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Transmutation of Distributions Using Generalized and Dual Generalized Order Statistics

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Abstract: Probability distributions are widespread across various fields, such as economics, science, and engineering. These distributions offer a way to represent complex occurrences in everyday life. The increasing complexity of these occurrences has led researchers to develop new techniques for creating probability distributions. This paper introduces several generalized methods for transforming probability distributions using generalized and dual generalized order statistics. These methods yield transmuted distributions and record-based transmuted distributions in particular. Important properties of these methods, including moments, quantiles, hazard rate, and entropy, have been investigated. The paper also delves into discussing maximum likelihood estimation of the parameters. The proposed generalized transmuted distributions have been analyzed based on the Weibull distribution as a baseline probability distribution, and real-data applications have also been included.

***Keywords*:** transmutation, generalized order statistics, dual generalized order statistics, moments, entropy, maximum likelihood estimations, Weibull distribution

MSC: 65L05, 34K06, 34K28

Abbreviation

GOS	Generalized Order Statistics
DGOS	Dual Generalized Order Statistics
GOSTWD	Generalized Order Statistics Transmuted Weibull Distribution
DGOSTWD	Dual Generalized Order Statistics Transmuted Weibull Distribution

1. Introduction

The probability distribution is crucial for understanding various life phenomena. Traditional probability models are being augmented with more adaptable distribution families to account for increasingly intricate data. Incorporating additional parameters into a given distribution is a recognized approach to creating more versatile new families of distributions. Shaw and Buckley [1] presented an innovative technique for introducing new parameters to a base

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distribution that enhances distributional flexibility, referred to as the quadratic rank transmuted distribution, for any baseline distribution $G(x)$ defined as follows:

$$F(x) = (1+\lambda)G(x) - \lambda G^2(x),\ \lambda \in [-1,\ 1]. \tag{1}$$

To develop the quadratic ranking transmutation map [2] focused on the first two-order statistics and subsequently expanded it to create a cubic ranking transmutation map. Significant research has been conducted on the transmuted family of distributions, as demonstrated in studies such as [3–6]. In [7], a family of distributions based on transmuted records was developed by applying a weighted sum of the distributions of the first two record values, following the theory of record values. They incorporated both upper and lower record values. They utilized a similar approach to represent the family of quadratic rank transmuted distributions of order 2, drawing from the distribution of the first upper record values of the base distribution $G(x)$. The density function for this transmuted family is as follows:

$$f_{R_2}(x) = g(x)[\pi + (1-\pi)\{-\ln(1-G(x))\}]. \tag{2}$$

In a similar manner, they obtained a dual record-based transmuted family of distributions of order two by using lower record values, as illustrated below:

$$f^*_{R_2}(x) = g(x)[\pi + (1-\pi)\{-\ln(G(x))\}]. \tag{3}$$

Various studies have been developed based on a record-based transmuted family of distributions; for instance, see [8–10].

2. Motivation and significance of the research

Probability distributions play a vital role in modeling certain random phenomena. In today's challenging world, more and more complex data appears, and more flexible probability distributions are needed to model these complex phenomena. Traditional distributions may not adequately capture the intricate structures found in real-world data, such as skewness, heavy tails, or dependence between observations. In this research, we have proposed a new family of distributions using a linear combination of generalized and dual generalized order statistics. The proposed family of distributions generalizes the transmuted family of distributions by [1] and the record-based transmuted distributions by [7]. These earlier families have provided valuable frameworks for modeling, but our approach introduces greater flexibility by leveraging order statistics, which enhances the ability to model extreme values and other complex data features. The proposed families of distributions are general in nature and can be used to obtain certain new families of distributions for different other choices of the parameters. This versatility allows researchers to tailor the distributions to various applications, such as reliability analysis, survival studies, and risk management. Furthermore, the theoretical properties of these distributions, including moments and survival functions, offer deeper insights into the behavior of complex phenomena.

3. Generalized and dual generalized order statistics

In this section, we have a brief discussion of generalized and dual generalized order statistics.

3.1 *The generalized order statistics*

The Generalized Order Statistics (GOS) provides a comprehensive framework for ordered random variables, as introduced by [11]. The probability density function for the rth GOS is described as follows:

$$f_{r\,:\,n,\,m,\,k}(x) = \frac{C_{r-1}}{(r-1)!} f(x)\,[1-F(x)]^{\gamma_r-1}\, g_m^{r-1}[F(x)]\,, \tag{4}$$

where, $C_{r-1} = \prod_{j=1}^{r} \gamma_r,\ r = 1,\ 2,\ 3,\ \ldots,\ n,\ \gamma_r = k + (n-r)(m+1)$ such that $\gamma_r \geq 1$ for all $r \in 1,\ 2,\ \ldots,\ n-1$, and $\gamma_n = k$. Also, on the unit interval $[0,\ 1)$ the function $g_m(x)$ is defined as:

$$g_m(x) = \begin{cases} \dfrac{1}{m+1}\left[1-(1-x)^{m+1}\right] & \text{if } m \neq -1; \\[2ex] -\ln(1-x) & \text{if } m = -1. \end{cases}$$

The distribution function of rth (GOS) is given by the following:

$$F_{X(r\,:\,n,\,m,\,k)}(x) = I_{\alpha[F(x)]}\left(r,\ \frac{\gamma_r}{m+1}\right),\ m \neq -1, \tag{5}$$

where, $I_{\alpha[F(x)]}$ is incomplete beta function ratio and $\alpha[F(x)] = 1 - \{1-F(x)\}^{m+1}$.

3.2 *The dual generalized order statistics*

The concept of Dual Generalized Order Statistics (DGOS) was proposed by [12] as a comprehensive framework for random variables sorted in reverse order. The probability density function for the rth DGOS is expressed as:

$$f_{r(d)\,:\,n,\,m,\,k}(x) = \frac{C_{r-1}}{(r-1)!} f(x)\,[F(x)]^{\gamma_r-1}\, g_m^{*r-1}[F(x)]\,, \tag{6}$$

where the function $g_m^*(x)$ is defined as:

$$g_m^*(x) = \begin{cases} \left(1-x^{m+1}\right)/(m+1)\,; & m \neq -1; \\ -\ln x; & m = -1. \end{cases}$$

The distribution function of rth DGOS is given as:

$$F_{r(d)\,:\,n,\,m,\,k}(x) = I_{\alpha^*[F(x)]}\left(\frac{\gamma_r}{m+1},\ r\right),\ m \neq -1, \tag{7}$$

where, $I_{\alpha^*[F(x)]}$ is the incomplete beta function ratio and $\alpha^*[F(x)] = \{F(x)\}^{m+1}$.

For further information on generalized and dual generalized ordered random variables, refer to [13].

4. Transmutation using generalized and dual generalized order statistics

The subsequent section introduces the Generalized Transmuted Families of Distributions through the use of (GOS) and (DGOS).

4.1 *Generalized order statistics transmuted family of distributions*

The cumulative distribution function of the proposed (GOS) family is given in the following theorem:

Theorem 1 Suppose that we have a sample of size two from a distribution with CDF $F(x)$ and pdf $f(x)$. Let $X_{1\,:\,2,\,m,\,k} = Min_{GOS}(X_1,\ X_2)$ and $X_{2\,:\,2,\,m,\,k} = Max_{GOS}(X_1,\ X_2)$ be the smallest GOS and the largest GOS, respectively. Then, the (GOS) transmutation map is given by:

$$F_Y(x) = 1 - \frac{k+m+1}{2(m+1)} \left\{ (1-\lambda)\,[1-F(x)]^k + \left(\lambda - \frac{k-m-1}{k+m+1} \right) [1-F(x)]^{k+m+1} \right\}, \tag{8}$$

with $\lambda \in [-1,\ 1]$, $k \geq 1$, and $m \neq -1$.

Proof. let X_1 and X_2 be independent and identically random variables with distribution $F(x)$. Now consider:

$X_{1\,:\,2,\,m,\,k} = Min_{GOS}(X_1,\ X_2)$ and $X_{2\,:\,2,\,m,\,k} = Max_{GOS}(X_1,\ X_2)$ are smallest GOS largest GOS respectively. The pdf of the smallest (1st) GOS:

$$f_{1\,:\,2,\,m,\,k}(x) = (k+m+1) f(x) [1-F(x)]^{k+m}.$$

The pdf of the largest GOS:

$$f_{2\,:\,2,\,m,\,k}(x) = \frac{k(k+m+1)}{(m+1)} f(x)\ [1-F(x)]^{k-1}\ [1 - \{1-F(x)\}^{m+1}].$$

The corresponding CDF_s:

$$= 1 - [1-F(x)]^{k+m+1}. \tag{9}$$

and

$$= 1 - \left(\frac{k+m+1}{m+1} \right) [1-F(x)]^k + \left(\frac{k}{m+1} \right) [1-F(x)]^{k+m+1}. \tag{10}$$

The weighted sum of the distribution of the first two (GOS) can be obtained by using equation (9) and equation (10) in a two-component mixture model:

$$F_Y(x) = \pi \times F_{X(1\,:\,2,\,m,\,k)}(x) + (1-\pi) \times F_{X(2\,:\,2,\,m,\,k)}(x),$$

where, $0 < \pi < 1$ is the mixing probability

$$= \pi \times \left\{1 - [1 - F(x)]^{k+m+1}\right\} + (1 - \pi)$$

$$\times \left\{1 - \left(\frac{k+m+1}{m+1}\right)[1 - F(x)]^{k} + \left(\frac{k}{m+1}\right)[1 - F(x)]^{k+m+1}\right\}$$

$$= \pi - \pi \times [1 - F(x)]^{k+m+1} + 1 - \left(\frac{k+m+1}{m+1}\right)[1 - F(x)]^{k} + \left(\frac{k}{m+1}\right)[1 - F(x)]^{k+m+1}$$

$$- \pi + \pi \times \left(\frac{k+m+1}{m+1}\right)[1 - F(x)]^{k} - \pi \times \left(\frac{k}{m+1}\right)[1 - F(x)]^{k+m+1}$$

$$= 1 - \left(\frac{k+m+1}{m+1} - \frac{k+m+1}{m+1}\pi\right)[1 - F(x)]^{k}$$

$$+ \left(\frac{k}{m+1} - \frac{k}{m+1}\pi - \pi\right)[1 - F(x)]^{k+m+1}$$

Put $\lambda = 2\pi - 1$, such that $-1 < \lambda < 1$. Hence equation (4) becomes:

$$F_y(x) = 1 - \left(\frac{k+m+1}{m+1} - \frac{k+m+1}{m+1}\left(\frac{1+\lambda}{2}\right)\right)[1 - F(x)]^{k}$$

$$+ \left(\frac{k}{m+1} - \frac{k}{m+1}\left(\frac{1+\lambda}{2}\right) - \left(\frac{1+\lambda}{2}\right)\right)[1 - F(x)]^{k+m+1}$$

$$= 1 - \frac{k+m+1}{m+1}\left(1 - \left(\frac{1+\lambda}{2}\right)\right)[1 - F(x)]^{k}$$

$$+ \left(\frac{2k - k(1+\lambda) - (m+1)(1+\lambda)}{2(m+1)}\right)[1 - F(x)]^{k+m+1}$$

$$= 1 - \frac{k+m+1}{2(m+1)}\left\{(1-\lambda)[1 - F(x)]^{k} + \left(\lambda - \frac{k-m-1}{k+m+1}\right)[1 - F(x)]^{k+m+1}\right\}$$

□

Definition 1 The density function of the (GOS) transmutation map is given by:

$$f_Y(x) = \frac{k+m+1}{2(m+1)} f(x)\left\{k(1-\lambda)[1 - F(x)]^{k-1} + ((k+m+1)\lambda - (k-m-1))[1 - F(x)]^{k+m}\right\}. \tag{11}$$

The transmutation using (GOS) provides the classical transmuted family and transmutation using record value as special cases. Especially in the case where $k = 1$ and $m = 0$, the equation (8) simplifies to the CDF of the quadratic rank transmuted, as provided by [1].

$$F_Y(x) = (1+\lambda)F(x) - \lambda F^2(x).$$

Additionally, for $k = 1$ and $m = -1$, the equation (8) simplifies to the record-based transmuted family of distributions as described in [7]:

$$f_{R_2}(x) = g(x)[\pi + (1-\pi)\{-\ln(1-G(x))\}].$$

4.2 *Dual generalized order statistics transmuted family of distributions*

The cumulative distribution function of (DGOS) transmuted family is presented in the following theorem:

Theorem 2 Suppose that we have a sample of size two from a distribution with CDF $F(x)$ and PDF $f(x)$. Let $X_{1\,:\,2,\,m,\,k} = Min_{DGOS}(X_1,\ X_2)$ and $X_{2\,:\,2,\,m,\,k} = Max_{DGOS}(X_1,\ X_2)$ be the smallest and largest (DGOS) respectively. Then, the (DGOS) transmutation map is given by:

$$F_Y(x) = \left(\frac{k+m+1}{2(m+1)}\right)\left\{(1-\lambda)\,[F(x)]^k + \left(\lambda - \frac{k-m-1}{k+m+1}\right)[F(x)]^{k+m+1}\right\}, \tag{12}$$

with $\lambda \in [-1,\ 1]$, $k \geq 1$, and $m \neq -1$.

