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Research Article

Energy Storage System Selection for AI-Controlled Microgrids Using Complex Hesitant Fuzzy MCDM Approach Based on Dombi Operators

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Abstract: The current definition of the Complex Hesitant Fuzzy Set (CHFS), derived from the Ramot form of complex numbers, cannot process information as in Tamir's complex fuzzy form. We have data with uncertainty and extra information that cannot be described by any other structure than Tamir's complex fuzzy form. Hence, in this article, we initiated the idea of CHFS based on Tamir's complex fuzzy form and established its operational laws. Since Decision-Making (DM) theory is central to nearly all disciplines, we have proposed a novel complex hesitant fuzzy Multi-Criterion Decision-Making (MCDM) model. This method can handle all sorts of real-life MCDM problems, where the data contains uncertainty, hesitancy, and extra fuzzy information. While developing this method, we also develop and apply Dombi aggregation operators in this manuscript. After that, we illustrate a case study that concerns energy storage system selection for AI-controlled microgrids and discuss how the theory we have developed can be applied to real-world challenges. Last, we confer on how this proposed theory is superior to other theories and why it should be adopted.

***Keywords*:** energy storage system, artificial intelligence, MCDM approach, complex hesitant fuzzy set

MSC: 90B50, 68U35

Abbreviation

FS	Fuzzy Set
HFS	Hesitant Fuzzy Set
CFS	Complex Fuzzy Set
CHFS	Complex Hesitant Fuzzy Set
CHFDWA AOs	Complex Hesitant Fuzzy Dombi Weighted Average Aggregation Operators
CHFDWG AOs	Complex Hesitant Fuzzy Dombi Weighted Geometric Aggregation Operators

1. Introduction

The Fuzzy Set (FS) theory was established by Zadeh [1] in 1965. In FS every element from the universal set U ranging from the unit interval [0, 1], FS only deals with Membership Grade (MG). FS has generalized the crisp set and provided

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a useful mathematical framework to decision-makers for handling ambiguity and uncertainty in real-life issues. After the development of FS theory, many researchers utilized them in different fields of life [2, 3]. FS plays a very important role in the field of Decision-Making (DM) procedures [4, 5]. Tanaka and Sugeno [6] made significant contributions to FS theory, particularly in the development of fuzzy logic systems and fuzzy inference systems, which have found wide applications in control systems and DM processes. Fuzzy control theory [7] makes use of fuzzy variables and fuzzy logic to create control systems that can deal with imprecise or uncertain input, providing flexibility and robustness while operating complex and nonlinear systems. FS has received a lot of consideration over the last few eras [8–10]. Also, the fuzzy cooperative approach was recently devised by Chen et al. [11] to evaluate the applicability of smart health. Hesitancy is seen in many real-life problems. Torra [12] developed Hesitant Fuzzy Sets (HFSs) to deal with such situations. HFSs are a powerful variant of FS that enables several positive ratings to be associated with any preference data for handling a challenging and complex situation. Using the benefits of HFS, Tan et al. [13] established the notion of HF Hamacher AOs for multi-criteria decision-making. He [14] discussed Typhoon disaster assessment based on Dombi HF information AOs. HF Dombi Archimedean AO [15] and HF Maclaurin symmetric mean AO [16] have been developed to discuss the analysis of HF in aggregation theory. Moreover, HF information AOs have been delivered by Xia and Xu [17]. Xia et al. [18] delivered the notion of some HF AOs and established their applications to group decision-making. Wei et al. [19] introduced the idea of HF coquet integral AOs and their applications to MCDM. HF power AOs have been delivered by Zhang [20]. Li et al. [21] discussed the consistency of HF linguistic preference relations. Wu et al. [22] discussed the connection of the numerical scale model with assessing attitudes and proposed its application to MADM. Sahoo et al. [23] initiated a review of the MCDM application to solve energy management problems. Similarly, Sahoo et al. [24] proposed a systematic review of the MCDM approach towards sustainable renewable energy developments. Complex-valued positive grades, which carry two-dimensional data in a specific set, are a representation of the concept of a Complex Fuzzy Set (CFS). The polar form of CFS was initially introduced by Ramot et al. [25]. Rehman [26] utilized the notion of CFS and proposed the probability AOs for the selection of a database management system. In the Polar form of CFS, MG belongs to the unit circle in a complex plan and it is a limited notion pointed out by Tamir et al. [27]. Tamir et al. [27] introduced the CF membership grade and MG belongs to the unit square in the complex plan rather than that of the unit circle in the complex plan. The Tamir's form of CFS is more advanced because it can handle more advanced data and, in this case, the chance of data loss can be reduced. Numerous investigators employed CFS across various domains. Yazdanbakhsh and Dick [28] established the systematic review using CFS and reasoning. The Complex Hesitant Fuzzy Set (CHFS) is a combination of the HFS and CFS. In CHFS theory, degrees of membership take on complex values and are represented in polar coordinates [29]. CHFS plays a significant role in DM problems [30] due to its capability to handle complex and uncertain information. When handling complicated and challenging data in real-world decision-making, CHFS is more potent than other theories like FS, CFS, and HFS. CHFS provides a flexible framework to represent these uncertainties, allowing DM to express their preferences more accurately. The polar form of CHFS deals with unit circles only to make it more accurate and easier to understand. We discuss the CHFS based on Tamir's CFS. Instead of using the old polar method, we now use CHFS using unit squares in the complex plan, which makes it simpler to draw and calculate. In this article:

(1) We introduce different AOs using the framework of CHFSs.

(2) We introduce here complex HFDWA (CHFDWA), complex HFDOWA (CHFDOWA) and complex HFDHWA (CHFDHWA) AOs. Similarly, we introduce CHFD geometric, complex HFDWG (CHFDWG), complex HFDOWG (CHFDOWG), and complex HFDHWG (CHFDHWG) AOs.

(3) To discuss the application of the proposed theory, we have defined an algorithm and utilized the initiated theory.

(4) The comparative analysis of the developed approach has been discussed to show the advancement of the introduced work.

The rest of the article is organized as follows; in section 2 we have discussed the fundamental notions. Section 2 is about the basic Dombi operational laws for CHHFNs. Section 4 is about the aggregation theory in which we have developed the notion of CHFDWA, CHFDWG, CHFDOWA, CHFDOWG, CHFDHA, and CHFDHG AOs. In section 5 we have discussed the algorithm and application of the introduced approach. Section 6 is about the comparative analysis of the delivered approach. In section 7, we have discussed the conclusion remarks.

2. Preliminaries

This section of the article is devoted to discussing some important notions and their operational laws that can help us define the fundamental idea of the proposed theory.

Definition 1 [12] A Hesitant Fuzzy Set (HFS) $\hat{H}$ is of the form

$$\hat{H} = \{(i, N_{\hat{H}_1}(i)) \mid i \in U\}$$

where $N_{\hat{H}_1}(i)$ is the set of different finite values in $[0, 1]$ representing the Membership Grades (MG) for each element $i \in U$. Further, the notation $\hat{H} = \{N_{\hat{H}}\}$ is called a Hesitant Fuzzy Number (HFN).

Definition 2 [12] Suppose $\hat{H}_1 = \{N_{\hat{H}_1}\}$ and $\hat{H}_2 = \{N_{\hat{H}_2}\}$ be two HFNs. The fundamental functions are defined as:

$$\hat{H}_1^C = \bigcup_{\hat{C}_1 \in \hat{H}_1} \{1 - \hat{C}_1\},$$

$$\hat{H}_1 \cup \hat{H}_2 = \bigcup_{\hat{C}_1 \in \hat{H}_1,\, \hat{C}_2 \in \hat{H}_2} \{\max(\hat{C}_1, \hat{C}_2)\},$$

$$\hat{H}_1 \cap \hat{H}_2 = \bigcup_{\hat{C}_1 \in \hat{H}_1,\, \hat{C}_2 \in \hat{H}_2} \{\min(\hat{C}_1, \hat{C}_2)\}.$$

Definition 3 [12] For any given three HFNs $\hat{H} = \{N\}$, $\hat{H}_1 = \{N_1\}$, $\hat{H}_2 = \{N_2\}$ and $\delta > 0$, the operational rules are given by:

$$\hat{H}_1 \oplus \hat{H}_2 = \bigcup_{\hat{C}_1 \in \hat{H}_1,\, \hat{C}_2 \in \hat{H}_2} \{\hat{C}_1 + \hat{C}_2 - \hat{C}_1\hat{C}_2\},$$

$$\hat{H}_1 \otimes \hat{H}_2 = \bigcup_{\hat{C}_1 \in \hat{H}_1,\, \hat{C}_2 \in \hat{H}_2} \{\hat{C}_1\hat{C}_2\},$$

$$\delta\hat{H} = \bigcup_{\hat{C} \in \hat{H}} \{1 - (1 - \hat{C})^{\delta}\},$$

$$\hat{H}^{\delta} = \bigcup_{\hat{C} \in \hat{H}} \{\hat{C}^{\delta}\}.$$

The idea of Tamir's Complex Fuzzy Set (CFS) is a more generalized form compared to Ramot's CFS, due to using the unit square in a complex plane for membership degrees. The notion of CFS defined by Tamir et al. [27] is given by:

Definition 4 [27] A CFS $\hat{H}$ is of the form

$$\hat{H} = \{(i, N_{\hat{H}}(i)) \mid i \in U\},$$

where $N_{\hat{H}}(i) = \hat{C}_{\hat{H}}(i) + i\tilde{T}_{\hat{H}}(i)$ and $\hat{H} = \{N_{\hat{H}} = \hat{C}_{\hat{H}} + i\tilde{T}_{\hat{H}}\} = \{\hat{C}_{\hat{H}} + i\tilde{T}_{\hat{H}}\}$ represents the Complex Fuzzy Number (CFN), where $i = \sqrt{-1}$ and $\hat{C}_{\hat{H}},\ \tilde{T}_{\hat{H}} \in [0,\ 1]$.

