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Research Article

Well-Posedness of the Dispersion-Generalized Modified Benjamin-Ono Equation in Generalized Fourier-Lebesgue Spaces

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Abstract: We consider the Cauchy problem of the dispersion-generalized modified Benjamin-Ono equation on the real line with low-regularity initial data. To this motivation, we apply a generalized Fourier-Lebesgue space $\widehat{M}^{s}_{r,\,q}(\mathbb{R})$, which serves as a unification of modulation spaces and Fourier-Lebesgue spaces. One of the key ingredients is an improved bilinear estimate and the well-posedness is obtained by perturbation arguments.

***Keywords*:** dispersion-generalized modified Benjamin-Ono equations, local well-posedness, modulation spaces, Fourier-Lebesgue spaces

MSC: 35E15, 35Q53, 35Q55

1. Introduction

In this paper, we study the Cauchy problem of the dispersion-generalized modified Benjamin-Ono equation (dgmBO) with low regularity initial data, and refine the previous works in [1, 2], which is a class of nonlinear dispersive partial differential equations that generalizes the classical modified Benjamin-Ono (mBO) equation by incorporating higher-order dispersion terms:

$$\begin{cases} \partial_t u + \partial_x D_x^{1+\alpha} u = \pm u^2 \partial_x u, \ (x,\, t) \in \mathbb{R}^2, \\ u(x,\, 0) = u_0(x) \end{cases} \tag{1}$$

where $0 \leq \alpha \leq 1$. The operator $D_x^{1+\alpha}$ is defined via Fourier transform with symbol $|\xi|^{1+\alpha}$. These equations arise as models describing the weakly nonlinear propagation of long waves in shallow channels with different dispersion effects, introduced by [3, 4]. For instance, modeling nonlinear waves in the Earth's magnetosphere and solar wind, analyzing the behavior of intense laser pulses in nonlinear optical materials, and studying the propagation of electrical signals in nerve fibers. See more examples in [1, 2, 5–8]. In particular, when $\alpha = 0$, it reduces to the modified Benjamin-Ono equation (mBO):

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$$\partial_t u + \mathscr{H}\partial_x^2 u = \pm u^2 \partial_x u, \tag{2}$$

where $\mathscr{H}$ is the Hilbert transform. When $\alpha = 1$, it is the modified KdV equation (mKdV):

$$\partial_t u + \partial_x^3 u = \pm u^2 \partial_x u. \tag{3}$$

This work is motivated by the study of low-regularity well-posedness of (1), as many physically relevant data exhibit extremely low regularity, including the Dirac measure and white noise. One of the significances of understanding the well-posedness (existence, uniqueness, and stability of solutions) helps in designing efficient numerical methods, for instance, the numerical study in [9, 10]. We aim to obtain the well-posedness of (1) for initial data in a space that is as large as possible. There are some appropriate initial data spaces in which the analysis of well-posedness has been conducted. We begin by presenting some choices for the initial data space as follows.

We first introduce the natural choice, Sobolev space $H^s(\mathbb{R})$, defined by the norm

$$\|f\|_{H^s(\mathbb{R})} := \left\| \langle \xi \rangle^s \widehat{f}(\xi) \right\|_{L^2_\xi} \tag{4}$$

where $s \in \mathbb{R}$ and $\langle \cdot \rangle := \left(1 + |\cdot|^2\right)^{1/2}$. In the Sobolev space H^s, Local Well-Posedness (LWP) for $s \geq 1/2$ when $\alpha = 0$ was shown in [7]; $s \geq 1/4$ when $\alpha = 1$ was obtained in [11] and $s \geq 1/2 - \alpha/4$ when $0 < \alpha < 1$ was proved in [1]. These results are optimal when requiring C^3 regularity or uniform continuity of the solution maps. However, global well-posedness with only continuity is still possible. For recent major breakthroughs, see [12] for the BO equation and [6] for the mKdV equation, where integrability plays a crucial role. However, when $0 < \alpha < 1$, the dgmBO equation (1) is not integrable; hence, the method there doesn't work for this situation.

