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Research Article

A Note on Left Cocyclic Modules

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Abstract: We explore some properties of cocyclic modules and rings. We establish when every cyclic module is cocyclic. In addition, we characterize the case in which every cocyclic module embeds in a free module. Finally, we give the necessary conditions for a class of cocyclic modules to be closed under quotients and projective covers.

***Keywords*:** cyclic module, cocyclic module, artinian principal ideal rings, left uniserial ring, left Köthe ring

MSC: 16D80, 16D10, 16P20

1. Introduction

In this work, we will consider associative rings with a unitary element and all the classes of left modules considered are closed under isomorphisms.

The left cyclic modules have been studied systematically in module theory. Since Barbara Osofsky proved in [1] that the condition: every left cyclic module is left injective is equivalent to the ring being semisimple, most of the research concern with left cyclic modules is related to the question: *Which rings satisfy that every left cyclic module is ...*? The excellent book [2] devoted to the left cyclic modules is an example of how the former question is in the heart of the recent literature concern with left cyclic modules. There are important rings defined by means of left cyclic modules, such as *left Köthe rings*, that is: rings in which every module is a direct sum of left cyclic modules. The historical development of these rings is well exposed in [3], in that paper the authors deal with the Köthe-Cohen-Kaplansky problem: *Are left Köthe rings artinian principal ideal rings*? This problem is an example of the fact that there are many interesting and difficult problems in the noncommutative ring theory related to left cyclic modules. Although the dual concept of left cyclic modules, that is, the *cocyclic modules* introduced by Maranda in [4] has not been developed as much as the left cyclic modules, there are some books and articles that have worked on those modules, such as [5, 6]. Recall that a left module M is called *cocyclic* if there exists a simple left module S such that $S \leq_e M$, see [5]. Finally, recall that a ring R is a left V-ring if every left simple module is injective.

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In Table 1, we illustrate some similarities and contrasts between cyclic and cocyclic modules and also use them to pose some open questions. Also note that some of the statements in the list are known and others have been proven in this work.

Table 1. Cyclic vs cocyclic

Cyclic side	Cocyclic side
Every module has cyclic submodule.	Every module has a cocyclic quotient.
Every cyclic module is finitely generated.	Every cocyclic module is finitely cogenerated.
R is semisimple if and only if every cyclic module is injective.	R is a left V-ring if and only if every cocyclic module is injective.
It is an open question if that being a left artinian ring is equivalent to a ring in which every cyclic module embeds in a free module.	A ring in which every cocyclic embeds in a free module is a left cogenerator ring.
R is a semisimple ring if and only if every cyclic module is projective.	R is a semisimple ring if and only if every cocyclic module is projective.
The class of cyclic modules is always closed under quotients.	The class of cocyclic modules is closed under quotients precisely when $E(S)$ (the injective hull of S) is uniserial and artinian for each simple module S.

In this article we will basically address three types of question: When do cocyclic modules behave like cyclic modules, when do cyclic modules behave like cocyclic modules, and what properties do cocyclic modules have that cyclic modules do not and vice versa?

2. When cocyclic modules behave like cyclic modules and vice versa

It is well known that the class of cyclic modules is closed under quotients, but the class of cocyclic modules is not always closed under quotients, as the following example states.

Example 1 Consider the commutative ring

$$R = \left\{ \begin{pmatrix} a & (x, y) \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z}_2,\ (x, y) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2 \right\}.$$

This ring have the following lattice of ideals as it can be seen in [7] (Figure 1).

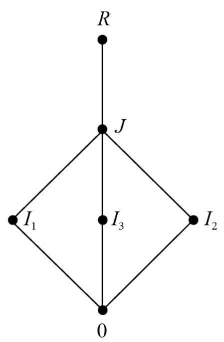

Figure 1. Hasse diagram of R

Then R is a commutative artinian ring with only one simple R-module S up to isomorphism. Then by [8] Theorem 2.13, there is a lattice anti-isomorphism between the lattice of ideals of R and the lattice of fully invariant submodules of $E(S)$. Hence the lattice of fully invariant submodules of $E(S)$ is (Figure 2):

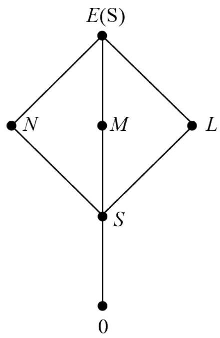

Figure 2. Hasse diagram of fully invariant submodules of $E(S)$

Then the quotient $E(S)/S$ has two non zero submodules with zero intersection. Hence $E(S)/S$ can not be cocyclic.

