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Research Article

Structure of Algebras Satisfying an ω-Polynomial Identity of Degree Six

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Abstract: This paper is devoted to the study of a class of commutative non-associative algebras characterized by the identity:

$$x^2x^4 = (1-\alpha)\omega(x)^2x^4 + \alpha\omega(x)^4x^2,$$

where $\alpha \in [0, 1]$. In this study, we strongly use the Peirce decomposition technique. This allowed us to determine the conditions for an algebra of this class to be Bernstein, principal train, or evolution.

Keywords: idempotent, Peirce decomposition, ω-polynomial identity, Bernstein algebras, train algebras, evolution algebra

MSC: 17A30, 17A36

Abbreviation

GCD Greatest Common Divisor

1. Introduction

In 1923, Serge Bernstein proved that the only law of heredity compatible with the principle of stationarity is Mendel's law. However, it was only in 1975 that Ph. Holgate proposed a rigorous algebraic definition of structures known as Bernstein algebras [1].

Let A be a commutative non-associative algebra over a commutative field $\mathbb{K}$, and $\omega : A \to \mathbb{K}$ a non-zero morphism of algebras. For any $x \in A$ and $n \in \mathbb{N}^*$, the principal power of x in A is defined as $x^1 = x$ and recursively as $x^{n+1} = x^n x$. The pair (A, ω) is called a weighted algebra. A weighted algebra (A, ω) is a Bernstein algebra, if for any $x \in A$, $(x^2)^2 = \omega(x)^2x^2$. (see [2–4] for more details).

The objective of this paper is to describe the main properties of commutative and non-associative algebras characterized by the identity:

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$$x^2x^4 = (1-\alpha)\omega(x)^2x^4 + \alpha\omega(x)^4x^2, \quad \forall x \in A \quad \text{where} \quad \alpha \in [0,\, 1]. \tag{1}$$

Unless explicitly stated otherwise, throughout this study, A is a weighted $\mathbb{C}$-algebra that satisfies (1), where $\mathbb{C}$ is the field of complex numbers.

2. Preliminaries

For an algebra A satisfying identity (1), an element x can be interpreted as a genotype of a given population. The successive powers x^i describe the genotypes of individuals from the i-th generation, obtained through i successive crossings of the initial population. So, identity (1) models a population where the genetic crossing between the second and fourth generations produces individuals with genetic traits in proportions $1-\alpha$ and α from the two populations. The real α could represent the frequency of individuals in the population resulting from the crossing between individuals from the second and fourth generations. When $\alpha = 0$, the crossing generates only individuals from the fourth generation. This phenomenon can be explained by the assumption that individuals from the second generation carry hidden alleles, while those from the fourth generation express visible alleles. Conversely, when $\alpha = 1$, only individuals from the second generation are produced. This scenario could be justified by assuming that, in this case, individuals from the fourth generation carry the hidden alleles, while those from the second generation possess the visible alleles. In A, an element e satisfying $e^2 = e$ is called idempotent and represents a state of equilibrium.

Proposition 2.1 Let x, y and z be elements of A. Then:

$$\begin{aligned}
&2(yx)x^4 + x^2(yx^3) + x^2(x(yx^2)) + 2x^2(x(x(xy))) \\
&= (1-\alpha)\left[2\omega(yx)x^4 + \omega(x)^2yx^3 + \omega(x)^2x(yx^2) + 2\omega(x)^2x(x(yx))\right] + \alpha\left[4\omega(yx^3)x^2 + 2\omega(x)^4yx\right];
\end{aligned} \tag{2}$$

$$\begin{aligned}
&2(yz)x^4 + 2(yx)(zx^3) + 2(yx)(x(zx^2)) + 4(yx)(x(x(zx))) + 2(zx)(yx^3) \\
&+ x^2(y(zx^2)) + 2x^2(y(x(xz))) + 2(zx)(x(yx^2)) + x^2(z(yx^2)) + 2x^2(x(y(zx))) \\
&+ 4(zx)(x(x(xy))) + 2x^2(z(x(xy))) + 2x^2(x(z(xy))) + 2x^2(x(x(zy))) \\
&= (1-\alpha)\left[2\omega(yz)x^4 + 2\omega(yx)zx^3 + 2\omega(yx)x(zx^2) + 4\omega(yx)x(x(xz))\right. \\
&+ 2\omega(zx)yx^3 + \omega(x)^2y(zx^2) + 2\omega(x)^2y(x(xz)) + 2\omega(xz)x(yx^2) + \omega(x)^2z(yx^2) + 2\omega(x)^2x(y(zx)) \\
&\left.+ 4\omega(zx)x(x(yx)) + 2\omega(x)^2z(x(yx)) + 2\omega(x)^2x(z(yx)) + 2\omega(x)^2x(x(yz))\right] \\
&+ \alpha\left[12\omega(y(zx^2))x^2 + 8\omega(yx^3)zx + 8\omega(zx^3)yx + 2\omega(x)^4yz\right].
\end{aligned} \tag{3}$$

Proof. The identities (2) and (3) are obtained by a partial linearization of order 1 and 2 respectively of (1) (The linearization technique is detailed in [5], 3. Linearization, p.174). □

The following example is an algebra that verifies the identity (1).

