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Research Article

Liouville Theorems and Gradient Estimates for Positive Solutions to $\Delta_p u + \Delta_q u + h(u) = 0$ on a Complete Manifold

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Abstract: In this paper, we use the Saloff-Coste Sobolev inequality and Nash-Moser iteration method to study the local and global behaviors of positive solutions to the nonlinear elliptic equation $\Delta_p u + \Delta_q u + h(u) = 0$ defined on a complete Riemannian manifold $(M,\ g)$ with Ricci lower bound, where $q \geq p > 1$ are constants and $\Delta_z u = \operatorname{div}\left(|\nabla u|^{z-2}\nabla u\right)$, with $z \in \{p,\ q\}$, is the usual z-Laplace operator. Under some assumptions on $h(u)$, we derive gradient estimates and Liouville type theorems for positive solutions to the above equation. In particular, we show that, if an entire positive solution u to $\Delta_p u + \Delta_q u = 0$ $(1 < p \leq q)$ on a complete non-compact Riemannian manifold M with non-negative Ricci curvature and $\dim M = n \geq 3$ satisfies

$$\lim_{M \ni x \to \infty} \frac{u(x)}{\mathrm{d}(x_0,\ x)} = 0$$

for some $x_0 \in M$, then u is a constant.

***Keywords*:** gradient estimate, Nash-Moser iteration, Liouville type theorem

MSC: 65L05, 34K06, 34K28

1. Introduction

Gradient estimate is a fundamental technique in the study of partial differential equations on a Riemannian manifold. Indeed, one can use gradient estimate to deduce Liouville type theorems [1–6], to derive Harnack inequalities [4, 6], to study the geometry of manifolds [4, 7–9], etc.

On the other hand, it is well-known that the Liouville theorem has had a huge impact across many fields, such as complex analysis, partial differential equations, geometry, probability, discrete mathematics and complex and algebraic geometry. The impact of the Liouville theorem has been even larger as the starting point of many further developments. For more details on the Liouville properties of harmonic functions and some related theory of function on a manifold we refer to an expository paper [10] written by Colding (see also [7]).

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In this paper, we are concerned with the following equation defined on a complete Riemannian manifold $(M,\ g)$ equipped with a metric g

$$\Delta_p u+\Delta_q u+h(u)=0, \tag{1}$$

where $q \geq p > 1$ are constants, $h \in C^1(\mathbb{R}^+)$ and $\Delta_z u = \operatorname{div}\left(|\nabla u|^{z-2}\nabla u\right)$, $z=p$ or q is the usual z-Laplace operator.

Since the content of this paper is closely concerned with double phase problems, we start with a short description on the background. The double-phase problem (1) is motivated by numerous models arising in mathematical physics. For instance, we can refer to Born-Infeld equation [11] that appears in electromagnetism:

$$-\operatorname{div}\left(\frac{\nabla u}{\sqrt{1-2|\nabla u|^2}}\right)=h(u) \ \text{ in } \ \Omega_0,$$

where $\Omega_0 \subset \mathbb{R}^n$ is a bounded domain with C^2-boundary $\partial\Omega_0$. Indeed, by the Taylor formula, we have

$$\frac{1}{\sqrt{1-2x}}=1+x+\frac{3}{2}x^2+\cdots+\frac{(2m-3)!!}{(m-1)!}x^{m-1}+\cdots.$$

Taking $x=|\nabla u|^2$ and adopting the first order approximation, we obtain problem (1) for $p=2$ and $q=4$. Furthermore, the m-th order approximation problem is driven by the multi-phase differential operator

$$-\Delta u-\Delta_4 u-\frac{3}{2}\Delta_6 u-\cdots-\frac{(2m-3)!!}{(m-1)!}\Delta_{2m}u.$$

Moreover, that class of equations comes, for example, from a general reaction-diffusion system:

$$u_t=\operatorname{div}[D(u)\nabla u]+c(x,\ u),$$

where $D(u)=(|\nabla u|^{p-2}+|\nabla u|^{q-2})$. This system has a wide range of applications in physics and related sciences, such as biophysics, plasma physics and chemical reaction design. In such applications, the function u describes a concentration, the first term on the right-hand side of the equation corresponds to the diffusion with a diffusion coefficient $D(u)$; whereas the second one is the reaction and relates to source and loss processes. Typically, in chemical and biological applications, the reaction term $c(x,\ u)$ is a polynomial of u with variable coefficients (see [12, 13]).

Some mathematicians studied the existence and multiplicity properties of solutions to these equations, in particular, one may pay attention to the equation (1). When $h(u)=0$, then equation (1) reduces to

$$\Delta_p u+\Delta_q u=0, \tag{2}$$

which has been studied by Bonheure and Rossi. In particular, in [14] they studied the behavior as $p\to+\infty$ of solutions $u_{p,\ q}$ to (2) in a bounded smooth domain $\Omega_1 \subset \mathbb{R}^n$ with a Lipschitz Dirichlet boundary datum $u=g$ on $\partial\Omega_1$. They showed that there is a uniform limit of a sub-sequence of solutions, that is, there is $p_j \to +\infty$ such that $u_{p_j,\, q} \to u_\infty$ uniformly in $\overline{\Omega}_1$.

In [15], Alreshidi et al. studied the existence of a positive solution to the $(p,\, q)$-Laplacian equation

$$\begin{cases} -\Delta_p u - \Delta_q u = \lambda_0 h_0(u), \ \text{in} \ \Omega_2, \\ u = 0, \qquad\qquad\qquad \text{on} \ \partial\Omega_2, \end{cases}$$

where Ω_2 is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega_2$, $h_0 : (0,\, +\infty) \longrightarrow \mathbb{R}$ is continuous, p-sublinear at ∞ and is allowed to be singular at 0, and $\lambda_0 > 0$ is a large parameter.

In [16], Alves and Figueiredo studied the multiplicity and concentration of solutions for the following problem with linear potential

$$\begin{cases} \varepsilon^p \Delta_p u + \varepsilon^q \Delta_q u - V(x)(|u|^{p-2}u + |u|^{q-2}u) + P_0(u) = 0, \ \text{on} \ \mathbb{R}^n, \\ u \in W^{1,\, p}(\mathbb{R}^n) \cap W^{1,\, q}(\mathbb{R}^n), \ u > 0, \end{cases}$$

where the authors assumed that the potential $V(x)$ satisfies the following global condition introduced by Rabinowitz [17] $0 < \inf_{x\in\mathbb{R}^n} V(x) < \liminf_{|x|\to\infty} V(x) < \infty$ and the nonlinear term $P_0(u)$ has C^1-smoothness with superlinear and subcritical growth. Cai and Rădulescu [18] also considered the existence of the following $(p,\, q)$-Laplacian equation:

$$-\Delta_p u - \Delta_q u + \lambda |u|^{p-2}u = f(u) \ \text{ in } \ \mathbb{R}^n$$

with L^p-constraint $\int_{\mathbb{R}^n} |u|^p = c^p$, where $1 < p < q < n$, $c > 0$ is a constant and $u \in W^{1,\, p}(\mathbb{R}^n) \cap W^{1,\, q}(\mathbb{R}^n)$. For more details we refer to [18] and references therein.

As for $\Delta_p u + \Delta_q u + h(u) = 0$, there is little literature to discuss their global and local behaviors of solutions. Very recently, the authors of this paper in [19] employed the Nash-Moser iteration method to study the gradient estimates for the nonnegative solutions to the following $\Delta_p u + \Delta_q u + au^s + bu^l = 0$ on a complete Riemannian manifold and obtain some Liouville theorems.

On the other hand, when $p = q$ the equation (1) reduces to the following $\Delta_p u + f(u) = 0$, where $f(u) = \frac{1}{2}h(u)$. For the sake of convenience and uniformization of notation, without confusions we still denote the above equation as $\Delta_p u + h(u) = 0$. Many mathematicians pay attention to the equation and great progress has been made. For details we refer to [3, 19–27] and references therein. For instance, as $h(u) = \lambda |u|^{p-2}u$ Lindqvist [28] proved the first eigenvalue $\lambda = \lambda_1$ for the equation $\Delta_p u + \lambda |u|^{p-2}u = 0$ defined on a bounded domain $\Omega \subset \mathbb{R}^n$ is simple in any bounded domain $\Omega \subset \mathbb{R}^n$. In the case M is an Euclidean space and $h(u)$ is a general real function, this equation was studied by Serrin and Zou in [29], and some Liouville theorems and universal estimates were established (also see [30]). Very recently, He, the first named author of this article and Wei [31] adopted a new way to employ the Nash-Moser iteration to study the gradient estimates for the quasilinear elliptic equation $\Delta_p u + au^q = 0$ on a complete Riemannian manifold.

In particular, it is worthy to point out that for the case $p = 2$ the following new estimate was obtained in [32]

$$\frac{|\nabla u|^2}{u^2} + au^{q-1} \leq \frac{2n}{2 - n\max\{0,\, q-1\}} \left[\frac{C_1^2(n-1)(1+\sqrt{\kappa}R) + C_2}{R^2} + 2\kappa + \frac{2nC_1^2}{(2+n\max\{0,\, q-1\})R^2} \right], \tag{3}$$

if the Ricci curvature of domain manifold satisfies $\mathrm{Ric}_g \geq -(n-1)\kappa$ and $q < \dfrac{n+2}{n}$. Obviously, this is a stronger estimate than the logarithmic gradient estimate (also see [33]). Wang-Wei [5] also derived Cheng-Yau type gradient estimates for positive solutions to $\Delta u + u^q = 0$ under the assumption $q \in \left(-\infty,\ \dfrac{n+1}{n-1} + \dfrac{2}{\sqrt{n(n-1)}}\right)$. Shortly after, He, the first named author of this article and Wei [31] extended the Cheng-Yau estimate to the range $q \in \left(-\infty,\ \dfrac{n+3}{n-1}\right)$. Very recently, Lu extended the estimate (3) in [32] to the value range $q \in \left(-\infty,\ \dfrac{n+3}{n-1}\right)$.

In the case $h(u) \equiv 0$, this equation is just p-Laplace equation. Wang and Zhang [9] obtained the logarithmic Cheng-Yau type gradient estimates (also see [8]) for p-harmonic functions on a complete Riemannian manifold $(M,\ g)$. Later, Chang et al. [34] introduced and studied an approximate solution of the p-Laplace equation $\Delta_p u = 0$ and a linearization $\mathscr{L}_\varepsilon$ of a perturbed p-Laplace operator. By deducing an $\mathscr{L}_\varepsilon$-type Bochner's formula and Kato type inequalities, they proved a Liouville type theorem for weakly p-harmonic functions with finite p-energy on a complete noncompact manifold M which supports a weighted Poincaré inequality and satisfies a curvature assumption. Moreover, they also established a Liouville type theorem for strongly p-harmonic functions with finite q-energy on Riemannian manifolds.

In fact, Gidas and Spruck [3] also considered the following equation $\Delta v + f(x,\ v) = 0$ and derived a Liouville theorem, Harnack inequality and singularity decay estimate if the nonlinear term $f(x,\ \cdot)$ fulfills some additional technique conditions. These results can be found in Theorems 6.1 and 6.3 in [3]. Actually, Lu also considered the equation defined on a manifold $(M,\ g)$ in [35]. Even in the case $f(x,\ v) \equiv f(v)$, one always needs to assume that $t^{-\alpha} f(t)$ is non-increasing on $(0,\ +\infty)$ for some $\alpha \in \left(1,\ \dfrac{n+3}{n-1}\right)$ if the domain manifold is not an Euclidean space. For more details we refer to [35].

Besides, Han et al. in [36] proved recently any solution $u \in C^1(B_R(o))$ to the following equation $\Delta_p u - |\nabla u|^q = 0$ with $q > p-1$ defined on a geodesic ball $B_R(o)$ of a complete Riemannian manifold $(M,\ g)$ with $\mathrm{Ric}_g \geq -(n-1)\kappa g$ satisfies

$$\sup_{B_{R/2}(o)} |\nabla u| \leq C(n,\ p,\ q) \left(\frac{1+\sqrt{\kappa} R}{R} \right)^{\frac{1}{q-p+1}}.$$

Inspired by [19, 31, 36], in the present paper we shall use the Nash-Moser iteration method to study the gradient estimate and the Liouville property of the above equation (1), defined on a complete Riemannian manifold, with a non-decreasing function $h(u) \in C^1(0,\ +\infty)$.

1.1 *Main results*

By a solution u of (1) in an (arbitrary) domain Ω_3, we mean a positive solution $u \in C^1(\Omega_3) \cap C^3(\widetilde{\Omega}_3)$, where $\widetilde{\Omega}_3 = \{x \in \Omega_3\ :\ |\nabla u(x)| \neq 0\}$. It is worth mentioning that, if the coefficients of (1) satisfy some suitable conditions, it is well-known that any solution of (1) satisfies $u \in C^{1,\ \alpha}(\Omega_3)$ for some $\alpha \in (0,\ 1)$ (for example, see [24, 25, 37]). Moreover, u is in fact smooth in $\widetilde{\Omega}_3$.

For the sake of simplicity, we define

$$\delta(z)=\begin{cases}2-z, & 1<z<2,\\ 0, & z\geq 2,\end{cases}\qquad \theta(z)=\begin{cases}z-1, & 1<z<2,\\ 1, & z\geq 2,\end{cases}$$

$$\lambda_z(\beta)=\beta(z-1)\left[1-\frac{z}{2}\beta-\frac{n}{8(1-\delta(z))}\beta(z-1)\right] \text{ and } \phi=\|u\|_{L^\infty(B(x_0,\, R))}.$$

By using the above notation, we set

$$\mathfrak{A}_0(p,\, q)=\left\{\beta\ :\ 0<\beta\leq\frac{4}{q-p+2},\ \lambda_p(\beta)>0 \text{ and } \lambda_q(\beta)>0\right\}$$

and

$$\mathfrak{B}_0(\beta,\, p,\, q)=\left\{t\ :\ t>\max\left\{1,\ 2-\frac{p}{2},\ \frac{(n-2)(q-p)}{2}-\frac{p}{2}\right\}\right.$$

and

$$\left.\frac{1}{\theta(z)t}(z-1)^2\left(\frac{z+p}{2}\right)^2\beta^2\leq\frac{\lambda_z(\beta)}{2} \text{ with } z\in\{p,\, q\}\right\}.$$

Now, we state our main results.

Theorem 1 Let $1<p\leq q$ and $(M,\ g)$ be an n-dim $(n\geq 3)$ complete manifold with $\mathrm{Ric}\geq-(n-1)\kappa$, where κ is a non-negative constant. Furthermore, $h\in C^1(\mathbb{R}^+)$ is a non-increasing function. Assume u is a positive solution to equation (1) on the geodesic ball $B(x_0,\ 2R)\subset M$. For any $\beta\in\mathfrak{A}_0(p,\ q)$ and $t_0\in\mathfrak{B}_0(\beta,\ p,\ q)$, then there exist

$$\zeta=1-\frac{n(q-p)}{4\beta_1}$$

where $\beta_1=\dfrac{n}{n-2}\cdot\dfrac{p+2t_0}{2}>0$, and a positive constant $\mathscr{C}^*=\mathscr{C}^*(n,\ p,\ q)$, such that the following estimate holds true

$$\sup_{B\left(x_0,\ \frac{R}{8}\right)}\frac{|\nabla u|^2}{u^\beta}\leq\mathscr{C}^*\left[\xi_*+\left(\xi_*\phi^{(1-\zeta)\beta}\right)^{\frac{1}{\zeta}}\right],$$

where

$$\xi_* = \exp\left\{C_n \frac{n(1+\sqrt{\kappa}R)}{2t_0+p}\right\}(1+\kappa R^2)^{\frac{n}{2\beta_1}}$$

$$\cdot\left[(1+\kappa R^2)^{\frac{2}{p+2t_0}}\left(\kappa+\frac{1}{R^2}\right)^{\frac{2}{q-p+2}}\lambda_q^{-\frac{2}{q-p+2}}\phi^{\frac{4}{q-p+2}-\beta}+\kappa^{\frac{q+2t_0+2}{p+2t_0}}R^{\frac{4}{p+2t_0}}\lambda_q^{-\frac{q+2t_0}{p+2t_0}}\phi^{\frac{2q+4t_0}{p+2t_0}-\beta}\right.$$

$$\left.+t_0R^{-\frac{2q+4t_0}{p+2t_0}}\lambda_q^{-\frac{q+2t_0}{p+2t_0}}\phi^{\frac{2q+4t_0}{p+2t_0}-\beta}+t_0^{\frac{2}{q-p+2}}R^{-\frac{4}{q-p+2}}\lambda_q^{-\frac{2}{q-p+2}}\phi^{\frac{4}{q-p+2}-\beta}\right].$$

Remark 1 Here we would like to give a remark of the above corollary.

• If we let $p = q = 2$, then $\mathfrak{A}_0(2,\ 2) = \left(0,\ \frac{8}{n+8}\right)$ and $\mathfrak{B}_0(\beta,\ 2,\ 2) = \left(\frac{64}{8-(n+8)\beta},\ +\infty\right)$.

For the sake of convenience, we define

$$\widetilde{\gamma}(\omega,\ p,\ q,\ \varepsilon_0) = \begin{cases} \dfrac{4}{q-p+2} - \varepsilon_0, & \omega < 1, \\ 2+\varepsilon_0, & \omega \geq 1. \end{cases}$$

By using Theorem 1, we can achieve the following conclusions.

Theorem 2 Let $1 < p \leq q$ and $(M,\ g)$ be a complete non-compact Riemannian manifold with non-negative Ricci curvature and $\dim M = n \geq 3$. Furthermore, $h \in C^1(\mathbb{R}^+)$ is a non-increasing function. If a positive solution u to equation (1) on M satisfies

$$\limsup_{M\ni x\to\infty} \frac{u(x)}{d(x_0,\ x)} \leq \omega$$

for some $x_0 \in M$, then u has the following gradient estimate

$$\sup_M |\nabla u|^2 \leq \mathscr{C}_2^* \omega^{\gamma_0},$$

where $\mathscr{C}_2^* = \mathscr{C}_2^*(n,\ p,\ q,\ \gamma_0)$ is a positive constant and $\gamma_0 \in \{\widetilde{\gamma}(\omega,\ p,\ q,\ \varepsilon_0) > 0\ :\ \varepsilon_0 > 0\}$.

From the above theorem, we can obtain directly the following conclusions on Liouville type properties of entire positive solutions of (1).

Theorem 3 Let $1 < p \leq q$ and $(M,\ g)$ be a complete non-compact Riemannian manifold with non-negative Ricci curvature and $\dim M = n \geq 3$. Furthermore, $h \in C^1(\mathbb{R}^+)$ is a non-increasing function. If a positive solution u to equation (1) on M satisfies

$$\lim_{M\ni x\to\infty} \frac{u(x)}{\mathrm{d}(x_0,\ x)} = 0$$

for some $x_0 \in M$, then either u is a trivial solution or u does not exist.

In fact, comparing with Theorem 3, we can achieve a more general conclusion by using Theorem 1. For the sake of convenience, we define

$$\widetilde{\Theta}(p,\, q) = \sup_{\beta \in \mathfrak{A}_0(p,\, q),\; t_0 \in \mathfrak{B}_0(\beta,\, p,\, q)} \frac{4t_0 + 2q}{4t_0 + 2q - \zeta\beta(2t_0 + p)} = \sup_{\beta \in \mathfrak{A}_0(p,\, q)} \frac{2}{2-\beta},$$

where $\zeta = \zeta(p,\, q,\, t_0)$ is defined in Theorem 1. By using the above notation, we draw the following conclusion:

Theorem 4 Let $1 < p \leq q$ and $(M,\, g)$ be a complete non-compact Riemannian manifold with non-negative Ricci curvature and $\dim M = n \geq 3$. Furthermore, $h \in C^1(\mathbb{R}^+)$ is a non-increasing function. If a positive solution u to equation (1) on M satisfies

$$\lim_{M \ni x \to \infty} \frac{u(x)}{(d(x_0,\, x))^{\Theta_0}} = 0$$

for some $x_0 \in M$ and $\Theta_0 \in \left(0,\, \widetilde{\Theta}(p,\, q)\right)$, then either u is a trivial solution or u does not exist.