Definition 5 [27] Let $\hat{H}_1 = \{\hat{C}_{\hat{H}_1} + i\tilde{T}_{\hat{H}_1}\}$ and $\hat{H}_2 = \{\hat{C}_{\hat{H}_2} + i\tilde{T}_{\hat{H}_2}\}$ be two CFNs. Then their basic operations are defined as:

$$\hat{H}_1^C = \{1 - \hat{C}_{\hat{H}_1} + i(1 - \tilde{T}_{\hat{H}_1})\},$$

$$\hat{H}_1 \cup \hat{H}_2 = \{\max(\hat{C}_{\hat{H}_1},\ \hat{C}_{\hat{H}_2}) + i\max(\tilde{T}_{\hat{H}_1},\ \tilde{T}_{\hat{H}_2})\},$$

$$\hat{H}_1 \cap \hat{H}_2 = \{\min(\hat{C}_{\hat{H}_1},\ \hat{C}_{\hat{H}_2}) + i\min(\tilde{T}_{\hat{H}_1},\ \tilde{T}_{\hat{H}_2})\}.$$

Definition 6 [31] Let δ be a parameter greater than 0 and $\varsigma,\ \tau \in [0,\ 1]$. The Dombi t-norm and t-conorm are defined as follows:

$$T(\varsigma,\ \tau) = \frac{1}{1 + \left[\left(\frac{1-\varsigma}{\varsigma}\right)^{\delta} + \left(\frac{1-\tau}{\tau}\right)^{\delta}\right]^{1/\delta}},$$

$$T^*(\varsigma,\ \tau) = 1 - \frac{1}{1 + \left[\left(\frac{\varsigma}{1-\varsigma}\right)^{\delta} + \left(\frac{\tau}{1-\tau}\right)^{\delta}\right]^{1/\delta}}.$$

3. Dombi operational rules based on complex hesitant fuzzy sets

The idea of CHFS has been proposed by Albaity et al. [30]. In this section of the article, we have discussed the basic operational laws for the CHFSs based on Dombi t-norm and t-conorm. The overall discussion is given by

Definition 7 [30] A CHFS $\hat{H}$ is of the form

$$\hat{H} = \{(i,\ N_{\hat{H}_1}(i)) \mid i \in \mathscr{U}\},\ \text{where} N_{\hat{H}}(i) = \{\mathscr{C}_{\hat{H}}(i) + i\mathscr{T}_{\hat{H}}(i)\}$$

where $N_{\hat{H}}(i)$ is the set of distinct finite complex values and $\mathscr{C}_{\hat{H}}(i),\ \mathscr{T}_{\hat{H}}(i) \in [0,\ 1]$. Note that

$$\hat{H} = \{N_{\hat{H}} = \mathscr{C}_{\hat{H}} + i\mathscr{T}_{\hat{H}}\} = \{\mathscr{C}_{\hat{H}} + i\mathscr{T}_{\hat{H}}\}$$

represents the CHFN, where $i = \sqrt{-1}$.

Definition 8 Let

$$\hat{H}_1 = \{\mathscr{C}_{\hat{H}_1} + i\mathscr{T}_{\hat{H}_1}\},\ \hat{H}_2 = \{\mathscr{C}_{\hat{H}_2} + i\mathscr{T}_{\hat{H}_2}\}$$

be two CHFNs. Then, their fundamental functions are described as:

$$\hat{H}_1^C = \bigcup_{\mathscr{C}_i,\ \mathscr{T}_i \in \hat{H}_1} \{1 - \mathscr{C}_i + i(1 - \mathscr{T}_i)\},$$

$$\hat{H}_1 \cup \hat{H}_2 = \bigcup_{\substack{\mathscr{C}_i,\ \mathscr{T}_i \in \hat{H}_1 \\ \mathscr{C}_j,\ \mathscr{T}_j \in \hat{H}_2}} \{\max\{\mathscr{C}_i,\ \mathscr{C}_j\} + i\max\{\mathscr{T}_i,\ \mathscr{T}_j\}\},$$

$$\hat{H}_1 \cap \hat{H}_2 = \bigcup_{\substack{\mathscr{C}_i,\ \mathscr{T}_i \in \hat{H}_1 \\ \mathscr{C}_j,\ \mathscr{T}_j \in \hat{H}_2}} \{\min\{\mathscr{C}_i,\ \mathscr{C}_j\} + i\min\{\mathscr{T}_i,\ \mathscr{T}_j\}\}.$$

Definition 9 The operations for any three CHFNs $\hat{H} = (\mathscr{C} + i\mathscr{T})$, $\hat{H}_1 = (\mathscr{C}_1 + i\mathscr{T}_1)$, $\hat{H}_2 = (\mathscr{C}_2 + i\mathscr{T}_2)$ and $\delta > 0$, are defined as follows:

$$\hat{H}_1 \oplus \hat{H}_2 = \bigcup_{\substack{\mathscr{C}_1,\ \mathscr{T}_1 \in \hat{H}_1 \\ \mathscr{C}_2,\ \mathscr{T}_2 \in \hat{H}_2}} \{(\mathscr{C}_1 + \mathscr{C}_2 - \mathscr{C}_1\mathscr{C}_2) + i(\mathscr{T}_1 + \mathscr{T}_2 - \mathscr{T}_1\mathscr{T}_2)\},$$

$$\hat{H}_1 \otimes \hat{H}_2 = \bigcup_{\substack{\mathscr{C}_1,\ \mathscr{T}_1 \in \hat{H}_1 \\ \mathscr{C}_2,\ \mathscr{T}_2 \in \hat{H}_2}} \{(\mathscr{C}_1\mathscr{C}_2) + i(\mathscr{T}_1\mathscr{T}_2)\},$$

$$\delta\hat{H} = \bigcup_{\mathscr{C},\ \mathscr{T} \in \hat{H}} \left\{(1 - (1 - \mathscr{C})^{\delta}) + i(1 - (1 - \mathscr{T})^{\delta})\right\},$$

$$\hat{H}^{\delta} = \bigcup_{\mathscr{C},\ \mathscr{T} \in \hat{H}} \left\{\mathscr{C}^{\delta} + i\mathscr{T}^{\delta}\right\}.$$

3.1 *Characteristic of Dombi aggregation operators*

Aggregation operators are the mathematical framework that can convert a huge amount of information into a single value that can further help to make a decision easily. Based on the Dombi t-norm and t-conorm, Dombi AOs provide flexible parameterized control over the aggregation process. Because these operators are smooth and can represent varying levels of strictness in aggregation, they are frequently utilized in fuzzy logic, decision-making, and multi-criteria analysis. The following are the main characteristics of Dombi AOs:

(1) Dombi AOs utilize the tunable parameter that can help to see the behaviors of the aggregation. For the different values of the parameter, we can analyze the results and decide the effects of the parameter on the results.

(2) Unlike the classical AOs, Dombi AOs provide a smooth transition between extreme cases.

(3) These AOs are more flexible and reliable. Accuracy in the results can be obtained in many decision-making problems by the utilization of Dombi AOs.

Definition 10 For any given three CHFNs $\hat{H} = (\mathscr{C} + i\mathscr{T})$, $\hat{H}_1 = (\mathscr{C}_1 + i\mathscr{T}_1)$, $\hat{H}_2 = (\mathscr{C}_2 + i\mathscr{T}_2)$, and $\delta > 0$, the Dombi operations are described below:

$$\hat{H}_1 \oplus \hat{H}_2 = \bigcup_{\substack{\mathscr{C}_1,\ \mathscr{T}_1 \in \hat{H}_1 \\ \mathscr{C}_2,\ \mathscr{T}_2 \in \hat{H}_2}} \left\{ \begin{array}{c} \left(1 - \dfrac{1}{1 + \left(\left(\dfrac{\mathscr{C}_1}{1-\mathscr{C}_1}\right)^{\delta} + \left(\dfrac{\mathscr{C}_2}{1-\mathscr{C}_2}\right)^{\delta}\right)^{1/\delta}}\right) + \\ i\left(1 - \dfrac{1}{1 + \left(\left(\dfrac{\mathscr{T}_1}{1-\mathscr{T}_1}\right)^{\delta} + \left(\dfrac{\mathscr{T}_2}{1-\mathscr{T}_2}\right)^{\delta}\right)^{1/\delta}}\right) \end{array} \right\},$$

$$\hat{H}_1 \otimes \hat{H}_2 = \bigcup_{\substack{\mathscr{C}_1,\ \mathscr{T}_1 \in \hat{H}_1 \\ \mathscr{C}_2,\ \mathscr{T}_2 \in \hat{H}_2}} \left\{ \left(\frac{1}{1 + \left(\left(\frac{1-\mathscr{C}_1}{\mathscr{C}_1}\right)^{\delta} + \left(\frac{1-\mathscr{C}_2}{\mathscr{C}_2}\right)^{\delta}\right)^{1/\delta}} \right) + i\left(\frac{1}{1 + \left(\left(\frac{1-\mathscr{T}_1}{\mathscr{T}_1}\right)^{\delta} + \left(\frac{1-\mathscr{T}_2}{\mathscr{T}_2}\right)^{\delta}\right)^{1/\delta}} \right) \right\},$$

$$\theta\hat{H} = \bigcup_{\mathscr{C},\ \mathscr{T} \in \hat{H}} \left\{ \left(1 - \frac{1}{1 + \left(\theta\left(\frac{\mathscr{C}}{1-\mathscr{C}}\right)^{\delta}\right)^{1/\delta}}\right) + i\left(1 - \frac{1}{1 + \left(\theta\left(\frac{\mathscr{T}}{1-\mathscr{T}}\right)^{\delta}\right)^{1/\delta}}\right) \right\},$$

$$\hat{H}^{\theta} = \bigcup_{\mathscr{C},\ \mathscr{T} \in \hat{H}} \left\{ \left(\frac{1}{1 + \left(\theta\left(\frac{1-\mathscr{C}}{\mathscr{C}}\right)^{\delta}\right)^{1/\delta}}\right) + i\left(\frac{1}{1 + \left(\theta\left(\frac{1-\mathscr{T}}{\mathscr{T}}\right)^{\delta}\right)^{1/\delta}}\right) \right\}.$$

Definition 11 For any given CHFN $\hat{H}$, the score function is defined as follows:

$$s(\hat{H}) = \frac{1}{2}\left(\frac{1}{\#\hat{H}} \sum_{\mathscr{C},\ \mathscr{T} \in \hat{H}} (\mathscr{C} + \mathscr{T})\right),$$

where $\#\hat{H}$ is the number of elements in $\hat{H}$. For any two CHFNs $\hat{H}_1$ and $\hat{H}_2$:

$$\text{if } s(\hat{H}_1) \succ s(\hat{H}_2), \text{ then } \hat{H}_1 \succ \hat{H}_2; \text{ if } s(\hat{H}_1) = s(\hat{H}_2), \text{ then } \hat{H}_1 \sim \hat{H}_2.$$

4. Complex hesitant fuzzy Dombi AOs

This section focuses on the CHF Dombi AOs, building on the previous section's study of CHF operations. This section introduces various aggregation operators, including CHFDWA, CHFDOWA, CHFDWG, CHFDOWG, CHFDHA, and CHFDHG. The article discusses the desirable features and gives an example of these operators. To discuss the applicability of Dombit-norm and t-conorm, it is necessary to define average and geometric AOs. The notion of CHFDWA and CHFDWG can weight CHF values. To cover these characteristics, the notions of CHFDWA and CHFDWG AOs are defined as follows:

Definition 12 Assuming a set of CHFNs $\hat{H}_j$ (where $j = 1, 2, 3, \ldots, n$), the CHFDWA operator is given below.

$$\text{CHFDWA}(\hat{H}_1, \hat{H}_2, \ldots, \hat{H}_n) = \bigoplus_{j=1}^{n} \tilde{w}_j \hat{H}_j = \tilde{w}_1 \hat{H}_1 \oplus \tilde{w}_2 \hat{H}_2 \oplus \cdots \oplus \tilde{w}_n \hat{H}_n$$

Then $\tilde{w} = (\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_n)^T$ is the Weight Vector (WV) of the CHFNs, $\tilde{w}_j \in [0, 1]$ and $\sum_{j=1}^{n} \tilde{w}_j = 1$.

Theorem 1 Assume that there is a collection of CHFNs $\hat{H}_j$ $(j = 1, 2, \ldots, n)$ with WV $\tilde{w} = (\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_n)^T$, then the aggregated CHFNs is:

$$\text{CHFDWA}(\hat{H}_1, \hat{H}_2, \ldots, \hat{H}_n) = \bigoplus_{j=1}^{k+1} \tilde{w}_j \hat{H}_j = \bigcup_{\substack{C_1, T_1 \in \hat{H}_1 \\ C_2, T_2 \in \hat{H}_2 \\ \vdots \\ C_n, T_n \in \hat{H}_n}} \left(\left(1 - \frac{1}{1 + \left(\sum_{j=1}^{n} \tilde{w}_j \left(\frac{C_j}{1 - C_j} \right)^{\delta} \right)^{1/\delta}} \right) + \iota \left(1 - \frac{1}{1 + \left(\sum_{j=1}^{n} \tilde{w}_j \left(\frac{T_j}{1 - T_j} \right)^{\delta} \right)^{1/\delta}} \right) \right)$$

Proof. We use mathematical induction.