Example 2.2 Let $A = (\langle a_1, a_2, a_3\rangle, \omega)$ be a weighted $\mathbb{C}$-algebra with the multiplication table defined as follows: $a_1^2 = a_1 + a_3$, $a_2^2 = a_3$, $a_1a_2 = \frac{1}{2}a_2 + a_3$, and $a_1a_3 = \delta a_3$, where $\delta = \frac{-\alpha - \mathrm{i}\sqrt{-\alpha^2+4(\alpha+1)}}{2}$ (products not mentioned are zero). The algebra homomorphism $\omega : A \to \mathbb{K}$ is defined such that $\omega(a_1) = 1$ and $\omega(a_2) = \omega(a_3) = 0$. For $x = \beta a_1 + \gamma a_2 + \mu a_3 \in A$, we have $x^2x^4 = (1-\alpha)\omega(x)^2x^4 + \alpha\omega(x)^4x^2$, i.e. A satisfies identity (1).

This algebra is not Bernstein algebra. Indeed, we have $(x^2)^2 - \omega(x)^2x^2 = [2\delta\beta^4 + 2\delta\beta^2\gamma^2 + 4\delta\beta^3\gamma + 2\delta(2\delta - 1)\beta^3\mu]a_3 \neq 0$ for some x. A non-zero idempotent of the algebra A is given by: $a_1 + \gamma a_2 + \frac{(1+\gamma)^2}{1-2\delta}a_3 \mid \gamma \in \mathbb{C}$.

In the following, we denote A as an algebra satisfying identity (1) and assume that A has a non-zero idempotent which we will write e.

Proposition 2.3 The Peirce decomposition of A is:

$$A = \mathbb{C}e \oplus A_0 \oplus A_{\frac{1}{2}} \oplus A_\delta \oplus A_{\overline{\delta}},$$

where $A_\mu = \{x \in \ker\omega \mid ex = \mu x\}$, $\mu \in \left\{0, \frac{1}{2}, \delta, \overline{\delta}\right\}$, $\delta = \frac{-\alpha - \mathrm{i}\sqrt{-\alpha^2+4(\alpha+1)}}{2}$ and $\overline{\delta} = \frac{-\alpha + \mathrm{i}\sqrt{-\alpha^2+4(\alpha+1)}}{2}$.

Proof. Consider the mapping $\ell_e : \ker\omega \to \ker\omega$, $x \mapsto ex$. Substituting $x = e$ into (2), we obtain:

$$\ell_e\left(\ell_e - \frac{1}{2}\right)\left(\ell_e^2 + \alpha\ell_e + \alpha + 1\right) = 0. \tag{4}$$

Setting $Q(t) = t\left(t - \frac{1}{2}\right)\left(t^2 + \alpha t + (\alpha+1)\right)$.

We have $Q(\ell_e) = 0$ and $Q(t) = t\left(t - \frac{1}{2}\right)\left(t - \delta\right)\left(t - \overline{\delta}\right)$. Consequently, the Peirce decomposition of A is: $A = \mathbb{C}e \oplus \ker Q(\ell_e) = \mathbb{C}e \oplus A_0 \oplus A_{\frac{1}{2}} \oplus A_\delta \oplus A_{\overline{\delta}}$, where $A_\mu = \{x \in \ker\omega \mid ex = \mu x\}$, $\mu \in \left\{0, \frac{1}{2}, \delta, \overline{\delta}\right\}$. □

In the remainder of our study, we will assume that $\alpha \neq \frac{1}{2}$, as this case has already been studied in [6, 7].

3. Structure

The theorem below provides the product of Peirce spaces of the algebra A.

Theorem 3.1 Let $A = \mathbb{C}e \oplus A_0 \oplus A_{\frac{1}{2}} \oplus A_\delta \oplus A_{\overline{\delta}}$, we have:

(i) $A_0A_0 \begin{cases} \subseteq A_0, & \text{if } \alpha = 0, \\ = \{0\}, & \text{otherwise}; \end{cases}$

(ii) $A_0A_{\frac{1}{2}} \subseteq A_{\frac{1}{2}} \oplus A_\delta \oplus A_{\overline{\delta}}$;

(iii) $A_0A_\delta \begin{cases} \subseteq A_{\overline{\delta}}, & \text{if } \alpha = 1, \\ = \{0\}, & \text{otherwise}; \end{cases}$

(iv) $A_0A_{\overline{\delta}} \begin{cases} \subseteq A_\delta, & \text{if } \alpha = 1, \\ = \{0\}, & \text{otherwise}; \end{cases}$

(v) $A_{\frac{1}{2}}A_{\frac{1}{2}} \subseteq A_0 \oplus A_\delta \oplus A_{\overline{\delta}}$;

(vi) $A_{\frac{1}{2}}A_\delta \subseteq A_0 \oplus A_{\frac{1}{2}} \oplus A_{\overline{\delta}}$;

(vii) $A_{\frac{1}{2}}A_{\overline{\delta}} \subseteq A_0 \oplus A_{\frac{1}{2}} \oplus A_\delta$;

(viii) $A_\delta A_\delta = \{0\}$;

(ix) $A_{\overline{\delta}}A_{\overline{\delta}} = \{0\}$;

