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Some Notes on Generalized *P*-Derivations with Ideals in Factor Rings

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Abstract: The main goal of this article is to delve deeper into the discussion of the commutativity of a factor ring R/P by analyzing certain differential identities that involve generalized P-derivations and P-multipliers connecting I to P. Here, I represents a non-zero ideal of an arbitrary ring R, and P is a prime ideal of R such that $P \subsetneq I$. Furthermore, we will explore some outcomes from our various theorems. To underscore the necessity of the primeness assumption in our theorems, we will provide some illustrative counterexamples.

Keywords: prime ideal, generalized P-derivation, P-multiplier, integral domain, factor ring

MSC: 16W25, 16N60, 16U80

1. Introduction

Throughout this paper, unless otherwise stated, let R be an associative ring with center $Z(R)$. A ring R is called prime if for all $a, b \in R$, $aRb = 0$ implies that either $a = 0$ or $b = 0$. On the other hand, R is called semiprime if $aRa = 0$, then $a = 0$ for any $a \in R$. A proper ideal P of a ring R is called prime ideal when $aRb \subseteq P$ for all $a, b \in R$ implies that either $a \in P$ or $b \in P$. Consequently, R is prime ring if and only if $\{0\}$ is prime ideal of R. A ring R is 2-torsion free if $2a = 0$ implies that $a = 0$ for any $a \in R$. For all $a, b \in R$, $[a, b] = ab - ba$ and $a \circ b = ab + ba$ represent of the Lie product and Jordan product, respectively.

By definition, a derivation is an additive mapping δ from R to itself that satisfies $\delta(ab) = \delta(a)b + b\delta(a)$ for all $a, b \in R$. A generalized derivation, on the other hand, is an additive mapping $\digamma$ from R to itself that satisfies $\digamma(ab) = \digamma(a)b + a\delta(b)$ for all $a, b \in R$, where δ is the associated derivation with $\digamma$. It is evident that every derivation is a generalized derivation, but the converse is not true in general. The additive mappings $\mathscr{H} : R \longrightarrow R$, defined by the rules $\mathscr{H}(ab) = \mathscr{H}(a)b$ and $\mathscr{H}(ab) = a\mathscr{H}(b)$ for all $a, b \in R$, is referred to as a left multiplier and right multiplier, respectively. If $\mathscr{H}$ is both a right and left multiplier, it is called a multiplier. It is clear that $\mathscr{H}$ is a generalized derivation associated with the derivation $\delta = 0$. Examples and counterexamples of these concepts can be found in the literature. The mapping $\delta : R \longrightarrow R$ is called P-additive if it satisfies $\delta(a+b) - \delta(a) - \delta(b) \in P$ for all $a, b \in R$, where P is a prime ideal. A P-additive mapping δ is called a P-derivation if it satisfies the relation $\delta(ab) - \delta(a)b - a\delta(b) \in P$ for all $a, b \in R$, where P is a prime ideal. A P-additive mapping $\digamma : R \longrightarrow R$ is called a generalized P-derivation associated with a P-derivation

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δ if it satisfies $\digamma(ab) - \digamma(a)b - a\delta(b) \in P$ for all $a,\, b \in R$, where P is a prime ideal. Additionally, assuming δ is a P-trivial (i.e., $\delta(R) \subseteq P$) in the last relation gives us a P-left multiplier concept, defined as $\mathscr{H}(ab) - \mathscr{H}(a)b \in P$ for all $a,\, b \in R$, where P is a prime ideal. The P-right multiplier is defined as $\mathscr{H}(ab) - a\mathscr{H}(b) \in P$ for all $a,\, b \in R$, where P is a prime ideal. Moreover, $\mathscr{H}$ is considered a P-multiplier if it is both a P-left and P-right-multiplier. It is clear that every generalized derivation is a generalized P-derivation, and that every left multiplier is also a P-left multiplier. For examples and counterexamples regarding the existence of these concepts, refer to [1].

The extensive body of literature on ring theory involves studying the behavior of rings or their relevant subsets under different types of derivations that satisfy various identities. In [2], Ashraf et al. proved that a prime ring R is commutative if it satisfies certain identities such as (i) $\delta(a) \circ \digamma(b) = 0$, (ii) $[\delta(a),\, \digamma(b)] = 0$, (iii) $[\delta(a),\, \digamma(b)] = [a,\, b]$, (iv) $[\delta(a),\, \digamma(b)] + [a,\, b] = 0$, (v) $\delta(a)\digamma(b) \pm ab \in Z(R)$ for all $a,\, b \in I$, where I is a nonzero ideal of R. Dhara et al. [3] continued to investigate the above identities and found that the semi-prime ring R contains a nonzero central ideal. In [4], Tiwari et al. proved that a prime ring R is commutative if it admits generalized derivations $\digamma$ and $\backepsilon$ satisfy any of the following identities for each $a, b \in I$: $\backepsilon(ab) \pm \digamma(a)\digamma(b) \pm ab \in Z(R)$, $\backepsilon(ab) \pm \digamma(a)\digamma(b) \pm ba \in Z(R)$, $\backepsilon(ab) \pm \digamma(b)\digamma(a) \pm ab \in Z(R)$, $\backepsilon(ab) \pm \digamma(b)\digamma(a) \pm ba \in Z(R)$, $\backepsilon(ab) \pm \digamma(b)\digamma(a) \pm [a,\, b] \in Z(R)$ for all $a, b \in I$, where I is a nonzero ideal of R. Previous studies have received extensive investigation from several researchers on various rings, such as prime and semiprime, or any appropriate subsets of them, such as ideal or Lie ideal. For more details, please refer to [5–7].