Assuming for $(n = 2)$:

$$\tilde{w}_1 \hat{H}_1 = \bigcup_{C_1, T_1 \in \hat{H}_1} \left\{ \left(1 - \frac{1}{1 + (\tilde{w}_1 (C_1/(1 - C_1))^{\delta})^{1/\delta}} \right) + \iota \left(1 - \frac{1}{1 + (\tilde{w}_1 (T_1/(1 - T_1))^{\delta})^{1/\delta}} \right) \right\}$$

$$\tilde{w}_2 \hat{H}_2 = \bigcup_{C_2, T_2 \in \hat{H}_2} \left\{ \left(1 - \frac{1}{1 + (\tilde{w}_2 (C_2/(1 - C_2))^{\delta})^{1/\delta}} \right) + \iota \left(1 - \frac{1}{1 + (\tilde{w}_2 (T_2/(1 - T_2))^{\delta})^{1/\delta}} \right) \right\}$$

Then

$$\acute{w}_1\hat{H}_1 \oplus \acute{w}_2\hat{H}_2$$

$$= \bigcup_{\substack{\check{C}_1,\ \tilde{T}_1 \in \hat{H}_1 \\ \check{C}_2,\ \tilde{T}_2 \in \hat{H}_2}} \left\{ \begin{array}{c} \left(1 - \dfrac{1}{1 + \left(\left(\dfrac{1 - \dfrac{1}{1 + \left(\acute{w}_1 \left(\dfrac{\check{C}_1}{1-\check{C}_1} \right)^{\delta} \right)^{1/\delta}}}{1 - 1 + \dfrac{1}{\left(\acute{w}_1 \left(\dfrac{\check{C}_1}{1-\check{C}_1} \right)^{\delta} \right)^{1/\delta}}} \right)^{\delta} + \left(\dfrac{1 - \dfrac{1}{1 + \left(\acute{w}_2 \left(\dfrac{\check{C}_2}{1-\check{C}_2} \right)^{\delta} \right)^{1/\delta}}}{1 - 1 + \dfrac{1}{\left(\acute{w}_2 \left(\dfrac{\check{C}_2}{1-\check{C}_2} \right)^{\delta} \right)^{1/\delta}}} \right)^{\delta} \right)^{1/\delta}} \right) + \\ \iota \left(1 - \dfrac{1}{1 + \left(\left(\dfrac{1 - \dfrac{1}{1 + \left(\acute{w}_1 \left(\dfrac{\tilde{T}_1}{1-\tilde{T}_1} \right)^{\delta} \right)^{1/\delta}}}{1 - 1 + \dfrac{1}{\left(\acute{w}_1 \left(\dfrac{\tilde{T}_1}{1-\tilde{T}_1} \right)^{\delta} \right)^{1/\delta}}} \right)^{\delta} + \left(\dfrac{1 - \dfrac{1}{1 + \left(\acute{w}_2 \left(\dfrac{\tilde{T}_2}{1-\tilde{T}_2} \right)^{\delta} \right)^{1/\delta}}}{1 - 1 + \dfrac{1}{\left(\acute{w}_2 \left(\frac{\tilde{T}_2}{1-\tilde{T}_2} \right)^{\delta} \right)^{1/\delta}}} \right)^{\delta} \right)^{1/\delta}} \right) \end{array} \right\}$$

If Eq. 1 holds $n = k$, that is

$$\bigoplus_{j=1}^{k} \tilde{w}_j \hat{H}_j = \bigcup_{C_1,\ T_1 \in \hat{H}_1,\ \ldots,\ C_k,\ T_k \in \hat{H}_k} \left\{ \begin{array}{c} \left(1 - \dfrac{1}{1 + \left(\sum_{j=1}^{k} \tilde{w}_j (C_j/(1-C_j))^{\delta} \right)^{1/\delta}} \right) + \\ \iota \left(1 - \dfrac{1}{1 + \left(\sum_{j=1}^{k} \tilde{w}_j (T_j/(1-T_j))^{\delta} \right)^{1/\delta}} \right) \end{array} \right\}$$

Now for $n = k+1$we have

$$\bigoplus_{j=1}^{k+1} \tilde{w}_j \hat{H}_j = \left(\bigoplus_{j=1}^{k} \tilde{w}_j \hat{H}_j\right) \oplus \left(\tilde{w}_{k+1} \hat{H}_{k+1}\right)$$

In other words, for $n = k+1$. Eq. 1 holds. So, for every n Eq. 1 holds

$$\bigoplus_{j=1}^{k} \tilde{w}_j \hat{H}_j = \bigcup_{C_1,\, T_1 \in \hat{H}_1,\, \ldots,\, C_k,\, T_k \in \hat{H}_k} \left\{ \begin{array}{c} \left(1 - \dfrac{1}{1 + \left(\sum_{j=1}^{k} \tilde{w}_j (C_j/(1-C_j))^{\delta}\right)^{1/\delta}}\right) + \\ \iota \left(1 - \dfrac{1}{1 + \left(\sum_{j=1}^{k} \tilde{w}_j (T_j/(1-T_j))^{\delta}\right)^{1/\delta}}\right) \end{array} \right\}$$

$$\bigoplus \bigcup_{C_{k+1},\, T_{k+1} \in \hat{H}_{k+1}} \left\{ \left(1 - \frac{1}{1 + \left(\tilde{w}_{k+1} (C_{k+1}/(1-C_{k+1}))^{\delta}\right)^{1/\delta}}\right) + \iota \left(1 - \frac{1}{1 + \left(\tilde{w}_{k+1} (T_{k+1}/(1-T_{k+1}))^{\delta}\right)^{1/\delta}}\right) \right\}$$

Example 1 Assume two CHFNs

$$\hat{H}_1 = \{0.5 + \iota 0.3,\ 0.8 + \iota 0.5\},\ \hat{H}_2 = \{0.2 + \iota 0.1,\ 0.3 + \iota 0.4,\ 0.5 + \iota 0.6\},$$

with

$$\acute{w} = \begin{pmatrix} 0.45 \\ 0.55 \end{pmatrix}$$

being the weight vector (WV) of the two CHFNs. The aggregate CHFN is for = 3

$$\text{CHFDWA}(\hat{H}_1,\ \hat{H}_2) = \bigcup_{\hat{C}_1,\, \hat{T}_1 \in \hat{H}_1,\ \hat{C}_2,\, \hat{T}_2 \in \hat{H}_2} \left\{ \begin{array}{c} \left(1 - \dfrac{1}{1 + \left(\sum_{j=1}^{2} \acute{w}_j \left(\dfrac{\hat{C}_j}{1-\hat{C}_j}\right)^{\delta}\right)^{1/\delta}}\right) + \\ \iota \left(1 - \dfrac{1}{1 + \left(\sum_{j=1}^{2} \acute{w}_j \left(\dfrac{\hat{T}_j}{1-\hat{T}_j}\right)^{\delta}\right)^{1/\delta}}\right) \end{array} \right\}$$

$$
=\left\{\begin{array}{l}
1-\dfrac{1}{1+\left(0.45\left(\dfrac{0.5}{1-0.5}\right)^{3}+0.55\left(\dfrac{0.2}{1-0.2}\right)^{3}\right)^{1/3}}+\iota\left(1-\dfrac{1}{1+\left(0.45\left(\dfrac{0.3}{1-0.3}\right)^{3}+0.55\left(\dfrac{0.1}{1-0.1}\right)^{3}\right)^{1/3}}\right),\\
1-\dfrac{1}{1+\left(0.45\left(\dfrac{0.5}{1-0.5}\right)^{3}+0.55\left(\dfrac{0.3}{1-0.3}\right)^{3}\right)^{1/3}}+\iota\left(1-\dfrac{1}{1+\left(0.45\left(\dfrac{0.3}{1-0.3}\right)^{3}+0.55\left(\dfrac{0.4}{1-0.4}\right)^{3}\right)^{1/3}}\right),\\
1-\dfrac{1}{1+\left(0.45\left(\dfrac{0.5}{1-0.5}\right)^{3}+0.55\left(\dfrac{0.5}{1-0.5}\right)^{3}\right)^{1/3}}+\iota\left(1-\dfrac{1}{1+\left(0.45\left(\dfrac{0.3}{1-0.3}\right)^{3}+0.55\left(\dfrac{0.6}{1-0.6}\right)^{3}\right)^{1/3}}\right),\\
1-\dfrac{1}{1+\left(0.45\left(\dfrac{0.8}{1-0.8}\right)^{3}+0.55\left(\dfrac{0.2}{1-0.2}\right)^{3}\right)^{1/3}}+\iota\left(1-\dfrac{1}{1+\left(0.45\left(\dfrac{0.5}{1-0.5}\right)^{3}+0.55\left(\dfrac{0.1}{1-0.1}\right)^{3}\right)^{1/3}}\right),\\
1-\dfrac{1}{1+\left(0.45\left(\dfrac{0.8}{1-0.8}\right)^{3}+0.55\left(\dfrac{0.3}{1-0.3}\right)^{3}\right)^{1/3}}+\iota\left(1-\dfrac{1}{1+\left(0.45\left(\dfrac{0.5}{1-0.5}\right)^{3}+0.55\left(\dfrac{0.4}{1-0.4}\right)^{3}\right)^{1/3}}\right),\\
1-\dfrac{1}{1+\left(0.45\left(\dfrac{0.8}{1-0.8}\right)^{3}+0.55\left(\dfrac{0.5}{1-0.5}\right)^{3}\right)^{1/3}}+\iota\left(1-\dfrac{1}{1+\left(0.45\left(\dfrac{0.5}{1-0.5}\right)^{3}+0.55\left(\dfrac{0.6}{1-0.6}\right)^{3}\right)^{1/3}}\right)
\end{array}\right\}
$$

$$= \begin{Bmatrix} 0.4354 + \imath 0.2485, \\ 0.4414 + \imath 0.3684, \\ 0.5000 + \imath 0.5529, \\ 0.7540 + \imath 0.4340, \\ 0.7541 + \imath 0.4593, \\ 0.7552 + \imath 0.5692 \end{Bmatrix}$$

Various aggregated CHFNs can be created when the parameter $\acute{\xi}$ has various values.

