right) + \sum_{j=1}^{n}\exp\left\{\alpha y_j - \frac{\beta}{y_j}\right\}\right]. \tag{27}$$

Then, the estimated parameters are given by

$$E(\hat{\Theta}_i) = \prod_{j\neq i}\int_{\Theta_j}\Theta_i\pi(\boldsymbol{\Theta}|X,\ Y)d\Theta_j,\ i\neq j,\ i=j=1,\ 2,\ 3. \tag{28}$$

5. Simulation study

In this study, we aimed to compare the performance of four different reliability estimation methods: $\hat{R}SML$ (Maximum Likelihood Estimation), $\hat{R}DML$ (Dual Maximum Likelihood Estimation), $\hat{R}SBE$ (Bayesian Estimation), and $\hat{R}DBE$ (Dual Bayesian Estimation). To evaluate their performance, we computed the estimated reliability values using both Maximum Likelihood Estimation (MLE) and Bayes procedures, based on the parameters $\alpha_1,\ \beta_1,\ \lambda_1,\ \alpha_2,\ \beta_2,$ and λ_2. The detailed steps followed in the analysis are outlined below: Procedure Sample Creation: A sample of the highest values was created by following specific steps and adhering to predefined parameters. This ensured that the samples represented a range of extreme values that would allow for robust testing of the estimation methods. Parameter Sets: The sets of parameters were selected such that they satisfied the condition of common values. Specifically, we used $(\alpha_1=\beta_1,\ \lambda_1=\alpha_2,\ \beta_2=\lambda_2)$. This condition allowed us to study the effects of symmetry in the model and its impact on the estimators' performance. True Reliability Calculation: The true values of the reliability R were calculated using the given values of the parameters $\alpha_1,\ \beta_1,\ \lambda_1,\ \alpha_2,\ \beta_2$, and λ_2. These true values served as a benchmark for comparing the accuracy of the estimated reliability. Parameter Pairs for Stress and Strength: The selected pairs of maximum stress and strength random variables were chosen as follows: (5, 5), (15, 15), (50, 50), and (100, 100). These pairs represent different levels of stress and strength combinations, enabling a diverse set of scenarios for estimation. Repetition of Calculations: To ensure the robustness of the results, all calculations were repeated 10, 000 times. This repetition allowed for a reliable assessment of the bias and variability of the estimators across multiple trials. Variation in Parameters: The calculations were then repeated with varying values of the parameters $\alpha_1,\ \beta_1,\ \lambda_1,\ \alpha_2,\ \beta_2$, and λ_2. This variation enabled a comprehensive evaluation of the performance of the estimators across different parameter configurations. The Maximum Likelihood Estimations of the reliability, denoted as $\hat{R}SML$ and $\hat{R}DML$, were obtained and summarized in Tables 2 and 3. These tables provide the estimated reliability values for the different parameter sets, sample sizes, and estimation methods.

Bayesian Estimation: Similarly, the Bayesian estimations of the reliability, denoted as $\hat{R}SBE$ and $\hat{R}DBE$, were computed and presented in Tables 4 and 5. These estimations were obtained using Bayes' procedure with priors on the parameters.

Table 2. The results using single Maximum likelihood

Exact R	(n, m)	$\hat{R}_{SML}$	BIAS	MSE
$\alpha = \alpha_1 = \alpha_2 = 0.00001$, $\beta = \beta_1 = \beta_2 = 0.001$ and $\lambda = \lambda_1 = \lambda_2 = 1$				
0.901	(5, 5)	0.925	0.022	0.0004929
	(15, 15)	0.903	-0.00005827	0.000000003396
	(50, 50)	0.924	0.021	0.0004810
	(100, 100)	0.900	-0.002609	0.000006808
$\alpha = \alpha_1 = \alpha_2 = 0.2$, $\beta = \beta_1 = \beta_2 = 0.3$ and $\lambda = \lambda_1 = \lambda_2 = 0.1$				
Exact R	(n, m)	$\hat{R}_{SML}$	BIAS	MSE
0.826	(5, 5)	0.867	0.042	0.001727
	(15, 15)	0.873	0.048	0.002274
	(50, 50)	0.800	0.026	0.0006513
	(100, 100)	0.790	0.006	0.0007223
$\alpha = \alpha_1 = \alpha_2 = 0.5$, $\beta = \beta_1 = \beta_2 = 0.4$ and $\lambda = \lambda_1 = \lambda_2 = 0.5$				
Exact R	(n, m)	$\hat{R}_{SML}$	BIAS	MSE
0.856	(5, 5)	0.973	0.117	0.014
	(15, 15)	0.915	0.059	0.003465
	(50, 50)	0.911	0.055	0.003037
	(100, 100)	0.800	0.056	0.003137