Some other scales of function spaces are used to lower the regularity in the literature. One possibility is to use the Fourier-Lebesgue space $\widehat{H}^s_r$, where

$$\|f\|_{\widehat{H}^s_r(\mathbb{R})} := \left\| \langle \xi \rangle^s \widehat{f}(\xi) \right\|_{L^{r'}_\xi}, \ 1/r + 1/r' = 1 \tag{5}$$

for $s \in \mathbb{R}$ and $1 \leq r \leq \infty$. We write $\widehat{L}^r = \widehat{H}^0_r$. LWP of mKdV in $\widehat{H}^s_r$ for $s \geq 1/2 - 1/2r$ and $4/3 < r \leq 2$ was first obtained by Grünrock [13] with the range of r later extended to $1 < r \leq 2$ in [14].

Another possibility is to use the modulation space $M^s_{r,\, q}(\mathbb{R})$, initially introduced by Feichtinger [15]. The equivalent norm for this space is defined as

$$\|f\|_{M^s_{r,\, q}(\mathbb{R})} := \left\| \langle k \rangle^s \|\Pi_k f\|_{L^r_x} \right\|_{\ell^q_k(\mathbb{Z})} \tag{6}$$

where $s \in \mathbb{R}$, $1 \leq r,\ q \leq \infty$, and Π_k is a Fourier projection operator adapted to the unit interval, as defined in (7). For more examples of the application of modulation space to various equations, see [16]. Utilizing the Fourier restriction norm method adapted to modulation spaces and an improved L^4 Strichartz estimate inspired by [17], LWP of mKdV in $M^s_{2,\, q}(\mathbb{R})$ for $s \geq 1/4$ and $2 \leq q \leq \infty$ was proved in [18, 19]. A simpler proof of dispersion generalized mBO equations in $M^s_{2,\, q}(\mathbb{R})$ for $s \geq 1/2 - \alpha/4$ was given in [2] by proving an improved bilinear Strichartz estimate.

In this paper, we employ a generalized Fourier-Lebesgue space $\widehat{M}^s_{r,\,q}$ that unifies Fourier-Lebesgue and Modulation spaces; see the definition in (8). This space, previously referred to as the Wiener amalgam space or Fourier-amalgam space, has been used in the study of other equations; see [20–24]. In particular, we have considered the mKdV equation ($\alpha = 1$) in this setting [25].

1.1 *Main results*

We begin by defining the generalized Fourier-Lebesgue space $\widehat{M}^s_{r,\,q}(\mathbb{R})$. Let $\varphi \in C_0^\infty(\mathbb{R})$ be a real-valued, non-negative, even, and radially-decreasing function satisfying supp$\varphi \subset [-8/5,\ 8/5]$ and $\varphi \equiv 1$ in $[-5/4,\ 5/4]$. Define $\eta(\xi) := \dfrac{\varphi(\xi)}{\sum_{k\in\mathbb{Z}} \varphi(\xi - k)}$. For $\lambda > 0$ and $m \in \mathbb{R}$, we introduce the following frequency projection operator, which is adapted to intervals centered at λm with length λ:

$$\widehat{(\Pi^\lambda_m f)}(\xi) := \eta(\xi/\lambda - m)\hat{f}(\xi) \tag{7}$$

We call it box decomposition with length λ in the frequency space. When $\lambda = 1$, we simply denote $\Pi^\lambda_m = \Pi_m$. For $s \in \mathbb{R}$, $1 \le r,\ q \le \infty$, the space $\widehat{M}^s_{r,\,q}(\mathbb{R})$ is defined with the following finite norm:

$$\|f\|_{\widehat{M}^s_{r,\,q}(\mathbb{R})} := \left\| \langle k\rangle^s \|\Pi_k f\|_{\widehat{L}^r_x} \right\|_{\ell^q_k(\mathbb{Z})}. \tag{8}$$

By the Hausdorff-Young theorem, we have the inclusion $M^s_{r,\,q} \subset \widehat{M}^s_{r,\,q}$ for $1 \le r \le 2$. Additionally, note the conventions $\widehat{M}^s_{r,\,r'} = \widehat{H}^s_r$ and $M^s_{2,\,q} = \widehat{M}^s_{2,\,q}$. These observations reveal that the Fourier-Lebesgue and modulation spaces can be encapsulated in this wider class of function spaces.