(x) $A_{\delta}A_{\overline{\delta}} \begin{cases} \subseteq A_0, & \text{if } \alpha = 0, \\ = \{0\}, & \text{otherwise.} \end{cases}$

Proof. Substituting $x = e$, $y \in A_{\mu}$, i.e. $ey = \mu y$ and $z \in A_{\beta}$, i.e. $ez = \beta z$ into identity (3), we obtain:

$$2e(yz)+2\mu\beta(yz)+2\mu\beta^2(yz)+4\mu\beta^3(yz)+2\mu\beta(yz)+\beta e(yz)+2\beta^2 e(yz)$$

$$+2\mu^2\beta(yz)+\mu e(yz)+2\beta e(e(yz))+4\mu^3\beta(yz)+2\mu^2 e(yz)+2\mu e(e(yz))+2e(e(e(yz)))$$

$$= (1-\alpha)[\beta yz+2\beta^2 yz+\mu yz+2\beta e(yz)+2\mu^2 yz+2\mu e(yz)+2e(e(yz))]+2\alpha(yz). \tag{5}$$

From this, we obtain:

$$2\ell_e^3(yz)+2\Big[\beta+\mu+\alpha-1\Big]\ell_e^2(yz)+\Big[2\big(\mu^2+\beta^2\big)+\big(\mu+\beta\big)\big(2\alpha-1\big)+2\Big]\ell_e(yz)$$

$$+\Big[\big(2\mu\beta+\alpha-1\big)\big[2\big(\mu^2+\beta^2\big)+\big(\mu+\beta\big)\big]+2\big(2\mu\beta-\alpha\big)\Big]Id_A(yz)=0. \tag{6}$$

Let $P_{(\mu,\,\beta)}(t)$ be the annihilating polynomial of the space $A_{\mu}A_{\beta}$, defined as:

$$P_{(\mu;\,\beta)}(t)=2t^3+2\big(\mu+\beta+\alpha-1\big)t^2+\big[2\big(\mu^2+\beta^2\big)+\big(\mu+\beta\big)\big(2\alpha-1\big)+2\big]t$$

$$+\Big[\big(2\mu\beta+\alpha-1\big)\big[2\big(\mu^2+\beta^2\big)+\big(\mu+\beta\big)\big]+2\big(2\mu\beta-\alpha\big)\Big], \tag{7}$$

with $\mu,\,\beta \in \left\{0,\,\frac{1}{2},\,\delta,\,\overline{\delta}\right\}$.

• For $\mu=\beta=0$, we have $P_{0,\,0}(t)=2t^3+2(\alpha-1)t^2+2t-2\alpha$. If $\alpha=0$, then $P_{0,\,0}(t)=t(2t^2-2t+2)$. Hence, $\gcd(Q(t),\,P_{0,\,0}(t))=t$ if $\alpha=0$, and $\gcd(Q(t),\,P_{0,\,0}(t))=1$ if $\alpha\neq 0$. Thus, $A_0A_0\subseteq A_0$ if $\alpha=0$, and $A_0A_0=0$ otherwise.

• For $\mu=0$, $\beta=\frac{1}{2}$, we have $P_{0,\,\frac{1}{2}}(t)=2t^3+(2\alpha-1)t^2+(\alpha+2)t-(\alpha+1)$. Factoring yields $P_{0,\,\frac{1}{2}}(t)=2\left(t-\frac{1}{2}\right)(t-\delta)\left(t-\overline{\delta}\right)$. Thus, $\gcd(Q(t),\,P_{0,\,\frac{1}{2}}(t))=\left(t-\frac{1}{2}\right)(t-\delta)\left(t-\overline{\delta}\right)$. We conclude that $A_0A_{\frac{1}{2}}\subseteq A_{\frac{1}{2}}\oplus A_{\delta}\oplus A_{\overline{\delta}}$.

• For $\mu=0$, $\beta=\gamma$, with $\gamma\in\{\delta,\,\overline{\delta}\}$, we have $P_{0,\,\gamma}(t)=2t^3+2(\gamma+\alpha-1)t^2-(2\alpha+\gamma)t-\big(2\alpha^2\gamma+2\alpha^2-3\alpha\gamma+2\alpha+\gamma-2\big)$. Here, $\gcd(Q(t),\,P_{0,\,\gamma}(t))=1$ if $\alpha\neq 1$ so $A_0A_{\gamma}=0$, and $\gcd(Q(t),\,P_{0,\,\gamma}(t))=(t-\overline{\gamma})$ if $\alpha=1$ then $A_0A_{\gamma}\subseteq A_{\overline{\gamma}}$ for $\gamma\in\{\delta,\,\overline{\delta}\}$.

Other results are obtained in the same way. □

Lemma 3.2 Suppose that $A=\mathbb{C}e\oplus A_{\frac{1}{2}}\oplus A_{\mu}$ and $\mu\in\{\delta,\,\overline{\delta}\}$. Then, for any $x_{\beta}\in A_{\beta}$, $\beta\in\left\{\delta,\,\frac{1}{2}\right\}$, we have:

(i) $x_{\frac{1}{2}}^3=0$;

(ii) $x_{\mu}\left(x_{\frac{1}{2}}x_{\mu}\right)=0$;

(iii) $x_{\frac{1}{2}}\left(x_{\frac{1}{2}}x_{\mu}\right)=0$;

(iv) $\left(x_{\frac{1}{2}}x_{\mu}\right)^2 = 0$;

(v) $x_{\frac{1}{2}}^2\left(x_{\frac{1}{2}}x_{\mu}\right) = 0$.