Remark 2 Here we would like to give several remarks of the above theorems.

• 1. Since $\widetilde{\Theta}(p,\, q) > 1$, we can know that Theorem 4 is more general than Theorem 3.

• 2. It is worthy to point out that we can achieve an estimate of $\frac{|\nabla u|^2}{u^\beta}$, which is similar to the estimate obtained in Theorem 2. One can easily achieve this estimate by using Theorem 1.

1.2 *Some further examples*

Next, we give some examples of the function $h(u)$ which fall into the above cases.

Example 1 Let $h(u) = 0$, then equation (1) reduces to

$$\Delta_p u + \Delta_q u = 0. \tag{4}$$

Let $1 < p \leq q$ and M ($\dim M \geq 3$) be a complete non-compact Riemannian manifold with non-negative Ricci curvature. If a positive solution u to equation (4) on M satisfies

$$\lim_{M \ni x \to \infty} \frac{u(x)}{\mathrm{d}(x_0,\, x)} = 0$$

for some $x_0 \in M$, then u is a trivial solution according to Theorem 3.

Example 2 Let $h(u) = -au^l$ and $al \geq 0$, then equation (1) reduces to

$$\Delta_p u + \Delta_q u = au^l. \tag{5}$$

Let $1 < p \leq q$ and M ($\dim M \geq 3$) be a complete non-compact Riemannian manifold with non-negative Ricci curvature. If a positive solution u to equation (5) on M satisfies

$$\lim_{M\ni x\to\infty}\frac{u(x)}{\mathrm{d}(x_0,\ x)}=0$$

for some $x_0\in M$, then either u is a trivial solution or u does not exist according to Theorem 3.

Example 3 If $p=q$ and $h(u)=-2ae^u+2be^{-u}$ with $a\geq 0$ and $b\geq 0$, then equation (1) reduces to

$$\Delta_p u-ae^u+be^{-u}=0. \tag{6}$$

Let $1<p$ and M ($\dim M\geq 3$) be a complete non-compact Riemannian manifold with non-negative Ricci curvature. If a positive solution u to equation (6) on M satisfies

$$\lim_{M\ni x\to\infty}\frac{u(x)}{\mathrm{d}(x_0,\ x)}=0$$

for some $x_0\in M$, then u is a trivial solution $c_0>0$ with $-ae^{c_0}+be^{-c_0}=0$ or (6) has no positive solution according to Theorem 4.

1.3 *Main ideas of proof and the organization of paper*

In order to give the gradient estimates, we consider the linearized operator of $\mathscr{L}_{p,\ q}$ of $(p,\ q)$-Laplace operator $\Delta_p+\Delta_q$ at a solution u, and let $\mathscr{L}_{p,\ q}$ act on an auxiliary function:

$$F(u)=\frac{|\nabla u|^2}{u^\beta}.$$

Then, we need to establish some suitable point-wise estimate of $\mathscr{L}_{p,\ q}(F)$ so that we can take a Nash-Moser iteration scheme to give the L^∞-norm of $F=\dfrac{|\nabla u|^2}{u^\beta}$. The Saloff-Coste's Sobolev inequalities play an important role in our arguments.

Our paper is organized as follows: In Section 2, we first recall some preliminaries and then establish some important lemmas, which will play a key role in the Nash-Moser iteration process. In Section 3, we prove some important gradient estimates, which will constitute the main body of this paper. Some necessary proofs of the main results will be given in Section 4.

2. Preliminaries

Throughout this paper, we denote $(M,\ g)$ an n-dim Riemannian manifold ($n\geq 3$), and ∇ the corresponding Levi-Civita connection. We denote the volume form $\mathrm{dvol}=\sqrt{\det(g_{ij})}\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_n$, where $(x_1,\ \cdots,\ x_n)$ is a local coordinate system, and for simplicity we usually omit the volume form of integral over M.

The z-Laplacian operator is defined by

$$\Delta_z u=\mathrm{div}\left(|\nabla u|^{z-2}\nabla u\right),$$

where z is a real number. We also need to recall the definition of weak solution to (1):

Definition 1 We say that $u \in C^1(M) \cap W^{1,\,q}_{loc}(M)$ is a weak solution of (1) if for all $\psi \in W^{1,\,p}_0(M) \cap W^{1,\,q}_0(M)$ we have

$$\int_M |\nabla u|^{p-2} \langle \nabla u,\, \nabla \psi \rangle + \int_M |\nabla u|^{q-2} \langle \nabla u,\, \nabla \psi \rangle = \int_M h(u)\psi.$$

Next, we recall the Saloff-Coste's Sobolev inequalities (see [38]), which shall play a key role in our proof of the main theorems.

Lemma 1 (Saloff-Coste [38]) Let $(M,\, g)$ be a complete n-dimensional manifold with $\mathrm{Ric} \geq -(n-1)\kappa$. For $n > 2$, there exists a positive constant C_n depending only on n, such that for all $B \subset M$ of radius R and volume V we have for $h_1 \in C^\infty_0(B)$

$$\|h_1\|^2_{L^{\frac{2n}{n-2}}(B)} \leq \exp\left\{C_n(1+\sqrt{\kappa}R)\right\} V^{-\frac{2}{n}} R^2 \int_B \left(|\nabla h_1|^2 + R^{-2} h_1^2\right).$$

For $n = 2$, the above inequality holds with n replaced by any fixed $n' > 2$.

Now we consider the linearisation operator $\mathscr{L}_z$ of z-Laplace operator:

$$\mathscr{L}_z(\psi) = \mathrm{div}\left[f^{\frac{z}{2}-1} A_z(\nabla\psi)\right], \tag{7}$$

where $f = |\nabla u|^2$, and

$$A_z(\nabla\psi) = \nabla\psi + (z-2) f^{-1} \langle \nabla\psi,\, \nabla u \rangle \nabla u. \tag{8}$$

We first derive a useful expression of $\mathscr{L}_z(f)$.

Lemma 2 The equality

$$\mathscr{L}_z(f) = \left(\frac{z}{2}-1\right) f^{\frac{z}{2}-2} |\nabla f|^2 + 2 f^{\frac{z}{2}-1} \left(|\nabla\nabla u|^2 + \mathrm{Ric}(\nabla u,\, \nabla u)\right) + 2\langle \nabla \Delta_z u,\, \nabla u \rangle \tag{9}$$

holds point-wisely in $\{x\ :\ f(x) > 0\ \}$.

Proof. By the definition of A_z in (8), we have

$$A_z(\nabla f) = \nabla f + (z-2) f^{-1} \langle \nabla f,\, \nabla u \rangle \nabla u. \tag{10}$$

Combining (7) and (10) together, we obtain

$$\begin{aligned}\mathscr{L}_z(f) = &\left(\frac{z}{2}-1\right) f^{\frac{z}{2}-2} |\nabla f|^2 + f^{\frac{z}{2}-1} \Delta f + (z-2)\left(\frac{z}{2}-2\right) f^{\frac{z}{2}-3} \langle \nabla f,\, \nabla u \rangle^2 \\ &+ (z-2) f^{\frac{z}{2}-2} \langle \nabla \langle \nabla f,\, \nabla u \rangle,\, \nabla u \rangle + (z-2) f^{\frac{z}{2}-2} \langle \nabla f,\, \nabla u \rangle \Delta u. \end{aligned} \tag{11}$$

On the other hand, by the definition of the z-Laplacian, we have

$$2\langle \nabla \Delta_z u,\, \nabla u\rangle = (z-2)\left(\frac{z}{2}-2\right) f^{\frac{z}{2}-3}\langle \nabla f,\, \nabla u\rangle^2 + (z-2) f^{\frac{z}{2}-2}\langle \nabla\langle \nabla f,\, \nabla u\rangle,\, \nabla u\rangle$$

$$+(z-2) f^{\frac{z}{2}-2}\langle \nabla f,\, \nabla u\rangle \Delta u + 2 f^{\frac{z}{2}-1}\langle \nabla \Delta u,\, \nabla u\rangle. \tag{12}$$

Combining (11) and (12) together, we obtain

$$\mathscr{L}_z(f) = \left(\frac{z}{2}-1\right) f^{\frac{z}{2}-2}|\nabla f|^2 + f^{\frac{z}{2}-1}\Delta f + 2\langle \nabla \Delta_z u,\, \nabla u\rangle - 2 f^{\frac{z}{2}-1}\langle \nabla \Delta u,\, \nabla u\rangle. \tag{13}$$

By (13) and the following Bochner formula

$$\frac{1}{2}\Delta f = |\nabla\nabla u|^2 + \mathrm{Ric}\,(\nabla u,\, \nabla u) + \langle \nabla \Delta u,\, \nabla u\rangle,$$

we have

$$\mathscr{L}_z(f) = \left(\frac{z}{2}-1\right) f^{\frac{z}{2}-2}|\nabla f|^2 + 2 f^{\frac{z}{2}-1}\left(|\nabla\nabla u|^2 + \mathrm{Ric}\,(\nabla u,\, \nabla u)\right) + 2\langle \nabla \Delta_z u,\, \nabla u\rangle.$$

□

Another tool that will be used in the later is the following lemma (for the proof see [39]):

Lemma 3 Suppose that $\psi_0(t)$ is a positive and bounded function, which is defined on $[T_0,\, T_1]$. If for all $T_0 \le t < s \le T_1$, ψ_0 satisfies

$$\psi_0(t) \le \theta \psi_0(s) + \frac{A}{(s-t)^{\alpha_0}} + B,$$

where $\theta < 1$, A, B and α_0 are some non-negative constants. Then, for any $T_0 \le \rho < \tau \le T_1$ there exists

$$\psi_0(\rho) \le C(\alpha_0,\, \theta)\left[\frac{A}{(\tau-\rho)^{\alpha_0}} + B\right],$$

where $C(\alpha_0,\, \theta)$ is a positive constant which depends only on α_0 and θ. Furthermore, if we set $\theta = \frac{1}{2}$ and let α_0 change in a bounded interval, then there exists a positive constant C_0 such that $C\left(\alpha_0,\, \frac{1}{2}\right) \le C_0$.

3. Gradient estimate

First, we need to give the pointwise estimate of $\mathscr{L}_{p,\,q}(F)$, where $F = \frac{f}{u^\beta}$ with the constant $\beta > 0$ and $\mathscr{L}_{p,\,q}$ is the linearized operator of p-Laplacian + q-Laplacian at u.

3.1 *Estimate for the linearized operator of p-Laplace + q-Laplace operator*

In this subsection we prove some gradient estimates of the positive solutions of (1). We begin with a point-wise estimate for $\mathscr{L}_{p,\,q}(F) = \mathscr{L}_p(F) + \mathscr{L}_q(F)$.

Lemma 4 The equality

$$\mathscr{L}_z(F) = u^{-\beta}\mathscr{L}_z(f) + \beta(\beta+1)(z-1)u^{-\beta-2}f^{\frac{z}{2}+1} - \beta\left(1+\frac{z}{2}\right)(z-1)u^{-\beta-1}f^{\frac{z}{2}-1}\langle\nabla f,\, \nabla u\rangle$$
$$-\beta(z-1)u^{-\beta-1}f^{\frac{z}{2}}\Delta u \tag{14}$$

holds point-wisely in $\{x \,:\, f(x) > 0\}$.

Proof. By the definition of A_z in (8), we have

$$A_z(\nabla F) = u^{-\beta}A_z(\nabla f) - \beta u^{-\beta-1}fA_z(\nabla u), \tag{15}$$

$$A_z(\nabla u) = (z-1)\nabla u \tag{16}$$

and

$$A_z(\nabla f) = \nabla f + (z-2)f^{-1}\langle\nabla u,\, \nabla f\rangle\nabla u. \tag{17}$$

Combining (7) and (15) together, we obtain

$$\mathscr{L}_z(F) = \mathrm{div}\left[u^{-\beta}f^{\frac{z}{2}-1}A_z(\nabla f)\right] - \beta\mathrm{div}\left[u^{-\beta-1}f^{\frac{z}{2}}A_z(\nabla u)\right]. \tag{18}$$

Direct computation shows that

$$\mathrm{div}\left[u^{-\beta}f^{\frac{z}{2}-1}A_z(\nabla f)\right] = -\beta u^{-\beta-1}f^{\frac{z}{2}-1}\langle A_z(\nabla f),\, \nabla u\rangle + u^{-\beta}\mathrm{div}\left[f^{\frac{z}{2}-1}A_z(\nabla f)\right] \tag{19}$$

and

$$\mathrm{div}\left[u^{-\beta-1}f^{\frac{z}{2}}A_z(\nabla u)\right] = -(\beta+1)u^{-\beta-2}f^{\frac{z}{2}}\langle A_z(\nabla u),\, \nabla u\rangle + \frac{z}{2}u^{-\beta-1}f^{\frac{z}{2}-1}\langle A_z(\nabla u),\, \nabla f\rangle$$
$$+u^{-\beta-1}f^{\frac{z}{2}}\mathrm{div}A_z(\nabla u). \tag{20}$$

Substituting (7) and (17) into (19), we have

$$\operatorname{div}\left[u^{-\beta} f^{\frac{z}{2}-1} A_z(\nabla f)\right] = -\beta(z-1)u^{-\beta-1} f^{\frac{z}{2}-1}\langle\nabla f,\ \nabla u\rangle + u^{-\beta}\mathscr{L}_z(f). \tag{21}$$

Substituting (16) into (20), we obtain

$$\operatorname{div}\left[u^{-\beta-1} f^{\frac{z}{2}} A_z(\nabla u)\right] = -(\beta+1)(z-1)u^{-\beta-2} f^{\frac{z}{2}+1} + \frac{z}{2}(z-1)u^{-\beta-1} f^{\frac{z}{2}-1}\langle\nabla f,\ \nabla u\rangle$$

$$+(z-1)u^{-\beta-1} f^{\frac{z}{2}}\Delta u. \tag{22}$$

Substituting (21) and (22) into (18), we finish the proof of Lemma 4. □

Using Lemma 2 and Lemma 4, we can establish the following lemma:

Lemma 5 Let u be a positive solution of equation (1) in $\Omega \subset M$. Set

$$\widetilde{\mathscr{L}_z}(F) = u^{-\beta}\left[\left(\frac{z}{2}-1\right) f^{\frac{z}{2}-2}|\nabla f|^2 + 2f^{\frac{z}{2}-1}\left(|\nabla\nabla u|^2 + \operatorname{Ric}(\nabla u,\ \nabla u)\right)\right]$$

$$+\beta(\beta+1)(z-1)u^{-\beta-2} f^{\frac{z}{2}+1} - \beta\left(1+\frac{z}{2}\right)(z-1)u^{-\beta-1} f^{\frac{z}{2}-1}\langle\nabla f,\ \nabla u\rangle$$

$$-\beta(z-1)u^{-\beta-1} f^{\frac{z}{2}}\Delta u. \tag{23}$$

Then, the following

$$\mathscr{L}_{p,\,q}(F) = \widetilde{\mathscr{L}_p}(F) + \widetilde{\mathscr{L}_q}(F) - 2u^{-\beta} f h'(u) \tag{24}$$

holds point-wisely in $\{x \in \Omega\ :\ f(x) > 0\}$.

Proof. Combining Lemmas 2 and 4 together, we can achieve that

$$\mathscr{L}_{p,\,q}(F) = u^{-\beta}\left[\left(\frac{p}{2}-1\right) f^{\frac{p}{2}-2}|\nabla f|^2 + 2f^{\frac{p}{2}-1}\left(|\nabla\nabla u|^2 + \operatorname{Ric}(\nabla u,\ \nabla u)\right) + 2\langle\nabla\Delta_p u,\ \nabla u\rangle\right]$$

$$+\beta(\beta+1)(p-1)u^{-\beta-2} f^{\frac{p}{2}+1} - \beta\left(1+\frac{p}{2}\right)(p-1)u^{-\beta-1} f^{\frac{p}{2}-1}\langle\nabla f,\ \nabla u\rangle - \beta(p-1)u^{-\beta-1} f^{\frac{p}{2}}\Delta u$$

$$+u^{-\beta}\left[\left(\frac{q}{2}-1\right) f^{\frac{q}{2}-2}|\nabla f|^2 + 2f^{\frac{q}{2}-1}\left(|\nabla\nabla u|^2 + \operatorname{Ric}(\nabla u,\ \nabla u)\right) + 2\langle\nabla\Delta_q u,\ \nabla u\rangle\right]$$

$$+\beta(\beta+1)(q-1)u^{-\beta-2} f^{\frac{q}{2}+1} - \beta\left(1+\frac{q}{2}\right)(q-1)u^{-\beta-1} f^{\frac{q}{2}-1}\langle\nabla f,\ \nabla u\rangle - \beta(q-1)u^{-\beta-1} f^{\frac{q}{2}}\Delta u.$$

Hence

$$\mathscr{L}_{p,\,q}(F) = u^{-\beta}\left[\left(\frac{p}{2}-1\right)f^{\frac{p}{2}-2}|\nabla f|^2 + 2f^{\frac{p}{2}-1}\left(|\nabla\nabla u|^2 + \mathrm{Ric}\,(\nabla u,\ \nabla u)\right)\right]$$

$$+\beta(\beta+1)(p-1)u^{-\beta-2}f^{\frac{p}{2}+1} - \beta\left(1+\frac{p}{2}\right)(p-1)u^{-\beta-1}f^{\frac{p}{2}-1}\langle\nabla f,\ \nabla u\rangle$$

$$-\beta(p-1)u^{-\beta-1}f^{\frac{p}{2}}\Delta u + u^{-\beta}\left[\left(\frac{q}{2}-1\right)f^{\frac{q}{2}-2}|\nabla f|^2 + 2f^{\frac{q}{2}-1}\left(|\nabla\nabla u|^2 + \mathrm{Ric}\,(\nabla u,\ \nabla u)\right)\right]$$

$$+\beta(\beta+1)(q-1)u^{-\beta-2}f^{\frac{q}{2}+1} - \beta\left(1+\frac{q}{2}\right)(q-1)u^{-\beta-1}f^{\frac{q}{2}-1}\langle\nabla f,\ \nabla u\rangle$$

$$-\beta(q-1)u^{-\beta-1}f^{\frac{q}{2}}\Delta u + 2u^{-\beta}\langle\nabla(\Delta_p u + \Delta_q u),\ \nabla u\rangle.$$

Substituting (1) into the above equality, we finish the proof of Lemma 5. □

In order to achieve the pointwise estimate of $\mathscr{L}_{p,\,q}(F)$, we need to estimate $\widetilde{\mathscr{L}_z}(F)$. Therefore, we begin with the following equality about $\widetilde{\mathscr{L}_z}(F)$.

Lemma 6 Let u be a positive solution of equation (1) in $\Omega \subset M$. Set

$$\widehat{\mathscr{L}_z}(f) = u^{-\beta}\left[\left(\frac{z}{2}-1\right)f^{\frac{z}{2}-2}|\nabla f|^2 + 2f^{\frac{z}{2}-1}\left(|\nabla\nabla u|^2 + \mathrm{Ric}\,(\nabla u, \nabla u)\right)\right]$$

$$-\beta(z-1)u^{-\beta-1}f^{\frac{z}{2}}\Delta u. \tag{25}$$

Then, the following

$$\widetilde{\mathscr{L}_z}(F) = \beta(z-1)\left(1-\frac{z}{2}\beta\right)u^{\frac{z}{2}\beta-2}F^{\frac{z}{2}+1} - \left(1+\frac{z}{2}\right)(z-1)\beta u^{\left(\frac{z}{2}-1\right)\beta-1}F^{\frac{z}{2}-1}\langle\nabla F,\ \nabla u\rangle + \widehat{\mathscr{L}_z}(f) \tag{26}$$

holds point-wisely in $\{x\in\Omega\ :\ f(x)>0\}$.