The next theorem tells us when the class of cocyclic modules is closed under quotients. Recall that a module M is uniform if every nonzero submodule is essential in M.

Theorem 1 The following statements are equivalent for a ring R:

(1) $E(S)$ is uniserial and artinian for each simple R-module S.

(2) The class of all cocyclic R-modules is closed under quotients.

(3) For each simple R-module S, the submodules of $E(S)$ are well-ordered with respect to inclusion.

Proof. $(1) \Rightarrow (2)$ Let M be a cocyclic module and $M \to L$ a nonzero epimorphism. Then there exists a simple module S such that $S \leq_e M \leq E(S)$. By hypothesis, M is artinian and uniserial and therefore L is also artinian and uniserial. Then there exists a simple submodule K of L. By linearity, $K \leq_e L$. Thus, L is cocyclic.

$(2) \Rightarrow (1)$ Let S be a simple module. Consider a nonzero epimorphism $E(S) \to L$. As $E(S)$ is cocyclic, it follows from the hypothesis that L is also cocyclic, so it is uniform. Thus, by [9] Proposition 2.7, $E(S)$ is uniserial. On the other hand, since L is cocyclic, we have that it is finitely cogenerated, so that, $E(S)$ is artinian.

$(1) \Rightarrow (3)$ As every nonempty family of submodules of $E(S)$ has a least element and $E(S)$ is uniserial, then $E(S)$ also has a least element.

$(3) \Rightarrow (1)$ It is clear. □

Also, it is well known that if a cyclic module has a projective cover, then such a cover must be cyclic. The projective cover of a cocyclic module is not always cocyclic, as the following example shows.

Example 2 Consider the commutative ring

$$R = \left\{ \begin{pmatrix} a & (x, y) \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z}_2,\, (x, y) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2 \right\}$$

of Example 1. Then R is an indecomposable commutative artinian ring with only one simple module S up to isomorphism. Then the projective cover $P(S)$ of S is isomorphic to R. Then $P(S)$ is not a cocyclic module. □

We are going to give necessary conditions for a projective cover of a cocyclic module to be cocyclic, but first we have to prove some lemmas.

Lemma 1 Let $f : M \to N$ and $g : N \to K$ be two epimorphisms. Suppose that gf is a superfluous epimorphism. Then $f : M \to N$ is superfluous.

Proof. Note that $f^{-1}(ker(g)) = ker(gf) << M$. So, as $ker(f) = f^{-1}(\{0\}) \leq f^{-1}(ker(g))$, then $ker(f) << M$. □

Lemma 2 If R is an artinian ring such that Re is cocyclic for each non-zero indecomposable idempotent e, then $P(S)$ is cocyclic and $S \cong Re/Je$ for every simple module S, with e an indecomposable idempotent element.

Proof. Let S be a simple R-module. As R is left artinian, then the simple R-module S has a projective cover. Also, $P(S) = Re$ for some idempotent element $e \in R$. As $End(P(S))$ is local, $P(S) = Re$ is left indecomposable, thus $P(S)$ is cocyclic. Also, if $\psi : P(S) \twoheadrightarrow S$ is a projective cover, then $ker(\psi) << Re$, so $ker(\psi) \leq Rad(Re)$ and $Rad(Re) = J(Re) = Je$. Therefore, $Ker(\psi) = Rad(Re)$. This implies $S \cong Re/Je$. □

Recall that a module M is MAX if every nonzero submodule of M has maximal submodules. Also, a ring R is MAX if every nonzero R-module is a MAX module. Equivalently, R is left MAX if every nonzero R-module has a simple quotient. Finally, a module M is coatomic if every proper submodule of M is contained in a maximal submodule of M.

Definition 1 A module is local if there exists a unique maximal submodule.

Lemma 3 If R is a left MAX ring, then every R-module is coatomic. Consequently, every local module has a greatest submodule.