Proof. Let's assume that $\mu = \delta$, then $A = \mathbb{C}e \oplus A_{\frac{1}{2}} \oplus A_{\delta}$. The multiplication table satisfies: $A_{\frac{1}{2}}A_{\frac{1}{2}} \subseteq A_{\delta}$, $A_{\frac{1}{2}}A_{\delta} \subseteq A_{\frac{1}{2}}$, and $A_{\delta}A_{\delta} = 0$. Let x be an element of A of weight 1, such that $x = e + ax_{\frac{1}{2}} + bx_{\delta}$, with for any $x_{\beta} \in A_{\beta}$, $\beta \in \left\{\delta, \frac{1}{2}\right\}$. We have:

$$x^2 = e + ax_{\frac{1}{2}} + 2\delta bx_{\delta} + a^2x_{\frac{1}{2}}^2 + 2abx_{\frac{1}{2}}x_{\delta},$$

$$x^3 = e + \left[ax_{\frac{1}{2}} + 2(1+\delta)abx_{\frac{1}{2}}x_{\delta} + a^3x_{\frac{1}{2}}^3 + 2ab^2x_{\delta}\left(x_{\frac{1}{2}}x_{\delta}\right)\right]_{\frac{1}{2}}$$

$$+ \left[\left(2\delta^2 + \delta\right)bx_{\delta} + (\delta+1)a^2x_{\frac{1}{2}}^2 + 2a^2bx_{\frac{1}{2}}\left(x_{\delta}x_{\frac{1}{2}}\right)\right]_{\delta},$$

$$x^4 = e + \left[ax_{\frac{1}{2}} + \left(\delta + \frac{3}{2}\right)a^3x_{\frac{1}{2}}^3 + 2\left(\delta^2 + \delta + 1\right)abx_{\frac{1}{2}}x_{\delta} + (2\delta+3)ab^2x_{\delta}\left(x_{\frac{1}{2}}x_{\delta}\right)\right.$$

$$\left. + \left(x_{\frac{1}{2}}^3x_{\delta} + 2x_{\frac{1}{2}}\left(x_{\frac{1}{2}}\left(x_{\frac{1}{2}}x_{\delta}\right)\right)\right)a^3b + 2ab^3x_{\delta}\left(x_{\delta}\left(x_{\frac{1}{2}}x_{\delta}\right)\right)\right]_{\frac{1}{2}} + \left[\left(2\delta a^3 + \delta^2 + \delta\right)bx_{\delta}.\right.$$

$$\left. + \left(\delta^2 + \delta + 1\right)a^2x_{\frac{1}{2}}^2 + a^4x_{\frac{1}{2}}^4 + 2(2\delta+1)a^2bx_{\frac{1}{2}}\left(x_{\frac{1}{2}}x_{\delta}\right) + 2a^2b^2x_{\frac{1}{2}}\left(x_{\delta}\left(x_{\frac{1}{2}}x_{\delta}\right)\right)\right]_{\delta}.$$

Using the identity $(1-\alpha)x^4 + \alpha x^2 = x^2x^4$, and equating the powers of a^kb^j, $0 \leq k, j \leq 3$, we obtain the identities of the lemma.

The proof for $\mu = \overline{\delta}$ follow the same logic as for $\mu = \delta$. □

Lemma 3.3 Let $A = \mathbb{C}e \oplus A_0 \oplus A_{\delta} \oplus A_{\overline{\delta}}$, with $\alpha = 0$. Then, for any $x_{\beta} \in A_{\beta}$, $\beta \in \{0, \delta, \overline{\delta}\}$, we have:

(i) $x_0^4 = 0$;

(ii) $2x_0^2(x_{\overline{\delta}}x_{\delta}) = x_0(x_0(x_{\delta}x_{\overline{\delta}}))$;

(iii) $x_0^2(x_0(x_0(x_{\delta}x_{\overline{\delta}}))) = 0$;

(iv) $(x_{\overline{\delta}}x_{\delta})(x_0(x_0(x_{\overline{\delta}}x_{\delta}))) = 0$;

(v) $(x_{\overline{\delta}}x_{\delta})^2 = 0$.

Proof. The proof adheres to the same structure as Lemma 3.2. □

In the following sub-paragraphs, we establish links between algebras verifying the identity (1) with some weighted algebras.