We now state the main result:

Theorem 1 Assume that $0 < \alpha \le 1$ and $1 < r \le 2$. Then the dgmBO equation (1) is locally well-posed in $\widehat{M}^s_{r,\,q}(\mathbb{R})$ for $s \ge \dfrac{1}{2} - \dfrac{\alpha}{2r}$ and $r' \le q \le q_{\alpha,\,r}$, where

$$q_{\alpha,\,r} = \begin{cases} \infty, & \alpha > \sqrt{r} - 1; \\[2ex] \dfrac{2r(r-1)(1+\alpha)}{r-(1+\alpha)^2}, & \alpha \le \sqrt{r} - 1. \end{cases}$$

Remark 1 Our method combines the Fourier restriction method from [14] with the ideas of setting up an improved bilinear Strichartz estimates initialized in [2]. We derive an improved bilinear estimate for (1), which is crucial for handling nonlinear interactions. We establish the local wellposedness of (1) by perturbation arguments in this wider class of function space for all cases except the endpoint $r = 1$. The endpoint case cannot be reached due to the linear estimate, as shown in Lemma 2, which fails to hold. This is a standard limitation when applying the contraction mapping principle.

Remark 2 Our result generalizes and improves previous well-posedness results in the literature, particularly those of [1] in $H^s(\mathbb{R})$, which corresponds to the case $q = r' = 2$ in $\widehat{M}^s_{r,\,q}(\mathbb{R})$, as well as the result of [2] in $M^s_{2,\,q}(\mathbb{R})$, where $r = 2$ in $\widehat{M}^s_{r,\,q}(\mathbb{R})$. The regularity restriction condition in [1] and [2] are both $s \ge 1/2 - \alpha/4$; see the regularity conditions in Figure 1.

Remark 3 The proposed framework in this paper provides a broader setting for analyzing dispersive equations with low regularity data.

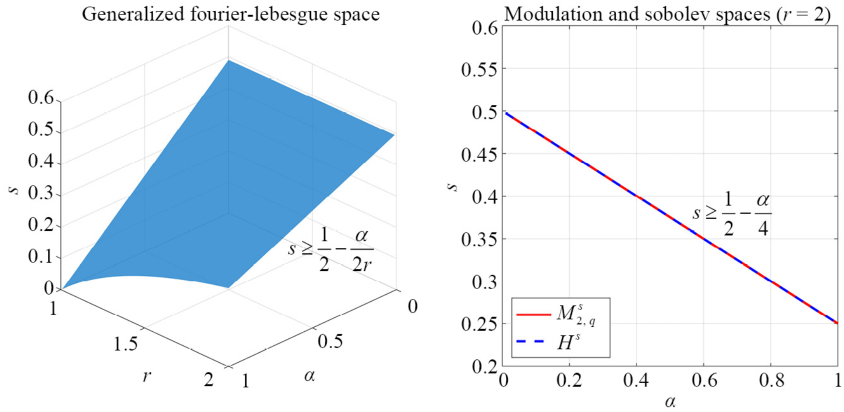

Figure 1. Regularity conditions in different spaces

2. Preliminary

In this paper, for $X, Y \in \mathbb{R}^{+}$, the notation $X \lesssim Y$ means there exists a constant $C > 0$ such that $X \leq CY$. $X \sim Y$ denotes the estimate $X \lesssim Y \lesssim X$. We use $X \ll Y$ to mean $X \leq cY$ for some small constant $0 < c < 0.1$. For $a \in \mathbb{R}$, the notation $a\pm$ represents $a \pm \delta$ for any sufficiently small $\delta > 0$. We use uppercase variables $\{N, N_1, N_2, \dots\}$ to denote dyadic numbers, while lowercase variables $\{i, j, k, l, n, \dots\}$ represent integers. The Fourier transform of u is denoted by $\widehat{u}$, which may represent $\mathscr{F}_x u$, $\mathscr{F}_t u$, or $\mathscr{F}_{t,x} u$, depending on the context.