Proof. Let M be a nonzero module and N a proper submodule of M. Then $M/N \neq 0$ and by hypothesis there exists a maximal submodule K/N of M/N. Hence $M/K \cong (M/N)/(K/N)$ is simple module, and consequently K is a maximal submodule of M. □

Theorem 2 If R is a left artinian ring, Re is left cocyclic for each indecomposable nonzero idempotent element $e \in R$ and M is a local cocyclic R-module, thus $P(M)$ exists and it is cocyclic.

Proof. Let M be a local cocyclic module and let N be the unique maximal submodule of M. Then $M/N = S$ is simple, and by Lemma 2, there exists an isomorphism $f : S \to Re/Je$ with e an indecomposable idempotent element of R and Re cocyclic. Hence, we have three epimorphisms $f^{-1}\pi_1 : Re \twoheadrightarrow M/N$ where $\pi_1 : Re \twoheadrightarrow Re/Je$, and $\pi : M \twoheadrightarrow M/N$. Since Re is projective, then there exists a morphism $\varphi : Re \to M$ such that the following diagram commutes (Figure 3):

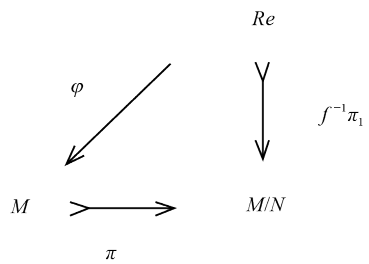

Figure 3. Projectivity of Re

We claim that, $\varphi : Re \to M$ is a projective cover of M. Indeed, as R is left artinian, it is perfect and by [10] Theorem 28.4 c, R is left MAX. Thus, by Lemma 3, N is the greatest proper submodule of M, so $N << M$. Then, as $\varphi(Re) + N = \varphi(Re) + Ker(\pi) = M$, we have $\varphi(Re) = M$, so $\varphi : Re \to M$ is an epimorphism. In addition, we know that $f^{-1}\pi_1$ is a superfluous epimorphism since π_1 is a superfluous epimorphism and f^{-1} is an isomorphism. Hence $\pi\varphi$ is a superfluous epimorphism. Then, by Lemma 1 $\varphi : Re \to M$ is superfluous. □

The next example shows that there exist rings that satisfy the hypothesis of Theorem 2.

Example 3 Consider the commutative ring

$$R = \frac{\mathbb{Z}_4[x]}{< x^2 >}$$

with $\bar{x} = u$ such that $u^2 = 0$. Then the lattice of ideals of this ring is (Figure 4):

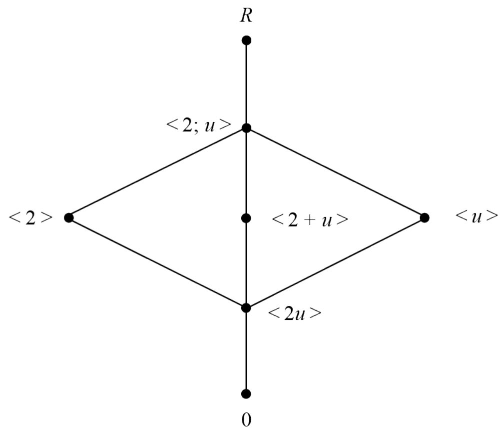

Figure 4. Hasse diagram of R

It is clear that R is a commutative cocyclic artinian module. Hence, the only direct summands are the trivial ones. Therefore, Re is cocyclic for each nonzero indecomposable idempotent e.

For left artinian rings such that $E(S)$ is noetherian for each simple module S we have been able to characterize when the class of cocyclic modules is closed under quotients and projective covers.

Theorem 3 For a left artinian ring R such that $E(S)$ is noetherian for each simple module S, the following statements are equivalent:

(1) The class of cocyclic modules is closed under quotients and projective covers.

(2) For each indecomposable idempotent $e \in R$, Re is cocyclic and $E(S)$ is uniserial and artinian.

Proof. (1) $\Rightarrow$ (2) By Theorem 1, $E(S)$ is artinian and uniserial. Now, consider an indecomposable idempotent element $e \in R$. As Re is projective, there exists a morphism $h : Re \to P(S)$ such that the following diagram commutes (Figure 5):

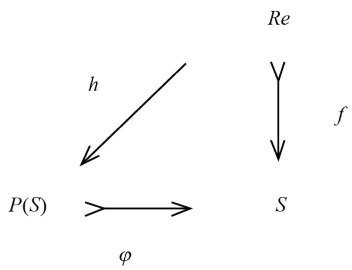

Figure 5. Projectivity of Re

Therefore $Ker(\varphi) + h(Re) = P(S)$, and consequently $P(S) = h(Re)$ since $Ker(\varphi)$ is superfluous in $P(S)$. Then $h : Re \to P(S)$ is an epimorphism. Hence $P(S)$ is a direct summand of Re. As Re is indecomposable, then $P(S) = Re$. Therefore Re is a cocyclic module.