Proof. The pdf of first DGOS:

$$f_{1_{(d)}\,:\,2,\,m,\,k}(x) = (k+m+1)f(x)[F(x)]^{k+m}.$$

The pdf of the second DGOS:

$$f_{2_{(d)}\,:\,2,\,m,\,k}(x) = \frac{k(k+m+1)}{(m+1)}f(x)\ [F(x)]^{k-1}\ [1-\{F(x)\}^{m+1}]$$

The corresponding CDF_s:

$$F_{X(1_{(d)}\,:\,2,\,m,\,k)}(x) = [F(x)]^{k+m+1} \tag{13}$$

and

$$F_{X(2_{(d)}\,:\,2,\,m,\,k)}(x) = \left(\frac{k+m+1}{m+1}\right)[F(x)]^k - \left(\frac{k}{m+1}\right)[F(x)]^{k+m+1} \tag{14}$$

Now, express the dual generalized order statistics transmuted family as a combination of the distribution of the first two (DGOS) by using equations (13) and (14) in a two-component mixture model.

$$F_y(x) = \pi \times F_{x(1_{(d)}\,:\,2,\,m,\,k)}(x) + (1-\pi) \times F_{x(2_{(d)}\,:\,2,\,m,\,k)}(x)$$

$$= \pi \times [F(x)]^{k+m+1}$$

$$+ (1-\pi) \times \left\{ \left(\frac{k+m+1}{m+1}\right) [F(x)]^{k} - \left(\frac{k}{m+1}\right) [F(x)]^{k+m+1} \right\}$$

$$= \pi \times [F(x)]^{k+m+1} + \left(\frac{k+m+1}{m+1}\right) [F(x)]^{k} - \left(\frac{k}{m+1}\right) [F(x)]^{k+m+1}$$

$$- \left(\frac{k+m+1}{m+1}\right) \pi\, [F(x)]^{k} + \left(\frac{k}{m+1}\right) \pi\, [F(x)]^{k+m+1}$$

$$= \left(\frac{k+m+1}{m+1}\right) (1-\pi)\, [F(x)]^{k} - \left(\frac{k}{m+1} - \frac{k}{m+1}\pi - \pi\right) [F(x)]^{k+m+1}$$

Put $\lambda = 2\pi - 1$, such that $-1 < \lambda < 1$.

$$F_y(x) = \left(\frac{k+m+1}{2(m+1)}\right) (1-\lambda)\, [F(x)]^{k} - \left(\frac{k}{m+1} - \frac{k(1+\lambda)}{2(m+1)} - \frac{(1+\lambda)}{2}\right) [F(x)]^{k+m+1}$$

$$= \left(\frac{k+m+1}{2(m+1)}\right) (1-\lambda)\, [F(x)]^{k} - \left(\frac{2k - k(1+\lambda) - (m+1)(1+\lambda}{2(m+1)}\right) [F(x)]^{k+m+1}$$

$$= \left(\frac{k+m+1}{2(m+1)}\right) \left\{ (1-\lambda)\, [F(x)]^{k} + \left(\lambda - \frac{k-m-1}{k+m+1}\right) [F(x)]^{k+m+1} \right\}$$

□

Definition 2 The (DGOS) transmutation map's density function is expressed as follows:

$$f_Y(x) = \frac{k+m+1}{2(m+1)} f(x) \left\{ k(1-\lambda)\, [F(x)]^{k-1} + ((k+m+1)\lambda - (k-m-1))\, [F(x)]^{k+m} \right\}. \tag{15}$$

Reversed order statistics are special cases of (DGOS), when $k = 1,\ m = 0$; hence, the CDF_s of minimum and maximum (DGOS) be:

$$F_{X(1_{(d)}\,:\,2,\,0,\,1)}(x) = [F(x)]^2 = F_{rev(1\,:\,2)}(x),$$

and

$$F_{X(2_{(d)}\,:\,2,\,0,\,1)}(x) = 1 - [1-F(x)]^2 = F_{rev(2\,:\,2)}(x).$$

Then equation (12) can be reduced to:

$$F_Y(x) = (1-\lambda)F(x) + \lambda F^2(x).$$

Which is the CDF of the Quadratic rank transmuted using reversed-order statistics.

5. Properties of the generalized transmuted families of distributions

This section explores the statistical properties of the newly proposed families.

5.1 *Properties of the generalized order statistics transmuted family of distributions*

5.1.1 *Reliability and hazard rate functions*

The reliability function of (GOS) transmuted family of distributions can be given directly using equation (8) by:

$$R_Y(x) = \frac{k+m+1}{2(m+1)}\left\{(1-\lambda)[1-F(x)]^k + \left(\lambda - \frac{k-m-1}{k+m+1}\right)[1-F(x)]^{k+m+1}\right\}. \tag{16}$$

The hazard rate function of distributions in the (GOS) transmuted family can be expressed from equations (11) and (16) as follows:

$$h_Y(x) = h(x)\frac{\left\{k(1-\lambda)[1-F(x)]^{k-1} + ((k+m+1)\lambda - (k-m-1))[1-F(x)]^{k+m}\right\}}{\left\{(1-\lambda)[1-F(x)]^{k-1} + \left(\lambda - \frac{k-m-1}{k+m+1}\right)[1-F(x)]^{k+m}\right\}}, \tag{17}$$

where $h(x)$ is the hazard rate function of the baseline distribution.

5.1.2 r^{th} *moments*

The r^{th} term can be determined using the probability-weighted moments of the base distribution $F(x)$, which is defined by [14] as follows:

Definition 3 The probability-weighted moments for any random variable X has CDF $F_X(x)$ and pdf $f_X(x)$ is given by:

$$\begin{aligned} M_{r,\,s,\,t} = E(x^r[F(X)]^s[1-F(x)]^t) &= \int_{-\infty}^{\infty} x^r[F(X)]^s[1-F(x)]^t f(x)dx \\ &= \int_0^1 (x(F))^r[F(X)]^s[1-F(x)]^t dF(x). \end{aligned} \tag{18}$$

Where, r, s and t are real numbers, $x(F)$ is the inverse of CDF.

Hence, the r^{th} moment for random variable X from $F_Y(x)$ and $f_Y(x)$ is given by:

$$\mu_r^{'} = \frac{k+m+1}{2(m+1)}\left\{k(1-\lambda)M_{r,\,0,\,k-1} + ((k+m+1)\lambda - (k-m-1))M_{r,\,0,\,k+m}\right\}. \tag{19}$$

5.1.3 *Moment generating function and characteristic function*

Using the Taylor's expansion of the moment-generating function and characteristic function along with equation (19), the moment generating function and characteristic function of the (GOS) transmuted distribution are given below, respectively:

$$M_X(t) = \sum_{r=0}^{\infty} \frac{(t)^r}{r!}\mu_r^{'}, \tag{20}$$

$$\phi_X(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!}\mu_r^{'}. \tag{21}$$

5.1.4 *Quantile function*

The quantile function of (GOS) transmuted family is given by:

$$\frac{k+m+1}{2(m+1)}\left\{\left(\lambda - \frac{k-m-1}{k+m+1}\right)[1-F(x)]^{k+m+1} + (1-\lambda)[1-F(x_p)]^k\right\} = 1-p, \tag{22}$$

where, $k \geq 1$, $m \in R$, $m \neq -1$ and $|\lambda| \leq 1$. This equation is determined by evaluating values of k, m, p, and λ then solved with respect to $Q(p)$.

5.1.5 *Entropies*

Entropy quantifies the unpredictability or disorder in a random variable X. Two commonly used entropy measures are the Rényi entropy and the Shannon entropies.

1- Rényi Entropy

The definition of the Rényi Entropy for a random variable with probability density function $f_Y(x)$ is given by the following expression:

$$I_R(\rho) = \frac{1}{1-\rho}\log\left(\int_{-\infty}^{\infty} f_Y^{\rho}(x)\right)\, dx, \tag{23}$$

2- Shannon entropy

The expression for the Shannon entropy of a random variable with probability density function $f_Y(x)$ is given as follows:

$$H = E(-\log f_Y(x)) = \int_{-\infty}^{\infty} \{-\log f_Y(x)\} f_Y(x)\, dx \tag{24}$$

5.2 *Properties of the dual generalized order statistics transmuted family of distributions*

5.2.1 *Reliability and hazard rate functions*

The reliability function of the (DGOS) transmuted family of distributions $R_Y(x)$ can be derived using equation (12) as shown below:

$$\frac{k+m+1}{2(m+1)}\left\{\frac{2(m+1)}{k+m+1}-(1-\lambda)\left[F(x)\right]^{k}-\left(\lambda-\frac{k-m-1}{k+m+1}\right)\left[F(x)\right]^{k+m+1}\right\}. \tag{25}$$

From equations (15) and (25), the hazard rate function of (DGOS) transmuted family of distributions is given by:

$$h_Y(x)=f(x)\left\{\frac{k(1-\lambda)\left[F(x)\right]^{k-1}+((k+m+1)\lambda-(k-m-1))\left[F(x)\right]^{k+m}}{\left\{\frac{2(m+1)}{k+m+1}-(1-\lambda)\left[F(x)\right]^{k}-\left(\lambda-\frac{k-m-1}{k+m+1}\right)\left[F(x)\right]^{k+m+1}\right\}}\right\}, \tag{26}$$

where $f(x)$ is the density function of the baseline distribution.

5.2.2 r^{th} *moments*

The r^{th} moment can be expressed using probability-weighted moments of the base distribution $F(x)$ and equation (15) as follows:

$$\mu_r'=\frac{k+m+1}{2(m+1)}\left\{k(1-\lambda)M_{r,\,k-1,\,0}+((k+m+1)\lambda-(k-m-1))M_{r,\,k+m,\,0}\right\}. \tag{27}$$

5.2.3 *Moment-generating function and characteristic function*

The moment generating function and characteristic function of the (DGOS) transmuted family is given in equations (20) and (21), respectively.

5.2.4 *Quantile function*

The quantile function of (DGOS) transmuted family is given by:

$$\left(\frac{(k+m+1)\lambda-(k-m-1)}{2(m+1)}\right)\left[F(x_p)\right]^{k+m+1}+\frac{(k+m+1)(1-\lambda)}{2(m+1)}\left(F(x_p)\right)^{k}=p, \tag{28}$$

5.2.5 *Rényi entropy and shannon entropy*

It can be computed directly by applying equations (23) and (24) to equation (15).

6. Maximum likelihood estimation of the parameters

Maximum Likelihood Estimation (MLE) of parameters is a crucial technique in estimation theory. There has been significant research on the MLE of parameters, particularly in recent studies, such as [15]. This section applies MLE to estimate the parameters for the transmuted (GOS) and (DGOS) transmuted families of distributions.