3.1 *Algebras verifying the identity (1) which are Bernstein*

Lemma 3.4 Let $A = \mathbb{C}e \oplus A_0 \oplus A_{\frac{1}{2}}$. Then, for any $x_{\beta} \in A_{\beta}$, $\beta \in \left\{0, \frac{1}{2}\right\}$, we have:

(1) If $\alpha = 0$:

(i) $x_{\frac{1}{2}}^3 = 0$;

(ii) $\left(x_{\frac{1}{2}}^2\right)^2 = 0$;

(iii) $x_0^4 = 0$;

(iv) $x_0x_{\frac{1}{2}}^2 = 2x_{\frac{1}{2}}\left(x_0x_{\frac{1}{2}}\right)$;
(v) $x_0^2x_{\frac{1}{2}} = 2x_0\left(x_0x_{\frac{1}{2}}\right)$;
(vi) $5\left(x_0x_{\frac{1}{2}}\right)^2 = 2x_0\left(x_0x_{\frac{1}{2}}^2\right) = 4x_0\left(x_{\frac{1}{2}}\left(x_0x_{\frac{1}{2}}\right)\right)$;
(vii) $\frac{1}{2}x_0^3x_{\frac{1}{2}} - 2x_0^2\left(x_0x_{\frac{1}{2}}\right) + 2x_0\left(x_0\left(x_0x_{\frac{1}{2}}\right)\right) = 0$;
(viii) $2x_{\frac{1}{2}}^2\left(x_0x_{\frac{1}{2}}\right) = x_{\frac{1}{2}}\left(x_{\frac{1}{2}}\left(x_0x_{\frac{1}{2}}\right)\right)$.
(2) If $\alpha \neq 0$ or ($\alpha = 0$ and $A_0^2 = 0$):
(i) $x_{\frac{1}{2}}^3 = 0$;
(ii) $x_0\left(x_0x_{\frac{1}{2}}\right) = 0$;
(iii) $x_{\frac{1}{2}}\left(x_0x_{\frac{1}{2}}\right) = 0$;
(iv) $x_{\frac{1}{2}}^2\left(x_0x_{\frac{1}{2}}\right) = 0$;
(v) $\left(x_0x_{\frac{1}{2}}\right)^2 = 0$.

Proof. The proof adheres to the same pattern as Lemma 3.2. □

The following result provides the necessary and sufficient conditions for an algebra satisfying identity (1) to be a Bernstein algebra.

Theorem 3.5 Let $A = \mathbb{C}e \oplus A_0 \oplus A_{\frac{1}{2}} \oplus A_\delta \oplus A_{\overline{\delta}}$. A is a Bernstein algebra if and only if $A_\delta = A_{\overline{\delta}} = A_0^2 = 0$.

Proof. Let's suppose that the algebra A satisfies identity (1) with $\alpha = 0$ and $A_\delta = A_{\overline{\delta}} = A_0^2 = 0$. Let $x = e + x_0 + x_{\frac{1}{2}}$ be an element of A such that $\omega(x) = 1$. Then: $x^2 = e + x_{\frac{1}{2}} + x_{\frac{1}{2}}^2 + 2\left(x_0x_{\frac{1}{2}}\right)$, $(x^2)^2 = e + x_{\frac{1}{2}} + x_{\frac{1}{2}}^2 + 2x_{\frac{1}{2}}^3 + 2x_0x_{\frac{1}{2}} + 4\left(x_0x_{\frac{1}{2}}\right)^2 + 4x_{\frac{1}{2}}\left(x_0x_{\frac{1}{2}}\right) + 4x_{\frac{1}{2}}^2\left(x_0x_{\frac{1}{2}}\right)$. Using the identities in part (2) of Lemma 3.4, we find that $(x^2)^2 - x^2 = 0$. Since the set of elements of weight 1 (i.e. such that $\omega(x) = 1$) is dense in A, by the Zariski topology (for its definition and main characteristics on spaces not necessarily of finite dimension, see McCrimmon in [8]), it follows that $(x^2)^2 - \omega(x)^2x^2 = 0$. Thus, A is a Bernstein algebra.

Conversely, if A is a Bernstein algebra, then $A = \mathbb{C}e \oplus A_0 \oplus A_{\frac{1}{2}}$ and $A_0A_0 \subseteq A_{\frac{1}{2}}$. Since A satisfies identity (1), the Peirce decomposition and the multiplication table of its Peirce subspaces imply that $A_\delta = A_{\overline{\delta}} = 0$ and $A_0A_0 = 0$.

For $\alpha \neq 0$, the proof is similar to the previous case. □

3.2 *Power-associativity of algebras verifying the identity (1)*

Proposition 3.6 Let $A = \mathbb{C}e \oplus A_0 \oplus A_{\frac{1}{2}} \oplus A_\delta \oplus A_{\overline{\delta}}$. Then the following conditions are equivalent:
(i) A is a Jordan algebra;
(ii) A is power-associative, i.e. $x^kx^j = x^{k+j}$, $\forall x \in A$, $\forall\, k,\, j \geq 1$;
(iii) $A_\delta = A_{\overline{\delta}} = A_0^2 = 0$;
(iv) A satisfies the equation $x^3 - \omega(x)x^2 = 0$.

Proof. Assume that A satisfies identity (1) with $\alpha \neq 0$. For (i) ⇒ (ii), see [9]. For (ii) ⇒ (iii), see [9]. For (iii) ⇒ (iv), since $A_\delta = A_{\overline{\delta}} = 0$ and $\alpha \in [0,\, 1]$, using the identities from Lemma 3.4, we have: $x^2 = e + bx_{\frac{1}{2}} + b^2x_{\frac{1}{2}}^2 + 2abx_0x_{\frac{1}{2}}$, $x^3 = e + bx_{\frac{1}{2}} + b^2x_{\frac{1}{2}}^2 + 2abx_0x_{\frac{1}{2}}$. Thus, $x^3 - x^2 = 0$, which leads to $x^3 - \omega(x)x^2 = 0$ according to Zariski topology. For (iv) ⇒ (i), see Corollary 4.4 and Corollary 4.5 in [10]. The proof is analogous if $\alpha = 0$. □

3.3 *Algebras verifying the identity (1) which are train*

Definition 3.7 A weighted $\mathbb{C}$-algebra $(A,\, \omega)$ is a train algebra of rank $n \geq 2$ if there exist scalars $\mu_i \in \mathbb{C}$ for $i \in \{1,\, \ldots,\, n-1\}$ such that:

$$x^n+\mu_1\omega(x)x^{n-1}+\mu_2\omega(x)^2x^{n-2}+\cdots+\mu_{n-1}\omega(x)^{n-1}x=0, \quad (8)$$

for all x in A, and n is the smallest integer satisfying this equation.