(2) $\Rightarrow$ (1) Let $\mathscr{C}$ be the class of cocyclic modules. By Theorem 1, $\mathscr{C}$ is closed under quotients. As $E(S)$ is artinian and noetherian for each simple module S, then $E(S)$ is of finite length for each simple module S. Hence each cocyclic module has a unique maximal submodule. Thus, by Theorem 2, the class $\mathscr{C}$ is closed under projective covers. □

It is clear that every nonzero module has a cyclic submodule. Also, it is easily seen that every R-module has a cyclic quotient if and only if the ring R is left MAX. It is known that every module has a cocyclic quotient, although we give a proof for the reader's convenience. Also, we prove that every nonzero R-module has a nonzero cocyclic submodule precisely when R is a left semiartinian ring.

Proposition 1 Every non zero module has a cocyclic quotient.

Proof. Let $0 \neq x \in M$ and consider the set:

$$\Gamma_x = \{N \leq M \mid x \notin N\}.$$

Then, using Zorn's Lemma we get a maximal element $L_x \in \Gamma_x$. It is easy to see that L_x is a maximal submodule of $Rx + L_x$, and consequently $(Rx + L_x)/L_x$ is a simple submodule of M/L_x. Suppose that L'/L_x is a non zero submodule of M/L_x. Then $L_x < L'$, and consequently $x \in L'$. Hence $Rx + L_x \leq L'$. Therefore $(Rx + L_x)/L_x \leq L'/L_x$ and $M/_x$ is a cocyclic quotient of M. □

Recall that a ring R is left semiartinian if every nonzero R-module has a simple submodule.

Proposition 2 The following statements are equivalent for a ring R:

(1) R is left semiartinian.

(2) Every nonzero module has a cocyclic submodule.

(3) Every injective module is the injective hull of a direct sum of cocyclic modules.

Proof. $(2) \Rightarrow (1)$ Let M be a nonzero module. Then M has a nonzero cocyclic submodule N. Then there exists a simple module S such that $S \leq_e N$. Therefore, M has a simple submodule.

$(2) \Rightarrow (3)$ This is by a standard use of Zorn's Lemma.

$(3) \Rightarrow (2)$ Let M be a nonzero module. Then $M \leq_e E(M) = E(\bigoplus_{i \in I} K_i)$, with K_i cocyclic for each $i \in I$. By the projection argument [11] 2.3.3, M has a nonzero submodule isomorphic to a submodule of K_j for some $j \in I$. Therefore M has a nonzero cocyclic submodule. □

At this stage, it is natural to ask when the cyclic modules and the cocyclic modules coincide. This question was answered in [12]. In that paper, the authors demonstrated the following:

Theorem 4 The classes of nonzero cyclic and cocyclic modules coincide if and only if R is a left uniserial and an artinian principal ideal ring.

We now characterize when every cyclic module is cocyclic and describe for which rings every cocyclic module is cyclic.

Theorem 5 The following statements are equivalent for a ring R:

(1) Every cyclic module is cocyclic.

(2) R is left artinian and left uniserial.

(3) Every nonempty subset of left ideals of R has least element.

(4) R is left semiartinian and left uniserial.

(5) The lattice of left ideals of R is as follows:

$$0 < J^{n-1} < J^{n-2} < \cdots < J < R$$

where J is the Jacobson radical and J^m is a principal ideal for every $m \in \mathbb{N}$.

Proof. $(1) \Rightarrow (2)$ Let M be a R-module and ${}_R R \to M$ an epimorphism. Then, by hypothesis, M is a cocyclic module, and thus it is finitely cogenerated, consequently R is left artinian. Also, as M is cocyclic, then it is uniform. Therefore by [9] Proposition 2.7, R is left uniserial.

$(2) \Rightarrow (1)$ Let C be a cyclic module. Then there exists an epimorphism $R \to C$. Therefore C is artinian and uniserial. As C is artinian, then it has a simple submodule S. Furthermore, by the linearity of C, we have that S is essential in C.