Recently, many authors have been investigating the commutativity of the factor ring R/P without imposing any conditions on a ring R, where P is a prime ideal. This is done through various appropriate additive mappings such as generalized derivation, left-multiplier, or automorphisms. For example, readers can refer to references [8–13].

In [14], Mohssine et al. studied the behavior of a quotient near-ring N/P when N admits a $(\alpha,\, \tau)$-P-derivation δ that satisfies certain identities where P is a prime ideal of N. In [15] Oukhtite et al. discuss the relationship between specific identities, including a pair of P-left multipliers, and the structure of a division ring R/P, where P is a prime ideal of any ring R. In [1], Sandhu et al. explore the relationship between a factor ring R/P and certain identities involving a mixture of a generalized P-derivation and P-left multipliers, where P is a prime ideal in any ring R.

Our current article focuses on expanding previous studies in analogous ways by utilizing differential identities that include a pair of generalized P-derivations and a P-multiplier, with elements in a non-zero ideal I. We will investigate the connection between these identities and the behavior of a factor ring R/P, without any restrictions on a ring R, where P is a prime ideal of R under the constraint $P \subsetneq I$. Moreover, we will derive several significant related consequences. Finally, we will include some examples to illustrate the importance of the primeness condition in our theorems.

2. Elementary results

In this article, the symbols $(\digamma,\, \delta)$ and $(\backepsilon,\, g)$ will denote generalized P-derivations associated with P-derivations δ and g, respectively. The symbol $\mathscr{H} : R \longrightarrow R$ will denote a P-multiplier of R. In this section, we will present some useful facts and lemmas that will be used frequently for the development of our theorems proof. Indeed, the following three facts and remark, unless indicated otherwise, will be used without explicit mention.

Fact 1 Let R be a ring with a nonzero ideal I and prime ideal P such that $P \subsetneq I$ and $aIb \subseteq P$ for $a,\, b \in R$, then either $a \in P$ or $b \in P$.

Fact 2 Let I be a non-zero left (or right) ideal of a ring R, and let P be a prime ideal of R provided that $P \subsetneq I$. If δ is a P-derivation of R such that $\delta(I)$ is contained in P, then $\delta(R)$ is also contained in P.

Fact 3 Let I be a non-zero left ideal of a ring R, and let P be a prime ideal of R such that $P \subsetneq I$ with $\delta(I) \subseteq P$. If $(\digamma,\, \delta)$ is a generalized P-derivation of R such that $\digamma(I)$ is contained in P, then $\digamma(R)$ is also contained in P.

Remark 1 [16, Remark] If P and I are two ideals of a given ring R, where P is a prime ideal such that $P \subsetneq I$, then R/P is a commutative integral domain *iff* $[a,\, b] \in P$ for every $a,\, b \in I$.

Lemma 1 [17, Lemma 3.10] In a prime ring R with a characteristic different from two, let $a,\, b \in R$ such that $a\tau b + b\tau a = 0$ for all $\tau \in R$. Then at least one of a or b must be zero.

The following lemma is a slight modification of the previous lemma, without imposing any constraints on a ring R.

Lemma 2 Let R be a ring, I be a nonzero ideal of R, and let P be a prime ideal such that $P \subsetneq I$. Suppose that $char(R/P) \neq 2$. If for fixed elements a and b in R, $aib + bia \in P$ for all $i \in I$, then $aib \in P$ and $bia \in P$ for all $i \in I$. Moreover, in this case, either $a \in P$ or $b \in P$.

Proof. The proof can be easily derived from [18, Lemma 3.1]. □

In [19], Posner proved the commutativity when a derivation δ commutes on a prime ring R. This result was further extended in several ways, including cases where the derivation commutes on a prime ideal. To facilitate the proof of our theorems, we will analyze the previous results in a more general context by examining the effect of a generalized P-derivation that commutes on a prime ideal P on the behavior of a factor ring R/P.

Lemma 3 Consider an arbitrary ring R equipped with a generalized P-derivation $(\digamma, \delta)$. Assume that P and I are ideals of R such that P is prime such that $P \subsetneq I$. If $[\digamma(a), a] \in P$ for all $a \in I$, then either $\delta(R) \subseteq P$ or R/P is a commutative integral domain.