Proposition 3.8 Suppose $A=\mathbb{C}e\oplus A_0\oplus A_{\bar{\delta}}\oplus A_\delta$ and $\mu\in\{\delta,\,\bar{\delta}\}$.

(1) For $\alpha=0$:

If $A_\mu=0$, then A satisfies the equation: $x^6-(1+\bar{\mu})\omega(x)x^5+\bar{\mu}\omega(x)^2x^4=0$.

(2) For $\alpha\neq 0$:

If $A_\mu=0$, then A satisfies the equation: $x^4-(1+\bar{\mu})\omega(x)x^3+\bar{\mu}\omega(x)^2x^2=0$.

Proof. For (1), assume $A=\mathbb{C}e\oplus A_0\oplus A_\delta$. Since $\alpha=0$, we have $A_0A_0\subseteq A_0$ and $A_0A_\delta=A_\delta A_\delta=0$. Let $x\in A$ such that $x=e+x_0+x_\delta$. Then: $x^2=e+x_0^2+2\delta x_\delta$, $x^2-x=x_0+x_0^2+(2\delta-1)x_\delta$, $x(x^2-x)=(2\delta^2-\delta)x_\delta-x_0^2+x_0^3$, $x(x(x^2-x))=\delta(2\delta^2-\delta)x_\delta-x_0^3+x_0^4$, $x(x(x(x^2-x)))=\delta^2(2\delta^2-\delta)x_\delta-x_0^4+x_0^5$. Since $x_0^4=0$ by identity (i) in Lemma 3.3, it follows that $x^6-(1+\delta)x^5+\delta x^4=0$. Using the Zariski topology, we deduce that for all $x\in A$: $x^6-(1+\delta)\omega(x)x^5+\delta\omega(x)^2x^4=0$. The proof for $A_\delta=0$ is similar to that $A_{\bar{\delta}}=0$.

For (2), the proof is similar to the previous case. □

Proposition 3.9 Let $A=\mathbb{C}e\oplus A_{\frac{1}{2}}\oplus A_{\bar{\delta}}\oplus A_\delta$. If $A_\mu=0$, then A satisfies the equation: $x^3-(1+\bar{\mu})\omega(x)x^2+\bar{\mu}\omega(x)^2x=0$, where $\mu\in\{\delta,\,\bar{\delta}\}$.

Proof. Suppose $A=\mathbb{C}e\oplus A_{\frac{1}{2}}\oplus A_\delta$. Let $x\in A$ such that $x=e+x_{\frac{1}{2}}+x_\delta$. Using the identities from Lemma 3.2, we have $x(x^2-x)=\delta(x^2-x)$. Using Zariski topology, it follows that for all $x\in A$, $x^3-(1+\delta)\omega(x)x^2+\delta\omega(x)^2x=0$. The proof for $A_\delta=0$ is analogous to the case of $A_{\bar{\delta}}=0$, relying on the identities from Lemma 3.2. □

Proposition 3.10 Let $A=\mathbb{C}e\oplus A_0\oplus A_{\frac{1}{2}}\oplus A_{\bar{\delta}}\oplus A_\delta$. Then A is a train algebra of rank 3 if and only if its train equation is of the form:

$$x^3-(1+\mu)\omega(x)x^2+\mu\omega(x)^2x=0, \quad \text{with} \quad \mu\in\{0,\,\delta,\,\bar{\delta}\}.$$

Proof. Let A be an algebra satisfying the identity (1). Assume that A is a train algebra of rank 3, and its train equation is given by:

$$x^3-(1+\mu)\omega(x)x^2+\mu\omega(x)^2x=0, \quad \text{where} \quad \mu\in\mathbb{K}. \quad (9)$$

A partial linearization of (9) yields:

$$yx^2+2x(xy)-(1+\mu)[\omega(y)x^2+2\omega(x)(xy)]+\mu[2\omega(xy)x+\omega(x)^2y]=0. \quad (10)$$

Let $y=x^4$ in (10), we have:

$$x^4x^2+2x^6-(1+\mu)[\omega(x)^4x^2+2\omega(x)x^5]+\mu[2\omega(x)^5x+\omega(x)^2x^4]=0. \quad (11)$$