Definition 13 Let us assume there is a group of CHFNs

$$\hat{H} j \ (j = 1, 2, 3, \ldots, n),$$

a CHFDWG operator is specified as given below:

$$\text{CHFDWG}(\hat{H}_1, \hat{H}_2, \ldots, \hat{H}_n) = \bigotimes_{j=1}^{n} \hat{H}_j^{\acute{w}_j} = \hat{H}_1^{\acute{w}_1} \otimes \hat{H}_2^{\acute{w}_2} \otimes \cdots \otimes \hat{H}_n^{\acute{w}_n}$$

Then $\acute{w} = (\acute{w}_1, \acute{w}_2, \ldots, \acute{w}_n)^T$ is the Weight Vector (WV) of the CHFNs, where

$$\acute{w}_j \in [0, 1], \sum_{j=1}^{n} \acute{w}_j = 1.$$

Theorem 2 Assume that there is, a group of CHFNs $\hat{H}_j$ $(j = 1, 2, 3, \ldots, n)$ with the WV of $\acute{w} = (\acute{w}_1, \acute{w}_2, \ldots, \acute{w}_n)^T$, then the aggregated CHFNs is

$$\operatorname{CHFDWG}(\hat{H}_1,\ \hat{H}_2,\ \ldots,\ \hat{H}_n)=\bigotimes_{j=1}^{n}\left(\hat{H}_j^{\acute{w}_j}\right)=\bigcup_{\substack{\check{C}_1,\ \tilde{T}_1\in\hat{H}_1\\ \check{C}_2,\ \tilde{T}_2\in\hat{H}_2\\ \vdots\\ \check{C}_n,\ \tilde{T}_n\in\hat{H}_n}}\left\{\begin{array}{c}\left(\dfrac{1}{1+\left(\sum\limits_{j=1}^{n}\acute{w}_j\left(\dfrac{1-\check{C}_j}{\check{C}_j}\right)^{\delta}\right)^{1/\delta}}\right)+\\ \iota\left(\dfrac{1}{1+\left(\sum\limits_{j=1}^{n}\acute{w}_j\left(\dfrac{1-\tilde{T}_j}{\tilde{T}_j}\right)^{\delta}\right)^{1/\delta}}\right)\end{array}\right\}$$

Proof. We use mathematical induction.
In the case of $n=2$,

$$\hat{H}_1^{\acute{w}_1}=\bigcup_{\check{C}_1,\ \tilde{T}_1\in\hat{H}_1}\left\{\begin{array}{c}\left(\dfrac{1}{1+\left(\acute{w}_1\left(\dfrac{1-\check{C}_1}{\check{C}_1}\right)^{\delta}\right)^{1/\delta}}\right)+\\ \iota\left(1-\dfrac{1}{1+\left(\acute{w}_1\left(\dfrac{1-\tilde{T}_1}{\tilde{T}_1}\right)^{\delta}\right)^{1/\delta}}\right)\end{array}\right\}$$

$$\hat{H}_2^{\acute{w}_2}=\bigcup_{\check{C}_2,\ \tilde{T}_2\in\hat{H}_2}\left\{\begin{array}{c}\left(\dfrac{1}{1+\left(\acute{w}_2\left(\dfrac{1-\check{C}_2}{\check{C}_2}\right)^{\delta}\right)^{1/\delta}}\right)+\\ \iota\left(1-\dfrac{1}{1+\left(\acute{w}_2\left(\dfrac{1-\tilde{T}_2}{\tilde{T}_2}\right)^{\delta}\right)^{1/\delta}}\right)\end{array}\right\}$$

Then

$$\hat{H}_1^{\acute{w}_1} \otimes \hat{H}_2^{\acute{w}_2} = \bigcup_{\check{C}_1,\ \tilde{T}_1 \in \hat{H}_1,\ \check{C}_2,\ \tilde{T}_2 \in \hat{H}_2} \left\{ \begin{array}{l} \left(\dfrac{1}{1 + \left(\left(\acute{w}_1 \left(\dfrac{1-\check{C}_1}{\check{C}_1} \right)^{\delta} \right) + \left(\acute{w}_2 \left(\dfrac{1-\check{C}_2}{\check{C}_2} \right)^{\delta} \right) \right)^{1/\delta}} \right) + \\ \iota \left(\dfrac{1}{1 + \left(\left(\acute{w}_1 \left(\dfrac{1-\tilde{T}_1}{\tilde{T}_1} \right)^{\delta} \right) + \left(\acute{w}_2 \left(\dfrac{1-\tilde{T}_2}{\tilde{T}_2} \right)^{\delta} \right) \right)^{1/\delta}} \right) \end{array} \right\}$$

$$= \bigcup_{\check{C}_1,\ \tilde{T}_1 \in \hat{H}_1,\ \check{C}_2,\ \tilde{T}_2 \in \hat{H}_2} \left\{ \begin{array}{l} \left(\dfrac{1}{1 + \left(\sum_{j=1}^{2} \acute{w}_j \left(\dfrac{1-\check{C}_j}{\check{C}_j} \right)^{\delta} \right)^{1/\delta}} \right) + \\ \iota \left(\dfrac{1}{1 + \left(\sum_{j=1}^{2} \acute{w}_j \left(\dfrac{1-\tilde{T}_j}{\tilde{T}_j} \right)^{\delta} \right)^{1/\delta}} \right) \end{array} \right\}$$

For $n = k$, Eq. 2 is valid, which means

$$\text{CHFDWA}(\hat{H}_1,\ \hat{H}_2,\ \ldots,\ \hat{H}_n) = \bigotimes_{j=1}^{k} \left(\hat{H}_j^{\acute{w}_j} \right) = \bigcup_{\substack{\check{C}_1,\ \tilde{T}_1 \in \hat{H}_1 \\ \check{C}_n,\ \vdots\ \tilde{T}_n \in \hat{H}_n}} \left\{ \begin{array}{l} \left(\dfrac{1}{1 + \left(\sum_{j=1}^{k} \acute{w}_j \left(\dfrac{1-\check{C}_j}{\check{C}_j} \right)^{\delta} \right)^{1/\delta}} \right) + \\ \iota \left(\dfrac{1}{1 + \left(\sum_{j=1}^{k} \acute{w}_j \left(\dfrac{1-\tilde{T}_j}{\tilde{T}_j} \right)^{\delta} \right)^{1/\delta}} \right) \end{array} \right\}$$

Now for $n = k + 1$

$$\bigotimes_{j=1}^{k+1}\left(\hat{H}_j^{\acute{w}_j}\right)=\left(\bigotimes_{j=1}^{k}\hat{H}_j^{\acute{w}_j}\right)\otimes\hat{H}_{k+1}^{\acute{w}_{k+1}}$$

$$=\bigcup_{\check{C}_1,\,\tilde{T}_1\in\hat{H}_1,\,\ldots,\,\check{C}_k,\,\tilde{T}_k\in\hat{H}_k}\left\{\begin{array}{c}\left(\dfrac{1}{1+\left(\sum_{j=1}^{k}\acute{w}_j\left(\dfrac{1-\check{C}_j}{\check{C}_j}\right)^{\delta}\right)^{1/\delta}}\right)+\\ \iota\left(\dfrac{1}{1+\left(\sum_{j=1}^{k}\acute{w}_j\left(\dfrac{1-\tilde{T}_j}{\tilde{T}_j}\right)^{\delta}\right)^{1/\delta}}\right)\end{array}\right\}$$

$$\otimes\bigcup_{\check{C}_{k+1},\,\tilde{T}_{k+1}\in\hat{H}_{k+1}}\left\{\begin{array}{c}\left(\dfrac{1}{1+\left(\acute{w}_{k+1}\left(\dfrac{1-\check{C}_{k+1}}{\check{C}_{k+1}}\right)^{\delta}\right)^{1/\delta}}\right)+\\ \iota\left(\dfrac{1}{1+\left(\acute{w}_{k+1}\left(\dfrac{1-\tilde{T}_{k+1}}{\tilde{T}_{k+1}}\right)^{\delta}\right)^{1/\delta}}\right)\end{array}\right\}$$

$$=\bigcup_{\check{C}_1,\,\tilde{T}_1\in\hat{H}_1,\,\ldots,\,\check{C}_{k+1},\,\tilde{T}_{k+1}\in\hat{H}_{k+1}}\left\{\begin{array}{c}\left(\dfrac{1}{1+\left(\sum_{j=1}^{k+1}\acute{w}_j\left(\dfrac{1-\check{C}_j}{\check{C}_j}\right)^{\delta}\right)^{1/\delta}}\right)+\\ \iota\left(\dfrac{1}{1+\left(\sum_{j=1}^{k+1}\acute{w}_j\left(\dfrac{1-\tilde{T}_j}{\tilde{T}_j}\right)^{\delta}\right)^{1/\delta}}\right)\end{array}\right\}$$

It is true that Eq. 2 for $n=k+1$. Therefore, Eq. 2 is true for every n. Then,

$$\text{CHFDWG}(\hat{H}_1,\, \hat{H}_2,\, \ldots,\, \hat{H}_n) = \bigotimes_{j=1}^{n} \left(\hat{H}_j^{\acute{w}_j}\right) = \bigcup_{\substack{\check{C}_1,\, \tilde{T}_1 \in \hat{H}_1 \\ \vdots \\ \check{C}_n,\, \tilde{T}_n \in \hat{H}_n}} \left\{ \begin{array}{c} \left(\dfrac{1}{1 + \left(\sum\limits_{j=1}^{n} \acute{w}_j \left(\dfrac{1 - \check{C}_j}{\check{C}_j} \right)^{\delta} \right)^{1/\delta}} \right) + \\ \iota \left(\dfrac{1}{1 + \left(\sum\limits_{j=1}^{n} \acute{w}_j \left(\dfrac{1 - \tilde{T}_j}{\tilde{T}_j} \right)^{\delta} \right)^{1/\delta}} \right) \end{array} \right\}$$

Example 2 Let us assume there are two CHFNs:

$$\hat{H}_1 = \{0.34 + \iota 0.51,\ 0.42 + \iota 0.53\},\ \hat{H}_2 = \{0.12 + \iota 0.41,\ 0.31 + \iota 0.43,\ 0.44 + \iota 0.60\},$$

and the weight vector is given by:

$$\acute{w} = \begin{pmatrix} 0.45 \\ 0.55 \end{pmatrix},$$

with aggregation parameter $\delta = 4$. Then, the accumulated CHFN using the CHFDWG operator is computed as:

$$\text{CHFDWG}(\hat{H}_1,\, \hat{H}_2) = \bigcup_{\substack{\hat{C}_1,\, \hat{T}_1 \in \hat{H}_1 \\ \hat{C}_2,\, \hat{T}_2 \in \hat{H}_2}} \left\{ \left(\frac{1}{1 + \left(\sum_{j=1}^{n} \acute{w}_j \left(\frac{1 - \hat{C}_j}{\hat{C}_j} \right)^{\delta} \right)^{1/\delta}} \right) + \iota \left(\frac{1}{1 + \left(\sum_{j=1}^{n} \acute{w}_j \left(\frac{1 - \hat{T}_j}{\hat{T}_j} \right)^{\delta} \right)^{1/\delta}} \right) \right\}$$