Let $\chi(\xi) = \varphi(\xi) - \varphi(2\xi)$. For dyadic number $N \in 2^{\mathbb{Z}}$, we define the Littlewood-Paley projectors: $\widehat{P_1 f}(\xi) := \varphi(\xi)\widehat{f}(\xi)$ and for $N > 1$

$$\widehat{P_N f}(\xi) := \chi(\xi/N)\widehat{f}(\xi), \quad \widehat{P_N^{\pm} f}(\xi) := \chi(\xi/N) 1_{\pm\xi \geq 0} \cdot \widehat{f}(\xi). \tag{9}$$

Let $\omega(\xi) = -\xi|\xi|^{1+\alpha}$ be the dispersion relation associated with equation (1) and $W_\alpha(t) = \mathscr{F}^{-1} e^{it\omega(\xi)} \mathscr{F}$ be the linear propagator. The Fourier-Lebesgue type Bourgain space $\widehat{X}_r^{s,b}$ associated with (1) is defined by the norm

$$\|u\|_{\widehat{X}_r^{s,b}} := \left\| \langle \xi \rangle^s \langle \tau - \omega(\xi) \rangle^b \widehat{u}(\xi, \tau) \right\|_{L^{r'}_{\tau,\xi}}. \tag{10}$$

When $s = b = 0$, we write $\widehat{X}_r^{s,b}$ as $\widehat{L}^r_{t,x}$ for simplicity. For the resolution space, we use a variant of the Bourgain space $\widehat{X}_r^{s,b}$ adapted to modulation spaces, given $s, b \in \mathbb{R}$, $1 \leq r, q \leq \infty$, whose norm is defined as

$$\|u\|_{\widehat{X}^{s,b}_{r,q}} := \left\| \|\Pi_k u\|_{\widehat{X}^{s,b}_r} \right\|_{\ell_k{}^q(\mathbb{Z})} \sim \left\| \langle k \rangle^s \left\| \eta(\xi - k) \langle \tau - \omega(\xi) \rangle^b \widehat{u}(\xi, \tau) \right\|_{L^{r'}_{\tau,\xi}} \right\|_{\ell_k{}^q(\mathbb{Z})}. \tag{11}$$

Then $\widehat{X}^{s,b}_{r,r'} = \widehat{X}^{s,b}_r$.

By slightly modifying the proof of Section 2 in [13] and Lemma 4.1 in [26], we have the following result which extends linear estimates in $\widehat{L}^r_x$ to the estimates in $\widehat{X}_r^{0,b}$:

Lemma 1 [Extension Lemma] Let Z be any space-time Banach space satisfying the time modulation estimate

$$\|g(t)F(t,\, x)\|_Z \le \|g\|_{L_t^\infty(\mathbb{R})}\|F(t,\, x)\|_Z \tag{12}$$

for any $F \in Z$ and $g \in L_t^\infty(\mathbb{R})$. Let $T : (h_1,\, \cdots,\, h_k) \to T(h_1,\, \cdots,\, h_k)$ be a spatial multilinear operator for which one has the estimate

$$\|T(W_\alpha(t)f_1,\, \cdots,\, W_\alpha(t)f_k)\|_Z \lesssim \prod_{j=1}^{k} \|f_j\|_{\widehat{L}_x^r} \tag{13}$$

for all $f_1,\, \cdots,\, f_k \in \widehat{L}_x^r$. Then for $b > 1/r$, we have the estimate

$$\|T\,(u_1,\, \cdots,\, u_k)\|_Z \lesssim_k \prod_{j=1}^{k} \|u_j\|_{\widehat{X}_r^{0,\, b}} \tag{14}$$

for all $u_1,\, \cdots,\, u_k \in \widehat{X}_r^{0,\, b}$.

With Lemma 1, we have the embedding for $b > 1/r$,

$$\widehat{X}_{r,\, q}^{s,\, b} \subset C(\mathbb{R};\, \widehat{M}_{r,\, q}^{s}). \tag{15}$$

For $1 \le q \le p \le \infty$, there holds $\|u\|_{\widehat{X}_{r,\, p}^{s,\, b}} \le \|u\|_{\widehat{X}_{r,\, q}^{s,\, b}}$ and

$$\|P_N u\|_{\widehat{X}_{r,\, q}^{s,\, b}} \lesssim N^{1/q-1/p}\|P_N u\|_{\widehat{X}_{r,\, p}^{s,\, b}}. \tag{16}$$