Since A satisfies the identity (1), we know: $x^2x^4=(1-\alpha)\omega(x)^2x^4+\alpha\omega(x)^4x^2$, and: $x^3=(1+\mu)\omega(x)x^2-\mu\omega(x)^2x$. Thus: $2x^6=2(1+\mu)\omega(x)x^5-2\mu\omega(x)^2x^4$. Substituting $2x^6$ and x^2x^4 into (11), we obtain:

$$(1-\alpha-\mu)\omega(x)^2x^4+(\alpha-\mu-1)\omega(x)^4x^2+2\mu\omega(x)^5x=0. \tag{12}$$

From the equality $x^3=(1+\mu)\omega(x)x^2-\mu\omega(x)^2x$, we have:

$$x^4=(1+\mu)\omega(x)x^3-\mu\omega(x)^2x^2=(1+\mu+\mu^2)\omega(x)^2x^2-\mu(\mu+1)\omega(x)^3x.$$

Substituting x^4 into (12), we get:

$$-\mu(\mu^2+\alpha\mu+\alpha+1)\omega(x)^4[x^2-\omega(x)x]=0. \tag{13}$$

Since A is a train algebra of rank 3, then

$$\mu(\mu^2+\alpha\mu+\alpha+1)\omega(x)^4=0.$$

For $\omega(x)\neq 0$, this implies: $\mu(\mu-\delta)(\mu-\overline{\delta})=0$ so $\mu=0$, $\mu=\delta$, or $\mu=\overline{\delta}$.

Conversely, assume A is a train algebra of rank 3 with the train equation:

$$x^3-(1+\mu)\omega(x)x^2+\mu\omega(x)^2x=0, \quad \text{where} \quad \mu\in\{0,\,\delta,\,\overline{\delta}\}. \tag{14}$$

A partial linearization of (14) yields:

$$yx^2+2x(yx)-(1+\mu)\omega(y)x^2-2(1+\mu)\omega(x)yx+2\mu\omega(xy)x+\mu\omega(x)^2y=0. \tag{15}$$

Let us substitute $y=x^4$ in (15):

$$x^4x^2+2x^6-(1+\mu)\omega(x)^4x^2-2(1+\mu)\omega(x)x^5+2\mu\omega(x)^5x+\mu\omega(x)^2x^4=0. \tag{16}$$

If $\mu=0$, equation (14) becomes $x^3-\omega(x)x^2=0\Rightarrow x^6=\omega(x)^5x$. Equation (15) reduces to: $x^4x^2+2x^6-\omega(x)^4x^2-2\omega(x)x^5=0\Rightarrow x^4x^2=\omega(x)^4x^2$. By replacing $x^2x^4=\omega(x)^4x^2$ in identity (1), we have:

$$\omega(x)^4x^2-(1-\alpha)\omega(x)^2x^4-\alpha\omega(x)^4x^2$$

$$=(1-\alpha)\omega(x)^2x^4-(1-\alpha)\omega(x)^2x^4$$

$$=(1-\alpha)\omega(x)^2[x^4-\omega(x)^2x^2]=0,$$

since $x^4 = \omega(x)^2 x^2$. Thus:

$$x^4x^2 - (1-\alpha)\omega(x)^2x^4 - \alpha\omega(x)^4x^2 = 0,$$

and therefore A satisfies identity (1).

If $\mu = \delta$, equation (15) becomes:

$$x^4x^2 + 2x^6 - (1+\delta)\omega(x)^4x^2 - 2(1+\delta)\omega(x)x^5 + 2\delta\omega(x)^5x + \delta\omega(x)^2x^4 = 0. \tag{17}$$

Thus:

$$x^4x^2 = -2x^6 + (1+\delta)\omega(x)^4x^2 + 2(1+\delta)\omega(x)x^5 - 2\delta\omega(x)^5x - \delta\omega(x)^2x^4.$$

From this:

$$x^2x^4 = -2x^6 + (1+\delta)[\omega(x)^4x^2 + 2\omega(x)x^5] - \delta[2\omega(x)^5x + \omega(x)^2x^4].$$

We have:

$$x^2x^6 = -2[x^6 - (1+\delta)\omega(x)x^5 + \delta\omega(x)^2x^4] + \delta\omega(x)^2x^4 + (1+\delta)\omega(x)^4x^2 - 2\delta\omega(x)^5x.$$

Using $x^3 - (1+\delta)\omega(x)x^2 + \delta\omega(x)^2x = 0$, it follows that $x^6 - (1+\delta)\omega(x)x^5 + \delta\omega(x)^2x^4 = 0$. Hence:

$$x^2x^4 = \delta\omega(x)^2x^4 + (1+\delta)\omega(x)^4x^2 - 2\delta\omega(x)^5x.$$

Using $x^4 = (1+\delta)\omega(x)x^3 - \delta\omega(x)^2x^2$, we get:

$$x^2x^4 = \delta\omega(x)^2[(1+\delta)\omega(x)x^3 - \delta\omega(x)^2x^2] + (1+\delta)\omega(x)^4x^2 - 2\delta\omega(x)^5x.$$

Simplifying further:

$$x^2x^4 - (1-\alpha)\omega(x)^2x^4 - \alpha\omega(x)^4x^2 = [\delta^2 + \alpha\delta + \alpha - 1]\omega(x)^3x^3 - [\delta^2 - 2\delta + \alpha\delta + \alpha - 1]\omega(x)^4x^2 - 2\delta\omega(x)^5x.$$

As a result:

$$x^2x^4 - (1-\alpha)\omega(x)^2x^4 - \alpha\omega(x)^4x^2 = -2\omega(x)^3[x^3 - (1+\delta)\omega(x)^2 + \delta\omega(x)x].$$

Assuming $\omega(x) \neq 0$, we have:

$$x^3 - (1+\delta)\omega(x)^2 + \delta\omega(x)x = 0,$$

and thus:

$$x^2x^4 - (1-\alpha)\omega(x)^2x^4 - \alpha\omega(x)^4x^2 = 0.$$

Therefore, A satisfies identity (1).