Table 3. The results using double Maximum likelihood

Exact R	(n, m)	$\hat{R}_{DML}$	BIAS	MSE
$\alpha_1 = 0.2,\ \beta_1 = 0.1,\ \lambda_1 = 1,\ \alpha_2 = 0.4,\ \beta_2 = 0.4,\ \lambda_2 = 0.5$				
0.854	(5, 5)	0.928	0.074	0.005445
	(15, 15)	0.915	0.061	0.00366
	(50, 50)	0.800	0.054	0.002946
	(100, 100)	0.830	0.024	0.000567
Exact R	(n, m)	$\hat{R}_{DML}$	BIAS	MSE
$\alpha_1 = 0.4,\ \beta_1 = 0.4,\ \lambda_1 = 0.5,\ \alpha_2 = 0.2,\ \beta_2 = 0.1,\ \lambda_2 = 1$				
0.940	(5, 5)	0.900	0.040	0.00161
	(15, 15)	0.964	0.0241	0.0005758
	(50, 50)	0.955	0.015	0.000225
	(100, 100)	0.949	0.009	0.000081
$\alpha_1 = 0.5,\ \beta_1 = 0.6,\ \lambda_1 = 0.7,\ \alpha_2 = 0.002,\ \beta_2 = 0.001,\ \lambda_2 = 1$				
Exact R	(n, m)	$\hat{R}_{DML}$	BIAS	MSE
0.948	(5, 5)	0.957	0.009654	0.00009319
	(15, 15)	0.900	0.048	0.002289
	(50, 50)	0.956	0.008422	0.00007094
	(100, 100)	0.950	0.002	0.00000411

The results were evaluated using two key performance metrics: bias and Mean Squared Error (MSE) of the estimated reliability values. These metrics were calculated for both Maximum Likelihood Estimation (MLE) and Bayesian Estimation (BE). A summary of the findings from the tables is as follows: Bias and MSE Behavior: As the sample size increased, both the bias and MSE of the estimates decreased across all methods. This is consistent with the general statistical principle that larger sample sizes tend to yield more accurate estimates. In all cases, the performance of the Bayes estimators ($\hat{R}SBE$ and $\hat{R}DBE$) was consistently superior to that of the corresponding MLE estimators ($\hat{R}SML$ and $\hat{R}DML$). The Bayes methods exhibited lower bias and MSE, indicating that they provided more accurate and stable reliability estimates, especially as the sample size grew.

Table 4. The results using single Bayesian estimation

$\alpha = \alpha_1 = \alpha_2 = 0.00001,\ \beta = \beta_1 = \beta_2 = 0.001,\ \lambda = \lambda_1 = \lambda_2 = 1$				
Exact R	$(n,\ m)$	$\hat{R}_{SBE}$	BIAS	MSE
0.901	(5, 5)	0.954	0.053	0.002809
	(15, 15)	0.942	0.041	0.002809
	(50, 50)	0.821	0.121	0.015
	(100, 100)	0.915	0.027	0.000729
$\alpha = \alpha_1 = \alpha_2 = 0.2,\ \beta = \beta_1 = \beta_2 = 0.3,\ \lambda = \lambda_1 = \lambda_2 = 0.1$				
Exact R	$(n,\ m)$	$\hat{R}_{SBE}$	BIAS	MSE
0.826	(5, 5)	0.877	0.051	0.002601
	(15, 15)	0.853	0.027	0.000729
	(50, 50)	0.842	0.016	0.000256
	(100, 100)	0.830	0.0004	0.000016
$\alpha = \alpha_1 = \alpha_2 = 0.5,\ \beta = \beta_1 = \beta_2 = 0.4,\ \lambda = \lambda_1 = \lambda_2 = 0.5$				
Exact R	$(n,\ m)$	$\hat{R}_{SBE}$	BIAS	MSE
0.856	(5, 5)	0.875	0.019	0.000361
	(15, 15)	0.965	0.109	0.012
	(50, 50)	0.859	0.0003	0.0000009
	(100, 100)	0.854	0.0002	0.0000004