6.1 *Maximum likelihood estimation of the generalized order statistics ransmuted parameters*

Let $X_1,\ X_2,\ \ldots.,\ X_n$ be a random sample of size n from the (GOS) transmuted family with transmutation parameter $\lambda \in [-1,\ 1]$. Let $\Theta = (k,\ m,\ \lambda,\ \xi^T)^T$ be the vector of the parameters $p \times 1$. The total log-likelihood function for Θ is given by:

$$l(\Theta) = \sum_{i=1}^{n} \log\left(\frac{k+m+1}{2(m+1)} f(x)\right)$$

$$+\sum_{i=1}^{n} \log\left[k(1-\lambda)[1-F(x)]^{k-1} + ((k+m+1)\lambda-(k-m-1))[1-F(x)]^{k+m}\right]. \tag{29}$$

The score functions $U(\Theta) = (U_k,\ U_m,\ U_\lambda,\ U_\xi)^T$ for $w = 1,\ \ldots,\ p$

$$U_k = \frac{\partial l(\Theta)}{\partial k} = \frac{n}{k+m+1} + \sum_{i=1}^{n} \log(1-F(x))$$

$$+\sum_{i=1}^{n} \frac{(1-\lambda)[1-F(x)]^{k-1}-(1-\lambda)[1-F(x)]^{k+m}}{\delta_1(x,\ k,\ m,\ \lambda,\ \xi)}, \tag{30}$$

$$U_m = \frac{\partial l(\Theta)}{\partial m} = \frac{-2kn}{2(m+1)(k+m+1)}$$

$$+\sum_{i=1}^{n} \frac{(\lambda+1)[1-F(x)]^{k+m} + ((k+m+1)\lambda-(k-m-1))[1-F(x)]^{k+m}\log(1-F(x))}{\delta_1(x,\ k,\ m,\ \lambda,\ \xi)}, \tag{31}$$

$$U_\lambda = \frac{\partial l(\Theta)}{\partial \lambda} = \sum_{i=1}^{n} \frac{(k+m+1)[1-F(x)]^{k+m} - k[1-F(x)]^{k-1}}{\delta_1(x,\ k,\ m,\ \lambda,\ \xi)}, \tag{32}$$

$$U_\xi = \frac{\partial l(\Theta)}{\partial \xi_w} = \frac{\dfrac{\partial f_Y(x_i,\ \xi)}{\partial \xi_w}}{f_Y(x_i,\ \xi)}$$

$$+\sum_{i=1}^{n} \frac{k(1-\lambda)\dfrac{\partial}{\partial \xi_w}[1-F(x)]^{k-1} + ((k+m+1)\lambda-(k-m-1))\dfrac{\partial}{\partial \xi_w}[1-F(x)]^{k+m}}{\delta_1(x,\ k,\ m,\ \lambda,\ \xi)}, \tag{33}$$

where,

$$\delta_1(x,\ k,\ m,\ \lambda,\ \xi) = k(1-\lambda)[1-F(x)]^{k-1} + ((k+m+1)\lambda-(k-m-1))[1-F(x)]^{k+m}$$

6.2 *Maximum likelihood estimation of the dual generalized order statistics transmuted parameters*

Let $X_1, X_2, \ldots., X_n$ be a random sample of size n from the (DGOS) transmuted family with transmutation parameter $\lambda \in [-1, 1]$. Let $\Theta = (k, m, \lambda, \xi^T)^T$ be the vector of the parameters $p \times 1$. The total log-likelihood function for Θ is given by:

$$l(\Theta) = \sum_{i=1}^{n} \log\left(\frac{k+m+1}{2(m+1)} f(x)\right) + \sum_{i=1}^{n} \log\left[k(1-\lambda)[F(x)]^{k-1} + ((k+m+1)\lambda - (k-m-1))[F(x)]^{k+m}\right], \tag{34}$$

The score functions $U(\Theta) = (U_k, U_m, U_\lambda, U_\xi)^T$ for $w = 1, \ldots, p$ are given by:

$$U_k = \frac{\partial l(\Theta)}{\partial k} = \frac{n}{k+m+1} + \sum_{i=1}^{n} \log(F(x)) + \sum_{i=1}^{n} \frac{(1-\lambda)[F(x)]^{k-1} - (1-\lambda)[F(x)]^{k+m}}{\delta_2(x, k, m, \lambda, \xi)}, \tag{35}$$

$$U_m = \frac{\partial l(\Theta)}{\partial m} = \frac{-2kn}{2(m+1)(k+m+1)} + \sum_{i=1}^{n} \frac{(\lambda+1)[F(x)]^{k+m} + ((k+m+1)\lambda - (k-m-1))[F(x)]^{k+m} \log(F(x))}{\delta_2(x, k, m, \lambda, \xi)}, \tag{36}$$

$$U_\lambda = \frac{\partial l(\Theta)}{\partial \lambda} = \sum_{i=1}^{n} \frac{(k+m+1)[F(x)]^{k+m} - k[F(x)]^{k-1}}{\delta_2(x, k, m, \lambda, \xi)}, \tag{37}$$

$$U_\xi = \frac{\partial l(\Theta)}{\partial \xi_w} = \frac{\frac{\partial f_Y(x_i, \xi)}{\partial \xi_w}}{f_Y(x_i, \xi)} + \sum_{i=1}^{n} \frac{k(1-\lambda)\frac{\partial}{\partial \xi_w}[F(x)]^{k-1} + ((k+m+1)\lambda - (k-m-1))\frac{\partial}{\partial \xi_w}[F(x)]^{k+m}}{\delta_2(x, k, m, \lambda, \xi)}, \tag{38}$$

where,

$$\delta_2(x, k, m, \lambda, \xi) = k(1-\lambda)[F(x)]^{k-1} + ((k+m+1)\lambda - (k-m-1))[F(x)]^{k+m}$$

And obtain the MLE $\hat{\Theta} = (\hat{k},\ \hat{m},\ \hat{\lambda},\ \hat{\xi}^T)^T$ of $\Theta = (k,\ m,\ \lambda,\ \xi^T)^T$ for both two transmuted families (GOS) and (DGOS) by maximizing the log-likelihood.

7. Generalized order statistics based transmuted Weibull distribution

This section focuses on the Generalized Order Statistics Transmuted Using the Weibull Distribution (GOSTWD), along with its statistical properties, as an application of the transmuted family of (GOS).

7.1 *Distribution and density functions*

Theorem 3 A random variable X is said to have (GOSTWD) with shape and scale parameters β and $\theta > 0$ respectively and the transmutation parameter $|\lambda| \le 1$, $k \ge 1$, $m \in R - \{-1\}$, if x has the distribution function as follows:

$$F_Y(x) = 1 - \frac{k+m+1}{2(m+1)} \left\{ (1-\lambda) \exp\left\{ -k \left(\frac{x}{\theta}\right)^{\beta} \right\} + \left(\lambda - \frac{k-m-1}{k+m+1}\right) \exp\left\{ -(k+m+1) \left(\frac{x}{\theta}\right)^{\beta} \right\} \right\}. \tag{39}$$

Proof. By applying equation (8), we directly obtain equation (39), where the CDF of the Weibull distribution is expressed as:

$$F_X(x) = 1 - \exp\left\{ -\left(\frac{x}{\theta}\right)^{\beta} \right\}.$$

□

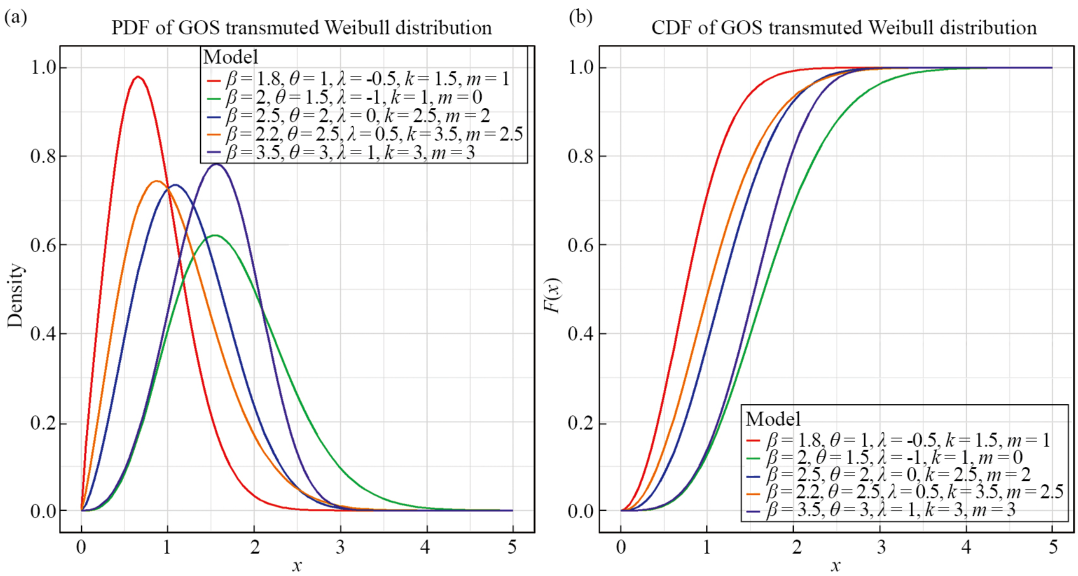


Figure 1. Plots of PDF and CDF of GOST Weibull distribution for different choices paramters

Definition 4 A random variable X is said to have (GOSTWD) with shape and scale parameters β and $\theta > 0$ respectively and the transmutation parameter $|\lambda| \le 1$, $k \ge 1$, $m \in R - \{-1\}$, if X has the density function as follows:

$$f_Y(x) = \frac{k+m+1}{2(m+1)} \frac{\beta}{\theta} \left(\frac{x}{\theta}\right)^{\beta-1} \exp\left\{-\left(\frac{x}{\theta}\right)^{\beta}\right\} \times \left\{k(1-\lambda)\exp\left\{-(k-1)\left(\frac{x}{\theta}\right)^{\beta}\right\}\right.$$
$$\left. +((k+m+1)\lambda-(k-m-1))\exp\left\{-(k+m)\left(\frac{x}{\theta}\right)^{\beta}\right\}\right\}. \tag{40}$$

Figure 1 shows the shape of (GOSTWD) *PDF* and *CDF* with different choices of parameters β, θ, and different values of k and m.

7.2 Statistical properties of generalized order statistics based transmuted Weibull distribution

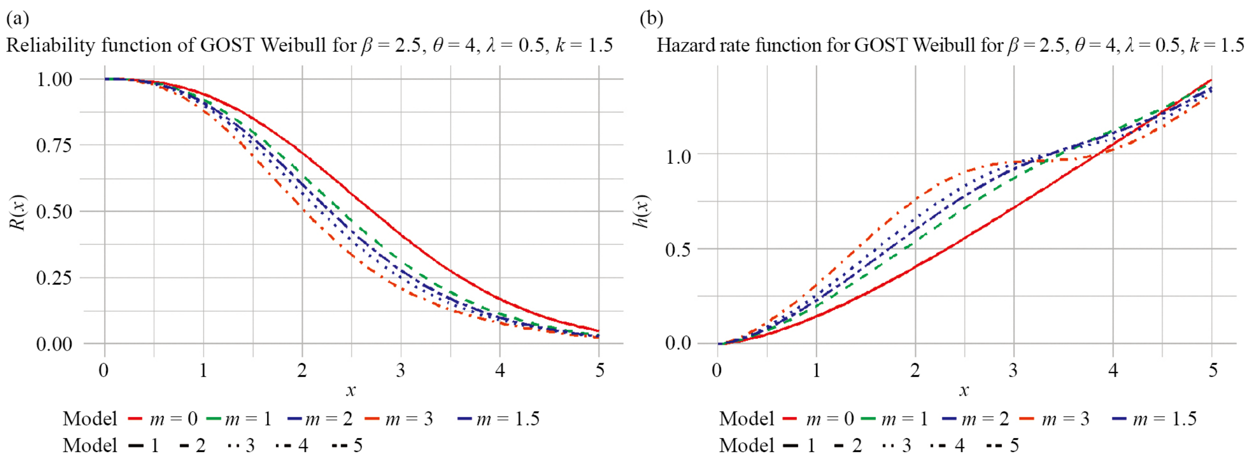
7.2.1 Reliability and hazard rate functions

The equations (16) and (17) provide the reliability and hazard rate functions for the (GOSTWD). These functions are given as follows:

$$R_Y(x, \beta, \theta, \lambda, k, m)$$
$$= \frac{k+m+1}{2(m+1)} \times \left\{(1-\lambda)\exp\left\{-k\left(\frac{x}{\theta}\right)^{\beta}\right\} + \left(\lambda - \frac{k-m-1}{k+m+1}\right)\exp\left\{-(k+m+1)\left(\frac{x}{\theta}\right)^{\beta}\right\}\right\}. \tag{41}$$

$$h_Y(x, \beta, \theta, \lambda, k, m)$$
$$= h(x) \times \frac{\left\{k(1-\lambda)\exp\left\{-(k-1)\left(\frac{x}{\theta}\right)^{\beta}\right\} + ((k+m+1)\lambda-(k-m-1))\exp\left\{-(k+m)\left(\frac{x}{\theta}\right)^{\beta}\right\}\right\}}{\left\{(1-\lambda)\exp\left\{-(k-1)\left(\frac{x}{\theta}\right)^{\beta}\right\} + \left(\lambda - \frac{k-m-1}{k+m+1}\right)\exp\left\{-(k+m)\left(\frac{x}{\theta}\right)^{\beta}\right\}\right\}} \tag{42}$$

where $h(x)$ is the hazard rate function of the Weibull distribution.