$$
=\left\{\begin{array}{l}
\dfrac{1}{\left(1+0.45\left(\dfrac{1-0.34}{0.34}\right)^4+0.55\left(\dfrac{1-0.12}{0.12}\right)^4\right)^{1/4}}+\iota\dfrac{1}{\left(1+0.45\left(\dfrac{1-0.51}{0.51}\right)^4+0.55\left(\dfrac{1-0.41}{0.41}\right)^4\right)^{1/4}},\\
\dfrac{1}{\left(1+0.45\left(\dfrac{1-0.34}{0.34}\right)^4+0.55\left(\dfrac{1-0.31}{0.31}\right)^4\right)^{1/4}}+\iota\dfrac{1}{\left(1+0.45\left(\dfrac{1-0.51}{0.51}\right)^4+0.55\left(\dfrac{1-0.43}{0.43}\right)^4\right)^{1/4}},\\
\dfrac{1}{\left(1+0.45\left(\dfrac{1-0.34}{0.34}\right)^4+0.55\left(\dfrac{1-0.44}{0.44}\right)^4\right)^{1/4}}+\iota\dfrac{1}{\left(1+0.45\left(\dfrac{1-0.51}{0.51}\right)^4+0.55\left(\dfrac{1-0.60}{0.60}\right)^4\right)^{1/4}},\\
\dfrac{1}{\left(1+0.45\left(\dfrac{1-0.42}{0.42}\right)^4+0.55\left(\dfrac{1-0.12}{0.12}\right)^4\right)^{1/4}}+\iota\dfrac{1}{\left(1+0.45\left(\dfrac{1-0.53}{0.53}\right)^4+0.55\left(\dfrac{1-0.41}{0.41}\right)^4\right)^{1/4}},\\
\dfrac{1}{\left(1+0.45\left(\dfrac{1-0.42}{0.42}\right)^4+0.55\left(\dfrac{1-0.31}{0.31}\right)^4\right)^{1/4}}+\iota\dfrac{1}{\left(1+0.45\left(\dfrac{1-0.53}{0.53}\right)^4+0.55\left(\dfrac{1-0.43}{0.43}\right)^4\right)^{1/4}},\\
\dfrac{1}{\left(1+0.45\left(\dfrac{1-0.42}{0.42}\right)^4+0.55\left(\dfrac{1-0.44}{0.44}\right)^4\right)^{1/4}}+\iota\dfrac{1}{\left(1+0.45\left(\dfrac{1-0.53}{0.53}\right)^4+0.55\left(\dfrac{1-0.60}{0.60}\right)^4\right)^{1/4}}
\end{array}\right\}
$$

$$
=\{0.1366+\iota 0.4373,\ 0.3214+\iota 0.4543,\ 0.3741+\iota 0.5442,\ 0.1367+\iota 0.4397,\ 0.3364+\iota 0.4575,\ 0.4302+\iota 0.5591\}.
$$

Various aggregated CHFNs can be established by changing the value of parameter δ.

The CHFDWA and CHFDWG operators consider the importance (weights) of CHF values, but do not address their ordered positions. To incorporate the ordered positions of CHFNs, the following operators are defined:

Definition 14 Let there be a group of CHFNs $\hat{H}_j$ $(j = 1, 2, \ldots, n)$, and let $\acute{w} = (\acute{w}_1, \acute{w}_2, \ldots, \acute{w}_n)^T$ be the corresponding weights such that $\acute{w}_j \in [0, 1]$ and $\sum_{j=1}^{n} \acute{w}_j = 1$. Let $\hat{H}_{\varphi(j)}$ denote the j-th largest CHF value. Then:

$$
\text{CHFDOWA}(\hat{H}_1, \hat{H}_2, \ldots, \hat{H}_n) = \bigoplus_{j=1}^{n}\left(\acute{w}_j \hat{H}_{\varphi(j)}\right) = \acute{w}_1\hat{H}_{\varphi(1)} \oplus \acute{w}_2\hat{H}_{\varphi(2)} \oplus \cdots \oplus \acute{w}_n\hat{H}_{\varphi(n)},
$$

$$
\text{CHFDOWG}(\hat{H}_1, \hat{H}_2, \ldots, \hat{H}_n) = \bigotimes_{j=1}^{n}\left(\hat{H}_{\varphi(j)}^{\acute{w}_j}\right) = \hat{H}_{\varphi(1)}^{\acute{w}_1} \otimes \hat{H}_{\varphi(2)}^{\acute{w}_2} \otimes \cdots \otimes \hat{H}_{\varphi(n)}^{\acute{w}_n}.
$$

Theorem 3 Assume that there is a set of CHFNs $\hat{H}_j$ $(j = 1, 2, 3, \ldots, n)$. Then, the aggregated CHFNs are determined by the CHFDOWA or CHFDOWG operator.

$$\mathrm{CHFDOWA}(\hat{H}_1, \hat{H}_2, \ldots, \hat{H}_n) = \bigcup_{\substack{\check{C}_{\phi(1)}, \tilde{T}_{\phi(1)} \in \hat{H}_{\phi(1)} \\ \check{C}_{\phi(2)}, \tilde{T}_{\phi(2)} \in \hat{H}_{\phi(2)} \\ \vdots \\ \check{C}_{\phi(n)}, \tilde{T}_{\phi(n)} \in \hat{H}_{\phi(n)}}} \left\{ \begin{array}{c} 1 - \dfrac{1}{\left(1 + \sum\limits_{j=1}^{n} \acute{w}_j \left(\dfrac{\check{C}_{\phi(j)}}{1 - \check{C}_{\phi(j)}}\right)^{\delta}\right)^{\frac{1}{\delta}}} + \\ \iota \left(1 - \dfrac{1}{\left(1 + \sum\limits_{j=1}^{n} \acute{w}_j \left(\dfrac{\tilde{T}_{\phi(j)}}{1 - \tilde{T}_{\phi(j)}}\right)^{\delta}\right)^{\frac{1}{\delta}}}\right) \end{array} \right\}$$

and

$$\mathrm{CHFDOWG}(\hat{H}_1, \hat{H}_2, \ldots, \hat{H}_n) = \bigcup_{\substack{\check{C}_{\phi(1)}, \tilde{T}_{\phi(1)} \in \hat{H}_{\phi(1)} \\ \check{C}_{\phi(2)}, \tilde{T}_{\phi(2)} \in \hat{H}_{\phi(2)} \\ \vdots \\ \check{C}_{\phi(n)}, \tilde{T}_{\phi(n)} \in \hat{H}_{\phi(n)}}} \left\{ \begin{array}{c} \dfrac{1}{\left(1 + \sum\limits_{j=1}^{n} \acute{w}_j \left(\dfrac{1 - \check{C}_{\phi(j)}}{\check{C}_{\phi(j)}}\right)^{\delta}\right)^{\frac{1}{\delta}}} + \\ \iota \left(\dfrac{1}{\left(1 + \sum\limits_{j=1}^{n} \acute{w}_j \left(\dfrac{1 - \tilde{T}_{\phi(j)}}{\tilde{T}_{\phi(j)}}\right)^{\delta}\right)^{\frac{1}{\delta}}}\right) \end{array} \right\}$$

Example 3 Let us assume there are three CHFNs:

$$\hat{H}_1 = \{0.2 + \iota 0.3,\ 0.4 + \iota 0.5\},\ \hat{H}_2 = \{0.2 + \iota 0.3,\ 0.6 + \iota 0.6,\ 0.8 + \iota 0.7\},\ \hat{H}_3 = \{0.4 + \iota 0.5\}$$

$$\acute{w} = (0.2,\ 0.3,\ 0.5)^T$$

is the weight vector of these CHFNs. Let the parameter $\delta = 4$. Then the score of each CHFN is calculated as:

$$s(\hat{H}_1) = \frac{1}{2}\left(\frac{0.2+0.3+0.4+0.5}{2}\right) = 0.35$$

$$s(\hat{H}_2) = \frac{1}{2}\left(\frac{0.2+0.3+0.6+0.6+0.8+0.7}{3}\right) = 0.53$$

$$s(\hat{H}_3) = \frac{0.4+0.5}{2} = 0.45$$

Since $s(\hat{H}_2) > s(\hat{H}_3) > s(\hat{H}_1)$, we have:

$$\hat{H}_{\phi(1)} = \hat{H}_2 = \{0.2+\imath 0.3,\ 0.6+\imath 0.6,\ 0.8+\imath 0.7\}$$

$$\hat{H}_{\phi(2)} = \hat{H}_3 = \{0.4+\imath 0.5\},$$

$$\hat{H}_{\phi(3)} = \hat{H}_1 = \{0.2+\imath 0.3,\ 0.4+\imath 0.5\}$$

According to the definition of the CHFDOWA operator, we have:

$$\text{CHFDOWA}(\hat{H}_1,\ \hat{H}_2,\ \hat{H}_3) = \bigcup_{\substack{\check{C}_{\phi(1)},\ \tilde{T}_{\phi(1)} \in \hat{H}_{\phi(1)} \\ \check{C}_{\phi(2)},\ \tilde{T}_{\phi(2)} \in \hat{H}_{\phi(2)} \\ \check{C}_{\phi(3)},\ \tilde{T}_{\phi(3)} \in \hat{H}_{\phi(3)}}} \left\{ \begin{array}{c} 1 - \dfrac{1}{\left(1 + \sum_{j=1}^{3} \acute{w}_j \left(\dfrac{\check{C}_{\phi(j)}}{1-\check{C}_{\phi(j)}}\right)^{\delta}\right)^{\frac{1}{\delta}}} + \\ \imath \left(1 - \dfrac{1}{\left(1 + \sum_{j=1}^{3} \acute{w}_j \left(\dfrac{\tilde{T}_{\phi(j)}}{1-\tilde{T}_{\phi(j)}}\right)^{\delta}\right)^{\frac{1}{\delta}}}\right) \end{array} \right\}$$

$$= \left\{ \begin{array}{l} 0.3329+\imath 0.4299,\ 0.3870+\imath 0.4866,\ 0.5044+\imath 0.5178, \\ 0.5098+\imath 0.5371,\ 0.7279+\imath 0.6125,\ 0.7280+\imath 0.6169 \end{array} \right\}$$

Similarly, for the CHFDOWG operator:

$$\text{CHFDOWG}(\hat{H}_1,\hat{H}_2,\hat{H}_3) = \bigcup_{\substack{\check{C}_{\phi(1)},\tilde{T}_{\phi(1)}\in\hat{H}_{\phi(1)}\\ \check{C}_{\phi(2)},\tilde{T}_{\phi(2)}\in\hat{H}_{\phi(2)}\\ \check{C}_{\phi(3)},\tilde{T}_{\phi(3)}\in\hat{H}_{\phi(3)}}} \left\{ \begin{array}{c} \dfrac{1}{\left(1+\sum\limits_{j=1}^{3}\acute{w}_j\left(\dfrac{1-\check{C}_{\phi(j)}}{\check{C}_{\phi(j)}}\right)^{\delta}\right)^{\frac{1}{\delta}}} + \\ \iota\left(\dfrac{1}{\left(1+\sum\limits_{j=1}^{3}\acute{w}_j\left(\dfrac{1-\tilde{T}_{\phi(j)}}{\tilde{T}_{\phi(j)}}\right)^{\delta}\right)^{\frac{1}{\delta}}}\right) \end{array} \right\}$$

$$= \left\{ \begin{array}{l} 0.2143+\iota 0.3183,\ 0.2684+\iota 0.3831,\ 0.2283+\iota 0.3363, \\ 0.4000+\iota 0.5109,\ 0.2286+\iota 0.3365,\ 0.4129+\iota 0.5134 \end{array} \right\}$$

Now we introduce some hybrid CHF Dombi aggregation operators.

The notion of CHFDWA and CHFDWG AOs can only weigh the CHF values, while CHFDOWA and CHFDOWG AOs can only consider the ordered positions. To incorporate both these characteristics in a unified structure, the CHFDHA and CHFDHG operators are defined as follows:

Definition 15 Let us assume there is a group of CHFNs $\hat{H}_j$ $(j = 1,\ 2,\ 3,\ \ldots,\ n)$, and

$$\acute{w} = (\acute{w}_1,\ \acute{w}_2,\ \ldots,\ \acute{w}_n)^T$$

is the weight vector (WV) of the CHFNs, where $\acute{w}_j \in [0,\ 1]$ and $\sum_{j=1}^{n}\acute{w}_j = 1$.