Proof. Since $F = u^{-\beta}f$, we can know that

$$\nabla f = \beta u^{\beta-1}F\nabla u + u^{\beta}\nabla F.$$

Hence

$$\langle\nabla f,\ \nabla u\rangle = u^{\beta}\langle\nabla F,\ \nabla u\rangle + \beta u^{\beta-1}Ff = u^{\beta}\langle\nabla F,\ \nabla u\rangle + \beta u^{2\beta-1}F^2. \tag{27}$$

Substituting (27) into (23), we obtain

$$\widetilde{\mathscr{L}_z}(F) = u^{-\beta}\left[\left(\frac{z}{2}-1\right)f^{\frac{z}{2}-2}|\nabla f|^2 + 2f^{\frac{z}{2}-1}\left(|\nabla\nabla u|^2 + \mathrm{Ric}\,(\nabla u,\ \nabla u)\right)\right] - \beta(z-1)u^{-\beta-1}f^{\frac{z}{2}}\Delta u$$

$$+\beta(z-1)\left(1-\frac{z}{2}\beta\right)u^{\frac{z}{2}\beta-2}F^{\frac{z}{2}+1} - \left(1+\frac{z}{2}\right)(z-1)\beta u^{\left(\frac{z}{2}-1\right)\beta-1}F^{\frac{z}{2}-1}\langle\nabla F,\ \nabla u\rangle. \tag{28}$$

Combining (25) and (28) together, we complete the proof of Lemma 6. □

Next, we need to consider the point-wise estimate of $\widetilde{\mathscr{L}_z}(F)$.

Lemma 7 Let u be a positive solution of equation (1) in $\Omega \subset M$ with $\mathrm{Ric} \geq -(n-1)\kappa$. Set

$$\delta(z) = \begin{cases} 2-z, \ 1<z<2, \\ \\ 0, \qquad z \geq 2. \end{cases} \tag{29}$$

Then, the following

$$\widetilde{\mathscr{L}_z}(F) \geq \beta(z-1)\left[1-\frac{z}{2}\beta - \frac{n}{8(1-\delta(z))}\beta(z-1)\right]u^{\frac{z}{2}\beta-2}F^{\frac{z}{2}+1} - 2(n-1)\kappa u^{\left(\frac{z}{2}-1\right)\beta}F^{\frac{z}{2}}$$

$$-\left(1+\frac{z}{2}\right)(z-1)\beta u^{\left(\frac{z}{2}-1\right)\beta-1}F^{\frac{z}{2}-1}\langle\nabla F,\ \nabla u\rangle \tag{30}$$

holds point-wisely in $\{x \in \Omega\ :\ f(x) > 0\}$.

Proof. Let $\{e_1,\ \cdots,\ e_n\}$ be an orthonormal frame of TM on a domain with $f \neq 0$ such that $e_1 = \dfrac{\nabla u}{|\nabla u|}$. We hence infer that

$$4\sum_{i=1}^{n} u_{1i}^2 = f^{-1}|\nabla f|^2. \tag{31}$$

By omitting some non-negative terms in $|\nabla\nabla u|^2$ and using Cauchy inequality, we arrive at

$$|\nabla\nabla u|^2 \geq \delta\sum_{i=1}^{n} u_{1i}^2 + (1-\delta)\sum_{i=1}^{n} u_{ii}^2 \geq \delta\sum_{i=1}^{n} u_{1i}^2 + \frac{1-\delta}{n}(\Delta u)^2, \tag{32}$$

where $\delta \in [0,\ 1)$. Substituting (31) and (32) into (25), we have

$$
\begin{aligned}
\widehat{\mathscr{L}_z}(f) \geq\, & u^{-\beta}\left\{\left(\frac{z}{2}-1\right) f^{\frac{z}{2}-2}|\nabla f|^2+2 f^{\frac{z}{2}-1}\left(\delta \sum_{i=1}^n u_{1 i}^2+\frac{1-\delta}{n}(\Delta u)^2+\operatorname{Ric}(\nabla u,\, \nabla u)\right)\right\} \\
& -\beta(z-1) u^{-\beta-1} f^{\frac{z}{2}} \Delta u \\
=\, & u^{-\beta}\left\{\left(\frac{z}{2}-1\right) f^{\frac{z}{2}-2}|\nabla f|^2+2 f^{\frac{z}{2}-1}\left(\frac{\delta}{4} f^{-1}|\nabla f|^2+\frac{1-\delta}{n}(\Delta u)^2+\operatorname{Ric}(\nabla u,\, \nabla u)\right)\right\} \\
& -\beta(z-1) u^{-\beta-1} f^{\frac{z}{2}} \Delta u \\
=\, & \left(\frac{z+\delta}{2}-1\right) u^{-\beta} f^{\frac{z}{2}-2}|\nabla f|^2+\frac{2(1-\delta)}{n} u^{-\beta} f^{\frac{z}{2}-1}(\Delta u)^2+2 u^{-\beta} f^{\frac{z}{2}-1} \operatorname{Ric}(\nabla u,\, \nabla u) \\
& -\beta(z-1) u^{-\beta-1} f^{\frac{z}{2}} \Delta u .
\end{aligned} \tag{33}
$$

Let

$$
\delta(z)= \begin{cases}2-z, & 1<z<2, \\ 0, & z \geq 2,\end{cases}
$$

we can know that $\delta \in[0,\, 1)$. Therefore, by (33), we obtain

$$
\widehat{\mathscr{L}_z}(f) \geq \frac{2(1-\delta)}{n} u^{-\beta} f^{\frac{z}{2}-1}(\Delta u)^2+2 u^{-\beta} f^{\frac{z}{2}-1} \operatorname{Ric}(\nabla u,\, \nabla u)-\beta(z-1) u^{-\beta-1} f^{\frac{z}{2}} \Delta u . \tag{34}
$$

By using the inequality $a^2-2 a b \geq-b^2$, we have

$$
\frac{2(1-\delta)}{n} u^{-\beta} f^{\frac{z}{2}-1}(\Delta u)^2-\beta(z-1) u^{-\beta-1} f^{\frac{z}{2}} \Delta u \geq-\frac{n}{8(1-\delta)} \beta^2(z-1)^2 u^{-\beta-2} f^{\frac{z}{2}+1} .
$$

Substituting the above inequality into (34), we can achieve that

$$
\widehat{\mathscr{L}_z}(f) \geq-\frac{n}{8(1-\delta)} \beta^2(z-1)^2 u^{-\beta-2} f^{\frac{z}{2}+1}+2 u^{-\beta} f^{\frac{z}{2}-1} \operatorname{Ric}(\nabla u,\, \nabla u) . \tag{35}
$$

Combining (26) and (35) together, we obtain

$$\widetilde{\mathscr{L}_z}(F) \geq \beta(z-1)\left(1-\frac{z}{2}\beta\right)u^{\frac{z}{2}\beta-2}F^{\frac{z}{2}+1}-\left(1+\frac{z}{2}\right)(z-1)\beta u^{\left(\frac{z}{2}-1\right)\beta-1}F^{\frac{z}{2}-1}\langle\nabla F,\ \nabla u\rangle$$

$$-\frac{n}{8(1-\delta)}\beta^2(z-1)^2u^{-\beta-2}f^{\frac{z}{2}+1}+2u^{-\beta}f^{\frac{z}{2}-1}\mathrm{Ric}\,(\nabla u,\ \nabla u)\,.$$

Using the facts $F=u^{-\beta}f$ and $\mathrm{Ric}\geq-(n-1)\kappa$, we can achieve that

$$\widetilde{\mathscr{L}_z}(F) \geq\ \beta(z-1)\left[1-\frac{z}{2}\beta-\frac{n}{8(1-\delta(z))}\beta(z-1)\right]u^{\frac{z}{2}\beta-2}F^{\frac{z}{2}+1}-2(n-1)\kappa u^{\left(\frac{z}{2}-1\right)\beta}F^{\frac{z}{2}}$$

$$-\left(1+\frac{z}{2}\right)(z-1)\beta u^{\left(\frac{z}{2}-1\right)\beta-1}F^{\frac{z}{2}-1}\langle\nabla F,\ \nabla u\rangle.$$

Thus, we finish the proof. □

We conclude this part by using Lemmas 5 and 7 to achieve the following corollary:

Corollary 1 Let $(M,\ g)$ be an n-dim $(n\geq 3)$ complete manifold with $\mathrm{Ric}\geq-(n-1)\kappa$, where κ is a non-negative constant. Assume u is a positive solution to equation (1) in $\Omega\subset M$. Set

$$\overline{\mathscr{L}}_z(F)=\beta(z-1)\left[1-\frac{z}{2}\beta-\frac{n}{8(1-\delta(z))}\beta(z-1)\right]u^{\frac{z}{2}\beta-2}F^{\frac{z}{2}+1}-2(n-1)\kappa u^{\left(\frac{z}{2}-1\right)\beta}F^{\frac{z}{2}}$$

$$-\left(1+\frac{z}{2}\right)(z-1)\beta u^{\left(\frac{z}{2}-1\right)\beta-1}F^{\frac{z}{2}-1}\langle\nabla F,\ \nabla u\rangle, \tag{36}$$

where $\delta(z)$ is defined in (29). If $h\in C^1(\mathbb{R}^+)$ is a non-increasing function, then the following

$$\mathscr{L}_{p,\ q}(F)\geq\overline{\mathscr{L}}_p(F)+\overline{\mathscr{L}}_q(F) \tag{37}$$

holds point-wisely in $\{x\in\Omega\ :\ f(x)>0\}$.

Proof. Since $u>0$ and $h\in C^1(\mathbb{R}^+)$ is a non-increasing function, we obtain

$$h'(u)\leq 0. \tag{38}$$

Combining (24) and (38) together, we have

$$\mathscr{L}_{p,\ q}(F)\geq\widetilde{\mathscr{L}_p}(F)+\widetilde{\mathscr{L}_q}(F). \tag{39}$$

By Lemma 7, we can achieve that

$$\widetilde{\mathscr{L}_z}(F) \geq \overline{\mathscr{L}}_z(F). \tag{40}$$

Combining (39) and (40) together, we finish the proof of Corollary 18. □

3.2 *Deducing the main integral inequality*

Now we choose a geodesic ball $\Omega = B_R(x_0) \subset M$. Since $\mathscr{L}_{p,\, q}(F) = \mathscr{L}_p(F) + \mathscr{L}_q(F)$, we consider $\mathscr{L}_z(F)$ firstly. If we choose a test function $\xi \cdot u^\lambda = F_\varepsilon^t \eta^2 \cdot u^\lambda$ where $\eta \in C_0^\infty(\Omega,\, \mathbb{R})$ is non-negative, $F_\varepsilon = (F-\varepsilon)^+$, $\varepsilon > 0$, $t > 1$ and $\lambda \in \mathbb{R}$ are to be determined later. It follows from (7) that

$$\begin{aligned}
\int_\Omega \mathscr{L}_z(F) \cdot \xi \cdot u^\lambda &= -\int_\Omega \left\langle \nabla(\xi u^\lambda),\, f^{\frac{z}{2}-1}\left[\nabla F + (z-2) f^{-1} \langle \nabla u,\, \nabla F\rangle \nabla u\right]\right\rangle \\
&= -\int_\Omega f^{\frac{z}{2}-1} u^\lambda \langle \nabla F,\, \nabla \xi\rangle - \lambda \int_\Omega u^{\lambda-1} f^{\frac{z}{2}-1} \langle \nabla F,\, \nabla u\rangle \xi \\
&\quad -(z-2)\int_\Omega f^{\frac{z}{2}-2} u^\lambda \langle \nabla F,\, \nabla u\rangle \langle \nabla u,\, \nabla \xi\rangle - (z-2)\lambda \int_\Omega f^{\frac{z}{2}-1} u^{\lambda-1} \langle \nabla F,\, \nabla u\rangle \xi \\
&= -\int_\Omega f^{\frac{z}{2}-1} u^\lambda \langle \nabla F,\, \nabla \xi\rangle - (z-1)\lambda \int_\Omega f^{\frac{z}{2}-1} u^{\lambda-1} \langle \nabla F,\, \nabla u\rangle \xi \\
&\quad -(z-2)\int_\Omega f^{\frac{z}{2}-2} u^\lambda \langle \nabla F,\, \nabla u\rangle \langle \nabla u,\, \nabla \xi\rangle.
\end{aligned}$$

Since $\xi = F_\varepsilon^t \eta^2$, we can achieve that

$$
\begin{aligned}
\int_{\Omega} \mathscr{L}_z(F) \cdot F_\varepsilon^t \eta^2 \cdot u^\lambda = & -\int_{\Omega} f^{\frac{z}{2}-1} u^\lambda \langle \nabla F,\ t F_\varepsilon^{t-1} \eta^2 \nabla F + 2 F_\varepsilon^t \eta \nabla \eta \rangle \\
& -(z-1)\lambda \int_{\Omega} f^{\frac{z}{2}-1} u^{\lambda-1} \langle \nabla F,\ \nabla u \rangle F_\varepsilon^t \eta^2 \\
& -(z-2) \int_{\Omega} f^{\frac{z}{2}-2} u^\lambda \langle \nabla F,\ \nabla u \rangle \langle \nabla u,\ t F_\varepsilon^{t-1} \eta^2 \nabla F + 2 F_\varepsilon^t \eta \nabla \eta \rangle \\
= & -t \int_{\Omega} u^{\left(\frac{z}{2}-1\right)\beta+\lambda} F^{\frac{z}{2}-1} F_\varepsilon^{t-1} |\nabla F|^2 \eta^2 - 2 \int_{\Omega} u^{\left(\frac{z}{2}-1\right)\beta+\lambda} F^{\frac{z}{2}-1} F_\varepsilon^t \langle \nabla F,\ \nabla \eta \rangle \eta \\
& -(z-1)\lambda \int_{\Omega} u^{\left(\frac{z}{2}-1\right)\beta+\lambda-1} F^{\frac{z}{2}-1} F_\varepsilon^t \langle \nabla F,\ \nabla u \rangle \eta^2 \\
& -(z-2) t \int_{\Omega} u^{\left(\frac{z}{2}-2\right)\beta+\lambda} F^{\frac{z}{2}-2} F_\varepsilon^{t-1} \langle \nabla F,\ \nabla u \rangle^2 \eta^2 \\
& -2(z-2) \int_{\Omega} u^{\left(\frac{z}{2}-2\right)\beta+\lambda} F^{\frac{z}{2}-2} F_\varepsilon^t \langle \nabla F,\ \nabla u \rangle \langle \nabla u,\ \nabla \eta \rangle \eta.
\end{aligned}
$$

Combining $\mathscr{L}_{p,\,q}(F) = \mathscr{L}_p(F) + \mathscr{L}_q(F)$, Corollary 1 and the above equality, we can achieve the following inequality.

$$
\begin{aligned}
&-t\int_\Omega u^{\left(\frac{p}{2}-1\right)\beta+\lambda}F^{\frac{p}{2}-1}F_\varepsilon^{t-1}|\nabla F|^2\eta^2-2\int_\Omega u^{\left(\frac{p}{2}-1\right)\beta+\lambda}F^{\frac{p}{2}-1}F_\varepsilon^{t}\langle\nabla F,\,\nabla\eta\rangle\eta\\
&-(p-1)\lambda\int_\Omega u^{\left(\frac{p}{2}-1\right)\beta+\lambda-1}F^{\frac{p}{2}-1}F_\varepsilon^{t}\langle\nabla F,\,\nabla u\rangle\eta^2\\
&-(p-2)t\int_\Omega u^{\left(\frac{p}{2}-2\right)\beta+\lambda}F^{\frac{p}{2}-2}F_\varepsilon^{t-1}\langle\nabla F,\,\nabla u\rangle^2\eta^2\\
&-2(p-2)\int_\Omega u^{\left(\frac{p}{2}-2\right)\beta+\lambda}F^{\frac{p}{2}-2}F_\varepsilon^{t}\langle\nabla F,\,\nabla u\rangle\langle\nabla u,\,\nabla\eta\rangle\eta\\
&-t\int_\Omega u^{\left(\frac{q}{2}-1\right)\beta+\lambda}F^{\frac{q}{2}-1}F_\varepsilon^{t-1}|\nabla F|^2\eta^2-2\int_\Omega u^{\left(\frac{q}{2}-1\right)\beta+\lambda}F^{\frac{q}{2}-1}F_\varepsilon^{t}\langle\nabla F,\,\nabla\eta\rangle\eta\\
&-(q-1)\lambda\int_\Omega u^{\left(\frac{q}{2}-1\right)\beta+\lambda-1}F^{\frac{q}{2}-1}F_\varepsilon^{t}\langle\nabla F,\,\nabla u\rangle\eta^2\\
&-(q-2)t\int_\Omega u^{\left(\frac{q}{2}-2\right)\beta+\lambda}F^{\frac{q}{2}-2}F_\varepsilon^{t-1}\langle\nabla F,\,\nabla u\rangle^2\eta^2\\
&-2(q-2)\int_\Omega u^{\left(\frac{q}{2}-2\right)\beta+\lambda}F^{\frac{q}{2}-2}F_\varepsilon^{t}\langle\nabla F,\,\nabla u\rangle\langle\nabla u,\,\nabla\eta\rangle\eta\\
\geq\ &\beta(p-1)\left[1-\frac{p}{2}\beta-\frac{n}{8(1-\delta(p))}\beta(p-1)\right]\int_\Omega u^{\frac{p}{2}\beta+\lambda-2}F^{\frac{p}{2}+1}F_\varepsilon^{t}\eta^2\\
&-2(n-1)\kappa\int_\Omega u^{\left(\frac{p}{2}-1\right)\beta+\lambda}F^{\frac{p}{2}}F_\varepsilon^{t}\eta^2-\left(1+\frac{p}{2}\right)(p-1)\beta\int_\Omega u^{\left(\frac{p}{2}-1\right)\beta+\lambda-1}F^{\frac{p}{2}-1}F_\varepsilon^{t}\langle\nabla F,\,\nabla u\rangle\eta^2\\
&+\beta(q-1)\left[1-\frac{q}{2}\beta-\frac{n}{8(1-\delta(q))}\beta(q-1)\right]\int_\Omega u^{\frac{q}{2}\beta+\lambda-2}F^{\frac{q}{2}+1}F_\varepsilon^{t}\eta^2\\
&-2(n-1)\kappa\int_\Omega u^{\left(\frac{q}{2}-1\right)\beta+\lambda}F^{\frac{q}{2}}F_\varepsilon^{t}\eta^2-\left(1+\frac{q}{2}\right)(q-1)\beta\int_\Omega u^{\left(\frac{q}{2}-1\right)\beta+\lambda-1}F^{\frac{q}{2}-1}F_\varepsilon^{t}\langle\nabla F,\,\nabla u\rangle\eta^2. \qquad (41)
\end{aligned}
$$

Next we need to treat the coefficients of the following terms on the right hand side of the above inequality

$$\int_\Omega u^{\frac{p}{2}\beta+\lambda-2}F^{\frac{p}{2}+1}F_\varepsilon^{t}\eta^2$$

and

$$\int_\Omega u^{\frac{q}{2}\beta+\lambda-2}F^{\frac{q}{2}+1}F_\varepsilon^t\eta^2.$$

For this we set

$$\lambda_z(\beta)=\beta(z-1)\left[1-\frac{z}{2}\beta-\frac{n}{8(1-\delta(z))}\beta(z-1)\right], \tag{42}$$

where $\delta(z)$ is the function defined in (29). It is easy to see that λ_z will be a positive number when β is small enough. Furthermore, for the sake of convenience we also set

$$\mathfrak{A}=\left\{\beta\ :\ \beta>0,\ \lambda_p(\beta)>0 \text{ and } \lambda_q(\beta)>0\right\}. \tag{43}$$