$(2) \Rightarrow (3)$ Consider a nonempty family of left ideas of R. As R is left artinian, then such family posses a minimal element, and by linearity of R, such element is in fact a least element.

(3) $\Rightarrow$ (4) This is clear.

(4) $\Rightarrow$ (1) Let M be a cyclic module. Then M is a quotient of R, and therefore M is uniserial and semiartinian, so that $S = soc(M) \leq_e M$, for some simple module S. Then M is cocyclic.

(5) $\Rightarrow$ (2) Clear.

(2) $\Rightarrow$ (5) Note that at this point, we have that the former conditions are equivalent. Now, observe that, since R is left uniserial, it contains a unique maximal ideal I, so R is local, and hence $I = J \neq 0$ where J is the Jacobson radical. Also, as a consequence of R being artinian, we get that J is nilpotent. So, we will show by induction on the nilpotency index of the Jacobson radical, that every ideal of R has the form J^m for some $m \in \mathbb{N}$. Indeed, let n be the nilpotency index of the radical. If $n = 2$, then J is a R/J-module. Furthermore, J is a semisimple R/J-module, since R/J is a division ring. Thus J is a semisimple R-module. Then linearity of R implies that J is simple. So, if we consider an ideal I of R, then either $I = J$ or $I = J^0 = R$. Now, for the inductive case J^{n-1}, we have that $J^{n-1} \neq 0$ hence, the same argument as above, proves that J^{n-1} is simple. So, if I is a nonzero ideal, we get that $J^{n-1} \leq I$ and therefore R/J^{n-1} is an artinian uniserial ring with nilpotency index less than n, with $I/J^n \leq R/J^n$. Then the inductive hypothesis yields $I = J^m$ for some $m \in \mathbb{N}$. It remains to show that I is cyclic. Let $x \in J$ such that $x \in J^2$. Then, linearity ensures that $J^2 < Rx \leq J$, and this implies that $J = Rx$. □

Proposition 3 Let R be a ring such that every cocyclic module is local in the sense of Definition 1. Then the following statements are equivalent:

(1) Every cocyclic module is cyclic.

(2) R is left MAX and $E(S)$ is noetherian for each simple module S.

Proof. (1) $\Rightarrow$ (2) By hypothesis $E(S)$ and all its submodules are cyclic. Then $E(S)$ is noetherian for each simple module S. Now, take a nonzero module M. By Proposition 1, M has a cocyclic quotient which is cyclic by hypothesis. Hence M has a simple quotient. Therefore, R is a left MAX ring.

(2) $\Rightarrow$ (1) Consider M a cocyclic module. Then there exists a simple module S such that $M \leq E(S)$. Therefore, M is finitely generated. That is $M = Rx_1 + \cdots + Rx_n$ for some $x_1, \ldots, x_n \in M$. Note that by an induction argument it is sufficient to prove that if $M = Rx_1 + Rx_2$, then $M = Rx_1$ or $M = Rx_2$. Indeed, as R is a left MAX ring, then by Lemma 3, M has a greatest proper submodule N. If $Rx_1 < M$ and $Rx_2 < M$, then $Rx_1 \leq N$ and $Rx_2 \leq N$. Consequently $M = N$, a contradiction. Therefore M is cyclic. □

There are some rings that are defined by means of cyclic modules such as Köthe rings. Recall that a ring R is left Köthe if every module is a direct sum of cyclic modules. Also a Köthe ring is a ring which is left and right Köthe ring. These rings are related to the left uniserial left artinian rings in which every cyclic module is cocyclic as the next theorem elucidates.

Theorem 6 For a left uniserial ring R the following conditions are equivalent:

1. R is a left Köthe ring.
2. R is a Köthe ring.
3. R is an artinian principal ideal ring.
4. R is left artinian and each cocyclic module is a cyclic module.

Proof. (1) $\Rightarrow$ (3) Let I be a two sided ideal of R. By Lemma 4, it follows that R/I is a left Köthe ring. Since R is left uniserial, so is ${}_{R/I}R/I$. Consequently, $E(R/I)$ is an indecomposable R/I-module. Moreover, $E(R/I)$ is a direct sum of cyclic R/I-modules, so $E(R/I)$ is cyclic. Furthermore, by [13] Theorem 4.4, R/I is left artinian. Then [14] Section 3, 2. Lemma implies that R/I is left autoinjective. Hence, R/I is a QF ring for each two sided ideal I of R. Therefore, by [15] Proposition 25.4.6B, R is an artinian principal ideal ring.