Proof. We have

$$[\digamma(a), a] \in P \quad \text{for all} \quad a \in I. \tag{1}$$

Linearizing the last equation, we get

$$[\digamma(a), b] + [\digamma(b), a] \in P \quad \text{for all} \quad a, b \in I. \tag{2}$$

Exchanging a with ab in Equation (2) and applying it, we obtain

$$a[\delta(b), b] + [a, b]\delta(b) \in P \quad \text{for all} \quad a, b \in I. \tag{3}$$

Putting $a = ca$ in Equation (3) and using it, we obtain

$$[c, b]I\delta(b) \subseteq P \quad \text{for all} \quad b, c \in I. \tag{4}$$

Fact 1 implies that either $[c, b] \in P$ or $\delta(b) \in P$ for each $c, b \in I$. If $[c, b] \in P$ for all $c, b \in I$, thus we get R/P is a commutative integral domain by Remark 1. On the other hand, if $\delta(b) \in P$ for all $b \in I$, that implies $\delta(I) \subseteq P$. Therefore, $\delta(R) \subseteq P$. □

In the lemma above, if $\digamma = \delta$, we can obtain an improved version of [1, Lemma 1] for a nonzero ideal I of a ring R as follows:

Corollary 1 Let δ be a P-derivation of a ring R such that $[\delta(a), a] \in P$ for all $a \in I$, where I and P are ideals of R with P begin prime and $P \subsetneq I$. Then $\delta(R) \subseteq P$ or R/P is a commutative integral domain.

3. Main results

Throughout this section unless mentioned otherwise, the map $id_R : R \longrightarrow R$ defined by $id_R(a) = a$ for all $a \in R$ will denote an identity of R. Additionally, we will assume that $0 \neq I$ and P are ideals of R, where P is a prime ideal such that $P \subsetneq I$.

Theorem 1 Let R be a ring equipped with generalized P-derivations $(\mathcal{F}, \delta)$ and $(\eth, g)$ such that $a \circ \eth(b) \pm \mathcal{F}(a \circ b) \in P$ for all $a, b \in I$. Then $(\eth \pm \mathcal{F})(R) \subseteq P$, or R/P is a commutative integral domain of characteristic 2.

Proof. Our initial hypothesis is stated as follows:

$$a \circ \eth(b) \pm \mathcal{F}(a \circ b) \in P, \quad \text{for all} \quad a, b \in I. \tag{5}$$

Substituting b by bc in Equation (5) and applying it, we get

$$-\eth(b)[a, c] - b[a, g(c)] + (a \circ b)g(c) \mp \mathcal{F}(b)[a, c] \mp b\delta([a, c]) \pm (a \circ b)\delta(c) \in P \quad \text{for all} \quad a, b, c \in I. \tag{6}$$

Taking $c = a$ in Equation (6), we get

$$-b[a, g(a)] + (a \circ b)g(a) \pm (a \circ b)\delta(a) \in P \quad \text{for all} \quad a, b \in I. \tag{7}$$

Replacing b with hb in Equation (7) and comparing the resulting equation with it, we obtain

$$[a, h]I(g(a) \pm \delta(a)) \subseteq P \quad \text{for all} \quad a, h \in I. \tag{8}$$

Using Fact 1, we can conclude that either $[a, h] \in P$ for all $a, h \in I$, hence R/P is a commutative integral domain, or $g(a) \pm \delta(a) \in P$ for all $a \in I$. Let's consider the case when:

$$g(a) \pm \delta(a) \in P \quad \text{for all} \quad a \in I. \tag{9}$$

This simplifies Equation (7) to $b[a, g(a)] \in P$ for all $a, b \in I$. The primeness of P, along with Corollary 1, implies that either R/P is a commutative integral domain or $g(R) \subseteq P$. If $g(R) \subseteq P$, then Equation (9) becomes $\delta(a) \in P$ for all $a \in I$. Therefore, Equation (6) simplifies to $(\eth(b) \pm \mathcal{F}(b))[a, c] \in P$ for all $a, b, c \in I$. For any $m \in I$, replacing a by ma in the last equation and using it, we get $(\eth(b) \pm \mathcal{F}(b))I[a, c] \subseteq P$ for all $a, b, c \in I$. Again, Fact 1 gives either R/P is a commutative integral domain, or $\eth(b) \pm \mathcal{F}(b) \in P$ for all $b \in I$. Therefore, $(\eth \pm \mathcal{F})(R) \subseteq P$.

Now let's examine the case when R/P is a commutative integral domain. This along with the initial hypothesis leads to $2\eth(b)a \pm 2\mathcal{F}(ba) \in P$ for all $a, b \in I$. By setting $a = ah$ in the last relation and using it, we can derive $2bI\delta(h) \in P$ for all $b, h \in I$. Fact 1 implies that either $2b \in P$ or $\delta(h) \in P$ for all $b, h \in I$. The first case forces $char(R/P) = 2$. If $char(R/P) \neq 2$, then by using the second case, we can conclude that $(\eth \pm \mathcal{F})(R) \subseteq P$. □

By setting $\eth$ equal to $\mathcal{F}$ and following arguments similar to the proof of Theorem 1, the following corollary can easily be derived:

Corollary 2 Let R be a ring equipped with a generalized P-derivation $(\mathcal{F}, \delta)$. If $char(R/P) \neq 2$, such that $a \circ \mathcal{F}(b) + \mathcal{F}(a \circ b) \in P$ for all $a, b \in I$, then $\mathcal{F}(R) \subseteq P$ or R/P is a commutative integral domain.