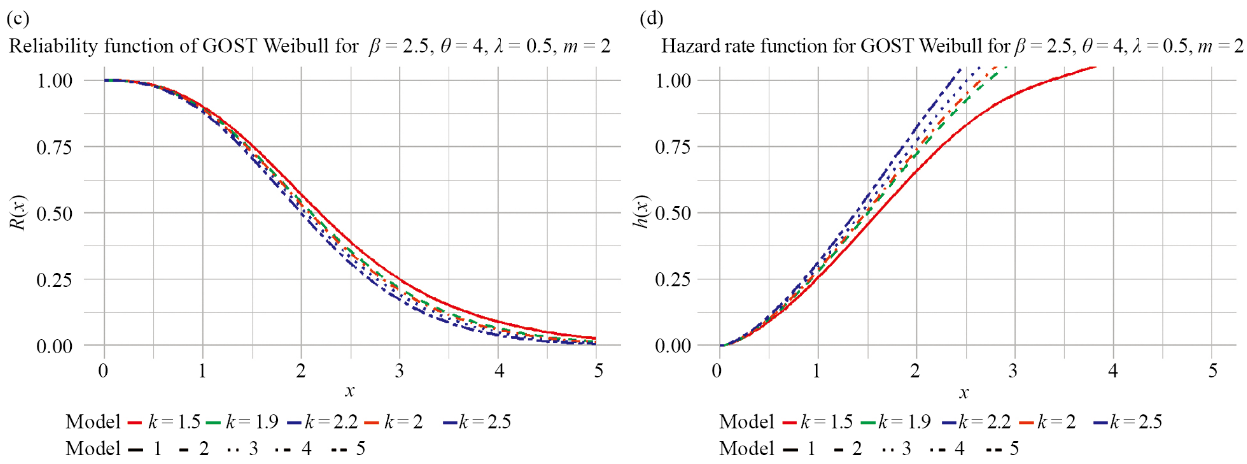


Figure 2. Plots of reliability and hazard rate functions of GOST Weibull distribution for different choices of parameters

Figure 2 shows the shape of (GOSTWD) reliability and hazard rate functions with different choices of parameters β, θ, and different values of k and m.

7.2.2 *The r^{th} moments*

The equation (19) provides the r^{th} moment for the random variable X with CDF $F_Y(x)$ and PDF $f_Y(x)$ as follows:

$$\mu_r' = E(x^r) = \frac{1}{2(m+1)}\theta^r\Gamma\left(\frac{r}{\beta}+1\right)\left[\frac{(k+m+1)(1-\lambda)}{(k)^{\frac{r}{\beta}}}+\frac{((k+m+1)\lambda-(k-m-1))}{(k+m+1)^{\frac{r}{\beta}}}\right]. \tag{43}$$

Definition 5 The mean of (GOSTWD) is given by setting $r = 1$ in equation (43) as follows:

$$\mu = E(x) = \frac{\theta}{2(m+1)}\Gamma\left(\frac{1}{\beta}+1\right)\left[\frac{(k+m+1)(1-\lambda)}{k^{\frac{1}{\beta}}}+\frac{((k+m+1)\lambda-(k-m-1))}{(k+m+1)^{\frac{1}{\beta}}}\right]. \tag{44}$$

7.2.3 *Moment generating function*

The moment-generating function (GOSTWD) is directly derived using equations (20) and (43) as shown below:

$$M_X(t) = E(e^{tx}) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\frac{\theta^r}{2(m+1)}\Gamma\left(\frac{r}{\beta}+1\right)\left[\frac{(k+m+1)(1-\lambda)}{k^{\frac{r}{\beta}}}+\frac{((k+m+1)\lambda-(k-m-1))}{(k+m+1)^{\frac{r}{\beta}}}\right]. \tag{45}$$

7.2.4 *Quantile function*

Using equation (22), the quantile function of (GOSTWD) is given by:

$$Q(p) = \left(\frac{(k+m+1)\lambda-(k-m-1)}{2(m+1)}\right)\exp\left\{-(k+m+1)\left(\frac{x_p}{\theta}\right)^{\beta}\right\}$$

$$+\frac{(k+m+1)(1-\lambda)}{2(m+1)}\exp\left\{-(k)\left(\frac{x_p}{\theta}\right)^{\beta}\right\} = 1-p,$$

7.2.5 *Rényi entropy*

Theorem 4 The Rényi Entropy of a random variable from (GOSTWD) is given by:

$$I_R(\rho) = \frac{\rho}{1-\rho}\log\left(\frac{k+m+1}{2(m+1)}\right) - \log\beta + \log\theta$$

$$+\frac{\rho}{1-\rho}\log(k(1-\lambda)) + \frac{1}{1-\rho}\log\left(\Gamma\left(\frac{(\rho-1)(\beta-1)}{\beta}+1\right)\right) + \frac{1}{1-\rho} \tag{46}$$

$$\times\log\left\{\sum_{i=0}^{\infty}\binom{\rho}{i}\left[\frac{(k+m+1)\lambda-(k-m-1)}{k(1-\lambda)}\right]^{i}(k\rho+mi+i-1)^{-\frac{(\rho-1)(\beta-1)}{\beta}+1}\right\}.$$

Proof. The Rényi Entropy of a random variable with pdf $f_Y(x)$ is defined by equation (23), hence the integral

$$\int_0^{\infty} f_Y^{\rho}(x)\, dx = \left(\frac{k+m+1}{2(m+1)}\right)^{\rho}\frac{\beta^{\rho}}{\theta^{\beta\rho}} \times \int_0^{\infty} x^{\rho(\beta-1)}$$

$$\times\left[k(1-\lambda)\exp\left\{-k\left(\frac{x}{\theta}\right)^{\beta}\right\} + ((k+m+1)\lambda-(k-m-1))\exp\left\{-(k+m+1)\left(\frac{x}{\theta}\right)^{\beta}\right\}\right]^{\rho} dx$$

where $\rho > 0,\ \rho \neq 1$ using Newton's generalized of the binomial theorem for any arbitrary non-negative integer, $n > 0,\ n \neq 1$.

$$(x+y)^n = \sum_{i=0}^{\infty}\binom{n}{i}x^{n-i}y^{i}$$

Let

$$n = \rho,\ x = k(1-\lambda)\exp\left\{-k\left(\frac{x}{\theta}\right)^{\beta}\right\},$$

$$y = ((k+m+1)\lambda-(k-m-1))\exp\left\{-(k+m+1)\left(\frac{x}{\theta}\right)^{\beta}\right\}$$

Then,

$$\left[k(1-\lambda)\exp\left\{-k\left(\frac{x}{\theta}\right)^{\beta}\right\}+((k+m+1)\lambda-(k-m-1))\exp\left\{-(k+m+1)\left(\frac{x}{\theta}\right)^{\beta}\right\}\right]^{\rho}$$

$$=\sum_{i=0}^{\infty}\binom{\rho}{i}\left(k(1-\lambda)\exp\left\{-k\left(\frac{x}{\theta}\right)^{\beta}\right\}\right)^{\rho-i}$$

$$\times\left(((k+m+1)\lambda-(k-m-1))\exp\left\{-(k+m+1)\left(\frac{x}{\theta}\right)^{\beta}\right\}\right)^{i}$$

$$=\sum_{i=0}^{\infty}\binom{\rho}{i}\left[\frac{(k+m+1)\lambda-(k-m-1)}{k(1-\lambda)}\right]^{i}[k(1-\lambda)]^{\rho}\left(\exp\left\{-\left(\frac{x}{\theta}\right)^{\beta}\right\}\right)^{(k\rho+mi+i)}$$

The integral will be:

$$=\sum_{i=0}^{\infty}\left(\frac{k+m+1}{2(m+1)}\right)^{\rho}\frac{\beta^{\rho}}{\theta^{\beta\rho}}\binom{\rho}{i}\left[\frac{(k+m+1)\lambda-(k-m-1)}{k(1-\lambda)}\right]^{i}[k(1-\lambda)]^{\rho}$$

$$\times\int_{0}^{\infty}x^{\rho(\beta-1)}\left(\exp\left\{-\left(\frac{x}{\theta}\right)^{\beta}\right\}\right)^{(k\rho+mi+i)}dx,$$

put $v=\left(\frac{k+m+1}{2(m+1)}\right)^{\rho}\binom{\rho}{i}\left[\frac{(k+m+1)\lambda-(k-m-1)}{k(1-\lambda)}\right]^{i}[k(1-\lambda)]^{\rho}$

$$=\sum_{i=0}^{\infty}v\frac{\beta^{\rho}}{\theta^{\beta\rho}}\int_{0}^{\infty}x^{\rho(\beta-1)}\exp\left\{-(k\rho+mi+i-1)\left(\frac{x}{\theta}\right)^{\beta}\right\},\ dx$$

by simplifying and integration

$$=\Gamma\left(\frac{(\rho-1)(\beta-1)}{\beta}+1\right)\sum_{i=0}^{\infty}v\frac{\beta^{\rho-1}\theta^{1-\rho}}{(k\rho+mi+i-1)^{\frac{(\rho-1)(\beta-1)}{\beta}+1}}$$

Then,

$$\frac{1}{1-\rho}\log\left(\int_{-\infty}^{\infty} f_Y^{\rho}(x)\right),\, dx$$

$$= \frac{\rho}{1-\rho}\log\left(\frac{k+m+1}{2(m+1)}\right) - \log\beta + \log\theta + \frac{\rho}{1-\rho}\log(k(1-\lambda))$$

$$+\frac{1}{1-\rho}\Gamma\left(\frac{(\rho-1)(\beta-1)}{\beta}+1\right)+\frac{1}{1-\rho}$$

$$\times\log\left\{\sum_{i=0}^{\infty}\binom{\rho}{i}\left[\frac{(k+m+1)\lambda-(k-m-1)}{k(1-\lambda)}\right]^{i}(k\rho+mi+i)^{-\frac{(\rho-1)(\beta-1)}{\beta}+1}\right\}$$

□

7.2.6 *Shannon entropy*

Using equations (24) and (40) the Shannon entropy of (GOSTWD) can be obtained directly.