Define

$$\begin{aligned} \|f\|_{\widehat{M}_{r,\, q}^{s,\, b}} &= \left\| \|f\|_{\widehat{M}_r^{s,\, b}} \right\|_{\ell_k^q} \\ &= \left\| \left(\int \langle\xi\rangle^{sr'}\langle\tau\rangle^{br'}|\eta(\xi-k)\widehat{f}(\xi,\, \tau)|^{r'}d\xi d\tau \right)^{1/r'} \right\|_{\ell_k^q} \end{aligned} \tag{17}$$

where the norm of the space $\widehat{M}_r^{s;\, b}$ is given by

$$\|f\|_{\widehat{M}_r^{s,\, b}} = \left(\int \langle\xi\rangle^{sr'}\langle\tau\rangle^{br'}|\eta(\xi-k)\widehat{f}(\xi,\, \tau)|^{r'}d\xi d\tau \right)^{1/r'}.$$

We then deduce that

$$\begin{aligned}\|W_\alpha(-\cdot)f\|_{\widehat{M}^{s,\,b}_{r,\,q}} &= \left\|\left(\int \langle\xi\rangle^{sr'}\langle\tau\rangle^{br'}\left|\eta(\xi-k)\mathscr{F}(W_\alpha(-\cdot)f)(\xi,\,\tau)\right|^{r'}d\xi d\tau\right)^{1/r'}\right\|_{\ell^q_k}\\ &= \left\|\left(\int \langle\xi\rangle^{sr'}\langle\tau-\phi(\xi)\rangle^{br'}|\eta(\xi-k)\widehat{f}(\xi,\,\tau)|^{r'}d\xi d\tau\right)^{1/r'}\right\|_{\ell^q_k} \\ &\sim \|f\|_{\widehat{X}^{s,\,b}_{r,\,q}}.\end{aligned} \tag{18}$$

By slightly modifying the proof of (2.17) and Lemma 2.2 in [13], we obtain the following linear estimates in $\widehat{X}^{s,\,b}_{r,\,q}$.

Lemma 2 [Linear estimates] (1) For $s,\ b \in \mathbb{R}$. Suppose that $1 \le r,\ q \le \infty$. Let $f \in \widehat{M}^s_{r,\,q}$. Then

$$\|\varphi(t)W_\alpha(t)f\|_{\widehat{X}^{s,\,b}_{r,\,q}} \lesssim \|f\|_{\widehat{M}^s_{r,\,q}}. \tag{19}$$

(2) Assume that $1<r<\infty$, $1\le q\le\infty$ and $b'+1\ge b\ge 0\ge b'>-1/r'$. Then

$$\left\|\varphi(t/T)\int_0^t W_\alpha(t-t')F(t')dt'\right\|_{\widehat{X}^{s,\,b}_{r,\,q}} \lesssim T^{b'+1-b}\|F\|_{\widehat{X}^{s,\,b'}_{r,\,q}} \tag{20}$$

for any $0<T\le 1$.

By standard iteration methods, the proof of Theorem 1 reduces to setting up the following trilinear estimate.

Proposition 1 Assume $0<\alpha\le 1$. Let $1<r\le 2$, $r'\le q\le q_{\alpha,\,r}$, and $s\ge s_{\alpha,\,r}$, then for $0<\varepsilon\ll 1$ we have

$$\|\varphi(t)\partial_x(u_1u_2u_3)\|_{\widehat{X}^{s,\,-1/r'+2\varepsilon}_{r,\,q}} \lesssim \prod_{j=1}^{3}\|u_j\|_{\widehat{X}^{s,\,1/r+\varepsilon}_{r,\,q}}. \tag{21}$$

3. Multilinear strichartz estimates

In this section, we review and prove some linear and multilinear Strichartz estimates that are useful to prove Proposition 1 in next section. By slightly modifying the proof of Corollary 3.6 in [13], we have

Lemma 3 [Linear Strichartz estimates] Suppose

$$\frac{2}{p}+\frac{1}{q}=\frac{1}{r}.$$

Then the estimate

$$\left\|D^{\alpha/p}W_\alpha(t)f\right\|_{L^p_tL^q_x}\lesssim\|f\|_{\widehat{L}^r_x} \tag{22}$$

holds if one of the following conditions is fulfilled:

$$
\begin{aligned}
&(i)\ 0 \le 1/p \le 1/4,\ 0 \le 1/q < 1/4,\\
&(ii)\ 1/4 \le 1/q \le 1/q + 1/p < 1/2,\\
&(iii)\ (p,\ q) = (\infty,\ 2).
\end{aligned}
\tag{23}
$$