Obviously, β belonging to $\mathfrak{A}$ means that β is small enough such that $\lambda_p(\beta)>0$ and $\lambda_q(\beta)>0$ simultaneously. Then, by rearranging (41) and keeping (42) in minds we can obtain the following

$$\begin{aligned}
&2(n-1)\kappa\int_\Omega u^{\left(\frac{p}{2}-1\right)\beta+\lambda}F^{\frac{p}{2}}F_\varepsilon^t\eta^2-2\int_\Omega u^{\left(\frac{p}{2}-1\right)\beta+\lambda}F^{\frac{p}{2}-1}F_\varepsilon^t\langle\nabla F,\ \nabla\eta\rangle\eta\\
&-2(p-2)\int_\Omega u^{\left(\frac{p}{2}-2\right)\beta+\lambda}F^{\frac{p}{2}-2}F_\varepsilon^t\langle\nabla F,\ \nabla u\rangle\langle\nabla u,\ \nabla\eta\rangle\eta\\
&+(p-1)\left[\left(1+\frac{p}{2}\right)\beta-\lambda\right]\int_\Omega u^{\left(\frac{p}{2}-1\right)\beta+\lambda-1}F^{\frac{p}{2}-1}F_\varepsilon^t\langle\nabla F,\ \nabla u\rangle\eta^2\\
&+2(n-1)\kappa\int_\Omega u^{\left(\frac{q}{2}-1\right)\beta+\lambda}F^{\frac{q}{2}}F_\varepsilon^t\eta^2-2\int_\Omega u^{\left(\frac{q}{2}-1\right)\beta+\lambda}F^{\frac{q}{2}-1}F_\varepsilon^t\langle\nabla F,\ \nabla\eta\rangle\eta\\
&-2(q-2)\int_\Omega u^{\left(\frac{q}{2}-2\right)\beta+\lambda}F^{\frac{q}{2}-2}F_\varepsilon^t\langle\nabla F,\ \nabla u\rangle\langle\nabla u,\ \nabla\eta\rangle\eta\\
&+(q-1)\left[\left(1+\frac{q}{2}\right)\beta-\lambda\right]\int_\Omega u^{\left(\frac{q}{2}-1\right)\beta+\lambda-1}F^{\frac{q}{2}-1}F_\varepsilon^t\langle\nabla F,\ \nabla u\rangle\eta^2\\
\geq{}& t\int_\Omega u^{\left(\frac{p}{2}-1\right)\beta+\lambda}F^{\frac{p}{2}-1}F_\varepsilon^{t-1}|\nabla F|^2\eta^2+(p-2)t\int_\Omega u^{\left(\frac{p}{2}-2\right)\beta+\lambda}F^{\frac{p}{2}-2}F_\varepsilon^{t-1}\langle\nabla F,\ \nabla u\rangle^2\eta^2\\
&+\lambda_p\int_\Omega u^{\frac{p}{2}\beta+\lambda-2}F^{\frac{p}{2}+1}F_\varepsilon^t\eta^2+t\int_\Omega u^{\left(\frac{q}{2}-1\right)\beta+\lambda}F^{\frac{q}{2}-1}F_\varepsilon^{t-1}|\nabla F|^2\eta^2\\
&+(q-2)t\int_\Omega u^{\left(\frac{q}{2}-2\right)\beta+\lambda}F^{\frac{q}{2}-2}F_\varepsilon^{t-1}\langle\nabla F,\ \nabla u\rangle^2\eta^2+\lambda_q\int_\Omega u^{\frac{q}{2}\beta+\lambda-2}F^{\frac{q}{2}+1}F_\varepsilon^t\eta^2.
\end{aligned} \tag{44}$$

Set

$$\mathfrak{L}_z = t\int_\Omega u^{\left(\frac{z}{2}-1\right)\beta+\lambda} F^{\frac{z}{2}-1} F_\varepsilon^{t-1} |\nabla F|^2 \eta^2 + (z-2)t\int_\Omega u^{\left(\frac{z}{2}-2\right)\beta+\lambda} F^{\frac{z}{2}-2} F_\varepsilon^{t-1} \langle \nabla F,\ \nabla u\rangle^2 \eta^2. \tag{45}$$

Then we need to consider the following two cases:

$$(i)\ z \geq 2; \quad (ii)\ 1 < z < 2.$$

For the case (i), from (45) we can easily know that

$$\mathfrak{L}_z \geq t\int_\Omega u^{\left(\frac{z}{2}-1\right)\beta+\lambda} F^{\frac{z}{2}-1} F_\varepsilon^{t-1} |\nabla F|^2 \eta^2. \tag{46}$$

For the case (ii), from (45) we can achieve that

$$\begin{aligned}
\mathfrak{L}_z &= t\int_\Omega u^{\left(\frac{z}{2}-1\right)\beta+\lambda} F^{\frac{z}{2}-1} F_\varepsilon^{t-1} |\nabla F|^2 \eta^2 - (2-z)t\int_\Omega u^{\left(\frac{z}{2}-2\right)\beta+\lambda} F^{\frac{z}{2}-2} F_\varepsilon^{t-1} \langle \nabla F,\ \nabla u\rangle^2 \eta^2 \\
&\geq t\int_\Omega u^{\left(\frac{z}{2}-1\right)\beta+\lambda} F^{\frac{z}{2}-1} F_\varepsilon^{t-1} |\nabla F|^2 \eta^2 - (2-z)t\int_\Omega u^{\left(\frac{z}{2}-2\right)\beta+\lambda} F^{\frac{z}{2}-2} F_\varepsilon^{t-1} |\nabla F|^2 f \eta^2 \\
&= (z-1)t\int_\Omega u^{\left(\frac{z}{2}-1\right)\beta+\lambda} F^{\frac{z}{2}-1} F_\varepsilon^{t-1} |\nabla F|^2 \eta^2.
\end{aligned} \tag{47}$$

Set

$$\theta(z) = \begin{cases} z-1, & 1 < z < 2; \\ 1, & z \geq 2. \end{cases} \tag{48}$$

Therefore, combining (45), (46), (47) and (48), we obtain

$$\mathfrak{L}_z \geq \theta(z)t\int_\Omega u^{\left(\frac{z}{2}-1\right)\beta+\lambda} F^{\frac{z}{2}-1} F_\varepsilon^{t-1} |\nabla F|^2 \eta^2. \tag{49}$$

Combining (44) and (49) together, we have

$$
\begin{aligned}
&2(n-1)\kappa\int_\Omega u^{\left(\frac{p}{2}-1\right)\beta+\lambda}F^{\frac{p}{2}}F_\varepsilon^t\eta^2-2\int_\Omega u^{\left(\frac{p}{2}-1\right)\beta+\lambda}F^{\frac{p}{2}-1}F_\varepsilon^t\langle\nabla F,\ \nabla\eta\rangle\eta\\
&-2(p-2)\int_\Omega u^{\left(\frac{p}{2}-2\right)\beta+\lambda}F^{\frac{p}{2}-2}F_\varepsilon^t\langle\nabla F,\ \nabla u\rangle\langle\nabla u,\ \nabla\eta\rangle\eta\\
&+(p-1)\left[\left(1+\frac{p}{2}\right)\beta-\lambda\right]\int_\Omega u^{\left(\frac{p}{2}-1\right)\beta+\lambda-1}F^{\frac{p}{2}-1}F_\varepsilon^t\langle\nabla F,\ \nabla u\rangle\eta^2\\
&+2(n-1)\kappa\int_\Omega u^{\left(\frac{q}{2}-1\right)\beta+\lambda}F^{\frac{q}{2}}F_\varepsilon^t\eta^2-2\int_\Omega u^{\left(\frac{q}{2}-1\right)\beta+\lambda}F^{\frac{q}{2}-1}F_\varepsilon^t\langle\nabla F,\ \nabla\eta\rangle\eta\\
&-2(q-2)\int_\Omega u^{\left(\frac{q}{2}-2\right)\beta+\lambda}F^{\frac{q}{2}-2}F_\varepsilon^t\langle\nabla F,\ \nabla u\rangle\langle\nabla u,\ \nabla\eta\rangle\eta\\
&+(q-1)\left[\left(1+\frac{q}{2}\right)\beta-\lambda\right]\int_\Omega u^{\left(\frac{q}{2}-1\right)\beta+\lambda-1}F^{\frac{q}{2}-1}F_\varepsilon^t\langle\nabla F,\ \nabla u\rangle\eta^2\\
\geq\ &\theta(p)t\int_\Omega u^{\left(\frac{p}{2}-1\right)\beta+\lambda}F^{\frac{p}{2}-1}F_\varepsilon^{t-1}|\nabla F|^2\eta^2+\lambda_p\int_\Omega u^{\frac{p}{2}\beta+\lambda-2}F^{\frac{p}{2}+1}F_\varepsilon^t\eta^2\\
&+\theta(q)t\int_\Omega u^{\left(\frac{q}{2}-1\right)\beta+\lambda}F^{\frac{q}{2}-1}F_\varepsilon^{t-1}|\nabla F|^2\eta^2+\lambda_q\int_\Omega u^{\frac{q}{2}\beta+\lambda-2}F^{\frac{q}{2}+1}F_\varepsilon^t\eta^2.
\end{aligned}
$$

By letting $\varepsilon\to 0^+$, we obtain

$$2(n-1)\kappa\int_{\Omega}u^{\left(\frac{p}{2}-1\right)\beta+\lambda}F^{t+\frac{p}{2}}\eta^{2}-2\int_{\Omega}u^{\left(\frac{p}{2}-1\right)\beta+\lambda}F^{t+\frac{p}{2}-1}\langle\nabla F,\ \nabla\eta\rangle\eta$$

$$-2(p-2)\int_{\Omega}u^{\left(\frac{p}{2}-2\right)\beta+\lambda}F^{t+\frac{p}{2}-2}\langle\nabla F,\ \nabla u\rangle\langle\nabla u,\ \nabla\eta\rangle\eta$$

$$+(p-1)\left[\left(1+\frac{p}{2}\right)\beta-\lambda\right]\int_{\Omega}u^{\left(\frac{p}{2}-1\right)\beta+\lambda-1}F^{t+\frac{p}{2}-1}\langle\nabla F,\ \nabla u\rangle\eta^{2}$$

$$+2(n-1)\kappa\int_{\Omega}u^{\left(\frac{q}{2}-1\right)\beta+\lambda}F^{t+\frac{q}{2}}\eta^{2}-2\int_{\Omega}u^{\left(\frac{q}{2}-1\right)\beta+\lambda}F^{\frac{q}{2}-1}F_{\varepsilon}^{t}\langle\nabla F,\ \nabla\eta\rangle\eta$$

$$-2(q-2)\int_{\Omega}u^{\left(\frac{q}{2}-2\right)\beta+\lambda}F^{t+\frac{q}{2}-2}\langle\nabla F,\ \nabla u\rangle\langle\nabla u,\ \nabla\eta\rangle\eta$$

$$+(q-1)\left[\left(1+\frac{q}{2}\right)\beta-\lambda\right]\int_{\Omega}u^{\left(\frac{q}{2}-1\right)\beta+\lambda-1}F^{t+\frac{q}{2}-1}\langle\nabla F,\ \nabla u\rangle\eta^{2}$$

$$\geq\theta(p)t\int_{\Omega}u^{\left(\frac{p}{2}-1\right)\beta+\lambda}F^{t+\frac{p}{2}-2}|\nabla F|^{2}\eta^{2}+\lambda_{p}\int_{\Omega}u^{\frac{p}{2}\beta+\lambda-2}F^{t+\frac{p}{2}+1}\eta^{2}$$

$$+\theta(q)t\int_{\Omega}u^{\left(\frac{q}{2}-1\right)\beta+\lambda}F^{t+\frac{q}{2}-2}|\nabla F|^{2}\eta^{2}+\lambda_{q}\int_{\Omega}u^{\frac{q}{2}\beta+\lambda-2}F^{t+\frac{q}{2}+1}\eta^{2}. \tag{50}$$

By absolute value inequality and Cauchy-inequality, we have

$$(z-1)\left[\left(1+\frac{z}{2}\right)\beta-\lambda\right]\int_{\Omega}u^{\left(\frac{z}{2}-1\right)\beta+\lambda-1}F^{t+\frac{z}{2}-1}\langle\nabla F,\ \nabla u\rangle\eta^{2}$$

$$\leq\left|(z-1)\left[\left(1+\frac{z}{2}\right)\beta-\lambda\right]\right|\int_{\Omega}u^{\left(\frac{z}{2}-1\right)\beta+\lambda-1}F^{t+\frac{z}{2}-1}|\nabla F|f^{\frac{1}{2}}\eta^{2}$$

$$=\left|(z-1)\left[\left(1+\frac{z}{2}\right)\beta-\lambda\right]\right|\int_{\Omega}u^{\left(\frac{z}{2}-\frac{1}{2}\right)\beta+\lambda-1}F^{t+\frac{z}{2}-\frac{1}{2}}|\nabla F|\eta^{2}$$

$$\leq\frac{\theta(z)}{4}t\int_{\Omega}u^{\left(\frac{z}{2}-1\right)\beta+\lambda}F^{t+\frac{z}{2}-2}|\nabla F|^{2}\eta^{2}+\frac{1}{\theta(z)t}(z-1)^{2}\left[\left(1+\frac{z}{2}\right)\beta-\lambda\right]^{2}\int_{\Omega}u^{\frac{z}{2}\beta+\lambda-2}F^{t+\frac{z}{2}+1}\eta^{2},$$

$$-2\int_{\Omega}u^{\left(\frac{z}{2}-1\right)\beta+\lambda}F^{t+\frac{z}{2}-1}\langle\nabla F,\ \nabla\eta\rangle\eta\leq2\int_{\Omega}u^{\left(\frac{z}{2}-1\right)\beta+\lambda}F^{t+\frac{z}{2}-1}|\nabla F||\nabla\eta|\eta$$

$$\leq\frac{\theta(z)}{4}t\int_{\Omega}u^{\left(\frac{z}{2}-1\right)\beta+\lambda}F^{t+\frac{z}{2}-2}|\nabla F|^{2}\eta^{2}+\frac{4}{\theta(z)t}\int_{\Omega}u^{\left(\frac{z}{2}-1\right)\beta+\lambda}F^{t+\frac{z}{2}}|\nabla\eta|^{2}$$

and

$$2(z-2)\int_{\Omega}u^{\left(\frac{z}{2}-2\right)\beta+\lambda}F^{t+\frac{z}{2}-2}\langle\nabla F,\ \nabla u\rangle\langle\nabla u,\ \nabla\eta\rangle\eta$$

$$\leq 2|z-2|\int_{\Omega}u^{\left(\frac{z}{2}-1\right)\beta+\lambda}F^{t+\frac{z}{2}-1}|\nabla F||\nabla\eta|\eta$$

$$\leq\frac{\theta(z)}{4}t\int_{\Omega}u^{\left(\frac{z}{2}-1\right)\beta+\lambda}F^{t+\frac{z}{2}-2}|\nabla F|^2\eta^2+\frac{4}{\theta(z)t}(z-2)^2\int_{\Omega}u^{\left(\frac{z}{2}-1\right)\beta+\lambda}F^{t+\frac{z}{2}}|\nabla\eta|^2.$$

Substituting the above three inequalities into (50), we obtain

$$\begin{aligned}
&\frac{\theta(p)}{4}t\int_{\Omega}u^{\left(\frac{p}{2}-1\right)\beta+\lambda}F^{t+\frac{p}{2}-2}|\nabla F|^2\eta^2+\lambda_p\int_{\Omega}u^{\frac{p}{2}\beta+\lambda-2}F^{t+\frac{p}{2}+1}\eta^2\\
&+\frac{\theta(q)}{4}t\int_{\Omega}u^{\left(\frac{q}{2}-1\right)\beta+\lambda}F^{t+\frac{q}{2}-2}|\nabla F|^2\eta^2+\lambda_q\int_{\Omega}u^{\frac{q}{2}\beta+\lambda-2}F^{t+\frac{q}{2}+1}\eta^2\\
\leq\ &2(n-1)\kappa\int_{\Omega}u^{\left(\frac{p}{2}-1\right)\beta+\lambda}F^{t+\frac{p}{2}}\eta^2+\frac{4}{\theta(p)t}\left[(p-2)^2+1\right]\int_{\Omega}u^{\left(\frac{p}{2}-1\right)\beta+\lambda}F^{t+\frac{p}{2}}|\nabla\eta|^2\\
&+\frac{1}{\theta(p)t}(p-1)^2\left[\left(1+\frac{p}{2}\right)\beta-\lambda\right]^2\int_{\Omega}u^{\frac{p}{2}\beta+\lambda-2}F^{t+\frac{p}{2}+1}\eta^2\\
&+2(n-1)\kappa\int_{\Omega}u^{\left(\frac{q}{2}-1\right)\beta+\lambda}F^{t+\frac{q}{2}}\eta^2+\frac{4}{\theta(q)t}\left[(q-2)^2+1\right]\int_{\Omega}u^{\left(\frac{q}{2}-1\right)\beta+\lambda}F^{t+\frac{q}{2}}|\nabla\eta|^2\\
&+\frac{1}{\theta(q)t}(q-1)^2\left[\left(1+\frac{q}{2}\right)\beta-\lambda\right]^2\int_{\Omega}u^{\frac{q}{2}\beta+\lambda-2}F^{t+\frac{q}{2}+1}\eta^2.
\end{aligned}\tag{51}$$

Now we choose t large enough such that

$$\frac{1}{\theta(z)t}(z-1)^2\left[\left(1+\frac{z}{2}\right)\beta-\lambda\right]^2\leq\frac{\lambda_z}{2},\ \ (z\in\{p,\ q\}).\tag{52}$$

It follows from (51) and (52) that

$$\frac{\theta(p)}{4}t\int_\Omega u^{(\frac{p}{2}-1)\beta+\lambda}F^{t+\frac{p}{2}-2}|\nabla F|^2\eta^2+\frac{\lambda_p}{2}\int_\Omega u^{\frac{p}{2}\beta+\lambda-2}F^{t+\frac{p}{2}+1}\eta^2$$

$$+\frac{\theta(q)}{4}t\int_\Omega u^{(\frac{q}{2}-1)\beta+\lambda}F^{t+\frac{q}{2}-2}|\nabla F|^2\eta^2+\frac{\lambda_q}{2}\int_\Omega u^{\frac{q}{2}\beta+\lambda-2}F^{t+\frac{q}{2}+1}\eta^2$$

$$\leq 2(n-1)\kappa\int_\Omega u^{(\frac{p}{2}-1)\beta+\lambda}F^{t+\frac{p}{2}}\eta^2+\frac{4}{\theta(p)t}\left[(p-2)^2+1\right]\int_\Omega u^{(\frac{p}{2}-1)\beta+\lambda}F^{t+\frac{p}{2}}|\nabla\eta|^2$$

$$+2(n-1)\kappa\int_\Omega u^{(\frac{q}{2}-1)\beta+\lambda}F^{t+\frac{q}{2}}\eta^2+\frac{4}{\theta(q)t}\left[(q-2)^2+1\right]\int_\Omega u^{(\frac{q}{2}-1)\beta+\lambda}F^{t+\frac{q}{2}}|\nabla\eta|^2. \tag{53}$$

By letting $\lambda=\beta\left(1-\dfrac{p}{2}\right)$ and omitting some non-negative terms in (53), we obtain

$$\frac{\theta(p)}{4}t\int_\Omega F^{t+\frac{p}{2}-2}|\nabla F|^2\eta^2+\frac{\lambda_q}{2}\int_\Omega u^{(\frac{q-p}{2}+1)\beta-2}F^{t+\frac{q}{2}+1}\eta^2$$

$$\leq 2(n-1)\kappa\int_\Omega F^{t+\frac{p}{2}}\eta^2+\frac{4}{\theta(p)t}\left[(p-2)^2+1\right]\int_\Omega F^{t+\frac{p}{2}}|\nabla\eta|^2$$

$$+2(n-1)\kappa\int_\Omega u^{\frac{q-p}{2}\beta}F^{t+\frac{q}{2}}\eta^2+\frac{4}{\theta(q)t}\left[(q-2)^2+1\right]\int_\Omega u^{\frac{q-p}{2}\beta}F^{t+\frac{q}{2}}|\nabla\eta|^2. \tag{54}$$