In [20, Theorem 3.5], Mamouni et al. discussed the relationship between a factor ring R/P and generalized derivations $(\mathcal{F}, \delta)$ and $(\eth, g)$ that satisfy the identity $\mathcal{F}(ab) + \eth(ba) \in P$ for all $a, b \in R$, where P is a prime ideal of R. Expanding on these findings, the following theorem further explores this discussion in a broader context by considering the identity $\mathcal{F}(ab) - \eth(ba) - \mathscr{H}[a, b] \in P$ for all $a, b \in I$, where $(\mathcal{F}, \delta)$ and $(\eth, g)$ are generalized P-derivations, $\mathscr{H}$ is a P-left multiplier, and I is a non-zero ideal of R such that $P \subsetneq I$.

Theorem 2 Let R be a ring equipped with generalized P-derivations $(\digamma, \delta)$, $(\eth, g)$ and a P-left multiplier $\mathscr{H}$. Then $\digamma(ab)+\eth(ba)-\mathscr{H}([a, b]) \in P$ for all $a, b \in I$ if and only if one of the following statements holds:

(i) $(\digamma-\mathscr{H})(R) \subseteq P$ and $(\eth+\mathscr{H})(R) \subseteq P$.

(ii) R/P is a commutative integral domain and $(\digamma+\eth)(R) \subseteq P$.

Proof. Clearly, if (i) or (ii) are true, the statement $\digamma(ab)+\eth(ba)-\mathscr{H}([a, b]) \in P$ holds for all $a, b \in I$. Conversely, let us assume that

$$\digamma(ab)+\eth(ba)-\mathscr{H}([a, b]) \in P \quad \text{for all} \quad a, b \in I. \tag{10}$$

This can be rewritten as

$$\digamma(a)b+a\delta(b)+\eth(b)a+bg(a)-\mathscr{H}([a, b]) \in P \quad \text{for all} \quad a, b \in I. \tag{11}$$

Substituting a by ac in Equation (11) and applying it, we get

$$\digamma(a)[c, b]+a\delta(c)b-a\delta(b)c+ac\delta(b)+bag(c)-\mathscr{H}(a)[c, b] \in P \quad \text{for all} \quad a, b, c \in I. \tag{12}$$

Taking $c=b$ in Equation (12), we obtain

$$ac\delta(b)+bag(b) \in P \quad \text{for all} \quad a, b \in I. \tag{13}$$

Replacing a by ma in Equation (13) and using it, we get

$$[b, m]ag(b) \in P \quad \text{for all} \quad a, b, m \in I. \tag{14}$$

That is, $[b, m]Ig(b) \subseteq P$ for all $b, m \in I$. By using Fact 1, we conclude that either $g(b) \in P$ or $[b, m] \in P$ for all $b, m \in I$. In the first case $g(R) \subseteq P$. Applying this in Equation (13), we obtain $aIb\delta(b) \subseteq P$ for all $a, b \in I$. Since I is a nonzero ideal of R, we get $b\delta(b) \in P$ for all $b \in I$. Linearizing the last equation, we get $b\delta(a)+a\delta(b) \in P$ for all $a, b \in I$. Replace a by am in the last relation and use it to get $ba\delta(m)+am\delta(b)-a\delta(b)m \in P$ for all $a, b, m \in I$. Again, replace a by ha in the last relation and use it, to get $[b, h]I\delta(m) \subseteq P$ for all $b, h, m \in I$. By using Fact 1, we conclude that either $\delta(R) \subseteq P$ or R/P is a commutative integral domain. If $\delta(R) \subseteq P$, then Equation (12) reduces to $\digamma(a)[c, b]-\mathscr{H}(a)[c, b] \in P$ for all $a, b, c \in I$. Replacing c by ck in the last relation and using it, we obtain $(\digamma(a)-\mathscr{H}(a))I[k, b] \subseteq P$ for all $a, b, k \in I$. By using Fact 1, we conclude that either $\digamma(a)-\mathscr{H}(a) \in P$ for all $a \in I$ or R/P is a commutative integral domain. In the first case, we have $(\digamma-\mathscr{H})(I) \subseteq P$, then Equation (11) can be rewritten as $(\digamma(a)-\mathscr{H}(a))b+(\eth(b)+\mathscr{H}(b))a=(\eth(b)+\mathscr{H}(b))a \in P$ $a, b \in I$. Since P is a prime and $P \subsetneq I$, we can conclude that $(\eth+\mathscr{H})(I) \subseteq P$ which implies that $(\eth+\mathscr{H})(R) \subseteq P$.