7.3 *Maximum likelihood estimation of generalized order statistics based transmuted Weibull distribution*

Let $X_1,\, X_2,\, ...,\, X_n$ be a random sample of size n from the (GOSTWD) with transmutation parameter $\lambda \in [-1,\, 1]$. Let $\Theta = (k,\, m,\, \lambda,\, \beta,\, \theta)^T$ be the vector of the parameters 5×1. The total log-likelihood function for Θ using equation equation (29)is given by:

$$l(\Theta) = \ln L = n\ln\left(\frac{k+m+1}{2(m+1)}\right) + n\ln\beta - n\beta\ln\theta + (\beta-1)\sum_{i=1}^{n}\ln x_i - \sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\beta}$$

$$+\sum_{i=1}^{n}\ln\left\{k(1-\lambda)\left(\exp\left\{-\left(\frac{x_i}{\theta}\right)^{\beta}\right\}\right)^{(k-1)} + ((k+m+1)\lambda-(k-m-1))\left(\exp\left\{-\left(\frac{x_i}{\theta}\right)^{\beta}\right\}\right)^{(k+m)}\right\}$$

The score function $U(\Theta) = (U_k,\, U_m,\, U_\lambda,\, U_\beta,\, U_\theta)^T$ is given in equations (30) to (33) by:

$$U_k = \frac{n}{k+m+1} - \sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\beta} + \sum_{i=1}^{n}\frac{(1-\lambda)\exp\left\{-(k-1)\left(\frac{x_i}{\theta}\right)^{\beta}\right\} - (1-\lambda)\exp\left\{-(k+m)\left(\frac{x_i}{\theta}\right)^{\beta}\right\}}{\delta_1(x,\, k,\, m,\, \lambda,\, \beta,\, \theta)},$$

$$U_m = \frac{-2kn}{2(m+1)(k+m+1)}$$

$$+\sum_{i=1}^{n}\frac{(\lambda+1)\exp\left\{-(k+m)\left(\frac{x_i}{\theta}\right)^{\beta}\right\}-\left(\frac{x_i}{\theta}\right)^{\beta}((k+m+1)\lambda-(k-m-1))\exp\left\{-(k+m)\left(\frac{x_i}{\theta}\right)^{\beta}\right\}}{\delta_1(x,\ k,\ m,\ \lambda,\ \beta,\ \theta)},$$

$$U_{\lambda}=\sum_{i=1}^{n}\frac{(k+m+1)\exp\left\{-(k+m)\left(\frac{x_i}{\theta}\right)^{\beta}\right\}-k\exp\left\{-(k-1)\left(\frac{x_i}{\theta}\right)^{\beta}\right\}}{\delta_1(x,\ k,\ m,\ \lambda,\ \beta,\ \theta)},$$

$$U_{\beta}=\frac{n}{\beta}-n\ln\theta+\sum_{i=1}^{n}\ln x_i-\sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\beta}\ln\left(\frac{x_i}{\theta}\right)-\sum_{i=1}^{n}\frac{k(k-1)(1-\lambda)\left(\frac{x_i}{\theta}\right)^{\beta}\ln\left(\frac{x_i}{\theta}\right)\exp\left\{-(k-1)\left(\frac{x_i}{\theta}\right)^{\beta}\right\}}{\delta_1(x,\ k,\ m,\ \lambda,\ \beta,\ \theta)}$$

$$-\sum_{i=1}^{n}\frac{(k+m)((k+m+1)\lambda-(k-m-1))\left(\frac{x_i}{\theta}\right)^{\beta}\ln\left(\frac{x_i}{\theta}\right)\exp\left\{-(k+m)\left(\frac{x_i}{\theta}\right)^{\beta}\right\}}{\delta_1(x,\ k,\ m,\ \lambda,\ \beta,\ \theta)},$$

$$U_{\theta}=-\frac{n\beta}{\theta}+\frac{\beta}{\theta}\sum_{i=1}^{n}\left(\frac{x}{\theta}\right)^{\beta}+\sum_{i=1}^{n}\frac{k(k-1)(1-\lambda)\left(\frac{\beta}{\theta}\right)\left(\frac{x_i}{\theta}\right)^{\beta}\exp\left\{-(k-1)\left(\frac{x_i}{\theta}\right)^{\beta}\right\}}{\delta_1(x,\ k,\ m,\ \lambda,\ \beta,\ \theta)}$$

$$+\sum_{i=1}^{n}\frac{(k+m)((k+m+1)\lambda-(k-m-1))\left(\frac{\beta}{\theta}\right)\left(\frac{x_i}{\theta}\right)^{\beta}\exp\left\{-(k+m)\left(\frac{x_i}{\theta}\right)^{\beta}\right\}}{\delta_1(x,\ k,\ m,\ \lambda,\ \beta,\ \theta)},$$

where,

$$\delta_1(x,\ k,\ m,\ \lambda,\ \beta,\ \theta)=k(1-\lambda)\exp\left\{-(k-1)\left(\frac{x_i}{\theta}\right)^{\beta}\right\}$$

$$+((k+m+1)\lambda-(k-m-1))\exp\left\{-(k+m)\left(\frac{x_i}{\theta}\right)^{\beta}\right\}$$

8. Dual generalized order statistics based transmuted Weibull distribution

In this part, we explore the statistical properties of (DGOS) based on Transmuted Weibull Distribution and demonstrate its application within the transmuted family of dual generalized order statistics.

8.1 *Distribution and density functions*

Theorem 5 A random variable X is said to have Dual Generalized Order Statistics Based Transmuted Weibull Distribution (DGOSTWD) with shape and scale parameters β and $\theta > 0$ respectively and the transmutation parameter $|\lambda| \leq 1,\ k \geq 1,\ m \in R-\{-1\}$ if x has the distribution function as follows:

$$F_Y(x) = \frac{k+m+1}{2(m+1)} \left\{ (1-\lambda) \left(1-\exp\left\{-\left(\frac{x}{\theta}\right)^{\beta}\right\}\right)^{k} + \left(\lambda - \frac{k-m-1}{k+m+1}\right) \left(1-\exp\left\{-\left(\frac{x}{\theta}\right)^{\beta}\right\}\right)^{k+m+1} \right\}. \quad (47)$$

Proof. From equation (12) we directly get the result by substitute of $F_X(x)$ of Weibull distribution. □

Definition 6 A random variable X is said to have (DGOSTWD) with shape and scale parameters β *and* $\theta > 0$ respectively and the transmutation parameter $|\lambda| \le 1$, $k \ge 1$, $m \in R - \{-1\}$ if X has the density function as follows:

$$f_Y(x) = \frac{k+m+1}{2(m+1)} \frac{\beta}{\theta} \left(\frac{x}{\theta}\right)^{\beta-1} \exp\left\{-\left(\frac{x}{\theta}\right)^{\beta}\right\}$$
$$\times \left\{ k(1-\lambda) \left(1-\exp\left\{-\left(\frac{x}{\theta}\right)^{\beta}\right\}\right)^{k-1} + ((k+m+1)\lambda - (k-m-1)) \left(1-\exp\left\{-\left(\frac{x}{\theta}\right)^{\beta}\right\}\right)^{k+m} \right\}. \quad (48)$$

Figure 3 shows the shape of (DGOSTWD) PDF and CDF with different choices of parameters β, θ, λ, k and m.

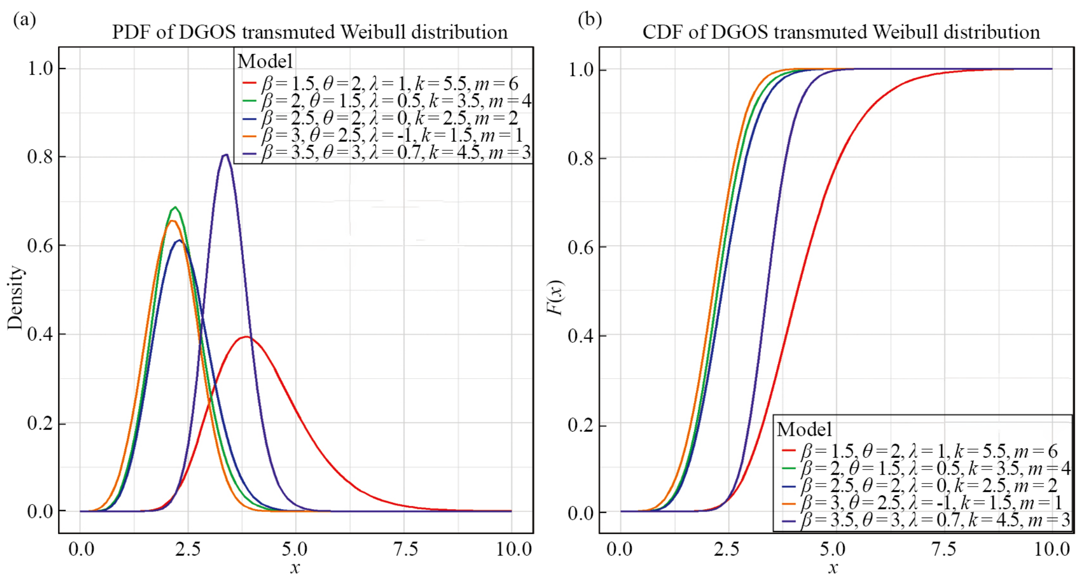


Figure 3. Plots of PDF and CDF of DGOST Weibull distribution for different choices paramters

8.2 *Statistical properties of dual generalized order statistics based transmuted Weibull distribution*

8.2.1 *Reliability and hazard rate functions*

The reliability and hazard rate functions of (DGOSTWD) are respectively given as follows using equations (25) and (26):

$$R_Y(x,\ \beta,\ \theta,\ \lambda,\ k,\ m) = \frac{k+m+1}{2(m+1)}\left\{\frac{2(m+1)}{k+m+1} - (1-\lambda)\left(1-\exp\left\{-\left(\frac{x}{\theta}\right)^{\beta}\right\}\right)^{k}\right.$$
$$\left. - \left(\lambda - \frac{k-m-1}{k+m+1}\right)\left(1-\exp\left\{-\left(\frac{x}{\theta}\right)^{\beta}\right\}\right)^{k+m+1}\right\}, \tag{49}$$

$$h_Y(x,\ \beta,\ \theta,\ \lambda,\ k,\ m)$$
$$= f(x) \times \frac{\left\{k(1-\lambda)\left(1-\exp\left\{-\left(\frac{x}{\theta}\right)^{\beta}\right\}\right)^{k-1} + ((k+m+1)\lambda - (k-m-1))\left(1-\exp\left\{-\left(\frac{x}{\theta}\right)^{\beta}\right\}\right)^{k+m}\right\}}{\left\{\frac{2(m+1)}{k+m+1} - (1-\lambda)\left(1-\exp\left\{-\left(\frac{x}{\theta}\right)^{\beta}\right\}\right)^{k} - \left(\lambda - \frac{k-m-1}{k+m+1}\right)\left(1-\exp\left\{-\left(\frac{x}{\theta}\right)^{\beta}\right\}\right)^{k+m+1}\right\}}. \tag{50}$$

where $f(x)$ is the density function of the Weibull distribution.