On the other hand, we have

$$|\nabla(F^{\frac{p}{4}+\frac{t}{2}}\eta)|^2=\left|\left(\frac{p+2t}{4}\right)F^{\frac{p}{4}+\frac{t}{2}-1}\eta\nabla F+F^{\frac{p}{4}+\frac{t}{2}}\nabla\eta\right|^2$$

$$\leq\frac{(p+2t)^2}{8}F^{\frac{p}{2}+t-2}|\nabla F|^2\eta^2+2F^{\frac{p}{2}+t}|\nabla\eta|^2. \tag{55}$$

Substituting (55) into (54) gives

$$\frac{2\theta(p)}{(p+2t)^2}t\int_\Omega|\nabla(F^{\frac{p}{4}+\frac{t}{2}}\eta)|^2+\frac{\lambda_q}{2}\int_\Omega u^{(\frac{q-p}{2}+1)\beta-2}F^{t+\frac{q}{2}+1}\eta^2$$

$$\leq 2(n-1)\kappa\int_\Omega F^{t+\frac{p}{2}}\eta^2+\left\{\frac{4}{\theta(p)t}\left[(p-2)^2+1\right]+\frac{4\theta(p)}{(p+2t)^2}t\right\}\int_\Omega F^{t+\frac{p}{2}}|\nabla\eta|^2$$

$$+2(n-1)\kappa\int_\Omega u^{\frac{q-p}{2}\beta}F^{t+\frac{q}{2}}\eta^2+\frac{4}{\theta(q)t}\left[(q-2)^2+1\right]\int_\Omega u^{\frac{q-p}{2}\beta}F^{t+\frac{q}{2}}|\nabla\eta|^2. \tag{56}$$

Noting that Saloff-Coste's Sobolev inequality implies

$$\exp\left\{-C_n(1+\sqrt{\kappa}R)\right\} V^{\frac{2}{n}} R^{-2} \left\| F^{\frac{p}{4}+\frac{t}{2}} \eta \right\|^2_{L^{\frac{2n}{n-2}}(\Omega)} \le \int_\Omega \left| \nabla \left(F^{\frac{p}{4}+\frac{t}{2}} \eta \right) \right|^2 + R^{-2} \int_\Omega F^{\frac{p}{2}+t} \eta^2, \tag{57}$$

we substitute (57) into (56) to obtain

$$\begin{aligned} &\frac{2\theta(p)}{(p+2t)^2} t \exp\left\{-C_n(1+\sqrt{\kappa}R)\right\} V^{\frac{2}{n}} R^{-2} \left\| F^{\frac{p}{4}+\frac{t}{2}} \eta \right\|^2_{L^{\frac{2n}{n-2}}(\Omega)} + \frac{\lambda_q}{2} \int_\Omega u^{\left(\frac{q-p}{2}+1\right)\beta-2} F^{t+\frac{q}{2}+1} \eta^2 \\ \le& \left[2(n-1)\kappa + \frac{2\theta(p)}{(p+2t)^2 R^2} t \right] \int_\Omega F^{t+\frac{p}{2}} \eta^2 + \left\{ \frac{4}{\theta(p)t} \left[(p-2)^2+1 \right] + \frac{4\theta(p)}{(p+2t)^2} t \right\} \int_\Omega F^{t+\frac{p}{2}} |\nabla \eta|^2 \\ &+ 2(n-1)\kappa \int_\Omega u^{\frac{q-p}{2}\beta} F^{t+\frac{q}{2}} \eta^2 + \frac{4}{\theta(q)t} \left[(q-2)^2+1 \right] \int_\Omega u^{\frac{q-p}{2}\beta} F^{t+\frac{q}{2}} |\nabla \eta|^2. \end{aligned} \tag{58}$$

By dividing the both sides of (58) by $\frac{2\theta(p)}{(p+2t)^2} t$, we obtain

$$\begin{aligned} &\exp\left\{-C_n(1+\sqrt{\kappa}R)\right\} V^{\frac{2}{n}} R^{-2} \left\| F^{\frac{p}{4}+\frac{t}{2}} \eta \right\|^2_{L^{\frac{2n}{n-2}}(\Omega)} + \frac{\lambda_q (p+2t)^2}{4\theta(p)t} \int_\Omega u^{\left(\frac{q-p}{2}+1\right)\beta-2} F^{t+\frac{q}{2}+1} \eta^2 \\ \le& \left[(n-1)\frac{(p+2t)^2}{\theta(p)t} \kappa + \frac{1}{R^2} \right] \int_\Omega F^{t+\frac{p}{2}} \eta^2 + \left\{ \frac{2(p+2t)^2}{\theta^2(p)t^2} \left[(p-2)^2+1 \right] + 2 \right\} \int_\Omega F^{t+\frac{p}{2}} |\nabla \eta|^2 \\ &+ (n-1)\frac{(p+2t)^2}{\theta(p)t} \kappa \int_\Omega u^{\frac{q-p}{2}\beta} F^{t+\frac{q}{2}} \eta^2 + \frac{2(p+2t)^2}{\theta(q)\theta(p)t^2} \left[(q-2)^2+1 \right] \int_\Omega u^{\frac{q-p}{2}\beta} F^{t+\frac{q}{2}} |\nabla \eta|^2. \end{aligned} \tag{59}$$

Set

$$\mu_1 = \max\left\{ \sup_{t\in[1,\,\infty)} \frac{2(p+2t)^2}{\theta^2(p)t^2} \left[(p-2)^2+1 \right] + 2,\ \sup_{t\in[1,\,\infty)} \frac{2(p+2t)^2}{\theta(q)\theta(p)t^2} \left[(q-2)^2+1 \right] \right\}, \tag{60}$$

$$\mu = \lambda_q \inf_{t\in[1,\,\infty)} \frac{(p+2t)^2}{4\theta(p)t^2} \quad \text{(see (42))}, \tag{61}$$

then we can see that μ_1 and μ are both finite positive constants. Combining (59), (60) and (61) yields

$$\exp\left\{-C_n(1+\sqrt{\kappa}R)\right\}V^{\frac{2}{n}}R^{-2}\left\|F^{\frac{p}{4}+\frac{t}{2}}\eta\right\|^2_{L^{\frac{2n}{n-2}}(\Omega)}+\mu t\int_\Omega u^{\left(\frac{q-p}{2}+1\right)\beta-2}F^{t+\frac{q}{2}+1}\eta^2$$

$$\leq\left[(n-1)\mu_1 t\kappa+\frac{1}{R^2}\right]\int_\Omega F^{t+\frac{p}{2}}\eta^2+\mu_1\int_\Omega F^{t+\frac{p}{2}}|\nabla\eta|^2$$

$$+(n-1)\mu_1 t\kappa\int_\Omega u^{\frac{q-p}{2}\beta}F^{t+\frac{q}{2}}\eta^2+\mu_1\int_\Omega u^{\frac{q-p}{2}\beta}F^{t+\frac{q}{2}}|\nabla\eta|^2. \tag{62}$$

Now we choose β small enough such that

$$\left(\frac{q-p}{2}+1\right)\beta-2\leq 0. \tag{63}$$

By combining (43) and (63) together we obtain

$$\beta\in\mathfrak{A}_0=\left\{\beta\ :\ 0<\beta\leq\frac{4}{q-p+2},\ \lambda_p(\beta)>0\ \text{ and }\ \lambda_q(\beta)>0\right\}, \tag{64}$$

where $\lambda_z(\beta)$ is defined in (42). Moreover, set

$$\phi=\|u\|_{L^\infty(\Omega)}. \tag{65}$$

Hence, combining (62), (64) and (65) we can know that

$$\exp\left\{-C_n(1+\sqrt{\kappa}R)\right\}V^{\frac{2}{n}}R^{-2}\left\|F^{\frac{p}{4}+\frac{t}{2}}\eta\right\|^2_{L^{\frac{2n}{n-2}}(\Omega)}+\mu t\phi^{\left(\frac{q-p}{2}+1\right)\beta-2}\int_\Omega F^{t+\frac{q}{2}+1}\eta^2$$

$$\leq\left[(n-1)\mu_1 t\kappa+\frac{1}{R^2}\right]\int_\Omega F^{t+\frac{p}{2}}\eta^2+\mu_1\int_\Omega F^{t+\frac{p}{2}}|\nabla\eta|^2$$

$$+(n-1)\mu_1 t\kappa\phi^{\frac{q-p}{2}\beta}\int_\Omega F^{t+\frac{q}{2}}\eta^2+\mu_1\phi^{\frac{q-p}{2}\beta}\int_\Omega F^{t+\frac{q}{2}}|\nabla\eta|^2. \tag{66}$$

3.3 L^{β_1}-bound of gradient in a geodesic ball with $\frac{3R}{4}$ radius

For the sake of simplicity, we set

$$\mathfrak{B}_0 = \left\{ t \ : \ t > \max\left\{1,\ 2 - \frac{p}{2},\ \frac{(n-2)(q-p)}{2} - \frac{p}{2}\right\}\right.$$

$$\left. \text{and } \ \frac{1}{\theta(z)t}(z-1)^2\left(\frac{z+p}{2}\right)^2\beta^2 \leq \frac{\lambda_z}{2} \ \text{ with } \ z \in \{p,\ q\} \right\}, \tag{67}$$

where $\theta(z)$ is defined in (48) and λ_z is defined in (42). By using the integral inequality (66), we can achieve the following lemma:

Lemma 8 Let $1 < p \leq q$ and $(M,\ g)$ be an n-dim $(n \geq 3)$ complete manifold with $\mathrm{Ric} \geq -(n-1)\kappa$, where κ is a non-negative constant. Furthermore, $h \in C^1(\mathbb{R}^+)$ is a non-increasing function. Assume u is a positive solution to equation (1) on the geodesic ball $B(x_0,\ 2R) \subset M$. Let $\beta \in \mathfrak{A}_0$ (see (64)), $t_0 \in \mathfrak{B}_0$ (see (67)) and $\beta_1 = \dfrac{n}{n-2}\dfrac{p+2t_0}{2}$, then there exists $\mathscr{C} = \mathscr{C}(n,\ p,\ q) > 0$ such that

$$\begin{aligned}
\|F\|_{L^{\beta_1}\left(B\left(x_0,\ \frac{3}{4}R\right)\right)} \leq{}& \mathscr{C}_0 \exp\left\{\frac{2C_n(1+\sqrt{\kappa}R)}{2t_0+p}\right\} V^{\frac{1}{\beta_1}} \\
&\cdot \Bigg[(1+\kappa R^2)^{\frac{2}{p+2t_0}} \left(\kappa + \frac{1}{R^2}\right)^{\frac{2}{q-p+2}} \lambda_q^{-\frac{2}{q-p+2}} \phi^{\frac{4}{q-p+2}-\beta} \\
&+ \kappa^{\frac{q+2t_0+2}{p+2t_0}} R^{\frac{4}{p+2t_0}} \lambda_q^{-\frac{q+2t_0}{p+2t_0}} \phi^{\frac{2q+4t_0}{p+2t_0}-\beta} + t_0 R^{-\frac{2q+4t_0}{p+2t_0}} \lambda_q^{-\frac{q+2t_0}{p+2t_0}} \phi^{\frac{2q+4t_0}{p+2t_0}-\beta} \\
&+ t_0^{\frac{2}{q-p+2}} R^{-\frac{4}{q-p+2}} \lambda_q^{-\frac{2}{q-p+2}} \phi^{\frac{4}{q-p+2}-\beta} \Bigg],
\end{aligned} \tag{68}$$

where V is the volume of geodesic ball $B_R(x_0)$, ϕ is defined in (65) and λ_q is defined in (42).

Proof. By (66), we can know that

$$\begin{aligned}
&\exp\left\{-C_n(1+\sqrt{\kappa}R)\right\} V^{\frac{2}{n}} R^{-2} \left\| F^{\frac{p}{4}+\frac{t_0}{2}} \eta \right\|^2_{L^{\frac{2n}{n-2}}(\Omega)} + \mu t_0 \phi^{\left(\frac{q-p}{2}+1\right)\beta-2} \int_\Omega F^{t_0+\frac{q}{2}+1} \eta^2 \\
\leq{}& \left[(n-1)\mu_1 t_0 \kappa + \frac{1}{R^2}\right] \int_\Omega F^{t_0+\frac{p}{2}} \eta^2 + \mu_1 \int_\Omega F^{t_0+\frac{p}{2}} |\nabla \eta|^2 \\
&+ (n-1)\mu_1 t_0 \kappa \phi^{\frac{q-p}{2}\beta} \int_\Omega F^{t_0+\frac{q}{2}} \eta^2 + \mu_1 \phi^{\frac{q-p}{2}\beta} \int_\Omega F^{t_0+\frac{q}{2}} |\nabla \eta|^2,
\end{aligned} \tag{69}$$

where $t_0 \in \mathfrak{B}_0$ and μ_1 is defined in (60). Denote

$$\widehat{\Omega}_1 = \left\{ x \in \Omega \, : \, F \geq \left\{ \left[4(n-1)\kappa \frac{\mu_1}{\mu} + \frac{4}{\mu t_0 R^2} \right] \phi^{2-\left(\frac{q-p}{2}+1\right)\beta} \right\}^{\frac{2}{q-p+2}} \right\},$$

then we have

$$\left[(n-1)\mu_1 t_0 \kappa + \frac{1}{R^2} \right] \int_{\widehat{\Omega}_1} F^{t_0+\frac{p}{2}} \eta^2 \leq \frac{\mu t_0}{4} \phi^{\left(\frac{q-p}{2}+1\right)\beta-2} \int_{\Omega} F^{t_0+\frac{q}{2}+1} \eta^2. \tag{70}$$

Set

$$\widehat{\Omega}_2 = \Omega \setminus \widehat{\Omega}_1 = \left\{ x \in \Omega \, : \, F < \left\{ \left[4(n-1)\kappa \frac{\mu_1}{\mu} + \frac{4}{\mu t_0 R^2} \right] \phi^{2-\left(\frac{q-p}{2}+1\right)\beta} \right\}^{\frac{2}{q-p+2}} \right\},$$

then we have

$$\begin{aligned} &\left[(n-1)\mu_1 t_0 \kappa + \frac{1}{R^2} \right] \int_{\widehat{\Omega}_2} F^{t_0+\frac{p}{2}} \eta^2 \\ \leq &\left[(n-1)\mu_1 t_0 \kappa + \frac{1}{R^2} \right] \left\{ \left[4(n-1)\kappa \frac{\mu_1}{\mu} + \frac{4}{\mu t_0 R^2} \right] \phi^{2-\left(\frac{q-p}{2}+1\right)\beta} \right\}^{\frac{p+2t_0}{q-p+2}} V, \end{aligned} \tag{71}$$

where V is the volume of $\Omega = B(x_0, R)$. Combining (70) and (71) together, we obtain

$$\begin{aligned} &\left[(n-1)\mu_1 t_0 \kappa + \frac{1}{R^2} \right] \int_{\Omega} F^{t_0+\frac{p}{2}} \eta^2 \\ \leq &\frac{\mu t_0}{4} \phi^{\left(\frac{q-p}{2}+1\right)\beta-2} \int_{\Omega} F^{t_0+\frac{q}{2}+1} \eta^2 \\ &+ \left[(n-1)\mu_1 t_0 \kappa + \frac{1}{R^2} \right] \left\{ \left[4(n-1)\kappa \frac{\mu_1}{\mu} + \frac{4}{\mu t_0 R^2} \right] \phi^{2-\left(\frac{q-p}{2}+1\right)\beta} \right\}^{\frac{p+2t_0}{q-p+2}} V. \end{aligned} \tag{72}$$

Denote

$$\widetilde{\Omega}_1 = \left\{ x \in \Omega \, : \, F \geq 4(n-1)\kappa \frac{\mu_1}{\mu} \phi^{2-\beta} \right\},$$

then we have

$$(n-1)\mu_1 t_0\kappa\phi^{\frac{q-p}{2}\beta}\int_{\widetilde{\Omega}_1} F^{t_0+\frac{q}{2}}\eta^2 \le \frac{\mu t_0}{4}\phi^{\left(\frac{q-p}{2}+1\right)\beta-2}\int_{\Omega} F^{t_0+\frac{q}{2}+1}\eta^2. \tag{73}$$

Set

$$\widetilde{\Omega}_2 = \Omega\setminus\widetilde{\Omega}_1 = \left\{x\in\Omega\ :\ F < 4(n-1)\kappa\frac{\mu_1}{\mu}\phi^{2-\beta}\right\},$$

then we have

$$(n-1)\mu_1 t_0\kappa\phi^{\frac{q-p}{2}\beta}\int_{\widetilde{\Omega}_2} F^{t_0+\frac{q}{2}}\eta^2 \le (n-1)\mu_1 t_0\kappa\phi^{\frac{q-p}{2}\beta}\left[4(n-1)\kappa\frac{\mu_1}{\mu}\phi^{2-\beta}\right]^{t_0+\frac{q}{2}} V. \tag{74}$$

Combining (73) and (74) together, we can know that

$$\begin{aligned}(n-1)\mu_1 t_0\kappa\phi^{\frac{q-p}{2}\beta}\int_{\Omega} F^{t_0+\frac{q}{2}}\eta^2 \le{}& \frac{\mu t_0}{4}\phi^{\left(\frac{q-p}{2}+1\right)\beta-2}\int_{\Omega} F^{t_0+\frac{q}{2}+1}\eta^2\\ &+(n-1)\mu_1 t_0\kappa\phi^{\frac{q-p}{2}\beta}\left[4(n-1)\kappa\frac{\mu_1}{\mu}\phi^{2-\beta}\right]^{t_0+\frac{q}{2}} V.\end{aligned} \tag{75}$$

We denote $\Omega_1 = B\left(x_0,\ \frac{3R}{4}\right)$ and choose $\eta_1\in C_0^\infty(\Omega)$ satisfying

$$0\le\eta_1\le 1,\quad \eta_1\equiv 1\ \text{ in }\ \Omega_1,\quad |\nabla\eta_1|\le\frac{C}{R}.$$

Let $\eta = \eta_1^{t_0+\frac{q}{2}+1}$. Then we have

$$\begin{aligned}\mu_1\int_{\Omega} F^{t_0+\frac{p}{2}}|\nabla\eta|^2 &= \mu_1\left(t_0+\frac{q}{2}+1\right)^2\int_{\Omega} F^{t_0+\frac{p}{2}}\eta_1^{2t_0+q}|\nabla\eta_1|^2\\ &\le \mu_1\left(t_0+\frac{q}{2}+1\right)^2\frac{C^2}{R^2}\int_{\Omega} F^{t_0+\frac{p}{2}}\eta_1^{2t_0+q}\end{aligned} \tag{76}$$

and

$$\begin{aligned}\mu_1\phi^{\frac{q-p}{2}\beta}\int_{\Omega} F^{t_0+\frac{q}{2}}|\nabla\eta|^2 &= \mu_1\phi^{\frac{q-p}{2}\beta}\left(t_0+\frac{q}{2}+1\right)^2\int_{\Omega} F^{t_0+\frac{q}{2}}\eta_1^{2t_0+q}|\nabla\eta_1|^2\\ &\le \mu_1\phi^{\frac{q-p}{2}\beta}\left(t_0+\frac{q}{2}+1\right)^2\frac{C^2}{R^2}\int_{\Omega} F^{t_0+\frac{q}{2}}\eta_1^{2t_0+q}.\end{aligned} \tag{77}$$