On the other hand, if $[b, m] \in P$ for all $b, m \in I$, then by using Remark 1, R/P is a commutative integral domain. Hence, Equation (13) becomes $(\delta+g)(I) \subseteq P$. That is, $(\delta+g)(R) \subseteq P$. From Equation (11), we can deduce that $(\digamma+\eth)(I) \subseteq P$. Therefore, we conclude that $(\digamma+\eth)(R) \subseteq P$. □

In Theorem 2, if we set $\digamma=\eth$ and follow similar arguments, we can prove the following corollary which requires the constraint $char(R/P) \neq 2$.

Corollary 3 Let R be a ring equipped with a generalized P-derivation $(\mathcal{F}, \delta)$ and a P-left multiplier $\mathscr{H}$ such that $char(R/P) \neq 2$. Then $\mathcal{F}(a \circ b) - \mathscr{H}([a, b]) \in P$ for all $a, b \in I$ if and only if one of the following statements holds:

(i) $(\mathcal{F} - \mathscr{H})(R) \subseteq P$ and $(\mathcal{F} + \mathscr{H})(R) \subseteq P$.

(ii) R/P is a commutative integral domain and $\mathcal{F}(R) \subseteq P$.

Theorem 3 Let R be a ring equipped with generalized P-derivations $(\mathcal{F}, \delta)$, $(\mathcal{G}, g)$ and a P-left multiplier $\mathscr{H}$. Then $\mathcal{F}(ab) - \mathcal{G}(ba) - \mathscr{H}([a, b]) \in P$ for all $a, b \in I$ if and only if one of the following statements holds:

(i) $(\mathcal{F} - \mathscr{H})(R) \subseteq P$ and $(\mathscr{H} - \mathcal{G})(R) \subseteq P$,

(ii) R/P is a commutative integral domain and $(\mathcal{F} - \mathcal{G})(R) \subseteq P$.

Proof. By following arguments and techniques similar to the proof of Theorem 2 with some necessary variations, it is easy to obtain the desired result. □

Corollary 4 Let R be a ring equipped with a generalized P-derivation $(\mathcal{F}, \delta)$ and a P-left multiplier $\mathscr{H}$. If $\delta \neq 0$, then $\mathcal{F}[a, b] - \mathscr{H}([a, b]) \in P$ for all $a, b \in I$ if and only if R/P is a commutative integral domain.

Proof. By replacing $\mathcal{G}$ with $\mathcal{F}$ in Theorem 3 and adjusting the proof of Theorem 2 accordingly, we can directly obtain the desired result. □

Sandhu et al. in [1] studied the behavior of the ring R/P with a characteristic that is not equal to 2. They considered a scenario where R admits a generalized P-derivation $(\mathcal{F}, \delta)$ and a P-multiplier $\mathscr{H}$ that satisfies the identity $[\mathcal{F}(a), \delta(b)] \pm \mathscr{H}([a, b]) \in P$ for all $a, b \in R$. Instead of the above identity and without the need to assume that the characteristic of R/P is not equal to 2, the main goal of our next theorem is to examine the effect of the identity $[\mathcal{F}(a), g(b)] \pm \mathscr{H}([a, b]) \in P$ for all $a, b \in I$ on the behavior of R/P where $(\mathcal{F}, \delta)$ is a generalized P-derivation, g is a P-derivation, and $\mathscr{H}$ is a P-multiplier.

Theorem 4 Let R be a ring equipped with a generalized P-derivation $(\mathcal{F}, \delta)$, P-derivation g and a P-multiplier $\mathscr{H}$. Then $[\mathcal{F}(a), g(b)] \pm \mathscr{H}([a, b]) \in P$ for all $a, b \in I$ if and only if one of the following holds:

(i) $\mathcal{F}(R) \subseteq P$ and $\mathscr{H}(R) \subseteq P$;

(ii) R/P is a commutative integral domain;

(iii) $g(R) \subseteq P$ and $\mathscr{H}(R) \subseteq P$.

Proof. If at least one of (i), (ii), or (iii) holds, then the statement is immediately satisfied.

To prove the converse side, let us assume

$$[\mathcal{F}(a), g(b)] \pm \mathscr{H}([a, b]) \in P \quad \text{for all} \quad a, b \in I. \tag{15}$$

Substituting a with ac in Equation (15) and applying it, we get

$$\mathcal{F}(a)[c, g(b)] + [a, g(b)]\delta(c) + a[\delta(c), g(b)] \pm \mathscr{H}(a)[c, b] \in P \quad \text{for all} \quad a, b, c \in I. \tag{16}$$

Setting $c = g(b)$ in Equation (16), we obtain

$$[a, g(b)]\delta(g(b)) + x[\delta(g(b)), g(b)] \pm \mathscr{H}(a)[g(b), b] \in P \quad \text{for all} \quad a, b \in I. \tag{17}$$

Again, substituting a by ha in Equation (17), we get

$$h[a, g(b)]\delta(g(b)) + [h, g(b)]a\delta(g(b)) + ha[\delta(g(b)), g(b)] \pm h\mathscr{H}(a)[g(b), b] \in P \quad \text{for all} \quad a, b, h \in I. \tag{18}$$