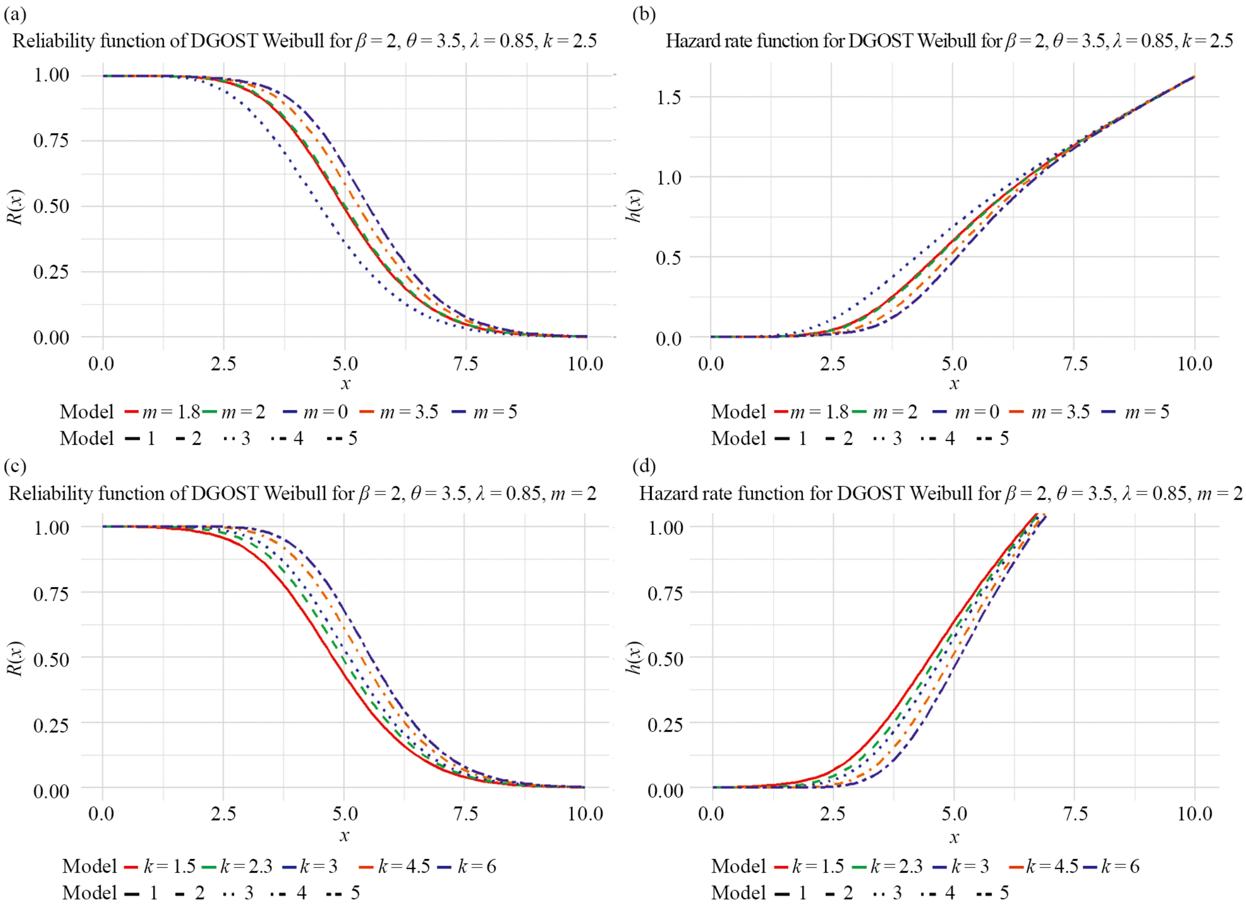


Figure 4. Plots of reliability and hazard rate functions of dual GOST Weibull distribution

Figure 4 shows the shape of Dual Generalized order statistics transmuted Weibull reliability and hazard rate functions with different choices of parameters β, θ, and different values of k and m.

8.2.2 r^{th} moments

The r^{th} moment for the random variable X with CDF $F_Y(x)$ and PDF $f_Y(x)$ is given by equation (27) as follows:

$$\mu_r' = \frac{k+m+1}{2(m+1)}\theta^r\Gamma\left(\frac{r}{\beta}+1\right)\times\left\{k(1-\lambda)\sum_{i=1}^{k-1}(-1)^i\binom{k-1}{i}\frac{1}{(i)^{\frac{r}{\beta}+1}}\right.$$
$$\left.+((k+m+1)\lambda-(k-m-1))\times\sum_{i=1}^{k+m}(-1)^i\binom{k+m}{i}\frac{1}{(i+1)^{\frac{r}{\beta}+1}}\right\}. \tag{51}$$

Definition 7 The mean of (DGOSTWD) is given by:

$$\mu = \frac{k+m+1}{2(m+1)}\theta\Gamma\left(\frac{1}{\beta}+1\right)\times\left\{k(1-\lambda)\sum_{i=1}^{k-1}(-1)^i\binom{k-1}{i}\frac{1}{(i)^{\frac{1}{\beta}+1}}\right.$$
$$\left.+((k+m+1)\lambda-(k-m-1))\sum_{i=1}^{k+m}(-1)^i\binom{k+m}{i}\frac{1}{(i+1)^{\frac{1}{\beta}+1}}\right\}$$

8.2.3 Moment generating function

The moment-generating function of (DGOSTWD) is derived from equations (20) and (51) and is given by:

$$M_X(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\frac{k+m+1}{2(m+1)}\theta^r\Gamma\left(\frac{r}{\beta}+1\right)\times\left\{k(1-\lambda)\sum_{i=1}^{k-1}(-1)^i\binom{k-1}{i}\frac{1}{(i)^{\frac{r}{\beta}+1}}\right.$$
$$\left.+((k+m+1)\lambda-(k-m-1))\sum_{i=1}^{k+m}(-1)^i\binom{k+m}{i}\frac{1}{(i+1)^{\frac{r}{\beta}+1}}\right\} \tag{52}$$

8.2.4 Quantile function

Using equation (28) the quantile function of (DGOSTWD) is given by:

$$Q(p) = \left(\frac{(k+m+1)\lambda-(k-m-1)}{2(m+1)}\right)\left(1-\exp\left\{-\left(\frac{x_p}{\theta}\right)^\beta\right\}\right)^{k+m+1}$$
$$+\frac{(k+m+1)(1-\lambda)}{2(m+1)}\left(1-\exp\left\{\left(\frac{x_p}{\theta}\right)^\beta\right\}\right)^k = p.$$

8.2.5 *Rényi entropy*

Theorem 6 The Rényi Entropy of the (DGOSTWD) is given by:

$$
\begin{aligned}
I_R(\rho) = & \frac{\rho}{1-\rho}\log\left(\frac{k+m+1}{2(m+1)}\right) - \log\beta - \beta\log\theta \\
& + \frac{\rho}{1-\rho}\log(k(1-\lambda)) + \frac{1}{1-\rho}\Gamma\left(\frac{(\rho-1)(\beta-1)}{\beta}+1\right) + \frac{1}{1-\rho} \\
& \times \log\left\{\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\binom{\rho}{i}\binom{\rho(k-1)+i(m+1)}{l}\right. \\
& \left.\times\left[\frac{(k+m+1)\lambda-(k-m-1)}{k(1-\lambda)}\right]^{i}(\rho+l+-1)^{-(\frac{(\rho-1)(\beta-1)}{\beta}+1)}\right\}.
\end{aligned} \tag{53}
$$

Proof. The Rényi Entropy of a random variable with pdf $f_Y(x)$ is defined by equation (23). Then:

$$
\begin{aligned}
& \int_0^{\infty} f_Y^{\rho}(x)\ dx \\
= & \left(\frac{k+m+1}{2(m+1)}\right)^{\rho}\left(\frac{\beta}{\theta}\right)^{\rho}\int_0^{\infty}\left(\frac{x}{\theta}\right)^{\rho(\beta-1)}\left(\exp\left\{-\left(\frac{x}{\theta}\right)^{\beta}\right\}\right)^{\rho} \\
& \times\left\{k(1-\lambda)\left(1-\exp\left\{-\left(\frac{x}{\theta}\right)^{\beta}\right\}\right)^{k-1}+((k+m+1)\lambda-(k-m-1))\left(1-\exp\left\{-\left(\frac{x}{\theta}\right)^{\beta}\right\}\right)^{k+m}\right\}^{\rho} dx,
\end{aligned}
$$

where $\rho > 0,\ \rho \neq 1$ using Newton's generalized of the binomial theorem for any arbitrary non-negative integer. Then,

$$
\begin{aligned}
& \left\{k(1-\lambda)\left(1-\exp\left\{-\left(\frac{x}{\theta}\right)^{\beta}\right\}\right)^{k-1}+((k+m+1)\lambda-(k-m-1))\left(1-\exp\left\{-\left(\frac{x}{\theta}\right)^{\beta}\right\}\right)^{k+m}\right\}^{\rho} \\
= & \sum_{i=0}^{\infty}\binom{\rho}{i}\left[\frac{(k+m+1)\lambda-(k-m-1)}{k(1-\lambda)}\right]^{i}[k(1-\lambda)]^{\rho}\left(1-\exp\left\{-\left(\frac{x}{\theta}\right)^{\beta}\right\}\right)^{\rho(k-1)+i(m+1)}
\end{aligned}
$$

Hence, the integral will be:

$$
=\sum_{i=0}^{\infty} v\frac{\beta^{\rho}}{\theta^{\beta\rho}}\int_0^{\infty} x^{\rho(\beta-1)}\left(\exp\left\{-\left(\frac{x}{\theta}\right)^{\beta}\right\}\right)^{(\rho)}\left(1-\exp\left\{-\left(\frac{x}{\theta}\right)^{\beta}\right\}\right)^{\rho(k-1)+i(m+1)} dx
$$

Such that, $v = \left(\frac{k+m+1}{2(m+1)}\right)^{\rho} \binom{\rho}{i} \left[\frac{(k+m+1)\lambda-(k-m-1)}{k(1-\lambda)}\right]^{i} [k(1-\lambda)]^{\rho}$

$$= \sum_{i=0}^{\infty} v \frac{\beta^{\rho-1}}{\theta^{\beta(\rho-1)}} \int_0^{\infty} x^{(\beta-1)(\rho-1)} (F)^{\rho(k-1)+i(m+1)} (1-F)^{(\rho-1)} dF$$

$$= \sum_{i=0}^{\infty} v \frac{\beta^{\rho-1}}{\theta^{\beta(\rho-1)}} M_{r,\ L,\ n}.$$

where $M_{r,\ L,\ n}$ is the probability-weighted moments given by equation (18) and $r = (\rho-1)(\beta-1)$, $L = \rho(k-1)+i(m+1)$ and $n = \rho-1$. Hence,

$$M_{r,\ L,\ n} = \int_0^1 (\theta(-log(1-F))^{\frac{1}{\beta}})^r [F(x)]^L [1-F(x)]^n dF(x)$$

$$= \theta^r \int_0^{\infty} (u)^{\frac{r}{\beta}} [1-\exp\{-u\}]^L \exp\{-un\} du.$$

Using Newton's generalized of binomial expansion for $[1-\exp\{-u\}]^L$ and complete integration:

$$= \theta^r \sum_{l=0}^{\infty} (-1)^l \binom{L}{l} \frac{1}{(l+n)^{\frac{r}{\beta}+1}} \Gamma\left(\frac{r}{\beta}+1\right)$$

$$= \theta^{(\rho-1)(\beta-1)} \times \sum_{l=0}^{\infty} (-1)^l \binom{\rho(k-1)+i(m+1)}{l} \frac{1}{(l+\rho-1)^{\frac{(\rho-1)(\beta-1)}{\beta}+1}} \Gamma\left(\frac{(\rho-1)(\beta-1)}{\beta}+1\right)$$

Hence, the integral will be:

$$= \sum_{i=0}^{\infty} v \frac{\beta^{\rho-1}}{\theta^{\beta(\rho-1)}} \times \sum_{l=0}^{\infty} (-1)^l \binom{\rho(k-1)+i(m+1)}{l} \frac{1}{(l+\rho-1)^{\frac{(\rho-1)(\beta-1)}{\beta}+1}} \Gamma\left(\frac{(\rho-1)(\beta-1)}{\beta}+1\right)$$

Substitute by v and take natural logarithm then the proof is complete. □

8.2.6 *Shannon entropy*

From equations (24) and (48) the Shannon entropy of the (DGOSTWD) directly given.