Moreover, by Lemma 1 we have for any $b > 1/r$,

$$
\left\| D^{\alpha/p} u \right\|_{L_t^p L_x^q} \lesssim \|u\|_{\widehat{X}_r^{0,\ b}}. \tag{24}
$$

In particular, for any $N \ge 1$, we have

$$
\|P_N u\|_{L_{t,\ x}^6} \lesssim N^{-\alpha/6} \|u\|_{\widehat{X}^{0,\ b}}, \tag{25}
$$

and

$$
\|P_N u\|_{L_t^8 L_x^4} \lesssim N^{-\alpha/8} \|u\|_{\widehat{X}^{0,\ b}}. \tag{26}
$$

Let the function $m : \mathbb{R}^2 \to \mathbb{R}$. We define a Coifman-Meyer bilinear multiplier, denoted by $B_m(f,\ g)$ as

$$
\mathscr{F}_x(B_m(f,\ g))(\xi) = \int m(\xi_1,\ \xi - \xi_1) \widehat{f}(\xi_1) \widehat{g}(\xi - \xi_1) d\xi_1. \tag{27}
$$

For dyadic number N, we define

$$
\begin{aligned}
&\mathscr{F}_x(\sigma_1(f,\ g))(\xi) = \int \varphi(2\xi_1 - \xi) \widehat{f}(\xi_1) \widehat{g}(\xi - \xi_1) d\xi_1,\\
&\mathscr{F}_x(\sigma_N(f,\ g))(\xi) = \int \chi\left((2\xi_1 - \xi)/N\right) \widehat{f}(\xi_1) \widehat{g}(\xi - \xi_1) d\xi_1,\ N \ge 2.
\end{aligned}
\tag{28}
$$

Then we obtain the following bilinear estimates.

Lemma 4 [Bilinear estimates] Let $\varphi(\xi_1,\ \xi - \xi_1) = \omega(\xi_1) + \omega(\xi - \xi_1)$ and

$$
\frac{|m(\xi_1,\ \xi - \xi_1)|}{\left|\partial_{\xi_1}[\varphi(\xi_1,\ \xi - \xi_1)]\right|^{1/r}} \sim 1. \tag{29}
$$

Then

$$\|B_m(W_\alpha(t)f,\ W_\alpha(t)g)\|_{\widehat{L}^r_{t,\ x}} \lesssim \|f\|_{\widehat{L}^r_x}\|g\|_{\widehat{L}^r_x}. \tag{30}$$

Moreover for $b > 1/r$, we have

$$\|B_m(u,\ v)\|_{\widehat{L}^r_{t,\ x}} \lesssim \|u\|_{\widehat{X}^{0,\ b}_r}\|v\|_{\widehat{X}^{0,\ b}_r}. \tag{31}$$

Proof. Using (27), we can get that

$$\begin{aligned}&\|B_m(W_\alpha(t)f,\ W_\alpha(t)g)\|_{\widehat{L}^r_{t,\ x}}\\ &=\left\|\int m(\xi_1,\ \xi-\xi_1)e^{it[\omega(\xi_1)+\omega(\xi-\xi_1)]}\widehat{f}(\xi_1)\widehat{g}(\xi-\xi_1)d\xi_1\right\|_{L^{r'}_\xi \widehat{L}^r_t}.\end{aligned} \tag{32}$$

By the change of variables $\tau = y(\xi_1) = \omega(\xi_1)+\omega(\xi-\xi_1)$ and $\tau = y(\xi_1)$, we obtain that