Left multiplying Equation (17) by h and comparing it with Equation (18), we deduce that

$$[h, g(b)]I\delta(g(b)) \subseteq P \quad \text{for all} \quad b, h \in I. \tag{19}$$

Using Fact 1, we conclude that either $[h, g(b)] \in P$ or $\delta(g(b)) \in P$ for all $b, h \in I$. If $\delta(g(b)) \in P$ for all $b \in I$, then substituting b with bm in the last relation and using it, we get $g(b)\delta(m) + \delta(b)g(m) \in P$ for all $b, m \in I$. Substituting m by ms in the last relation and using it, we get $g(b)m\delta(s) + \delta(b)mg(s) \in P$ for all $b, m, s \in I$. By putting $s = b$ in the last relation, we get $g(b)m\delta(b) + \delta(b)mg(b) \in P$ for all $b, m \in I$. Utilizing Lemma 2, we get either $\delta(b) \in P$ or $g(b) \in P$ for all $b \in I$. In the first case, we have $\delta(b) \in P$ for all $b \in I$ and hence Equation (16) reduces to $\mathcal{F}(a)[c, g(b)] \pm \mathcal{H}(a)[c, b] \in P$ for all $a, b, c \in I$. Letting $b = c$, the last relation becomes $\mathcal{F}(a)[b, g(b)] \in P$ for all $a, b \in I$. For any $m \in I$, replacing a by am we arrive at $\mathcal{F}(a)I[b, g(b)] \subseteq P$ for all $a, b \in I$. Fact 1 gives either $\mathcal{F}(a) \in P$ for all $a \in I$ or $[b, g(b)] \in P$ for all $b \in I$. In the case of $\mathcal{F}(a) \in P$ for all $a \in I$, Equation (16) reduces to $\mathcal{H}(a)[c, b] \in P$ for all $a, b, c \in I$. For any $h \in I$ replacing c with ch in the last equation and using it, we arrive at $\mathcal{H}(a)I[h, b] \in P$ for all $a, b, h \in I$. Using Fact 1, we get either $\mathcal{H}(R) \subseteq P$ or R/P is a commutative integral domain, by using Remark 1. In the case of $[b, g(b)] \in P$ for all $b \in I$, by using Corollary 1, we get either $g(R) \subseteq P$ or R/P is a commutative integral domain. If $g(R) \subseteq P$, then Equation (16) becomes $\mathcal{H}(a)[c, b] \in P$ and thus as discussed above we can obtain the desired result. □

Remark 1 In Theorem 4, we assume that $char(R/P) \neq 2$, and we set $g = \delta$ and $I = R$. Under these conditions, we can directly obtain [1, Theorem 14].

Theorem 5 Let R be a ring equipped with a generalized P-derivation $(\mathcal{F}, \delta)$, a P-derivation g and a P-multiplier $\mathcal{H}$. If $char(R/P) \neq 2$, then $[\mathcal{F}(a), \delta(b)] \pm [\mathcal{H}(a), g(b)] \in P$ for all $a, b \in I$ if and only if one of the following holds:

(i) $\delta(R) \subseteq P$ and $g(R) \subseteq P$;

(ii) $\delta(R) \subseteq P$ and $\mathcal{H}(R) \subseteq P$;

(iii) R/P is a commutative integral domain.

Proof. If at least one of (i) or (ii) or (iii) holds, then the statement is immediately satisfied.

Therefore, we shall assume

$$[\mathcal{F}(a), \delta(b)] \pm [\mathcal{H}(a), g(b)] \in P \quad \text{for all} \quad a, b \in I. \tag{20}$$

Substituting a by ac in Equation (20) and applying it, we get

$$\mathcal{F}(a)[c, \delta(b)] + [a, \delta(b)]\delta(c) + a[\delta(c), \delta(b)] \pm \mathcal{H}(a)[c, g(b)] \in P \quad \text{for all} \quad a, b, c \in I. \tag{21}$$

In particular, setting $c = \delta(b)$ in Equation (21), we get

$$[a, \delta(b)]\delta^2(b) + a[\delta^2(b), \delta(b)] \pm \mathcal{H}(a)[\delta(b), g(b)] \in P \quad \text{for all} \quad a, b \in I. \tag{22}$$

Again, substituting a by ha in Equation (22), we get

$$h[a, \delta(b)]\delta^2(b) + [h, \delta(b)]a\delta^2((b) + ha[\delta^2(b), \delta(b)] \pm h\mathcal{H}(a)[\delta(b), g(b)] \in P \quad \text{for all} \quad a, b, h \in I. \tag{23}$$

By left multiplying Equation (22) by h and comparing it with Equation (23), we can derive:

$$[h, \delta(b)]I\delta^2(b) \subseteq P \quad \text{for all} \quad b, h \in I. \tag{24}$$