8.3 *Maximum likelihood estimation of dual generalized order statistics based transmuted Weibull distribution*

Let $X_1,\ X_2,\ \ldots,\ X_n$ be a random sample of size n from the (DGOSTWD) with transmutation parameter $\lambda \in [-1,\ 1]$. Let $\Theta = (k,\ m,\ \lambda,\ \beta,\ \theta)^T$ be the 5×1 parameter vector. The total log-likelihood function for Θ is given by:

$$l(\Theta) = \ln L = n\ln\left(\frac{k+m+1}{2(m+1)}\right) + n\ln\beta - n\beta\ln\theta + (\beta-1)\sum_{i=1}^{n}\ln x_i - \sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\beta}$$

$$+\sum_{i=1}^{n}\ln\left\{k(1-\lambda)\left(1-\exp\left\{-\left(\frac{x_i}{\theta}\right)^{\beta}\right\}\right)^{k-1}\right.$$

$$\left.+((k+m+1)\lambda-(k-m-1))\left(1-\exp\left\{-\left(\frac{x_i}{\theta}\right)^{\beta}\right\}\right)^{k+m}\right\}$$

The score function $U(\Theta) = (U_k,\ U_m,\ U_\lambda,\ U_\beta,\ U_\theta)^T$ are given in equations (35) to (38) by:

$$U_k = \frac{n}{k+m+1} + \sum_{i=1}^{n}\log\left[1-\exp\left\{-\left(\frac{x_i}{\theta}\right)^{\beta}\right\}\right]$$

$$+\sum_{i=1}^{n}\frac{(1-\lambda)\left[1-\exp\left\{-\left(\frac{x_i}{\theta}\right)^{\beta}\right\}\right]^{k-1} - (1-\lambda)\left[1-\exp\left\{-\left(\frac{x_i}{\theta}\right)^{\beta}\right\}\right]^{k+m}}{\delta_2(x,\ k,\ m,\ \lambda,\ \beta,\ \theta)},$$

$$U_m = \frac{-2kn}{2(m+1)(k+m+1)} + \sum_{i=1}^{n}\frac{(\lambda+1)\left[1-\exp\left\{-\left(\frac{x_i}{\theta}\right)^{\beta}\right\}\right]^{k+m}}{\delta_2(x,\ k,\ m,\ \lambda,\ \beta,\ \theta)}$$

$$+\sum_{i=1}^{n}\frac{((k+m+1)\lambda-(k-m-1))\left[1-\exp\left\{-\left(\frac{x_i}{\theta}\right)^{\beta}\right\}\right]^{k+m}\log\left(1-\exp\left\{-\left(\frac{x_i}{\theta}\right)^{\beta}\right\}\right)}{\delta_2(x,\ k,\ m,\ \lambda,\ \beta,\ \theta)},$$

$$U_\lambda = \sum_{i=1}^{n}\frac{(k+m+1)\left(1-\exp\left\{-\left(\frac{x_i}{\theta}\right)^{\beta}\right\}\right)^{k+m} - k\left(1-\exp\left\{-\left(\frac{x_i}{\theta}\right)^{\beta}\right\}\right)^{k-1}}{\delta_2(x,\ k,\ m,\ \lambda,\ \beta,\ \theta)},$$

$$U_\beta = \frac{n}{\beta} - n\ln\theta - \sum_{i=1}^{n}\ln x_i - \sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\beta}\ln\left(\frac{x_i}{\theta}\right)$$

$$+\sum_{i=1}^{n}\frac{k(k-1)(1-\lambda)\left(\frac{x_i}{\theta}\right)^{\beta}\ln\left(\frac{x_i}{\theta}\right)\exp\left\{-\left(\frac{x_i}{\theta}\right)^{\beta}\right\}\left(1-\exp\left\{-\left(\frac{x_i}{\theta}\right)^{\beta}\right\}\right)^{k-2}}{\delta_2(x,\ k,\ m,\ \lambda,\ \beta,\ \theta)}$$

$$+\sum_{i=1}^{n}\frac{(k+m)((k+m+1)\lambda-(k-m-1))\left(\frac{x_i}{\theta}\right)^{\beta}\ln\left(\frac{x_i}{\theta}\right)\exp\left\{-\left(\frac{x_i}{\theta}\right)^{\beta}\right\}\left(1-\exp\left\{-\left(\frac{x_i}{\theta}\right)^{\beta}\right\}\right)^{k+m-1}}{\delta_2(x,\ k,\ m,\ \lambda,\ \beta,\ \theta)},$$

$$U_\theta = -\frac{n\beta}{\theta} + \frac{\beta}{\theta}\sum_{i=1}^{n}\left(\frac{x}{\theta}\right)^\beta + \sum_{i=1}^{n}\frac{k(k-1)(1-\lambda)\left(\frac{\beta}{\theta}\right)\left(\frac{x_i}{\theta}\right)^\beta \exp\left\{-\left(\frac{x_i}{\theta}\right)^\beta\right\}\left(1-\exp\left\{-\left(\frac{x_i}{\theta}\right)^\beta\right\}\right)^{k-2}}{\delta_2(x,\, k,\, m,\, \lambda,\, \beta,\, \theta)}$$

$$+\sum_{i=1}^{n}\frac{(k+m)((k+m+1)\lambda-(k-m-1))\left(\frac{\beta}{\theta}\right)\left(\frac{x_i}{\theta}\right)^\beta \exp\left\{-\left(\frac{x_i}{\theta}\right)^\beta\right\}\left(1-\exp\left\{-\left(\frac{x_i}{\theta}\right)^\beta\right\}\right)^{k+m-1}}{\delta_2(x,\, k,\, m,\, \lambda,\, \beta,\, \theta)},$$

where,

$$\delta_2(x,\, k,\, m,\, \lambda,\, \beta,\, \theta) = k(1-\lambda)\left(1-\exp\left\{-\left(\frac{x_i}{\theta}\right)^\beta\right\}\right)^{k-1}$$

$$+((k+m+1)\lambda-(k-m-1))\left[1-\exp\left\{-\left(\frac{x_i}{\theta}\right)^\beta\right\}\right]^{k+m}$$

9. Simulation studies

This section introduces the Maximum likelihood estimation of parameters for both previously proposed transmuted families using Simulation studies.

9.1 *Generalized order statistics based transmuted Weibull simulation study*

A simulation study was performed by selecting random samples of sizes 50, 75, 200, and 500 from the (GOSTWD) with parameters $\beta = 3,\ \theta = 2,\ \lambda = 0.5,\ k = 2$, and $m = 2.5$. For each sample, the parameters of the GOSTWD were estimated. This process was carried out 10,000 times to calculate the average estimates and the Mean Square Errors (MSEs). The findings are summarized in Table 1. The table indicates that the estimated values of the parameters closely match the actual values. Additionally, the estimated MSEs tend to decrease consistently as the sample sizes increase. This indicates the effectiveness of the estimation method used.

Table 1. MLE estimation of the (GOSTWD) and corresponding averages of MSEs

Sample size	Estimate					MSEs				
	β	θ	λ	k	m	β	θ	λ	k	m
50	3.3248	1.7953	0.3543	1.5236	3.6692	0.5046	0.0857	0.1835	1.1047	13.0615
75	3.2026	1.8702	0.5151	1.4612	3.0404	0.2835	0.05801	0.1429	0.7003	6.6163
200	3.1004	1.7288	0.3965	1.2837	2.0803	0.1255	0.0964	0.12914	0.8004	2.4888
500	3.0175	2.0052	0.5115	1.9026	2.6788	0.02828	0.0215	0.05088	0.1243	0.4257

9.2 *Dual generalized order statistics based transmuted Weibull simulation study*

The simulation study involved sampling randomly from the (DGOSTWD) with sample sizes of 50, 75, 150, and 300, using parameters set at $\beta = 1.5,\ \theta = 2,\ \lambda = 0.6,\ k = 2$, and $m = 3$. For each drawn sample, we estimate the parameters of the (DGOSTWD). This procedure was performed 10,000 times, allowing us to compute the averages of the estimates and their Mean Square Errors (MSE). The findings are summarized in Table 2. The table indicates that the estimated parameter values are in close proximity to the true values. Furthermore, it is observed that the estimated MSEs progressively improve as the sample sizes increase. This demonstrates the efficacy of the estimation technique.

Table 2. MLE estimation of the (DGOSTWD) and corresponding averages of MSEs

Sample size	Estimate					MSEs				
	β	θ	λ	k	m	β	θ	λ	k	m
50	1.7645	2.2531	0.5005	2.1156	3.1325	0.3505	0.4851	0.2243	8.7297	2.19656
75	1.7605	2.2876	0.4545	1.8882	3.0895	0.3087	0.4660	0.2104	3.1067	2.0068
150	1.7260	2.3177	0.4886	1.8251	3.0388	0.2196	0.4608	0.2040	0.5814	2.0103
300	1.6479	2.2059	0.5077	1.8912	2.9385	0.1308	0.2814	0.1119	0.3871	2.0523

10. Real data applications

This section provides real-data applications for both previously propsed transmuted families using two real data sets.

10.1 *Forced expiratory volume data*

The Forced Expiratory Volume (FEV) for 654 children smokers, obtained from [16]. The summary statistics of the data set are presented in Table 3.

Table 3. Summary statistics for selected data sets

Data	Min	Q1	Median	Mean	Q3	Max
FEV	0.791	1.981	2.547	2.637	3.119	5.793
GNI	0.00710	0.06168	0.15720	0.23979	0.33924	1.69070

We have considered several distributions to evaluate the performance of the proposed GOSTW and DGOSTW distributions, including the Weibull distribution developed by [17], the Transmuted Weibull of Quadratic Rank (QRTW) by [18], and the Transmuted Weibull distribution Based on Records (RBTW) by [19]. Table 4 presents the computed values of various selection criteria such as log-likelihood, Akaike's Information Criterion (AIC), corrected Akaike's Information Criterion (AICc), Bayesian Information Criterion (BIC), and p-values from goodness-of-fit tests, including Kolmogorov-Smirnov (Dn), Anderson-Darling (A2), and Cramér-von Mises (W2). The log-likelihood and test statistics, with high p-values compared to the significance level of 0.05, indicate a favorable alignment for the GOSTW and DGOSTW distributions.

Table 4. Selection criteria values obtained for selected models

Distribution	LogLike	AIC	AICc	BIC	Dn	A2	W2	Estimated parameters
Weibull	-832.53	1,669.06	1,669.08	1,678.03	0.0427	0.0052	0.02401	$\beta = 3.21$ $\theta = 2.94$
QRTW	-836.6430	1,679.286	1,679.32	1,692.74	0.0002	0.000001	0.00003	$\beta = 1.9$ $\theta = 2.17$ $\lambda = -1$
RBTW	-816.0475	1,638.095	1,638.13	1,651.54	0.3656	0.0944	0.2008	$\beta = 2.221$ $\theta = 2.0535$ $\lambda = 1$
GOSTW	-811.0461	1,632.092	1,632.1848	1,654.51	0.6159	0.228	0.3614	$\beta = 2.4649$ $\theta = 2.5846$ $\lambda = -1$ $k = 1.0336$ $m = 1.3560$
DGOSTW	-805.6908	1,621.382	1,621.47	1,643.79	0.7766	0.6986	0.8047	$\beta = 1.6675$ $\theta = 1.7019$ $\lambda = -0.4030$ $k = 4.4463$ $m = 9.9953$

The above table clearly shows that the proposed GOSTW and DGOSTW distributions fit the data reasonably well compared to the other competing distributions. We have conducted the likelihood ratio test alongside the Akaike weights to compare the various distributions. The results of the likelihood ratio tests; alongside the p-values; are given in Table 5, whereas the results of the Akaike weights, see [20] , are given in Table 6, below:

Table 5. Likelihood ratio test for forced expiratory volume data

Distributions	Hyppthesis	Statistic ω	p-value
GOSTW vs Weibull	$H_o: m = 0,\ k = 1,\ \lambda = 0$ vs H_o is not true	42.9678	< 0.00001
GOSTW vs QRTW	$H_o: m = 0,\ k = 1$ vs H_o is not true	51.1938	< 0.0001
GOSTW vs RBTW	$H_o: m = -1,\ k = 1$ vs H_o is not true	10.0028	< 0.0067
DGOSTW vs Weibull	$H_o: m = 0,\ k = 1,\ \lambda = 0$ vs H_o is not true	53.6784	< 0.00001
DGOSTW vs QRTW	$H_o: m = 0,\ k = 1$ vs H_o is not true	61.9044	< 0.00001
DGOSTW vs RBTW	$H_o: m = -1,\ k = 1$ vs H_o is not true	20.7134	< 0.00001

Table 6. Akaike weights for forced expiratory volume data

GOSTW vs Others		DGOSTW vs Others		All	
Distribution	Akaike weight	Distribution	Akaike weight	Distribution	Akaike weight
Weibull	< 0.000001	Weibull	< 0.000001	Weibull	< 0.000001
QRTW	0.000001	QRTW	0.000001	QRTW	0.000001
RBTW	0.047358	RBTW	0.000235	RBTW	0.000234
GOSTW	0.952642	DGOSTW	0.999765	GOSTW	0.004701
				DGOSTW	0.995065

The results of the above table indicate that the proposed GOSTW and DGOSTW distributions are a better fit as compared with the other distributions. The results of Table 6 indicate that the Akaike weights for the proposed distributions are much higher as compared with the other distributions and hence the proposed distributions are the better fit to the data.