(3) $\Rightarrow$ (2) This follows from Köthe's Theorem.

(2) $\Rightarrow$ (1) Clear.

(4) $\Leftrightarrow$ (3) Follows from Theorems 5 and 4. □

Moreover, the class of cyclic R-modules coincides with the class of cocyclic R-modules precisely when R is a left uniserial left Köthe ring as we will see in Theorem 7. But first we are going to introduce some lemmas.

Lemma 4 If R is a left Köthe ring, then R/I is a left Köthe ring for each two sided ideal I of R.

Proof. Let ${}_{R/I}M$ be a left R/I-module. Hence M is a R-module such that $IM = \{0\}$. Also, since R is left Köthe, $M = \bigoplus_{j\in J} C_j$ with C_j a left cyclic R-module for each $j \in J$. Thus $\bigoplus_{j\in J} IC_j = I(\bigoplus_{j\in J} C_j) = IM = \{0\}$. Then $IC_j = \{0\}$ for each $j \in J$, concluding that C_j is a left cyclic R/I-module. Therefore, R/I is a left Köthe ring for each ideal I of R. □

Lemma 5 If R is a ring such that every left module is a direct sum of cocyclic modules, then R is left artinian.

Proof. As every nonzero module is a direct sum of cocyclic modules, every nonzero module has nonzero socle. Equivalently each nonzero module has essential socle. Now, take Rx a cyclic module. Then $Rx = \bigoplus_{i=1}^{n} C_i$ with C_i cocyclic for each $i \in \{1, \ldots, n\}$ as being finitely generated. Then

$$\operatorname{soc}(Rx) = \operatorname{soc}\bigoplus_{i=1}^{n} C_i = \bigoplus_{i=1}^{n}\operatorname{soc}(C_i) = \bigoplus_{i=1}^{n} S_i$$

with S_i a simple module for each $i \in \{1, \cdots, n\}$. Then every left cyclic module has left finitely generated essential socle. Hence, each left cyclic is finitely cogenerated. Therefore R is left artinian. □

Theorem 7 The following statements are equivalent for a ring R:

1. The classes of nonzero cyclic R-modules and cocyclic R-modules coincide,
2. R is a left uniserial left Köthe ring,
3. R is a left uniserial and an artinian principal ideal ring,
4. R is a left uniserial ring such that every left module is a direct sum of left cocyclic modules.

Proof. (1) ⇔ (2) ⇔ (3) They follow from Theorem 6.

(3) ⇒ (4) By Köthe's theorem we see that R is a left Köthe ring. Also, Theorem 6 implies that the class of nonzero cyclic modules and cocyclic modules coincide. Therefore, every module is a direct sum of cocyclic modules.

(4) ⇒ (3) By hypothesis every cyclic module is a direct sum of cocyclic modules. Hence, every cyclic module has a simple submodule. Also, as R is uniserial, every cyclic module is uniserial, so the uniqueness of each submodule simple holds. Therefore, every cyclic module is cocyclic. This implies, by Theorem 6, that R is uniserial and an artinian principal ideal ring. □

It is well known that if each module embeds in a free module, then the ring R has to be QF. In recent years, rings (*CF rings*) with the property that all of it's cyclic modules embed in a free module have been studied in [2]. Also have been studied rings (*FGF rings*) in which every finitely generated module embeds in a free module, see [2]. Now we will study the dual properties of the CF and FGF rings, i.e., *every cocyclic module embeds in a free module* and *every finitely cogenerated module embeds in a free module* and conclude that these dual properties are equivalent.

Theorem 8 The following statements are equivalent for a ring R:

(1) Every cocyclic module embeds in a free module.

(2) The injective hull of every simple module is projective.

(3) R is a cogenerator.

(4) Every finitely cogenerated module embeds in a free module.

(5) The injective hull of every finitely cogenerated module is projective.

Proof. (1) ⇒ (2) By hypothesis, $E(S)$ embeds in a free module. Then $E(S)$ is a direct summand of a free module, so, it is projective.