By using Fact 1, we can deduce that either $[h, \delta(b)] \in P$ or $\delta^2(b) \in P$ for all $b, h \in I$. If $\delta^2(b) \in P$ for all $b \in I$, then substituting b by bm and using it, we find $\delta(b)\delta(m) + \delta(b)\delta(m) = 2\delta(b)\delta(m) \in P$ for all $m, b \in I$. Given that $char(R/P) \neq 2$, we can infer that $\delta(b)\delta(m) \in P$ for all $b, m \in I$. Replacing m by ms in the last relation and using it, we get $\delta(b)I\delta(s) \subseteq P$ for all $b, s \in I$. Therefore, Fact 1 implies $\delta(R) \subseteq P$. In this scenario, Equation (21) becomes $\mathscr{H}(a)[c, g(b)] \in P$ for all $a, b, c \in I$. For any $m \in I$, replacing c by mc in the last equation and using it, we can conclude $\mathscr{H}(a)I[c, g(b)] \subseteq P$ for all $a, b, c \in I$. By using Fact 1, we obtain $\mathscr{H}(a) \in P$ for all $a \in I$ or $[c, g(b)] \in P$ for all $b, c \in I$. The former case leads to $\mathscr{H}(R) \subseteq P$. In the latter case, by using Corollary 1, we deduce that $g(R) \subseteq P$ or R/P is a commutative integral domain.

On the other hand, if $[h, \delta(b)] \in P$ for all $b, h \in I$, then by using Corollary 1, we can infer that either $\delta(R) \subseteq P$ or R/P is a commutative integral domain. If $\delta(R) \subseteq P$, then as discussed before, the desired result can be obtained. □

Corollary 5 Let R be a ring equipped with a generalized P-derivation $(\digamma, \delta)$, a P-derivation g. If $char(R/P) \neq 2$, then $[\digamma(a), \delta(b)] \in P$ for all $a, b \in I$ if and only if R/P is a commutative integral domain or $\delta(R) \subseteq P$.

Proof. By setting $g = \delta$ in the identity imposed in the previous theorem, we obtain $[\digamma(a), \delta(b)] \pm [\mathscr{H}(a), \delta(b)] \in P$ for all $a, b \in I$. Replacing $\digamma$ with $\digamma \mp \mathscr{H}$ in the previous identity, we obtain $[\digamma(a), \delta(b)] \in P$ for all $a, b \in I$. Following arguments similar to those used in the proof of Theorem 1 with some minor modifications, we can easily obtain the desired conclusion. □

Theorem 6 Let R be a ring equipped with generalized P-derivations $(\digamma, \delta)$, $(\eth, g)$, and a P-multiplier $\mathscr{H}$. Then $\eth(ab) \pm g(a)\digamma(b) \pm [\mathscr{H}(a), b] \in P$ for all $a, b \in I$ if and only if one of the following holds:

(i) $g(R) \subseteq P$, $\mathscr{H}(R) \subseteq P$, and $\eth(R) \subseteq P$;

(ii) R/P is a commutative integral domain, $g(R) \subseteq P$, and $\eth(R) \subseteq P$.

Proof. Obviously if either (i) or (ii) holds, then the given statement holds.

Therefore, let us assume

$$\eth(ab) \pm g(a)\digamma(b) \pm [\mathscr{H}(a), b] \in P \quad \text{for all} \quad a, b \in I. \tag{25}$$

Equation (25) can be restated as:

$$\eth(a)b + ag(b) \pm g(a)\digamma(b) \pm [\mathscr{H}(a), b] \in P \quad \text{for all} \quad a, b \in I. \tag{26}$$

Substituting b by bc in Equation (26) and applying it, we obtain

$$abg(c) + g(a)b\delta(c) + b[\mathscr{H}(a), c] \in P \quad \text{for all} \quad a, b, c \in I. \tag{27}$$

Again, substituting b by mb in Equation (27) and applying it, we get

$$[a, m]bg(c) + [g(a), m]b\delta(c) \in P \quad \text{for all} \quad a, b, c, m \in I. \tag{28}$$

Taking $m = g(a)$, we get $[a, g(a)]bg(c) \in P$ for all $a, b, c \in I$. Replacing b by bc in the last relation and utilizing it, we find $[a, g(a)]b[c, g(c)] \in P$ for all $a, b, c \in I$. That is, $[a, g(a)]I[c, g(c)] \subseteq P$ for all $a, c \in I$. Primeness of P together with Corollary 1, implies that either R/P is a commutative integral domain or $g(c) \in P$ for all $c \in I$.

Suppose $g(c) \in P$ for all $c \in I$. Then Equation (27) is reduced to $[\mathscr{H}(a), c] \in P$ for all $a, c \in I$. Replacing a by ah in the last relation and using it, we get $\mathscr{H}(a)[h, c] \in P$ for all $a, c, h \in I$. For any $b \in I$, replacing a by ab, we obtain $\mathscr{H}(a)I[h, c] \subseteq P$ for all $a, c, h \in I$. Hence, Fact 1 gives R/P is a commutative integral domain or $\mathscr{H}(a) \in P$ for all $a \in I$. It is easy to see that both cases along with Equation (26), yield $\backepsilon(a)b \in P$ for all $a, b \in I$. Utilizing the primeness of P with the hypothesis that $P \subsetneq I$, we obtain $\backepsilon(I) = \backepsilon(RI) = \backepsilon(R)I \subseteq P$. Therefore, we can deduce that $\backepsilon(R) \subseteq P$.