Since

$$t_0 + \frac{p}{2} \le t_0 + \frac{q}{2},$$

we obtain

$$\eta_1^{2t_0+p} \ge \eta_1^{2t_0+q}. \tag{78}$$

Combining (76) and (78) together, we can achieve that

$$\mu_1 \int_\Omega F^{t_0+\frac{p}{2}} |\nabla\eta|^2 \le \mu_1 \left(t_0 + \frac{q}{2} + 1\right)^2 \frac{C^2}{R^2} \int_\Omega F^{t_0+\frac{p}{2}} \eta_1^{2t_0+p}. \tag{79}$$

By Hölder's inequality, (79) can be written as

$$\begin{aligned}\mu_1 \int_\Omega F^{t_0+\frac{p}{2}} |\nabla\eta|^2 &\le \mu_1 \left(t_0 + \frac{q}{2} + 1\right)^2 \frac{C^2}{R^2} \left(\int_\Omega F^{t_0+\frac{q}{2}+1} \eta_1^{2t_0+q+2}\right)^{\frac{2t_0+p}{2t_0+q+2}} V^{\frac{q-p+2}{2t_0+q+2}} \\ &= \mu_1 \left(t_0 + \frac{q}{2} + 1\right)^2 \frac{C^2}{R^2} \left(\int_\Omega F^{t_0+\frac{q}{2}+1} \eta^2\right)^{\frac{2t_0+p}{2t_0+q+2}} V^{\frac{q-p+2}{2t_0+q+2}}.\end{aligned} \tag{80}$$

By using Young's inequality, we can write (80) as

$$\begin{aligned}\mu_1 \int_\Omega F^{t_0+\frac{p}{2}} |\nabla\eta|^2 &\le \frac{\mu t_0}{4} \phi^{\left(\frac{q-p}{2}+1\right)\beta-2} \int_\Omega F^{t_0+\frac{q}{2}+1} \eta^2 \\ &\quad + \frac{q-p+2}{2t_0+q+2} \left[\mu_1 \left(t_0 + \frac{q}{2} + 1\right)^2 \frac{C^2}{R^2}\right]^{\frac{2t_0+q+2}{q-p+2}} \left[\frac{4(2t_0+p)}{(2t_0+q+2)\mu t_0} \phi^{2-\left(\frac{q-p}{2}+1\right)\beta}\right]^{\frac{p+2t_0}{q-p+2}} V.\end{aligned} \tag{81}$$

We take a similar argument to the proof of (81) to derive from (77) that

$$\begin{aligned}&\mu_1 \phi^{\frac{q-p}{2}\beta} \int_\Omega F^{t_0+\frac{q}{2}} |\nabla\eta|^2 \\ &\le \frac{\mu t_0}{4} \phi^{\left(\frac{q-p}{2}+1\right)\beta-2} \int_\Omega F^{t_0+\frac{q}{2}+1} \eta^2 \\ &\quad + \frac{2}{2t_0+q+2} \left[\mu_1 \phi^{\frac{q-p}{2}\beta} \left(t_0 + \frac{q}{2} + 1\right)^2 \frac{C^2}{R^2}\right]^{\frac{2t_0+q+2}{2}} \left[\frac{4(2t_0+q)}{(2t_0+q+2)\mu t_0} \phi^{2-\left(\frac{q-p}{2}+1\right)\beta}\right]^{\frac{2t_0+q}{2}} V.\end{aligned} \tag{82}$$

Substituting (72), (75), (81) and (82) into (69) gives

$$\exp\left\{-C_n(1+\sqrt{\kappa}R)\right\}V^{\frac{2}{n}}R^{-2}\left\|F^{\frac{p}{4}+\frac{t_0}{2}}\eta\right\|^2_{L^{\frac{2n}{n-2}}(\Omega)}$$

$$\leq\left[(n-1)\mu_1 t_0\kappa+\frac{1}{R^2}\right]\left\{\left[4(n-1)\kappa\frac{\mu_1}{\mu}+\frac{4}{\mu t_0 R^2}\right]\phi^{2-\left(\frac{q-p}{2}+1\right)\beta}\right\}^{\frac{p+2t_0}{q-p+2}}V$$

$$+(n-1)\mu_1 t_0\kappa\phi^{\frac{q-p}{2}\beta}\left[4(n-1)\kappa\frac{\mu_1}{\mu}\phi^{2-\beta}\right]^{t_0+\frac{q}{2}}V$$

$$+\frac{q-p+2}{2t_0+q+2}\left[\mu_1\left(t_0+\frac{q}{2}+1\right)^2\frac{C^2}{R^2}\right]^{\frac{2t_0+q+2}{q-p+2}}\left[\frac{4(2t_0+p)}{(2t_0+q+2)\mu t_0}\phi^{2-\left(\frac{q-p}{2}+1\right)\beta}\right]^{\frac{p+2t_0}{q-p+2}}V$$

$$+\frac{2}{2t_0+q+2}\left[\mu_1\phi^{\frac{q-p}{2}\beta}\left(t_0+\frac{q}{2}+1\right)^2\frac{C^2}{R^2}\right]^{\frac{2t_0+q+2}{2}}\left[\frac{4(2t_0+q)}{(2t_0+q+2)\mu t_0}\phi^{2-\left(\frac{q-p}{2}+1\right)\beta}\right]^{\frac{2t_0+q}{2}}V. \tag{83}$$

Taking power of $\dfrac{2}{2t_0+p}$ of the both sides of (83) respectively, we obtain

$$\|F\|_{L^{\beta_1}\left(B\left(x_0,\ \frac{3}{4}R\right)\right)}\leq\left\|F^{\frac{p}{4}+\frac{t_0}{2}}\eta\right\|^{\frac{4}{2t_0+p}}_{L^{\frac{2n}{n-2}}(\Omega)}$$

$$\leq\exp\left\{\frac{2C_n(1+\sqrt{\kappa}R)}{2t_0+p}\right\}\left\{\left[(n-1)\mu_1 t_0 R^2\kappa+1\right]\left\{\left[4(n-1)\kappa\frac{\mu_1}{\mu}+\frac{4}{\mu t_0 R^2}\right]\phi^{2-\left(\frac{q-p}{2}+1\right)\beta}\right\}^{\frac{p+2t_0}{q-p+2}}\right.$$

$$+(n-1)\mu_1 t_0\kappa\phi^{\frac{q-p}{2}\beta}\left[4(n-1)\kappa\frac{\mu_1}{\mu}\phi^{2-\beta}\right]^{t_0+\frac{q}{2}}R^2+\frac{q-p+2}{2t_0+q+2}\left[\mu_1\left(t_0+\frac{q}{2}+1\right)^2\frac{C^2}{R^2}\right]^{\frac{2t_0+q+2}{q-p+2}}$$

$$\cdot\left[\frac{4(2t_0+p)}{(2t_0+q+2)\mu t_0}\phi^{2-\left(\frac{q-p}{2}+1\right)\beta}\right]^{\frac{p+2t_0}{q-p+2}}R^2+\frac{2}{2t_0+q+2}\left[\mu_1\phi^{\frac{q-p}{2}\beta}\left(t_0+\frac{q}{2}+1\right)^2\frac{C^2}{R^2}\right]^{\frac{2t_0+q+2}{2}}$$

$$\left.\cdot\left[\frac{4(2t_0+q)}{(2t_0+q+2)\mu t_0}\phi^{2-\left(\frac{q-p}{2}+1\right)\beta}\right]^{\frac{2t_0+q}{2}}R^2\right\}^{\frac{2}{2t_0+p}}V^{\frac{1}{\beta_1}},$$

where $\beta_1=\dfrac{n}{n-2}\dfrac{p+2t_0}{2}$.

By using the inequality $(a_1+a_2+a_3+a_4)^b\leq 4^b(a_1^b+a_2^b+a_3^b+a_4^b)$ $(a_i\geq 0,\ b>0)$, we have

$$\|F\|_{L^{\beta_1}\left(B\left(x_0,\frac{3}{4}R\right)\right)} \leq \exp\left\{\frac{2C_n(1+\sqrt{\kappa}R)}{2t_0+p}\right\} V^{\frac{1}{\beta_1}} 4^{\frac{2}{2t_0+p}}$$

$$\cdot\left\{ \left[(n-1)\mu_1 t_0 R^2\kappa+1\right]^{\frac{2}{2t_0+p}} \left\{\left[4(n-1)\kappa\frac{\mu_1}{\mu}+\frac{4}{\mu t_0 R^2}\right]\phi^{2-\left(\frac{q-p}{2}+1\right)\beta}\right\}^{\frac{2}{q-p+2}}\right.$$

$$+\left[(n-1)\mu_1 t_0 R^2\kappa\phi^{\frac{q-p}{2}\beta}\right]^{\frac{2}{2t_0+p}}\left[4(n-1)\kappa\frac{\mu_1}{\mu}\phi^{2-\beta}\right]^{\frac{2t_0+q}{2t_0+p}}+\left(\frac{q-p+2}{2t_0+q+2}R^2\right)^{\frac{2}{2t_0+p}}$$

$$\cdot\left[\mu_1\left(t_0+\frac{q}{2}+1\right)^2\frac{C^2}{R^2}\right]^{\frac{2t_0+q+2}{q-p+2}\frac{2}{2t_0+p}}\left[\frac{4(2t_0+p)}{(2t_0+q+2)\mu t_0}\phi^{2-\left(\frac{q-p}{2}+1\right)\beta}\right]^{\frac{2}{q-p+2}}$$

$$\left.+\left(\frac{2R^2}{2t_0+q+2}\right)^{\frac{2}{2t_0+p}}\left[\mu_1\phi^{\frac{q-p}{2}\beta}\left(t_0+\frac{q}{2}+1\right)^2\frac{C^2}{R^2}\right]^{\frac{2t_0+q+2}{2t_0+p}}\left[\frac{4(2t_0+q)}{(2t_0+q+2)\mu t_0}\phi^{2-\left(\frac{q-p}{2}+1\right)\beta}\right]^{\frac{2t_0+q}{2t_0+p}}\right\}$$

$$:=\exp\left\{\frac{2C_n(1+\sqrt{\kappa}R)}{2t_0+p}\right\} V^{\frac{1}{\beta_1}} 4^{\frac{2}{2t_0+p}}(I_1+I_2+I_3+I_4). \tag{84}$$

For I_1, we have

$$I_1=\left[(n-1)\mu_1 t_0 R^2\kappa+1\right]^{\frac{2}{2t_0+p}}\left\{\left[4(n-1)\kappa\frac{\mu_1}{\mu}+\frac{4}{\mu t_0 R^2}\right]\phi^{2-\left(\frac{q-p}{2}+1\right)\beta}\right\}^{\frac{2}{q-p+2}}$$

$$=\left[(n-1)\mu_1 t_0 R^2\kappa+1\right]^{\frac{2}{2t_0+p}}\left[4(n-1)\kappa\mu_1+\frac{4}{t_0R^2}\right]^{\frac{2}{q-p+2}}\mu^{-\frac{2}{q-p+2}}\phi^{\frac{4}{q-p+2}-\beta}$$

$$\leq\left[(n-1)\mu_1 t_0 R^2\kappa+t_0\right]^{\frac{2}{2t_0+p}}\left[4(n-1)\kappa\mu_1+\frac{4}{R^2}\right]^{\frac{2}{q-p+2}}\mu^{-\frac{2}{q-p+2}}\phi^{\frac{4}{q-p+2}-\beta}$$

$$\leq\left[(n-1)\mu_1 t_0+t_0\right]^{\frac{2}{2t_0+p}}\left(1+R^2\kappa\right)^{\frac{2}{2t_0+p}}\left[4(n-1)\mu_1+4\right]^{\frac{2}{q-p+2}}\left(\kappa+\frac{1}{R^2}\right)^{\frac{2}{q-p+2}}$$

$$\cdot\mu^{-\frac{2}{q-p+2}}\phi^{\frac{4}{q-p+2}-\beta}. \tag{85}$$

Noticing that

$$\lim_{t_0\to+\infty}\left[(n-1)\mu_1 t_0+t_0\right]^{\frac{2}{2t_0+p}}=1,$$

we can know that

$$\sup_{t_0 \in [1,\ +\infty)} \left[(n-1)\mu_1 t_0 + t_0\right]^{\frac{2}{2t_0+p}} < +\infty. \tag{86}$$

Combining (85) and (86) together, we obtain

$$I_1 \leq \mathscr{C}_{I_1} \left(1+R^2\kappa\right)^{\frac{2}{2t_0+p}} \left(\kappa + \frac{1}{R^2}\right)^{\frac{2}{q-p+2}} \cdot \mu^{-\frac{2}{q-p+2}} \phi^{\frac{4}{q-p+2}-\beta}, \tag{87}$$

where $\mathscr{C}_{I_1}$ is a positive constant which depends only on n, p and q. Similarly, we have

$$I_2 \leq \mathscr{C}_{I_2} \kappa^{\frac{q+2t_0+2}{p+2t_0}} R^{\frac{4}{p+2t_0}} \mu^{-\frac{q+2t_0}{p+2t_0}} \phi^{\frac{2q+4t_0}{p+2t_0}-\beta}, \tag{88}$$

$$I_3 \leq \mathscr{C}_{I_3} t_0^{\frac{2}{q-p+2}} R^{-\frac{4}{q-p+2}} \mu^{-\frac{2}{q-p+2}} \phi^{\frac{4}{q-p+2}-\beta}, \tag{89}$$

$$I_4 \leq \mathscr{C}_{I_4} t_0 R^{-\frac{2q+4t_0}{p+2t_0}} \mu^{-\frac{q+2t_0}{p+2t_0}} \phi^{\frac{2q+4t_0}{p+2t_0}-\beta}, \tag{90}$$

where $\mathscr{C}_{I_i}(i=2,\ 3,\ 4)$ are positive constants which depend only on n, p and q. Substituting (87), (88), (89) and (90) into (84), we have

$$\begin{aligned}\|F\|_{L^{\beta_1}\left(B\left(x_0, \frac{3}{4}R\right)\right)} \leq \exp\left\{\frac{2C_n(1+\sqrt{\kappa}R)}{p+2t_0}\right\} 4^{\frac{2}{2t_0+p}} V^{\frac{1}{\beta_1}} \\ \cdot \Bigg[\mathscr{C}_{I_1}(1+\kappa R^2)^{\frac{2}{2t_0+p}} \left(\kappa+\frac{1}{R^2}\right)^{\frac{2}{q-p+2}} \mu^{-\frac{2}{q-p+2}} \phi^{\frac{4}{q-p+2}-\beta} \\ + \mathscr{C}_{I_2} \kappa^{\frac{q+2t_0+2}{p+2t_0}} R^{\frac{4}{p+2t_0}} \mu^{-\frac{q+2t_0}{p+2t_0}} \phi^{\frac{2q+4t_0}{p+2t_0}-\beta} + \mathscr{C}_{I_4} t_0 R^{-\frac{2q+4t_0}{p+2t_0}} \mu^{-\frac{q+2t_0}{p+2t_0}} \phi^{\frac{2q+4t_0}{p+2t_0}-\beta} \\ + \mathscr{C}_{I_3} t_0^{\frac{2}{q-p+2}} R^{-\frac{4}{q-p+2}} \mu^{-\frac{2}{q-p+2}} \phi^{\frac{4}{q-p+2}-\beta}\Bigg].\end{aligned}$$

Set

$$\mathscr{C} = \max\left\{\mathscr{C}_{I_1},\ \mathscr{C}_{I_2},\ \mathscr{C}_{I_3},\ \mathscr{C}_{I_4}\right\} 4^{\frac{2}{2+p}},$$

and we have

$$\|F\|_{L^{\beta_1}\left(B\left(x_0,\frac{3}{4}R\right)\right)} \leq \mathscr{C} \exp\left\{\frac{2C_n(1+\sqrt{\kappa}R)}{p+2t_0}\right\} V^{\frac{1}{\beta_1}}$$

$$\cdot \left[(1+\kappa R^2)^{\frac{2}{2t_0+p}} \left(\kappa + \frac{1}{R^2}\right)^{\frac{2}{q-p+2}} \mu^{-\frac{2}{q-p+2}} \phi^{\frac{4}{q-p+2}-\beta}\right.$$

$$+\kappa^{\frac{q+2t_0+2}{p+2t_0}} R^{\frac{4}{p+2t_0}} \mu^{-\frac{q+2t_0}{p+2t_0}} \phi^{\frac{2q+4t_0}{p+2t_0}-\beta} + t_0 R^{-\frac{2q+4t_0}{p+2t_0}} \mu^{-\frac{q+2t_0}{p+2t_0}} \phi^{\frac{2q+4t_0}{p+2t_0}-\beta}$$

$$\left. + t_0^{\frac{2}{q-p+2}} R^{-\frac{4}{q-p+2}} \mu^{-\frac{2}{q-p+2}} \phi^{\frac{4}{q-p+2}-\beta}\right], \tag{91}$$

where $\mathscr{C}$ is a positive constant which depends only on n, p and q. Combining (61) and (91) together, we finish the proof of Lemma 8. □

3.4 *Moser iteration for positive solutions of (1)*

By using the integral inequality (66), we can achieve the following lemma:

Lemma 9 Let $1 < p \leq q$ and $(M,\ g)$ be an n-dim $(n \geq 3)$ complete manifold with $\mathrm{Ric} \geq -(n-1)\kappa$, where κ is a non-negative constant. Furthermore, $h \in C^1(\mathbb{R}^+)$ is a non-increasing function. Assume u is a positive solution to equation (1) on the geodesic ball $B(x_0,\ 2R) \subset M$. For any $\beta \in \mathfrak{A}_0$ (see (64)) and $t_0 \in \mathfrak{B}_0$ (see (67)), then there exist $\beta_1 = \dfrac{n}{n-2}\dfrac{p+2t_0}{2}$, $\zeta = 1 - \dfrac{n(q-p)}{4\beta_1}$ and a positive constant $\mathscr{C}_1 = \mathscr{C}_1(n,\ p,\ q)$, such that for any $0 < s < \rho < \dfrac{1}{2}$ the following estimate holds true

$$\|F\|_{L^\infty(B(x_0,\ sR))} \leq \mathscr{C}_1 \left[\xi_1 + \left(\xi_1 \phi^{\frac{(q-p)n\beta}{4\beta_1}}\right)^{\frac{1}{\zeta}}\right] \frac{1}{(\rho-s)^{\frac{n}{\beta_1 \zeta}}}, \quad \text{(see (65))}$$

where

$$\xi_1 = \exp\left\{C_n \frac{n(1+\sqrt{\kappa}R)}{2\beta_1}\right\} (1+\kappa R^2)^{\frac{n}{2\beta_1}} \|F\|_{L^{\beta_1}\left(B\left(x_0,\ \frac{R}{2}\right)\right)} V^{-\frac{1}{\beta_1}}.$$

Proof. By omitting the second non-negative term in (66), we obtain

$$\exp\left\{-C_n(1+\sqrt{\kappa}R)\right\}V^{\frac{2}{n}}R^{-2}\left\|F^{\frac{p}{4}+\frac{t}{2}}\eta\right\|^2_{L^{\frac{2n}{n-2}}(\Omega)}$$

$$\leq \left[(n-1)\mu_1 t\kappa+\frac{1}{R^2}\right]\int_\Omega F^{t+\frac{p}{2}}\eta^2+\mu_1\int_\Omega F^{t+\frac{p}{2}}|\nabla\eta|^2$$

$$+(n-1)\mu_1 t\kappa\phi^{\frac{q-p}{2}\beta}\int_\Omega F^{t+\frac{q}{2}}\eta^2+\mu_1\phi^{\frac{q-p}{2}\beta}\int_\Omega F^{t+\frac{q}{2}}|\nabla\eta|^2. \tag{92}$$