(2) ⇒ (1) Let M be a cocyclic module. Then there exists a simple module S such that $M \leq E(S)$. By hypothesis, $E(S)$ is a direct summand of a free module, and hence M embeds in a free module.

(2) ⇒ (4) Let M be a finitely cogenerated module. Then $soc(M) = \bigoplus_{i=1}^{n} S_i \leq_e M$, so $E(\bigoplus_{i=1}^{n} S_i)$= $\bigoplus_{i=1}^{n} E(S_i) = E(M)$. Since each $E(S_i)$ is projective, $E(M)$ is also projective, and thus it embeds in a free module.

(5) ⇒ (3) Let us prove that the injective hull of every simple module embeds in R. Let S be a simple module. By hypothesis $E(S)$ is projective since it is finitely cogenerated, so $E(S)$ embeds in a free module. Then there exists a monomorphism $\varphi : E(S) \to R^{(X)}$ for some set X. Then there exists $\pi_j : R^{(X)} \to R$ with $\pi_j \neq 0$ such that $\pi_j\varphi(S) \neq 0$. Therefore, $\pi_j\varphi$ is a monomorphism.

$(3) \Rightarrow (2)$ As R is a left cogenerator ring, then $E(S)$ embeds in R for each simple module S. Therefore, $E(S)$ is projective for each simple module S.

$(4) \Rightarrow (5)$ Let M be a finitely cogenerated module. Then $E(M)$ is finitely cogenerated: indeed, let $S \leq E(M)$ be a simple module. Then $S \cap M \neq 0$, so $S \leq M$, thus $soc(E(M)) = soc(M)$. Hence, by hypothesis $E(M)$ embeds in a free module and, consequently, is a projective module. □

Finally, we observe some similarities between the cyclic and cocyclic modules with respect to the pretorsion classes and the free pretorsion classes. Recall that a class $\mathscr{C}$ of modules is pretorsion if it is closed under quotients and direct sums. Also, a class $\mathscr{C}$ is free torsion if it is closed under submodules and direct products. Recall that a class $\mathscr{C}$ is determined by its cyclic modules if for each module M : $M \in \mathscr{C}$ if and only if $Rx \in \mathscr{C}$ for each $x \in \mathscr{C}$. The next proposition is a well known result:

Proposition 4 The following statements are equivalent:

(1) Every pretorsion class is closed under submodules.

(2) Every pretorsion class is determined by its cyclic modules.

We now establish a definition:

Definition 2 A class $\mathscr{C}$ is determined by its cocyclic modules if for each module M : $M \in \mathscr{C}$ if and only if every cocyclic quotient of M is in the class $\mathscr{C}$.

Again, the relation between cyclic and cocyclic modules is in some cases dual, as the following proposition shows:

Proposition 5 The following statements are equivalent:

(1) Every free pretorsion class is closed under quotients.

(2) Every free pretorsion class is determined by its cocyclic modules.

Proof. $(1) \Rightarrow (2)$ Let $\mathscr{C}$ be a free pretorsion class. If $M \in \mathscr{C}$, then by hypothesis every cocyclic quotient belongs to $\mathscr{C}$. Now, suppose that every cocyclic quotient of a module M belongs to $\mathscr{C}$. As in the proof of Proposition 1, for each $x \neq 0$ there exists a maximal submodule L_x such that $x \notin L_x$ and M/L_x is a cocyclic quotient for each nonzero $x \in M$. Then consider the morphism

$$f : M \to \prod_{0 \neq x \in M} M/L_x$$

defined by $f(m) = (m + L_x)_{0 \neq x \in M}$. Let $x \in Ker(f) = \bigcap_{0 \neq x \in M} L_x$. If $y \in Ker(f)$ and $y \neq 0$, then $y \in L_x$ for each nonzero $x \in M$, in particular $y \in L_y$, which is absurd. Then f is a monomorphism, and consequently $M \in \mathscr{C}$.

$(2) \Rightarrow (1)$ Let $\mathscr{C}$ be a free pretorsion class and $f : M \to L$ an epimorphism with $M \in \mathscr{C}$. Then each cocyclic quotient of L is a cocyclic quotient of M. Hence each cocyclic quotient of L is in $\mathscr{C}$. Then $L \in \mathscr{C}$ by hypothesis. Therefore $\mathscr{C}$ is closed under quotients. □

Conflict of interest

The authors declare no conflicts of interest.

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