Table 5. The results using double Bayesian estimation

$\alpha_1 = 0.2,\ \beta_1 = 0.1,\ \lambda_1 = 1,\ \alpha_2 = 0.4,\ \beta_2 = 0.4,\ \lambda_2 = 0.5$				
Exact R	$(n,\ m)$	$\hat{R}_{DBE}$	BIAS	MSE
0.854	(5, 5)	0.897	0.043	0.001849
	(15, 15)	0.882	0.028	0.000784
	(50, 50)	0.921	0.067	0.004489
	(100, 100)	0.866	0.012	0.000144
$\alpha_1 = 0.4,\ \beta_1 = 0.4,\ \lambda_1 = 0.5,\ \alpha_2 = 0.2,\ \beta_2 = 0.1,\ \lambda_2 = 1$				
Exact R	$(n,\ m)$	$\hat{R}_{DBE}$	BIAS	MSE
0.94	(5, 5)	0.999	0.059	0.003481
	(15, 15)	0.976	0.036	0.001296
	(50, 50)	0.956	0.016	0.000256
	(100, 100)	0.945	0.0005	0.000025
$\alpha_1 = 0.5,\ \beta_1 = 0.6,\ \lambda_1 = 0.7,\ \alpha_2 = 0.002,\ \beta_2 = 0.001,\ \lambda_2 = 1$				
Exact R	$(n,\ m)$	$\hat{R}_{DBE}$	BIAS	MSE
0.948	(5, 5)	0.934	0.014	0.000196
	(15, 15)	0.968	0.020	0.00004
	(50, 50)	0.956	0.0008	0.000064
	(100, 100)	0.939	0.0009	0.000081

6. Application for stress strength model (electrical insulations)

In two different experiments, the failure time of electrical insulation was measured in seconds as the insulation was exposed to continuously increasing voltage pressure. A total of 25 different electrical insulation samples of each type were tested and their results were recorded-see Table 6.

Table 6. The failure times of the different electrical insulation

		X					Y		
2.1	3.4	3.2	2.3	3.8	4.2	2.6	1.7	3.2	2.3
3.2	7.4	6.1	4.9	5.7	2.4	3.5	4.5	3.7	1.3
4.3	3.8	4.3	5.4	3.7	3.7	7.5	7.8	5.4	4.5
5.2	2.4	4.4	3.4	2.5	5.4	3.6	3.6	3.4	8.6
6.2	1.7	8.3	2.5	4.5	7.5	7.5	3.2	7.6	4.6

Table 7 presents the MLE and Bayesian estimates of R for this data. It can be inferred that the Bayesian estimator outperforms the MLE for this data set.

Table 7. Point estimates of R by MLE and BE

Method	MLE	BE
Bais	0.013	0.0015
MSE	0.00821	0.0009
$\hat{R}$	0.8937	0.9639

7. Conclusion

In this paper, we introduced Maximum Likelihood Estimation (MLE) and Bayesian Estimation (BE) methods for estimating parameters in various scenarios where both X and Y are assumed to follow the exponential flexible Weibull extension distribution. A simulation study was conducted to evaluate these methods. Moreover, we have examined two actual data sets to demonstrate the practicality of the exponential flexible Weibull extension distribution. It was observed that this model offers a superior fit to the data. Both simulation results and real-world applications indicate that the Bayesian estimator outperforms the maximum likelihood estimator across various sample sizes. Additionally, the maximum likelihood method yields satisfactory results as the sample size increases.

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Availability of data and materials

All data exists in the paper with its related reference.

Contribution statement

The authors contribute equally to this paper.

Conflict of interest

The authors state no conflict of interest.

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