Also,

$$\acute{w} = (\acute{w}_1,\ \acute{w}_2,\ \ldots,\ \acute{w}_n)^T$$

is the related weight vector of the accumulated documents, satisfying $\acute{w}_j \in [0,\ 1]$ and $\sum_{j=1}^{n}\acute{w}_j = 1$.

Then, the definitions of the CHFDHA and CHFDHG operators are given as follows:

CHFDHA operator:

$$\text{CHFDHA}(\hat{H}_1,\ \hat{H}_2,\ \ldots,\ \hat{H}_n) = \bigoplus_{j=1}^{n}\acute{w}_j\ \dot{\hat{H}}_{\phi(j)} = \acute{w}_1\ \dot{\hat{H}}_{\phi(1)} \oplus \acute{w}_2\ \dot{\hat{H}}_{\phi(2)} \oplus \cdots \oplus \acute{w}_n\ \dot{\hat{H}}_{\phi(n)}$$

where $\dot{\hat{H}}_{\phi(j)}$ is the i-th largest element of

$$\dot{\hat{H}} = n\acute{w}_j\hat{H}_j,\ (j = 1,\ 2,\ \ldots,\ n)$$

CHFDHG operator:

$$\text{CHFDHG}(\hat{H}_1,\ \hat{H}_2,\ \ldots,\ \hat{H}_n) = \bigotimes_{j=1}^{n} \left(\ddot{H}_{\phi(j)}\right)^{\acute{w}_j} = \left(\ddot{H}_{\phi(1)}\right)^{\acute{w}_1} \otimes \left(\ddot{H}_{\phi(2)}\right)^{\acute{w}_2} \otimes \cdots \otimes \left(\ddot{H}_{\phi(n)}\right)^{\acute{w}_n}$$

where $\ddot{H}_{\phi(j)}$ is the j-th largest element of

$$\ddot{H}_j = \hat{H}_j^{n\acute{w}_j},\ (j = 1,\ 2,\ \ldots,\ n)$$

Theorem 4 Suppose there is a group of CHFNs $\hat{H}_j$ (where $j = 1,\ 2,\ 3,\ \ldots,\ n$). The aggregated CHFNs, established using the CHFDHA and CHFDHG operators, are as follows:

CHFDHA operator:

$$\text{CHFDHA}(\hat{H}_1,\ \hat{H}_2,\ \ldots,\ \hat{H}_n) = \bigcup_{\substack{\dot{C}_{\phi(1)},\ \dot{T}_{\phi(1)} \in \dot{H}_{\phi(1)} \\ \dot{C}_{\phi(2)},\ \dot{T}_{\phi(2)} \in \dot{H}_{\phi(2)} \\ \vdots \\ \dot{C}_{\phi(n)},\ \dot{T}_{\phi(n)} \in \dot{H}_{\phi(n)}}} \left\{ \begin{array}{l} \left(1 - \dfrac{1}{\left(1 + \sum_{j=1}^{n} \acute{w}_j \left(\dfrac{\dot{C}_{\phi(j)}}{1 - \dot{C}_{\phi(j)}}\right)^{\delta}\right)^{1/\delta}}\right) + \\ \iota \left(1 - \dfrac{1}{\left(1 + \sum_{j=1}^{n} \acute{w}_j \left(\dfrac{\dot{T}_{\phi(j)}}{1 - \dot{T}_{\phi(j)}}\right)^{\delta}\right)^{1/\delta}}\right) \end{array} \right\}$$

CHFDHG operator:

$$\text{CHFDHG}(\hat{H}_1,\ \hat{H}_2,\ \ldots,\ \hat{H}_n) = \bigcup_{\substack{\ddot{C}_{\phi(1)},\ \ddot{T}_{\phi(1)} \in \ddot{H}_{\phi(1)} \\ \ddot{C}_{\phi(2)},\ \ddot{T}_{\phi(2)} \in \ddot{H}_{\phi(2)} \\ \vdots \\ \ddot{C}_{\phi(n)},\ \ddot{T}_{\phi(n)} \in \ddot{H}_{\phi(n)}}} \left\{ \begin{array}{l} \left(\dfrac{1}{\left(1 + \sum_{i=1}^{n} \acute{w}_i \left(\dfrac{1 - \ddot{C}_{\phi(i)}}{\ddot{C}_{\phi(i)}}\right)^{\delta}\right)^{1/\delta}}\right) + \\ \iota \left(\dfrac{1}{\left(1 + \sum_{i=1}^{n} \acute{w}_i \left(\dfrac{1 - \ddot{T}_{\phi(i)}}{\ddot{T}_{\phi(i)}}\right)^{\delta}\right)^{1/\delta}}\right) \end{array} \right\}$$

Here, $\dot{\hat{H}}_{\phi(j)}$ is the j-th largest of $\dot{\hat{H}} = n\acute{w}_j\hat{H}_j$, and $\ddot{\hat{H}}_{\phi(j)}$ is the j-th largest of $\ddot{\hat{H}}_j = \hat{H}_j^{n\acute{w}_j}$ for $j = 1, 2, \ldots, n$.

Example 4 Let us assume there are three CHFNs

$$\hat{H}_1 = \{0.2 + \imath 0.4,\ 0.3 + \imath 0.5\},\ \hat{H}_2 = \{0.4 + \imath 0.3,\ 0.5 + \imath 0.6,\ \ 0.7 + \imath 0.7\},\ \hat{H}_3 = \{0.4 + \imath 0.6\},$$

with weight vector

$$\acute{w} = (0.1,\ 0.4,\ 0.5)^T,\ \text{and related weight vector } \acute{w} = (0.2,\ 0.3,\ 0.5)^T.$$

Let the accumulation parameter be $\delta = 4$.

On one hand, we compute:

$$\dot{\hat{H}}_1 = \left\{ \begin{array}{l} (1 - \dfrac{1}{(1 + 3 \cdot 0.1 \cdot (0.2/(1-0.2))^4)^{1/4}}) + \imath(1 - \dfrac{1}{(1 + 3 \cdot 0.1 \cdot (0.4/(1-0.4))^4)^{1/4}}), \\ (1 - \dfrac{1}{(1 + 3 \cdot 0.1 \cdot (0.3/(1-0.3))^4)^{1/4}}) + \imath(1 - \dfrac{1}{(1 + 3 \cdot 0.1 \cdot (0.5/(1-0.5))^4)^{1/4}}) \end{array} \right\}$$

$$= \{0.1561 + \imath 0.3304,\ 0.2408 + \imath 0.4253\}$$

$$\dot{\hat{H}}_2 = \{0.4111 + \imath 0.3097,\ 0.5114 + \imath 0.6109,\ 0.7095 + \imath 0.7095\} \text{ (similarly computed)}$$

$$\dot{\hat{H}}_3 = \{0.4246 + \imath 0.6241\}$$

We compute their score functions:

$$s(\dot{\hat{H}}_1) = \frac{1}{2} \cdot \frac{0.1561 + 0.3304 + 0.2408 + 0.4253}{2} = 0.2884,$$

$$s(\dot{\hat{H}}_2) = \frac{1}{2} \cdot \frac{0.4111 + 0.3097 + 0.5114 + 0.6109 + 0.7095 + 0.7095}{3} = 0.5437,$$

$$s(\dot{\hat{H}}_3) = \frac{1}{2}(0.4246 + 0.6241) = 0.5244.$$

Since $s(\dot{\hat{H}}_2) > s(\dot{\hat{H}}_3) > s(\dot{\hat{H}}_1)$, we have:

$$\dot{\hat{H}}_{\phi(1)} = \dot{\hat{H}}_2,\ \dot{\hat{H}}_{\phi(2)} = \dot{\hat{H}}_3,\ \dot{\hat{H}}_{\phi(3)} = \dot{\hat{H}}_1$$

Based on the definition of CHFDHA, we get:

$$\text{CHFDHA}(\hat{H}_1,\ \hat{H}_2,\ \hat{H}_3) = \bigcup_{\substack{\acute{C}_{\phi(1)},\ \acute{T}_{\phi(1)} \in \dot{H}_{\phi(1)} \\ \acute{C}_{\phi(2)},\ \acute{T}_{\phi(2)} \in \dot{H}_{\phi(2)} \\ \acute{C}_{\phi(3)},\ \acute{T}_{\phi(3)} \in \dot{H}_{\phi(3)}}} \left\{ \begin{array}{c} \left(1 - \dfrac{1}{(1 + \sum\limits_{j=1}^{3} \acute{w}_j (\dfrac{\acute{C}_{\phi(j)}}{1 - \acute{C}_{\phi(j)}})^{\rho})^{1/\rho}} \right) + \\ \iota \left(1 - \dfrac{1}{(1 + \sum\limits_{j=1}^{3} \acute{w}_j (\dfrac{\acute{T}_{\phi(j)}}{1 - \acute{T}_{\phi(j)}})^{\rho})^{1/\rho}} \right) \end{array} \right\}$$

$$= \{0.3783 + \iota 0.5523,\ 0.3802 + \iota 0.5556,\ 0.4312 + \iota 0.5781,\ 0.4319 + \iota 0.5801,$$

$$0.6210 + \iota 0.6367,\ 0.6210 + \iota 0.6374\}$$

On the other hand, compute:

$$\ddot{\hat{H}}_1 = \{0.2525 + \iota 0.4739,\ 0.3667 + \iota 0.5747\},$$

$$\ddot{\hat{H}}_2 = \{0.3891 + \iota 0.2905,\ 0.4886 + \iota 0.5890,\ 0.6903 + \iota 0.6903\},$$

$$\ddot{\hat{H}}_3 = \{0.3759 + \iota 0.5754\}$$

$$s(\ddot{\hat{H}}_1) = 0.4170,\ s(\ddot{\hat{H}}_2) = 0.5230,\ s(\ddot{\hat{H}}_3) = 0.4757$$

Ordering gives:

$$\ddot{\hat{H}}_{\phi(1)} = \ddot{\hat{H}}_2,\ \ddot{\hat{H}}_{\phi(2)} = \ddot{\hat{H}}_3,\ \ddot{\hat{H}}_{\phi(3)} = \ddot{\hat{H}}_1$$

Then we compute CHFDHG:

$$\text{CHFDHG}(\hat{H}_1,\ \hat{H}_2,\ \hat{H}_3) = \bigcup_{\substack{\acute{C}_{\phi(1)},\ \acute{T}_{\phi(1)} \in \ddot{H}_{\phi(1)} \\ \acute{C}_{\phi(2)},\ \acute{T}_{\phi(2)} \in \ddot{H}_{\phi(2)} \\ \acute{C}_{\phi(3)},\ \acute{T}_{\phi(3)} \in \ddot{H}_{\phi(3)}}} \left\{ \begin{array}{c} \left(\dfrac{1}{1 + \left(\sum\limits_{j=1}^{3} \acute{w}_j \left(\dfrac{1 - \acute{C}_{\phi(j)}}{\acute{C}_{\phi(j)}} \right)^{\rho} \right)^{1/\rho}} \right) + \\ \imath \left(\dfrac{1}{1 + \left(\sum\limits_{j=1}^{3} \acute{w}_j \left(\dfrac{1 - \acute{T}_{\phi(j)}}{\acute{T}_{\phi(j)}} \right)^{\rho} \right)^{1/\rho}} \right) \end{array} \right\}$$

$$= \{0.2822 + \imath 0.0001,\ 0.3733 + \imath 0.0001,\ 0.2833 + \imath 0.4672,\ 0.3810 + \imath 0.4950,$$

$$0.2836 + \imath 0.3920,\ 0.3830 + \imath 0.3986\}$$

5. Complex hesitant fuzzy MADM

To illustrate the CHF MADM issues, the following assumptions are used. Let $\tilde{A} = \{\tilde{A}_1,\ \tilde{A}_2,\ \ldots,\ \tilde{A}_m\}$ is a discrete collection of alternatives and $C = \{C_1,\ C_2,\ \ldots,\ C_n\}$ is a collection of attributes. If the DMs provide several values for the alternative $\tilde{A}_i$ under the state of nature C_j with secrecy, these values can be measured as CHFNs $\tilde{H}_{ij}$. When two DM supply the same value, the value only shows up once in $\tilde{H}_{ij}$. Assume that the CHF decision matrix is represented as the decision matrix $\tilde{H} = (\tilde{H}_{ij})_{m \times n}$, where $\tilde{H}_{ij}$ $(i = 1,\ 2,\ \ldots,\ m,\ \ j = 1,\ 2,\ \ldots,\ n)$ are represented as CHFNs. Our approach to addressing the MADM issues with CHF information is based on the models mentioned above. These stages are part of the method.