Now, we denote $r_m = sR+\dfrac{\tau-s}{\alpha^{m-1}}R$ and $\Omega_m = B(x_0,\ r_m)$, where $\alpha=\sqrt{\dfrac{n}{n-2}}$ and $0<s<\tau<\dfrac{1}{2}$; and then choose $\eta_m\in C_0^\infty(\Omega_m)$ satisfying

$$0\leq\eta_m\leq 1,\quad \eta_m\equiv 1\ \text{ in }\ B(x_0,\ r_{m+1}),\quad |\nabla\eta_m|\leq\frac{C\alpha^m}{(\tau-s)R}.$$

Substituting η by η_m in (92), we can easily verify that

$$\exp\left\{-C_n(1+\sqrt{\kappa}R)\right\}V^{\frac{2}{n}}R^{-2}\left\|F^{\frac{p}{4}+\frac{t}{2}}\right\|^2_{L^{\frac{2n}{n-2}}(\Omega_{m+1})}$$

$$\leq \left[(n-1)\mu_1 t\kappa+\frac{1}{R^2}\right]\int_{\Omega_m} F^{t+\frac{p}{2}}+\mu_1\frac{C^2\alpha^{2m}}{(\tau-s)^2R^2}\int_{\Omega_m} F^{t+\frac{p}{2}}$$

$$+(n-1)\mu_1 t\kappa\phi^{\frac{q-p}{2}\beta}\int_{\Omega_m} F^{t+\frac{q}{2}}+\mu_1\phi^{\frac{q-p}{2}\beta}\frac{C^2\alpha^{2m}}{(\tau-s)^2R^2}\int_{\Omega_m} F^{t+\frac{q}{2}}. \tag{93}$$

From (93) we take a computation to derive

$$\exp\left\{-C_n(1+\sqrt{\kappa}R)\right\}V^{\frac{2}{n}}R^{-2}\left\|F^{\frac{p}{4}+\frac{t}{2}}\right\|^2_{L^{\frac{2n}{n-2}}(\Omega_{m+1})}$$

$$\leq \left[(n-1)\mu_1 t\kappa+\frac{1}{R^2}\right]\int_{\Omega_m} F^{t+\frac{p}{2}}+(n-1)\mu_1 t\kappa\phi^{\frac{q-p}{2}\beta}\|F\|^{\frac{q-p}{2}}_{L^\infty(B(x_0,\ \tau R))}\int_{\Omega_m} F^{t+\frac{p}{2}}$$

$$+\mu_1\frac{C^2\alpha^{2m}}{(\tau-s)^2R^2}\int_{\Omega_m} F^{t+\frac{p}{2}}+\mu_1\phi^{\frac{q-p}{2}\beta}\frac{C^2\alpha^{2m}}{(\tau-s)^2R^2}\|F\|^{\frac{q-p}{2}}_{L^\infty(B(x_0,\ \tau R))}\int_{\Omega_m} F^{t+\frac{p}{2}}$$

$$\leq \left[(n-1)\mu_1 t\kappa+\frac{1}{R^2}+\mu_1\frac{C^2\alpha^{2m}}{(\tau-s)^2R^2}\right]\left(1+\phi^{\frac{q-p}{2}\beta}\|F\|^{\frac{q-p}{2}}_{L^\infty(B(x_0,\ \tau R))}\right)\int_{\Omega_m} F^{t+\frac{p}{2}}.$$

Next, we choose $\beta_1 = \left(t_0 + \frac{p}{2}\right) \frac{n}{n-2}$ and $\beta_{m+1} = \frac{n\beta_m}{n-2}$, and let $t = t_m$ such that $t_m + \frac{p}{2} = \beta_m$. Then it follows that

$$
\begin{aligned}
&\exp\left\{-C_n(1+\sqrt{\kappa}R)\right\} V^{\frac{2}{n}} R^{-2} \left(\int_{\Omega_{m+1}} F^{\beta_{m+1}}\right)^{\frac{n-2}{n}} \\
&\leq \left[(n-1)\mu_1 t_m \kappa + \frac{1}{R^2} + \mu_1 \frac{C^2 \alpha^{2m}}{(\tau-s)^2 R^2}\right] \left(1 + \phi^{\frac{q-p}{2}\beta} \|F\|^{\frac{q-p}{2}}_{L^\infty(B(x_0,\ \tau R))}\right) \int_{\Omega_m} F^{\beta_m}.
\end{aligned} \tag{94}
$$

Taking power of $\frac{1}{\beta_m}$ on the both sides of (94), we obtain

$$
\begin{aligned}
\|F\|_{L^{\beta_{m+1}}(\Omega_{m+1})} &\leq \exp\left\{\frac{C_n(1+\sqrt{\kappa}R)}{\beta_m}\right\} V^{-\frac{2}{n\beta_m}} \left[(n-1)\mu_1 t_m \kappa R^2 + 1 + \mu_1 \frac{C^2\alpha^{2m}}{(\tau-s)^2}\right]^{\frac{1}{\beta_m}} \\
&\quad \cdot \left(1 + \phi^{\frac{q-p}{2}\beta} \|F\|^{\frac{q-p}{2}}_{L^\infty(B(x_0,\ \tau R))}\right)^{\frac{1}{\beta_m}} \|F\|_{L^{\beta_m}(\Omega_m)}.
\end{aligned} \tag{95}
$$

Keeping the definition of t_m and $\alpha = \sqrt{\frac{n}{n-2}}$ in mind, from (95) we can deduce that

$$
\begin{aligned}
\|F\|_{L^{\beta_{m+1}}(\Omega_{m+1})} &\leq \exp\left\{\frac{C_n(1+\sqrt{\kappa}R)}{\beta_m}\right\} V^{-\frac{2}{n\beta_m}} \left(1 + \phi^{\frac{q-p}{2}\beta} \|F\|^{\frac{q-p}{2}}_{L^\infty(B(x_0,\ \tau R))}\right)^{\frac{1}{\beta_m}} \|F\|_{L^{\beta_m}(\Omega_m)} \\
&\quad \cdot \left[(n-1)\mu_1 \left(t_0 + \frac{p}{2}\right) \kappa R^2 \left(\frac{n}{n-2}\right)^m + 1 + \mu_1 \frac{C^2}{(\tau-s)^2} \left(\frac{n}{n-2}\right)^m\right]^{\frac{1}{\beta_m}} \\
&\leq \exp\left\{\frac{C_n(1+\sqrt{\kappa}R)}{\beta_m}\right\} V^{-\frac{2}{n\beta_m}} \left(1 + \phi^{\frac{q-p}{2}\beta} \|F\|^{\frac{q-p}{2}}_{L^\infty(B(x_0,\ \tau R))}\right)^{\frac{1}{\beta_m}} \\
&\quad \cdot \left[(n-1)\mu_1 \left(t_0 + \frac{p}{2}\right) \kappa R^2 + 1 + \mu_1 \frac{C^2}{(\tau-s)^2}\right]^{\frac{1}{\beta_m}} \left(\frac{n}{n-2}\right)^{\frac{m}{\beta_m}} \|F\|_{L^{\beta_m}(\Omega_m)}.
\end{aligned}
$$

Noting

$$
0 < s < \tau < \frac{1}{2}, \quad \sum_{m=1}^{\infty} \frac{1}{\beta_m} = \frac{n}{2\beta_1} \text{ and } \sum_{m=1}^{\infty} \frac{m}{\beta_m} = \frac{n^2}{4\beta_1},
$$

we have

$$\|F\|_{L^\infty(B(x_0,\, sR))} \leq \exp\left\{C_n \frac{n(1+\sqrt{\kappa}R)}{2\beta_1}\right\} \left(1+\phi^{\frac{q-p}{2}\beta}\|F\|^{\frac{q-p}{2}}_{L^\infty(B(x_0,\, \tau R))}\right)^{\frac{n}{2\beta_1}} \left(\frac{n}{n-2}\right)^{\frac{n^2}{4\beta_1}}$$

$$\cdot\left[(n-1)\mu_1\left(t_0+\frac{p}{2}\right)\kappa R^2+1+\mu_1\frac{C^2}{(\tau-s)^2}\right]^{\frac{n}{2\beta_1}}\|F\|_{L^{\beta_1}(B(x_0,\, \tau R))}V^{-\frac{1}{\beta_1}}$$

$$\leq \exp\left\{C_n \frac{n(1+\sqrt{\kappa}R)}{2\beta_1}\right\}\left(\frac{n}{n-2}\right)^{\frac{n^2}{4\beta_1}}\left[(n-1)\mu_1\left(t_0+\frac{p}{2}\right)\kappa R^2+1+\mu_1 C^2\right]^{\frac{n}{2\beta_1}}$$

$$\cdot\frac{1}{(\tau-s)^{\frac{n}{\beta_1}}}2^{\frac{n}{2\beta_1}}\left(1+\phi^{\frac{n(q-p)\beta}{4\beta_1}}\|F\|^{\frac{n(q-p)}{4\beta_1}}_{L^\infty(B(x_0,\, \tau R))}\right)\|F\|_{L^{\beta_1}\left(B\left(x_0,\, \frac{R}{2}\right)\right)}V^{-\frac{1}{\beta_1}}. \tag{96}$$

By (96), we can know that there exists a positive constant $\widetilde{\mathscr{C}}=\widetilde{\mathscr{C}}(n,\ p,\ q)$ such that

$$\|F\|_{L^\infty(B(x_0,\, sR))} \leq \widetilde{\mathscr{C}}\exp\left\{C_n \frac{n(1+\sqrt{\kappa}R)}{2\beta_1}\right\}(1+\kappa R^2)^{\frac{n}{2\beta_1}}\|F\|_{L^{\beta_1}\left(B\left(x_0,\, \frac{R}{2}\right)\right)}V^{-\frac{1}{\beta_1}}$$

$$\cdot\left(1+\phi^{\frac{n(q-p)\beta}{4\beta_1}}\|F\|^{\frac{n(q-p)}{4\beta_1}}_{L^\infty(B(x_0,\, \tau R))}\right)\frac{1}{(\tau-s)^{\frac{n}{\beta_1}}}. \tag{97}$$

Denote

$$\xi_1=\exp\left\{C_n \frac{n(1+\sqrt{\kappa}R)}{2\beta_1}\right\}(1+\kappa R^2)^{\frac{n}{2\beta_1}}\|F\|_{L^{\beta_1}\left(B\left(x_0,\, \frac{R}{2}\right)\right)}V^{-\frac{1}{\beta_1}}$$

and

$$\widetilde{\xi}_1=\xi_1\phi^{\frac{(q-p)n\beta}{4\beta_1}}.$$

From (97), we obtain

$$\|F\|_{L^\infty(B(x_0,\, sR))} \leq \widetilde{\mathscr{C}}\xi_1\frac{1}{(\tau-s)^{\frac{n}{\beta_1}}}+\widetilde{\mathscr{C}}\widetilde{\xi}_1\|F\|^{\frac{n(q-p)}{4\beta_1}}_{L^\infty(B(x_0,\, \tau R))}\frac{1}{(\tau-s)^{\frac{n}{\beta_1}}}. \tag{98}$$

Denote

$$\zeta_1 = \frac{n(q-p)}{4\beta_1} \text{ and } \zeta_2 = 1 - \zeta_1.$$

Applying Young's inequality and $p < q$ for the second and third integral on the right hand side of (98) we readily obtain

$$\widetilde{\mathscr{C}}\widetilde{\xi}_1 \|F\|_{L^\infty(B(x_0,\, \tau R))}^{\zeta_1} \frac{1}{(\tau - s)^{\frac{n}{\beta_1}}} \leq \frac{1}{2}\|F\|_{L^\infty(B(x_0,\, \tau R))} + \zeta_2 (2\zeta_1)^{\frac{\zeta_1}{\zeta_2}} \left(\widetilde{\mathscr{C}}\widetilde{\xi}_1\right)^{\frac{1}{\zeta_2}} \frac{1}{(\tau - s)^{\frac{n}{\beta_1 \zeta_2}}}. \tag{99}$$

Combining (98) and (99) together, we can see that there holds

$$\|F\|_{L^\infty(B(x_0,\, sR))} \leq \frac{1}{2}\|F\|_{L^\infty(B(x_0,\, \tau R))} + \widetilde{\mathscr{C}}\xi_1 \frac{1}{(\tau - s)^{\frac{n}{\beta_1}}} + \zeta_2 (2\zeta_1)^{\frac{\zeta_1}{\zeta_2}} \left(\widetilde{\mathscr{C}}\widetilde{\xi}_1\right)^{\frac{1}{\zeta_2}} \frac{1}{(\tau - s)^{\frac{n}{\beta_1 \zeta_2}}}. \tag{100}$$

Furthermore, by using (98), we can know that the above inequality is also true if $p = q$. Since

$$t_0 \in \mathfrak{B}_0 \text{ (see (67))},$$

we can know that $0 \leq \zeta_1 < \frac{1}{2}$. Hence, it is not difficult to see

$$\sup_{0 \leq \zeta_1 < \frac{1}{2}} \zeta_2 (2\zeta_1)^{\frac{\zeta_1}{\zeta_2}} < +\infty. \tag{101}$$

From (100) and (101), we obtain

$$\begin{aligned}
\|F\|_{L^\infty(B(x_0,\, sR))} &\leq \frac{1}{2}\|F\|_{L^\infty(B(x_0,\, \tau R))} + \overline{\mathscr{C}}\xi_1 \frac{1}{(\tau - s)^{\frac{n}{\beta_1}}} + \overline{\mathscr{C}}\widetilde{\xi}_1^{\frac{1}{\zeta_2}} \frac{1}{(\tau - s)^{\frac{n}{\beta_1 \zeta_2}}} \\
&\leq \frac{1}{2}\|F\|_{L^\infty(B(x_0,\, \tau R))} + \overline{\mathscr{C}}\left(\xi_1 + \widetilde{\xi}_1^{\frac{1}{\zeta_2}}\right) \frac{1}{(\tau - s)^{\frac{n}{\beta_1 \zeta_2}}},
\end{aligned}$$

where $\overline{\mathscr{C}} = \overline{\mathscr{C}}(n,\, p,\, q)$ is a positive constant. Setting $\psi_0(y) = \|F\|_{L^\infty(B(x_0,\, yR))}$, we obtain

$$\psi_0(s) \leq \frac{1}{2}\psi_0(\tau) + \overline{\mathscr{C}}\left(\xi_1 + \widetilde{\xi}_1^{\frac{1}{\zeta_2}}\right) \frac{1}{(\tau - s)^{\frac{n}{\beta_1 \zeta_2}}}.$$

Moreover, we have

$$\lim_{t_0 \to +\infty} \frac{n}{\beta_1 \zeta_2} = 0.$$

Then, from Lemma 3 we infer that there exists a positive constant $\mathscr{C}_1 = \mathscr{C}_1(n,\ p,\ q)$ such that for any $0 < s < \rho < \frac{1}{2}$, there holds true

$$\psi_0(s) \leq \mathscr{C}_1 \left(\xi_1 + \widetilde{\xi}_1^{\frac{1}{\zeta_2}} \right) \frac{1}{(\rho - s)^{\frac{n}{\beta_1 \zeta_2}}}.$$

By letting $\zeta_2 = \zeta$, from the above arguments we obtain the required estimate and finish the proof of Lemma 9. □

From Lemma 9 we conclude the following corollary:

Corollary 2 Let $1 < p \leq q$ and $(M,\ g)$ be an n-dim $(n \geq 3)$ complete manifold with $\mathrm{Ric} \geq -(n-1)\kappa$, where κ is a non-negative constant. Furthermore, $h \in C^1(\mathbb{R}^+)$ is a non-increasing function. Assume u is a positive solution to equation (1) on the geodesic ball $B(x_0,\ 2R) \subset M$. For any $\beta \in \mathfrak{A}_0$ (see (64)) and $t_0 \in \mathfrak{B}_0$ (see (67)), then there exist $\beta_1 = \frac{n}{n-2}\frac{p+2t_0}{2}$, $\zeta = 1 - \frac{n(q-p)}{4\beta_1}$ and a positive constant $\mathscr{C}_2 = \mathscr{C}_2(n,\ p,\ q)$, such that the following estimate holds true

$$\|F\|_{L^\infty\left(B\left(x_0,\ \frac{R}{8}\right)\right)} \leq \mathscr{C}_2 \left[\xi_1 + \left(\xi_1 \phi^{\frac{(q-p)n\beta}{4\beta_1}} \right)^{\frac{1}{\zeta}} \right], \quad \text{(see (65))}$$

where

$$\xi_1 = \exp\left\{ C_n \frac{n(1+\sqrt{\kappa}R)}{2\beta_1} \right\} (1+\kappa R^2)^{\frac{n}{2\beta_1}} \|F\|_{L^{\beta_1}\left(B\left(x_0,\ \frac{R}{2}\right)\right)} V^{-\frac{1}{\beta_1}}.$$

Proof. By Lemma 9, we can know that

$$\|F\|_{L^\infty(B(x_0,\ sR))} \leq \mathscr{C}_1 \left[\xi_1 + \left(\xi_1 \phi^{\frac{(q-p)n\beta}{4\beta_1}} \right)^{\frac{1}{\zeta}} \right] \frac{1}{(\rho - s)^{\frac{n}{\beta_1 \zeta}}}.$$

Setting $\rho = \frac{1}{4}$ and $s = \frac{1}{8}$, we can obtain that

$$\|F\|_{L^\infty\left(B\left(x_0,\ \frac{R}{8}\right)\right)} \leq \mathscr{C}_1 \left[\xi_1 + \left(\xi_1 \phi^{\frac{(q-p)n\beta}{4\beta_1}} \right)^{\frac{1}{\zeta}} \right] 8^{\frac{n}{\beta_1 \zeta}}. \tag{102}$$

Furthermore, we note that

$$\lim_{t_0 \to +\infty} 8^{\frac{n}{\beta_1 \zeta}} = 1. \tag{103}$$

By (102) and (103), we can obtain the required conclusion and complete the proof of Corollary 2. □

4. Proof of main theorem

Proof of Theorem 1

Proof. By Lemma 8, we can know that for any $\beta \in \mathfrak{A}_0(p,\ q)$ and $t_0 \in \mathfrak{B}_0(\beta,\ p,\ q)$, we have the following estimate

$$\|F\|_{L^{\beta_1}\left(B\left(x_0,\ \frac{R}{2}\right)\right)} \leq \|F\|_{L^{\beta_1}\left(B\left(x_0,\ \frac{3}{4}R\right)\right)} \leq \mathscr{C}_0 \exp\left\{\frac{2C_n(1+\sqrt{\kappa}R)}{2t_0+p}\right\} V^{\frac{1}{\beta_1}}$$

$$\cdot \left[(1+\kappa R^2)^{\frac{2}{p+2t_0}} \left(\kappa + \frac{1}{R^2}\right)^{\frac{2}{q-p+2}} \lambda_q^{-\frac{2}{q-p+2}} \phi^{\frac{4}{q-p+2}-\beta} + \kappa^{\frac{q+2t_0+2}{p+2t_0}} R^{\frac{4}{p+2t_0}} \lambda_q^{-\frac{q+2t_0}{p+2t_0}} \phi^{\frac{2q+4t_0}{p+2t_0}-\beta}\right.$$

$$\left. + t_0 R^{-\frac{2q+4t_0}{p+2t_0}} \lambda_q^{-\frac{q+2t_0}{p+2t_0}} \phi^{\frac{2q+4t_0}{p+2t_0}-\beta} + t_0^{\frac{2}{q-p+2}} R^{-\frac{4}{q-p+2}} \lambda_q^{-\frac{2}{q-p+2}} \phi^{\frac{4}{q-p+2}-\beta}\right], \tag{104}$$

where $F = \dfrac{|\nabla u|^2}{u^\beta}$, $\mathscr{C}_0 = \mathscr{C}_0(n,\ p,\ q) > 0$ and λ_q is defined in (42). By Corollary 2, we can know that there exist $\beta_1 = \dfrac{n}{n-2}\dfrac{p+2t_0}{2}$, $\zeta = 1 - \dfrac{n(q-p)}{4\beta_1}$ and a positive constant $\mathscr{C}_2 = \mathscr{C}_2(n,\ p,\ q)$, such that the following estimate holds true

$$\|F\|_{L^\infty\left(B\left(x_0,\ \frac{R}{8}\right)\right)} \leq \mathscr{C}_2 \left[\xi_1 + \left(\xi_1 \phi^{\frac{(q-p)n\beta}{4\beta_1}}\right)^{\frac{1}{\zeta}}\right], \tag{105}$$

where

$$\xi_1 = \exp\left\{C_n \frac{n(1+\sqrt{\kappa}R)}{2\beta_1}\right\} (1+\kappa R^2)^{\frac{n}{2\beta_1}} \|F\|_{L^{\beta_1}\left(B\left(x_0,\ \frac{R}{2}\right)\right)} V^{-\frac{1}{\beta_1}}. \tag{106}$$