For $\mu = \overline{\delta}$, the proof is analogous to the case $\mu = \delta$. □

3.4 *Evolution algebras satisfying the identity (1)*

Definition 3.11 A commutative finite-dimensional $\mathbb{C}$-algebra is called an evolution algebra if it admits a basis $B = \{e_1, \cdots, e_n\}$ such that:

$$e_i^2 = \sum_{j=1}^{n} a_{ij}e_j \quad \text{and} \quad e_ie_j = 0 \quad \text{for} \quad 1 \leq i \neq j \leq n. \tag{18}$$

Such a basis is called a natural basis (see [11]). A finite-dimensional weighted evolution algebra (Υ, ω) admits a natural basis $B = \{e_1, \cdots, e_n\}$ with a multiplication table (see [12], Corollary 3.4.) defined by:

$$e_1^2 = e_1 + \sum_{k=2}^{n} a_{1k}e_k, \quad e_i^2 = \sum_{k=2}^{n} a_{ik}e_k, \quad \text{with} \quad \omega(e_1) = 1, \quad \omega(e_i) = 0 \tag{19}$$

for $2 \leq i \leq n$.

Proposition 3.12 A finite-dimensional evolution algebra (Υ, ω) with natural basis B and multiplication defined by (19) satisfies identity (1) if the following conditions hold:

(i) $(e_1^2)^2 = e_1^2$;

(ii) $e_1^2e_i^2 = \alpha e_i^2$ for $2 \leq i \leq n$;

(iii) $e_1^2(e_1^2e_i) = (1-\alpha)e_1^2e_i$ for $2 \leq i \leq n$;

(iv) $e_j^2(e_1^2e_i) = 0$ for $2 \leq i, j \leq n$;

(v) $e_1^2(e_j(e_1^2e_i)) = (1-\alpha)e_j(e_1^2e_i)$ for $2 \leq i, j \leq n$;

(vi) $e_1^2(e_k(e_ie_j^2)) + e_j^2(e_k(e_1^2e_i)) = (1-\alpha)e_k(e_ie_j^2)$ for $2 \leq i, j, k \leq n$;

(vii) $e_l^2(e_k(e_ie_j^2)) = 0$ for $2 \leq i, j, k, l \leq n$.

Proof. Let Υ be an evolution algebra satisfying (i) to (vii), with natural basis B and $x = x_1e_1 + \sum_{i=2}^{n} x_ie_i \in \Upsilon$. We have:

$$x^2 = x_1^2e_1^2 + \sum_{i=2}^{n} x_i^2e_i^2,$$

$$x^4 = x_1^4 e_1^2 + \sum_{i=2}^{n} x_1^3 x_i e_1^2 e_i + \sum_{i,\,j=2}^{n} x_1^2 x_i x_j e_j (e_1^2 e_i) + \sum_{i,\,j,\,k=2}^{n} x_i x_j^2 x_k e_k (e_i e_j^2),$$

$$x^2 x^4 = x_1^6 e_1^2 e_1^2 + \sum_{i=2}^{n} x_1^5 x_i e_1^2 (e_1^2 e_i) + \sum_{i=2}^{n} x_1^4 x_i^2 e_1^2 e_i^2$$

$$+ \sum_{i,\,j=2}^{n} x_1^4 x_i x_j e_1^2 (e_j (e_1^2 e_i)) + \sum_{i,\,j=2}^{n} x_1^3 x_i x_j^2 e_j^2 (e_1^2 e_i)$$

$$+ \sum_{i,\,j,\,k=2}^{n} x_1^2 x_i x_j^2 x_k [e_1^2 (e_k (e_i e_j^2)) + e_j^2 (e_k (e_1^2 e_i))] + \sum_{i,\,j,\,k,\,l=2}^{n} x_i x_j^2 x_k x_l^2 e_l^2 (e_k (e_i e_j^2))$$

and

$$(1-\alpha)\omega(x)^2 x^4 + \alpha\omega(x)^4 x^2 = (1-\alpha)\left[x_1^6 e_1^2 + \sum_{i=2}^{n} x_1^5 x_i e_1^2 e_i + \sum_{i,\,j=2}^{n} x_1^4 x_i x_j e_j (e_1^2 e_i)\right.$$

$$\left. + \sum_{i,\,j,\,k=2}^{n} x_1^2 x_i x_j^2 x_k e_k (e_i e_j^2)\right] + \alpha \left[x_1^6 e_1^2 + \sum_{i=2}^{n} x_1^4 x_i^2 e_i^2\right].$$

Since the identities (i) to (vii) are satisfied, it follows that $x^2 x^4 = (1-\alpha)\omega(x)^2 x^4 + \alpha\omega(x)^4 x^2$. □

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Conflict of interest

The authors declare no competing financial interest.

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