5.1 *Algorithm of proposed study*

Step 1 Collect the information offered by the decision analyst in the form of CHFSs.

Step 2 Utilize the CHFDWA operator:

$$\tilde{H}_i = \text{CHFDWA}(\tilde{H}_{i1},\ \tilde{H}_{i2},\ \ldots,\ \tilde{H}_{in}) = \bigcup_{(\check{C}_{ij},\ \acute{T}_{ij} \in \tilde{H}_{ij})} \left\{ \begin{array}{c} \left(1 - \dfrac{1}{1 + \left(\sum_{j=1}^{n} \acute{w}_j \left(\dfrac{\check{C}_{ij}}{1 - \check{C}_{ij}} \right)^{\delta} \right)^{1/\delta}} \right) + \\ \imath \left(1 - \dfrac{1}{1 + \left(\sum_{j=1}^{n} \acute{w}_j \left(\dfrac{\acute{T}_{ij}}{1 - \acute{T}_{ij}} \right)^{\delta} \right)^{1/\delta}} \right) \end{array} \right\},\ i = 1,\ 2,\ \ldots,\ m$$

Or the CHFDWG operator:

$$\tilde{H}_i = \mathrm{CHFDWG}(\tilde{H}_{i1},\ \tilde{H}_{i2},\ \ldots,\ \tilde{H}_{in}) = \bigcup_{(\check{C}_{ij},\ \acute{T}_{ij} \in \tilde{H}_{ij})} \left\{ \begin{array}{c} \left(\dfrac{1}{1 + \left(\sum_{j=1}^{n} \acute{w}_j \left(\dfrac{1-\check{C}_{ij}}{\check{C}_{ij}} \right)^{\delta} \right)^{1/\delta}} \right) + \\ \iota \left(\dfrac{1}{1 + \left(\sum_{j=1}^{n} \acute{w}_j \left(\dfrac{1-\acute{T}_{ij}}{\acute{T}_{ij}} \right)^{\delta} \right)^{1/\delta}} \right) \end{array} \right\},\ i = 1,\ 2,\ \ldots,\ m$$

to derive the alternative C_i, total preference values of $\tilde{H}_i$ $(i = 1,\ 2,\ \ldots,\ m)$, where

$$s(\tilde{H}_i) = \frac{1}{n} \sum_{j=1}^{n} s(\tilde{H}_{ij}).$$

Step 3 Determine the total CHF preference value $\tilde{H}_i$ $(i = 1,\ 2,\ \ldots,\ m)$ by calculating the scores $s(\tilde{H}_i)$ to order each alternative and thereafter select the best option.

Step 4 Utilize $s(\tilde{H}_i)$, rank each alternative, and select the best option.

Step 5 End.

5.2 *Advantages of defined algorithm over existing algorithm*

The algorithms defined in [15–18] are based on HFSs. We can notice that HFS can cover the hesitancy of information. However, these existing algorithms can never discuss the CHF values as discussed in the defined algorithm. The defined algorithm can discuss the two-dimensional information as well as the hesitancy of the information. In the case of existing notions, the loss of data increases due to the limitation of the structure of HFS. On the other hand, the introduced notions can cover this data loss and provide more space for decision-makers to take their assessment in the form of CHFS. As a whole, we can say that the defined algorithm is more flexible and reliable as compared to the existing algorithm. Additionally, the flow chart of the defined algorithm is given in Figure 1.

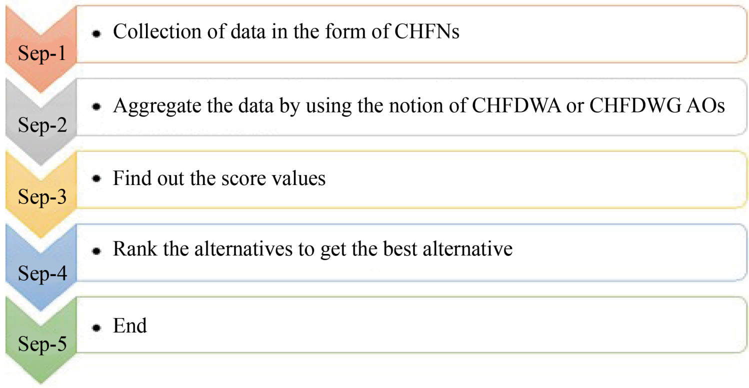

Figure 1. Flow chart of the proposed algorithm

5.3 *Case study*

The shift of power systems from centralized and fossil-based systems to decentralized and renewable-based systems has accelerated the emergence of AI-controlled microgrids. A large industrial park in the manufacturing segment with 150 acres and 25 industrial buildings and complexes is experiencing power quality and reliability issues as well as the integration of renewable energy. It is currently facing voltage fluctuations during the peak industrial processes and has not been able to efficiently incorporate renewable energy into the facility. Currently, the industrial park uses about $4.2 million in electricity consumption, and the power demand is expected to rise by 15% within the next five years, therefore, the management has opted to install the AI-controlled microgrid system. The microgrid will incorporate a significant amount of renewable generation, such as solar photovoltaic (2.5 MW) and wind (1.5 MW) generation; the variability of these sources necessitates the need for advanced energy storage to support the microgrid. The facility has a maximum demand of 3.8 MW and a mean daily energy demand of 45 MWh, with notable load fluctuations during the day owing to industrial applications. Present-day power quality problems lead to 120 hours of lost production time, which equates to about $800,000 in lost revenues. The implementation of an appropriate energy storage system is crucial for:

• Ensuring the stability of power quality and the frequency of its supply in the industry standards.

• Supporting the ability to buy cheap at certain times of the day and sell expensive at other, more demand-intensive times.

• To ensure smooth islanding operation during grid disturbances.

• Enhancing compatibility with the new higher-level AI control algorithms for predictive mode.

• To cater to future expansion plans and also to increase the generation of renewable energy.

• Reducing operating expenses and enhancing the stability of the entire system

Through market analysis and technical advice, decision-makers have defined five different types of energy storage systems, outlined in Table 1.

Table 1. Energy storage system descriptions

Symbols	Energy storage systems	Description
$\mathring{A}_1$	Lithium-ion Battery Banks	High-performance battery systems with new-generation cell chemistry and enhanced battery control circuits. Has a very high energy density of 200-300 Wh/kg, a short response time of less than 100 ms, and reliability in grid-scale applications. It has integrated thermal management, enhanced state of change estimation, and smart cell equalization. Delivers cycling efficiency of 85-95% with a calendar life of 10-15 years and cycle life of 3,000-5,000 cycles at 80% depth of discharge.
$\mathring{A}_2$	Flow Batteries	New generation vanadium redox flow battery systems with power and energy rating in parallel. It has a modular design which makes it easy to expand the capacity by increasing the volume of the electrolyte. Provides an extended operational lifetime of over 20 years, a deep discharge capability of up to 100% depth of discharge, and low capacity loss. Such as complex electrolyte control and the development of new membranes for higher efficiency (75-85%). Suits large loads with a focus on energy time shift and long-length use.
$\mathring{A}_3$	Flywheel Energy Storage	Advanced composite material and magnetic bearings mechanical storage with high performance. With response times less than 20ms and cycle efficiency higher than 90%. Includes vacuum isolation of the motor and stator, active magnetic bearing regulation, and high-performance power converters for connection to the power grid. This technology shows a very high cycle life, low maintenance, and very good frequency control. Most suitable for use in high power and low duration circuits or systems.
$\mathring{A}_4$	Compressed Air Energy Storage (CAES)	A mass storage system that uses underground cavities or above-ground tanks for air compression. Includes adiabatic compression with thermal energy storage enhancement for efficiency of 70-75%. Includes variable speed compression/expansion units, advanced controls, and multiple pressure systems. Has a long service life of more than 30 years with low degradation and high energy storage capacity. Ideal for large-scale energy storage and long-term storage.
$\mathring{A}_5$	Hybrid Supercapacitor Systems	A new generation of hybrid system that integrates high power supercapacitor with new generation battery technology. Incorporates advanced power controlling features for efficient utilization of available power supplies and quick switch performance. The supercapacitor component shows a high power density of more than 10,000 W/kg, moderate energy density between 10-20 Wh/kg, and an almost infinite cycle life. These are sophisticated thermal control systems and smart power delivery systems. Suitable for use in applications that need both power and energy handling capabilities.

To assess these alternatives adequately, the experts have developed three key assessment criteria presented in Table 2 that cover the economic, technical, and operational requirements for effective microgrid management.

Table 2. Key assessment criteria

Symbol	Criteria	Description
c_1	Lifecycle Cost	Full economic cost including the first cost or capital cost (CAPEX), annual or recurring cost for operation and maintenance (OPEX), replacement cost, and finally the cost of disposal at the end of the structure's useful life. Encompasses a specific evaluation of how system efficiency affects operating costs, warranty terms and conditions, and possible sources of income from grid services. Expressed in $/kWh of the system lifetime with degradation effects and capacity augmentation requirements taken into account. It also considers installation costs, space, and the possible effects that this will have on the company's insurance.
c_2	Technical Performance	Round trip efficiency, response time characteristics, cycle life under different operating conditions, and flexibility in depth of discharge. Includes power quality improvement features, overload capacity, self-discharge rates, and stability of the performance under different environmental conditions. Takes into account reliability parameters, failure types, and features of redundancy. These are supplemented with an assessment of monitoring capabilities, fault detection systems, and maintenance requirements.
c_3	Grid Integration Flexibility	The evaluation of the system's capability to deliver multiple grid services concurrently, communicate with advanced AI control schemes, and respond to different levels of renewable power integration. Assesses communication schemes, controls interface performance, and handling of changes in set-point. This entails assessment of black start capacity, ability to island and grid forming. Takes into account the issues of extensibility, compatibility with future technologies, and the possibility to enter new grid service markets.