Combining (104) and (106) together, we can know that

$$\xi_1 \leq \mathscr{C}_0 \exp\left\{C_n \frac{n(1+\sqrt{\kappa}R)}{2t_0+p}\right\} (1+\kappa R^2)^{\frac{n}{2\beta_1}}$$

$$\cdot \left[(1+\kappa R^2)^{\frac{2}{p+2t_0}} \left(\kappa + \frac{1}{R^2}\right)^{\frac{2}{q-p+2}} \lambda_q^{-\frac{2}{q-p+2}} \phi^{\frac{4}{q-p+2}-\beta} + \kappa^{\frac{q+2t_0+2}{p+2t_0}} R^{\frac{4}{p+2t_0}} \lambda_q^{-\frac{q+2t_0}{p+2t_0}} \phi^{\frac{2q+4t_0}{p+2t_0}-\beta}\right.$$

$$\left. + t_0 R^{-\frac{2q+4t_0}{p+2t_0}} \lambda_q^{-\frac{q+2t_0}{p+2t_0}} \phi^{\frac{2q+4t_0}{p+2t_0}-\beta} + t_0^{\frac{2}{q-p+2}} R^{-\frac{4}{q-p+2}} \lambda_q^{-\frac{2}{q-p+2}} \phi^{\frac{4}{q-p+2}-\beta}\right]. \tag{107}$$

Substituting (107) and $\frac{(q-p)n\beta}{4\beta_1} = (1-\zeta)\beta$ into (105), we obtain the required estimate and finish the proof of Theorem 1. □

Proof of Theorem 2

Proof. Since $(M,\ g)$ is a complete non-compact Riemannian manifold with non-negative Ricci curvature and $\dim M = n \geq 3$, we can achieve that for any $\beta \in \mathfrak{A}_0(p,\ q)$ and $t_0 \in \mathfrak{B}_0(\beta,\ p,\ q)$, then there exist $\beta_1 = \frac{n}{n-2}\frac{p+2t_0}{2}$, $\zeta = 1 - \frac{n(q-p)}{4\beta_1}$ and a positive constant $\mathscr{C}^* = \mathscr{C}^*(n,\ p,\ q)$, such that the following estimate holds true

$$\sup_{B\left(x_0,\ \frac{R}{8}\right)} \frac{|\nabla u|^2}{u^\beta} \leq \mathscr{C}^* \left[\overline{\xi}_* + \left(\overline{\xi}_* \phi^{(1-\zeta)\beta}\right)^{\frac{1}{\zeta}}\right], \tag{108}$$

where

$$\overline{\xi}_* = \exp\left\{\frac{C_n n}{2t_0 + p}\right\} \left[\left(1 + t_0^{\frac{2}{q-p+2}}\right) \lambda_q^{-\frac{2}{q-p+2}} R^{-\frac{4}{q-p+2}} \phi^{\frac{4}{q-p+2} - \beta} + t_0 \lambda_q^{-\frac{q+2t_0}{p+2t_0}} R^{-\frac{2q+4t_0}{p+2t_0}} \phi^{\frac{2q+4t_0}{p+2t_0} - \beta}\right]. \tag{109}$$

Multiplying $\sup_{B(x_0,\ R/8)} u$ on both sides of (108), we obtain

$$\sup_{B\left(x_0,\ \frac{R}{8}\right)} \frac{|\nabla u|^2}{u^\beta} \sup_{B\left(x_0,\ \frac{R}{8}\right)} u^\beta \leq \mathscr{C}^* \left[\overline{\xi}_* + \left(\overline{\xi}_* \phi^{(1-\zeta)\beta}\right)^{\frac{1}{\zeta}}\right] \sup_{B\left(x_0,\ \frac{R}{8}\right)} u^\beta. \tag{110}$$

On the other hand, we have

$$\sup_{B\left(x_0,\ \frac{R}{8}\right)} |\nabla u|^2 = \sup_{B\left(x_0,\ \frac{R}{8}\right)} \left(\frac{|\nabla u|^2}{u^\beta} \cdot u^\beta\right) \leq \sup_{B\left(x_0,\ \frac{R}{8}\right)} \frac{|\nabla u|^2}{u^\beta} \sup_{B\left(x_0,\ \frac{R}{8}\right)} u^\beta \tag{111}$$

and

$$\sup_{B\left(x_0,\ \frac{R}{8}\right)} u^\beta \leq \sup_{B(x_0,\ R)} u^\beta = \phi^\beta. \tag{112}$$

Substituting (111) and (112) into (110), we can achieve that

$$\sup_{B\left(x_0,\ \frac{R}{8}\right)} |\nabla u|^2 \leq \mathscr{C}^* \left[\overline{\xi}_* + \left(\overline{\xi}_* \phi^{(1-\zeta)\beta}\right)^{\frac{1}{\zeta}}\right] \phi^\beta. \tag{113}$$

Set

$$\xi_1^* = \overline{\xi}_* \phi^\beta \ \text{ and } \ \xi_2^* = \left(\overline{\xi}_* \phi^{(1-\zeta)\beta}\right)^{\frac{1}{\zeta}} \phi^\beta. \tag{114}$$

Substituting (114) into (113), we obtain

$$\sup_{B\left(x_0,\ \frac{R}{8}\right)} |\nabla u|^2 \leq \mathscr{C}^* \left(\xi_1^* + \xi_2^*\right). \tag{115}$$

Combining (109) and (114) together, we obtain

$$\xi_1^* = \exp\left\{\frac{C_n n}{2t_0 + p}\right\} \left[\left(1 + t_0^{\frac{2}{q-p+2}}\right) \lambda_q^{-\frac{2}{q-p+2}} \left(\frac{\phi}{R}\right)^{\frac{4}{q-p+2}} + t_0 \lambda_q^{-\frac{q+2t_0}{p+2t_0}} \left(\frac{\phi}{R}\right)^{\frac{2q+4t_0}{p+2t_0}}\right] \tag{116}$$

and

$$\xi_2^* = \left(\overline{\xi}_* \phi^{(1-\zeta)\beta}\right)^{\frac{1}{\zeta}} \phi^{\beta} = \left(\overline{\xi}_* \phi^{(1-\zeta)\beta} \phi^{\beta\zeta}\right)^{\frac{1}{\zeta}} = \left(\overline{\xi}_* \phi^{\beta}\right)^{\frac{1}{\zeta}} = \left(\xi_1^*\right)^{\frac{1}{\zeta}}. \tag{117}$$

Since

$$\limsup_{M \ni x \to \infty} \frac{u(x)}{\mathrm{d}(x_0,\ x)} \leq \omega, \tag{118}$$

we can know that for any $\varepsilon_1 > 0$, there exists a positive constant G_0, such that for any $R = d(x_0,\ x) > G_0$, the following estimate holds true

$$\frac{\phi}{R} = \frac{\sup_{B(x_0,\ R)} u(x)}{R} = \sup_{B(x_0,\ R)} \frac{u(x)}{R} \leq \omega + \varepsilon_1. \tag{119}$$

Then we need to consider the following two cases:

$$(i)\ \omega < 1; \quad (ii)\ \omega \geq 1.$$

(i) Let $0 < \varepsilon_1 < \dfrac{1-\omega}{2}$, we can know that $\omega + \varepsilon_1 < 1$. Combining (116) and (119) together, we can know that for any $R > G_0$, the following estimate holds true

$$\xi_1^* \leq \exp\left\{\frac{C_n n}{2t_0 + p}\right\} \left[\left(1 + t_0^{\frac{2}{q-p+2}}\right) \lambda_q^{-\frac{2}{q-p+2}} (\omega + \varepsilon_1)^{\frac{4}{q-p+2}} + t_0 \lambda_q^{-\frac{q+2t_0}{p+2t_0}} (\omega + \varepsilon_1)^{\frac{2q+4t_0}{p+2t_0}}\right]. \tag{120}$$

Denote

$$\gamma(\omega,\ p,\ q,\ t_0,\ \beta) = \min\left\{\frac{4}{q-p+2},\ \frac{4t_0+2q}{2t_0+p}\right\} = \frac{4}{q-p+2} \tag{121}$$

and

$$\widetilde{\gamma}(\omega,\ p,\ q,\ \varepsilon_0) = \frac{4}{q-p+2} - \varepsilon_0.$$

For any $\gamma_0 \in \{\widetilde{\gamma}(\omega,\ p,\ q,\ \varepsilon_0) > 0\ :\ \varepsilon_0 > 0\}$, we can find $\beta = \beta^* \in \mathfrak{A}_0(p,\ q)$, $\varepsilon_0 = \varepsilon_0^* > 0$ and $t_0 = t_0^* \in \mathfrak{B}_0(\beta^*,\ p,\ q)$, such that

$$\gamma_0 = \gamma(\omega,\ p,\ q,\ t_0^*,\ \beta^*) - \varepsilon_0^* > 0. \tag{122}$$

Combining (121), (122) and (120), we can know that for any $R > G_0$, the following estimate holds true

$$\xi_1^* \leq \exp\left\{\frac{C_n n}{2t_0^* + p}\right\}\left[\left(1 + t_0^{*\,\frac{2}{q-p+2}}\right)\lambda_q^{-\frac{2}{q-p+2}}(\omega+\varepsilon_1)^{\gamma_0} + t_0^*\lambda_q^{-\frac{q+2t_0^*}{p+2t_0^*}}(\omega+\varepsilon_1)^{\gamma_0}\right]. \tag{123}$$

Combining (115), (117) and (123), we can know that for any $R > G_0$, the following estimate holds true

$$\sup_{B\left(x_0,\ \frac{R}{8}\right)} |\nabla u|^2 \leq \widetilde{\mathscr{C}}_1^*\,(\omega+\varepsilon_1)^{\gamma_0},$$

where $\widetilde{\mathscr{C}}_1^* = \widetilde{\mathscr{C}}_1^*(n,\ p,\ q,\ \gamma_0)$ is a positive constant. Letting $R \longrightarrow +\infty$, we can achieve the following estimate

$$\sup_M |\nabla u|^2 \leq \widetilde{\mathscr{C}}_1^*\,\omega^{\gamma_0}.$$

Hence, we complete the proof of case (i).

(ii) Since its proof is similar to (i) and we omit it.

From the above arguments we obtain the required estimate and finish the proof of Theorem 2. □

Proof of Theorem 3

Proof. By Theorem 2, we can know that

$$\sup_M |\nabla u|^2 \leq 0.$$

Therefore, we can know that u is a positive constant on M. Hence, if there exist a positive constant a such that $h(a) = 0$, then u is a trivial solution. Otherwise, u does not exist. □

Proof of Theorem 4

Proof. Since $(M,\ g)$ is a complete non-compact Riemannian manifold with non-negative Ricci curvature and $\dim M = n \geq 3$, we can achieve that for any $\beta \in \mathfrak{A}_0(p,\ q)$ and $t_0 \in \mathfrak{B}_0(\beta,\ p,\ q)$, then there exist $\beta_1 = \dfrac{n}{n-2}\dfrac{p+2t_0}{2}$, $\zeta = 1 - \dfrac{n(q-p)}{4\beta_1}$ and a positive constant $\mathscr{C}^* = \mathscr{C}^*(n,\ p,\ q)$, such that the following estimate holds true

$$\sup_{B\left(x_0,\ \frac{R}{8}\right)} \frac{|\nabla u|^2}{u^\beta} \leq \mathscr{C}^* \left[\overline{\xi}_* + \left(\overline{\xi}_* \phi^{(1-\zeta)\beta}\right)^{\frac{1}{\zeta}}\right], \tag{124}$$

where

$$\overline{\xi}_* = \exp\left\{\frac{C_n n}{2t_0+p}\right\} \left[\left(1+t_0^{\frac{2}{q-p+2}}\right)\lambda_q^{-\frac{2}{q-p+2}} R^{-\frac{4}{q-p+2}} \phi^{\frac{4}{q-p+2}-\beta} + t_0 \lambda_q^{-\frac{q+2t_0}{p+2t_0}} R^{-\frac{2q+4t_0}{p+2t_0}} \phi^{\frac{2q+4t_0}{p+2t_0}-\beta}\right]. \tag{125}$$

By (125), we can know that

$$\overline{\xi}_* \phi^{(1-\zeta)\beta} = \exp\left\{\frac{C_n n}{2t_0+p}\right\} \left[\left(1+t_0^{\frac{2}{q-p+2}}\right)\lambda_q^{-\frac{2}{q-p+2}} R^{-\frac{4}{q-p+2}} \phi^{\frac{4}{q-p+2}-\zeta\beta} + t_0 \lambda_q^{-\frac{q+2t_0}{p+2t_0}} R^{-\frac{2q+4t_0}{p+2t_0}} \phi^{\frac{2q+4t_0}{p+2t_0}-\zeta\beta}\right]. \tag{126}$$

Denote

$$o_1 = \frac{4}{4-\beta(q-p+2)},\ \ o_2 = \frac{4t_0+2q}{4t_0+2q-\beta(2t_0+p)},$$

$$o_3 = \frac{4}{4-\zeta\beta(q-p+2)},\ \ o_4 = \frac{4t_0+2q}{4t_0+2q-\zeta\beta(2t_0+p)}.$$

Hence

$$R^{-\frac{4}{q-p+2}} \phi^{\frac{4}{q-p+2}-\beta} = \left(\frac{\phi}{R^{o_1}}\right)^{\frac{4}{q-p+2}-\beta},$$

$$R^{-\frac{2q+4t_0}{p+2t_0}} \phi^{\frac{2q+4t_0}{p+2t_0}-\beta} = \left(\frac{\phi}{R^{o_2}}\right)^{\frac{2q+4t_0}{p+2t_0}-\beta},$$

$$R^{-\frac{4}{q-p+2}} \phi^{\frac{4}{q-p+2}-\zeta\beta} = \left(\frac{\phi}{R^{o_3}}\right)^{\frac{4}{q-p+2}-\zeta\beta},$$

$$R^{-\frac{2q+4t_0}{p+2t_0}} \phi^{\frac{2q+4t_0}{p+2t_0}-\zeta\beta} = \left(\frac{\phi}{R^{o_4}}\right)^{\frac{2q+4t_0}{p+2t_0}-\zeta\beta}. \tag{127}$$

Furthermore, set

$$\Theta(p,\ q,\ t_0,\ \beta) = \min\{o_1,\ o_2,\ o_3,\ o_4\} = o_4$$

and

$$\widetilde{\Theta}(p,\ q) = \sup_{\beta\in\mathfrak{A}_0(p,\ q),\ t_0\in\mathfrak{B}_0(\beta,\ p,\ q)} \Theta(p,\ q,\ t_0,\ \beta).$$

Hence, we can know that

$$\widetilde{\Theta}(p,\ q) = \sup_{\beta\in\mathfrak{A}_0(p,\ q),\ t_0\in\mathfrak{B}_0(\beta,\ p,\ q)} \frac{4t_0+2q}{4t_0+2q-\zeta\beta(2t_0+p)} = \sup_{\beta\in\mathfrak{A}_0(p,\ q)} \frac{2}{2-\beta}.$$

Since $\Theta_0 \in \left(0,\ \widetilde{\Theta}(p,\ q)\right)$, we can know that there exists $\beta = \beta^* \in \mathfrak{A}_0(p,\ q)$ and $t_0 = t_0^* \in \mathfrak{B}_0(\beta^*,\ p,\ q)$, such that

$$\Theta_0 < \Theta(p,\ q,\ t_0^*,\ \beta^*). \tag{128}$$

Since

$$\lim_{M\ni x\to\infty} \frac{u(x)}{(\mathrm{d}(x_0,\ x))^{\Theta_0}} = 0, \tag{129}$$

we can know that for any $0 < \varepsilon_1 < 1$, there exists a positive constant $G_0 > 1$, such that for any $R = d(x_0,\ x) > G_0$, the following estimate holds true

$$\frac{\phi}{R^{\Theta_0}} = \frac{\sup\limits_{B(x_0,\ R)} u(x)}{R^{\Theta_0}} = \sup_{B(x_0,\ R)} \frac{u(x)}{R^{\Theta_0}} \le \varepsilon_1. \tag{130}$$

Combining (125), (126), (127), (128) and (130), we can know that for any $R = d(x_0,\ x) > G_0$, the following two estimates hold true

$$\begin{aligned}
\overline{\xi}_* &\le \exp\left\{\frac{C_n n}{2t_0^*+p}\right\}\left[\left(1+t_0^{*\frac{2}{q-p+2}}\right)\lambda_q^{-\frac{2}{q-p+2}}\left(\frac{\phi}{R^{\Theta_0}}\right)^{\frac{4}{q-p+2}-\beta^*} + t_0^*\lambda_q^{-\frac{q+2t_0^*}{p+2t_0^*}}\left(\frac{\phi}{R^{\Theta_0}}\right)^{\frac{2q+4t_0^*}{p+2t_0^*}-\beta^*}\right] \\
&\le \exp\left\{\frac{C_n n}{2t_0^*+p}\right\}\left[\left(1+t_0^{*\frac{2}{q-p+2}}\right)\lambda_q^{-\frac{2}{q-p+2}}\varepsilon_1^{\frac{4}{q-p+2}-\beta^*} + t_0^*\lambda_q^{-\frac{q+2t_0^*}{p+2t_0^*}}\varepsilon_1^{\frac{2q+4t_0^*}{p+2t_0^*}-\beta^*}\right]
\end{aligned} \tag{131}$$

and

$$
\begin{aligned}
\overline{\xi}_* \phi^{(1-\zeta)\beta} &\leq \exp\left\{\frac{C_n n}{2t_0^* + p}\right\} \left[\left(1 + t_0^{*\frac{2}{q-p+2}}\right) \lambda_q^{-\frac{2}{q-p+2}} \left(\frac{\phi}{R^{\Theta_0}}\right)^{\frac{4}{q-p+2} - \zeta\beta^*} + t_0^* \lambda_q^{-\frac{q+2t_0^*}{p+2t_0^*}} \left(\frac{\phi}{R^{\Theta_0}}\right)^{\frac{2q+4t_0^*}{p+2t_0^*} - \zeta\beta^*} \right] \\
&\leq \exp\left\{\frac{C_n n}{2t_0^* + p}\right\} \left[\left(1 + t_0^{*\frac{2}{q-p+2}}\right) \lambda_q^{-\frac{2}{q-p+2}} \varepsilon_1^{\frac{4}{q-p+2} - \zeta\beta^*} + t_0^* \lambda_q^{-\frac{q+2t_0^*}{p+2t_0^*}} \varepsilon_1^{\frac{2q+4t_0^*}{p+2t_0^*} - \zeta\beta^*} \right].
\end{aligned}
\tag{132}
$$

Combining (124), (131), (132) and letting $R \longrightarrow +\infty$, we obtain

$$
\sup_M \frac{|\nabla u|^2}{u^{\beta^*}} \leq 0.
$$

Therefore, we complete the proof of the Theorem 4. □

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Conflict of interest

Authors declare that they have no conflict of interest.

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