On the other hand, the case when R/P is a commutative integral domain together with Equation (27), yields

$$ag(c) + g(a)\delta(c) \in P \quad \text{for all} \quad a, c \in I. \tag{29}$$

Replacing a by am in the last equation and using it, we arrive at $g(m)I\delta(c) \subseteq P$ for all $m, c \in I$. By using Fact 1, we get either $\delta(c) \in P$ for all $c \in I$ or $g(m) \in P$ for all $m \in I$. If $\delta(c) \in P$ for all $c \in I$, then from Equation (29), we find that $g(m) \in P$ for all $m \in I$. Therefore, using Equation (26) and following the same discussion in the last paragraph of the previous case, the desired conclusion can be obtained. □

If we consider the identity $\backepsilon(ab) \pm \delta(a)\digamma(b) \pm [\mathscr{H}(a), b] \in P$ for all $a, b \in I$ and follow arguments similar to the proof of Theorem 6, we can easily prove the following theorem:

Theorem 7 Let R be a ring equipped with generalized P-derivations $(\digamma, \delta)$, $(\backepsilon, g)$, and a P-multiplier $\mathscr{H}$. Then $\backepsilon(ab) \pm \delta(a)\digamma(b) \pm [\mathscr{H}(a), b] \in P$ for all $a, b \in I$ if and only if one of the following holds:

(i) $\delta(R) \subseteq P$, $g(R) \subseteq P$, $\mathscr{H}(R) \subseteq P$, and $\backepsilon(R) \subseteq P$;

(ii) R/P is a commutative integral domain, $\delta(R) \subseteq P$, $g(R) \subseteq P$, and $\backepsilon(R) \subseteq P$.

Finally, the following examples confirm the necessity of the primeness of P in our various theorems.

Example 1 Consider $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in \mathbb{C} \right\}$, where $\mathbb{C}$ be a ring of complex numbers. $I = \left\{ \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$ and $P = \{0\}$. Define

$$\digamma \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \delta \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\backepsilon \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad g \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Assuming that $\mathscr{H} = id_R$ is a two sided multiplier, it is easy to verify that I is a nonzero ideal of R. Additionally, $\digamma$ and $\backepsilon$ are generalized derivations of R associated with derivations δ and g, respectively. It can also be verified that R satisfies the identities in Theorems 1-7. However, R/P is noncommutative, $g(R) \nsubseteq P$, $\delta(R) \nsubseteq P$, $\digamma(R) \nsubseteq P$, $\backepsilon(R) \nsubseteq P$, $\mathscr{H}(R) \nsubseteq P$, $(\digamma \pm \mathscr{H})(R) \nsubseteq P$, $(\backepsilon + \mathscr{H})(R) \nsubseteq P$, $(\mathscr{H} - \backepsilon)(R) \nsubseteq P$, and $(\digamma \pm \backepsilon)(R) \nsubseteq P$. Since $\begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \{0\}$, but neither $\begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ nor $\begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is in P, it is evident that P is not a prime ideal of R. Therefore, the assumption that P is prime in Theorems 1-7 is necessary.

Example 2 Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{H} \right\}$, where $\mathbb{H}$ be a Hamilton ring and let $I = \left\{ \begin{pmatrix} 0 & 2b \\ 0 & 0 \end{pmatrix} \right\}$ and $P = \left\{ \begin{pmatrix} 0 & 6b \\ 0 & 0 \end{pmatrix} \right\}$. Define $\digamma = \delta : R \longrightarrow R$ by $\digamma \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ and $\eth = g : R \longrightarrow R$ by $\eth \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = \begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix}$. Assuming that $\mathscr{H} = id_R$ is a two-sided multiplier of R, it is easy to checking that I is a nonzero ideal of R. Additionally, $\digamma$ and $\eth$ are generalized derivations associated with derivations δ and g, respectively. It can also be verified that R satisfies the identities in Theorems 1-7. However, R/P is noncommutative, $g(R) \nsubseteq P$, $\delta(R) \nsubseteq P$, $\digamma(R) \nsubseteq P$, $\eth(R) \nsubseteq P$, $\mathscr{H}(R) \nsubseteq P$, $(\digamma \pm \mathscr{H})(R) \nsubseteq P$, $(\eth + \mathscr{H})(R) \nsubseteq P$, $(\mathscr{H} - \eth)(R) \nsubseteq P$, and $(\digamma \pm \eth)(R) \nsubseteq P$. Since $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2a \\ 0 & 0 \end{pmatrix} \in P$, but neither $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ nor $\begin{pmatrix} 0 & 2a \\ 0 & 0 \end{pmatrix}$ is in P, it is evident that P is not a prime ideal of R. Therefore, the assumption that P is prime in Theorems 1-7 is necessary.

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Conflict of interest

The authors declare no conflicts of interest.

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