Now we utilize the proposed algorithm to choose the best alternative using the WV $\acute{w} = (0.24,\ 0.34,\ 0.42)^T$ and $\delta = 4$ given by the decision expert. The steps of the solution are devised below.

Step 1 Assume that the information offered by the decision analyst is based on the CHFSs in Table 3.

Table 3. Complex hesitant fuzzy decision matrix

	C_1	C_2	C_3
A_1	$\{(0.23 + \iota\ 0.41)\}$	$\{(0.71 + \iota\ 0.28), (0.45 + \iota\ 0.34)\}$	$\{(0.14 + \iota\ 0.27), (0.47 + \iota\ 0.76), (0.12 + \iota\ 0.21)\}$
A_2	$\{(0.42 + \iota\ 0.52), (0.44 + \iota\ 0.57)\}$	$\{(0.28 + \iota\ 0.58), (0.43 + \iota\ 0.64)\}$	$\{(0.46 + \iota\ 0.18)\}$
A_3	$\{(0.34 + \iota\ 0.24), (0.14 + \iota\ 0.53)\}$	$\{(0.43 + \iota\ 0.41), (0.40 + \iota\ 0.63), (0.38 + \iota\ 0.47)\}$	$\{(0.33 + \iota\ 0.71)\}$
A_4	$\{(0.44 + \iota\ 0.57), (0.54 + \iota\ 0.66)\}$	$\{(0.44 + \iota\ 0.79), (0.48 + \iota\ 0.76)\}$	$\{(0.45 + \iota\ 0.64)\}$
A_5	$\{(0.34 + \iota\ 0.97), (0.22 + \iota\ 0.36), (0.15 + \iota\ 0.41)\}$	$\{(0.24 + \iota\ 0.39)\}$	$\{(0.53 + \iota\ 0.68), (0.42 + \iota\ 0.36)\}$

Step 2 Utilize the CHFDWA operator or CHFDWG operator to aggregate the data given in Table 3. The aggregated results are given in Table 4.

Table 4. Complex hesitant fuzzy decision matrix

	CHFDWA	CHFDWG
A_1	$\begin{pmatrix}(0.6515+0.3409),\\(0.6527+0.7183),\\(0.6515+0.3362),\\(0.3854+0.3525),\\(0.4451+0.7183),\\(0.3853+0.3486)\end{pmatrix}$	$\begin{pmatrix}(0.1665+0.2867),\\(0.2969+0.3337),\\(0.1441+0.2402),\\(0.1665+0.3024),\\(0.2967+0.3912),\\(0.1441+0.2451)\end{pmatrix}$
A_2	$\begin{pmatrix}(0.4243+0.5281),\\(0.4426+0.5815),\\(0.4293+0.5425),\\(0.4464+0.5878)\end{pmatrix}$	$\begin{pmatrix}(0.3315+0.2142),\\(0.3478+0.2142),\\(0.3323+0.2142),\\(0.3435+0.2142)\end{pmatrix}$
A_3	$\begin{pmatrix}(0.3843+0.6637),\\(0.3648+0.6730),\\(0.3536+0.6642),\\(0.3774+0.6651),\\(0.3552+0.6742),\\(0.3419+0.6656)\end{pmatrix}$	$\begin{pmatrix}(0.3515+0.3363),\\(0.3483+0.3108),\\(0.3452+0.3097),\\(0.1878+0.4700),\\(0.1878+0.5991),\\(0.1601+0.5214)\end{pmatrix}$
A_4	$\begin{pmatrix}(0.4444+0.7451),\\(0.4600+0.7144),\\(0.4840+0.7468),\\(0.4928+0.7177)\end{pmatrix}$	$\begin{pmatrix}(0.4440+0.6288),\\(0.4557+0.6280),\\(0.4582+0.6683),\\(0.4434+0.6668)\end{pmatrix}$
A_5	$\begin{pmatrix}(0.4777+0.9577),\\(0.3777+0.9577),\\(0.4763+0.6316),\\(0.3707+0.3719)\\(0.4762+0.6319),\\(0.3701+0.3791),\end{pmatrix}$	$\begin{pmatrix}(0.2873+0.4551),\\(0.4015+0.3876),\\(0.2553+0.4077),\\(0.2903+0.3685),\\(0.1962+0.4301),\\(0.1960+0.3754),\end{pmatrix}$

Table 5. The score values of alternatives

	A_1	A_2	A_3	A_4	A_5
CHFDWA	0.4989	0.4978	0.5153	0.6001	0.5399
CHFDWG	0.2513	0.3002	0.3463	0.5491	0.3776

Step 4 Utilize $s(\tilde{\tilde{H}}_i)$, rank each alternative, and select the best option. Table 6 shows the ranking results.

Table 6. Ranking results

	Ordering
CHFDWA	$A_4 > A_5 > A_3 > A_1 > A_2$
CHFDWG	$A_4 > A_5 > A_3 > A_2 > A_1$

Step 5 End.

Hence, according to CHFDWA AO A_4 is the best alternative and A_4 is the best alternative according to CHFDWG AOs.

6. Comparative analysis

This section compares the invented work to existing methodologies to show its superiority and correctness. We cannot distinguish between superior and inferior without comparison. We apply previously published concepts to the proposed theory and analyze the results, as shown in Table 7.

First, we make a list of selected theories:

• MADM using HF Dombi-Archimedean weighted (HFDAW) AOs by Liu et al. [15].

• MADM using HF Maclaurin Symmetric Mean Operators (MSMO) and Its Application to MADM by Liu et al. [16].

• HF information aggregation in DM by Xia and Xu [17].

• HF AOs with their application in-group DM by Xia et al. [18].

• Probability Complex Fuzzy AOs by Rehman [26].

The selected theories mainly focus on aggregation, including DAW AOs, MSMO, HF information AOs, HF AOs, and CF AOs. We compare these AOs to our proposed theory for Dombi AOs. We use Dombi AOs to calculate CHFNs score values, and then use the mentioned operators to compare the results.

Based on Table 7, we conclude that all proposed theories do not align with our data and findings. We find that these theories are unable to accommodate our terms and conditions, since all the existing operators can only manage the hesitancy and cannot accommodate two-dimensional information in complex form.

Table 7. Comparative analysis of proposed work

Theories	$s(\acute{A}_1)$	$s(\acute{A}_2)$	$s(\acute{A}_3)$	$s(\acute{A}_4)$	$s(\acute{A}_5)$	Ranking
Liu et al. [15]	×××	×××	×××	×××	×××	×××
Liu et al. [16]	×××	×××	×××	×××	×××	×××
Xia and Xu [17]	×××	×××	×××	×××	×××	×××
Xia et al. [18]	×××	×××	×××	×××	×××	×××
Rehman [26]	×××	×××	×××	×××	×××	×××
Proposed work (CHFDWA)	0.4989	0.4978	0.5153	0.6001	0.5399	$\acute{A}_4 > \acute{A}_5 > \acute{A}_3 > \acute{A}_1 > \acute{A}_2$
Proposed work (CHFDWG)	0.7487	0.6998	0.6537	0.4509	0.6624	$\acute{A}_1 > \acute{A}_2 > \acute{A}_5 > \acute{A}_3 > \acute{A}_4$

Moreover, the theory of MADM methods based on CF AOs for the selection of data management system by Rehman [26] can solve all complex information but cannot handle hesitancy. Hence, the developed approach is more general and it can discuss the more advanced information. The above comparison shows that all the above-discussed theories in Table 7 cannot solve our CHFS information because our information contains all the following characteristics hesitancy, complex numbers data, and two-dimensional information. Therefore, the above comparison shows the effectiveness and superiority of our work. Additionally, the model of comparative analysis is given in Figure 2.

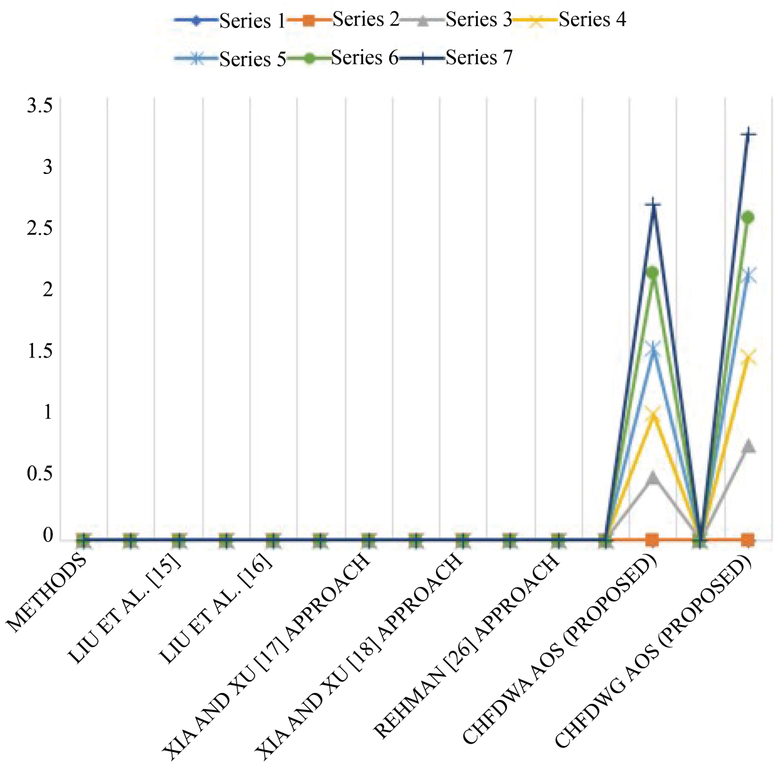

Figure 2. Geometrical model of data given in Table 7

7. Conclusion

The notion of CHFS based on Tamir's CFS idea is a more reliable and accurate idea to discover uncertain and ambiguous information. In this case, the chance of data loss decreases and this approach provides more space to decision makers to handle the MADM problems more effectively. Since Decision-Making (DM) theory is central to nearly all disciplines, we propose a novel complex hesitant fuzzy Multi-Criteria Decision-Making (MCDM) model. The proposed notion of CHFDWA, CHFDOWA, CHFDWG, and CGFDOWG AOs can aggregate the CHFS data and provide an efficient method to discuss the MCDM problems. This method can handle all sorts of real-life MCDM problems, where the data contains uncertainty, hesitancy, and extra fuzzy information. The case study concerns energy storage system selection for AI-controlled microgrids and shows how the theory we have developed can be applied to real-world challenges. Last, we see how this proposed theory is superior to other theories and why it should be adopted.

7.1 *Limitation and future research directions*

The defined notions are limited because defined notions discuss only two-dimensional information about membership grade and cover hesitancy. However, if the decision-makers want to discuss the non-membership grade in the form of hesitant information, then the developed notions fail to hold. In the future, we can apply these notions to the energy sector as discussed in [32, 33] to revolutionize this sector by using the delivered approaches. Moreover, an MCDM approach can be defined based on the proposed notions to discuss the superiority of the delivered approach as given in [34]. Some new aggregation operators can be defined like Sugeno-Weber AOs [35] based on the developed notions. These notions can be extended to complex fuzzy rough structures as given in [36, 37].

Data availability

All data generated during this study are included in this published article.

Ethics declaration statement

The authors state that this is their original work, and it has not been submitted to or under consideration in any other journal simultaneously.

Conflict of interest

The authors declare that they have no conflict of interest regarding the publication of this manuscript.

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