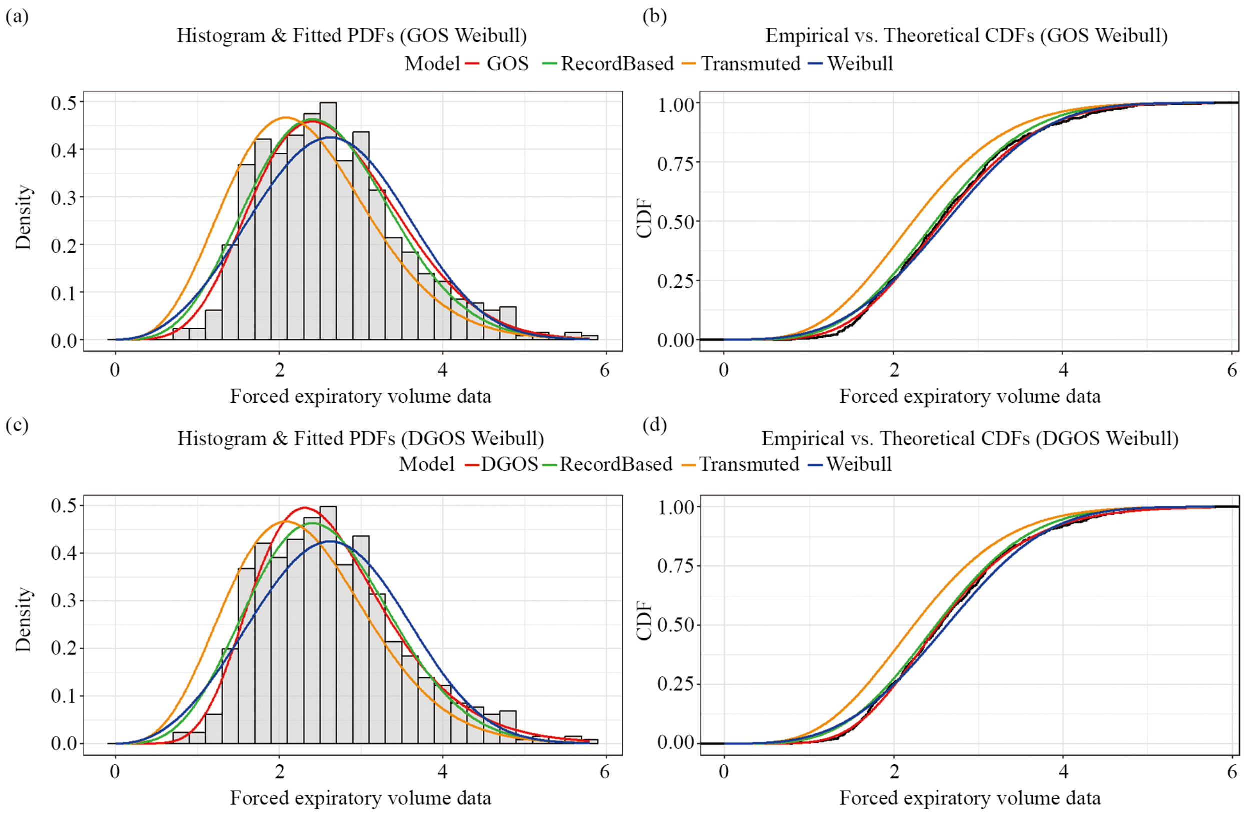


Figure 5. Estimated PDF and CDF for proposed GOSTW and DGOSTW along with selected models, using Forced Expiratory Volume (FEV) data set

The estimated PDF and CDF of GOSTW and DGOSTW distributions along with the selected models for the Forced Expiratory Volume data set are presented in the Figure 5.

10.2 *Gross national income data*

The Gross National Income (GNI) in hundred thousand dollars for 193 countries in the world for the year 2020 and can be accessed on the Human Development Reports website [21]. Summary statistics of the data set are presented in Table 3. To investigate the performance of new GOSTW and DGOSTW distributions, we have considered Weibull, QRTW, RBTW distributions. Table 7 contains the values calculated for the selection criteria Log-likelihood, AIC, AICc, BIC, and *p*-values of the goodness-of-fit test such as Dn, A2 and W2. It is to be noted that, the number of estimated parameters has a huge impact on the value of AIC, AICc and BIC. The obtained p-values of the selection criteria show the conformation in favor of GOSTW and DGOSTW distributions compared to the significance level of 0.05.

Table 7. Selection criteria values obtained for selected models

Distribution	LogLike	AIC	AICc	BIC	Dn	A2	W2	Estimated parameters
Weibull	80.3358	-156.6716	-156.6084	-150.1462	0.7116	0.3567	0.4213	$\beta = 1.0111$ $\theta = 0.2409$
QRTW	81.4187	-156.8374	-156.7104	-147.0493	0.8540	0.4394	0.5276	$\beta = 1.0716$ $\theta = 0.2985$ $\lambda = 0.3774$
RBTW	81.6191	-157.2382	-157.1112	-147.4501	0.7119	0.3569	0.4216	$\beta = 1.011$ $\theta = 0.2408$ $\lambda = 0.0004$
GOSTW	84.8325	-159.6650	-159.3441	-143.3515	0.7033	0.5122	0.7423	$\beta = 1.0664$ $\theta = 0.7203$ $\lambda = 0.4451$ $k = 2.3367$ $m = 1.8813$
DGOSTW	85.4510	-160.9020	-160.5811	-144.5885	0.6869	0.5745	0.6739	$\beta = 0.5695$ $\theta = 0.0642$ $\lambda = 0.0643$ $k = 3.2857$ $m = 1.6021$

The above table clearly shows that the proposed GOSTW and DGOSTW distributions fit the data reasonably well compared to the other competing distributions. We have also conducted the likelihood ratio test to compare the various distributions. The results of the likelihood ratio tests; alongside the p-values; are given in Table 8, whereas the results of the Akaike weights are given in Table 9, below:

Table 8. Likelihood ratio tests for the gross national income data

Distributions	Hyppthesis	Statistic ω	p-value
GOSTW vs Weibull	$H_o: m = 0,\ k = 1,\ \lambda = 0$ vs H_o is not true	8.9934	0.02938
GOSTW vs QRTW	$H_o: m = 0,\ k = 1$ vs H_o is not true	6.8276	0.03292
GOSTW vs RBTW	$H_o: m = -1,\ k = 1$ vs H_o is not true	6.4268	0.04022
DGOSTW vs Weibull	$H_o: m = 0,\ k = 1,\ \lambda = 0$ vs H_o is not true	10.2303	0.01671
DGOSTW vs QRTW	$H_o: m = 0,\ k = 1$ vs H_o is not true	8.0645	0.01773
DGOSTW vs RBTW	$H_o: m = -1,\ k = 1$ vs H_o is not true	7.6637	0.02167

Table 9. Akaike weights for gross national income data

GOSTW vs Others		DGOSTW vs Others		All	
Distribution	Akaike weight	Distribution	Akaike weight	Distribution	Akaike weight
Weibull	0.126890	Weibull	0.085435	Weibull	0.061836
QRTW	0.137857	QRTW	0.092820	QRTW	0.067181
RBTW	0.168447	RBTW	0.113416	RBTW	0.082088
GOSTW	0.566807	DGOSTW	0.708329	GOSTW	0.276219
				DGOSTW	0.512676

The results of the above table indicate that the proposed GOSTW and DGOSTW distributions are a better fit than the other distributions at a 0.05 level of significance. The results of Table 9 indicate that the Akaike weights for the proposed distributions are relatively higher as compared with the other distributions and hence the proposed distributions are the better fit to the data. The estimated PDF and CDF of GOSTW and DGOSTW distribution. with the selected models for the (GNI) Data set are presented in Figure 6.

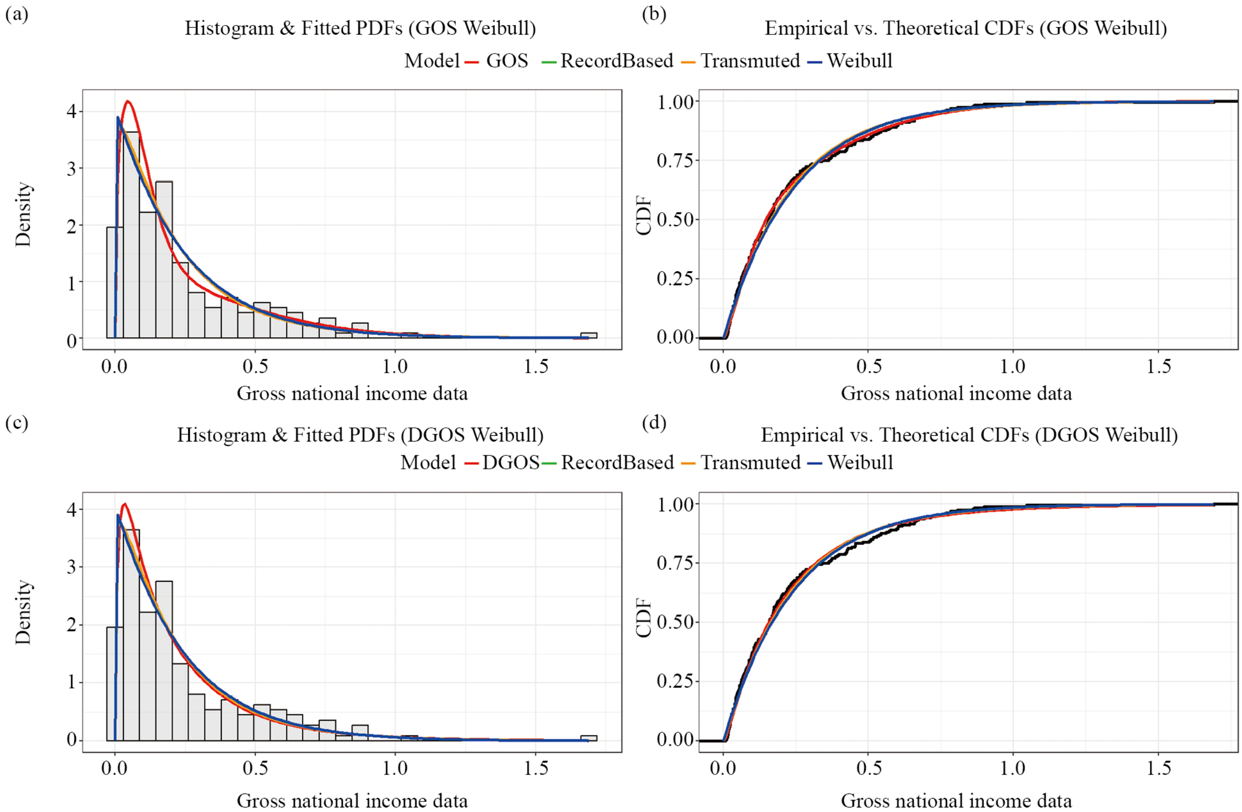


Figure 6. Estimated PDF and CDF for proposed GOSTW and DGOSTW along with selected models,using the Gross National Income (GNI) data set

Conflict of interest

The authors declare no competing financial interest.

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