$$\begin{aligned}\text{RHS of (32)} &= \left\|\int m(\xi_1,\ \xi-\xi_1)e^{it\tau}\widehat{f}(\xi_1)\widehat{g}(\xi-\xi_1)\frac{d\xi_1}{d\tau}d\tau\right\|_{L^{r'}_\xi \widehat{L}^r_t}\\
&= \left\|\left(\int \left|m(\xi_1,\ \xi-\xi_1)\widehat{f}(\xi_1)\widehat{g}(\xi-\xi_1)\right|^{r'}|y'(\xi_1)|^{-r'}\frac{d\tau}{d\xi_1}d\xi_1\right)^{1/r'}\right\|_{L^{r'}_\xi}\\
&= \left\|m(\xi_1,\ \xi-\xi_1)\widehat{f}(\xi_1)\widehat{g}(\xi-\xi_1)|y'(\xi_1)|^{-1/r}\right\|_{L^{r'}_\xi L^{r'}_{\xi_1}}\\
&\lesssim \left\||\widehat{f}|^{r'} * |\widehat{g}|^{r'}\right\|^{1/r'}_{L^1_\xi} \lesssim \|f\|_{\widehat{L}^r_x}\|g\|_{\widehat{L}^r_x},\end{aligned}$$

where we used the assumption (29) in the last second inequality and Young's inequality in the last inequality. □

With Lemma 4, we have

Corollary 1 (1) Let $N_1,\ N_2 \geq 1$ be dyadic numbers and $N_1 \gg N_2$. Then for $b > 1/r$

$$\|P_{N_1}uP_{N_2}v\|_{\widehat{L}^r_{t,\ x}} \lesssim N_1^{-(1+\alpha)/r}\|P_{N_1}u\|_{\widehat{X}^{0,\ b}_r}\|P_{N_2}v\|_{\widehat{X}^{0,\ b}_r}. \tag{33}$$

(2) Suppose that $k_1,\ k_2 \in \mathbb{Z}$ and $|k_1+k_2|,\ |k_1-k_2| \geq 2$. Then for $b > 1/r$

$$\left\|\Pi_{k_1}u\Pi_{k_2}v\right\|_{\widehat{L}^r_{t,\ x}} \lesssim B(k_1,\ k_2,\ \alpha)\left\|\Pi_{k_1}u\right\|_{\widehat{X}^{0,\ b}_r}\left\|\Pi_{k_2}v\right\|_{\widehat{X}^{0,\ b}_r}, \tag{34}$$

where $B(k_1,\ k_2,\ \alpha) = \max\{|k_1+k_2|,\ |k_1-k_2|\}^{-\alpha/r}\min\{|k_1+k_2|,\ |k_1-k_2|\}^{-1/r}$.

After a temporal localization, we can improve (31) as follows. It is crucial in the proof of Proposition 1.

Lemma 5 [Improved bilinear estimate] Assume $N_1 \sim N_2 \gg N_3 \geq 1$ and

$$m_{\pm}(\xi_1,\ \xi_2) = \chi_{N_3}(\xi_1 \pm \xi_2)\chi_{N_1}(\xi_1)\chi_{N_2}(\xi_2).$$

Then for $r' \leq q \leq \infty$ and $b > 1/r$ we have

$$\begin{aligned} &\left\|B_{m_-}(u,\ v)\right\|_{\widehat{L}^r_{t,\ x}} + \left\|\varphi(t)B_{m_+}(u,\ v)\right\|_{\widehat{L}^r_{t,\ x}} \\ &\lesssim N_1^{-\alpha/r}N_1^{1/r'-2/q}N_3^{1/r'-1/r}\|u\|_{\widehat{X}^{0,\ b}_{r,\ q}}\|v\|_{\widehat{X}^{0,\ b}_{r,\ q}}. \end{aligned} \tag{35}$$

Proof. We begin by considering the case with m_-. We decompose m_- as

$$\begin{aligned} m_-(\xi_1,\ \xi_2) &= \sum_j \sum_{j_1,\ j_2 \in \mathbb{Z}} \chi_{N_3}(\xi_1-\xi_2)\chi_{N_1}(\xi_1)\chi_{N_2}(\xi_2) \\ &\quad \cdot \chi\left(\frac{j_1N_3+\xi_1}{N_3}\right)\chi\left(\frac{j_2N_3+\xi_2}{N_3}\right)\chi\left(\frac{jN_3+\xi_1+\xi_2}{N_3}\right) \\ &= \sum_j m_-^j(\xi_1,\ \xi_2). \end{aligned} \tag{36}$$

By the support properties of the functions, we know that in the above summations $|j_1-j_2| \leq 3$ and $|j-j_1-j_2| \leq 5$, j takes $O(N_1/N_3)$ values. Hence for the fixed j, we have $|j_i - j/2| \leq 10$, $i = 1,\ 2$. For simplicity, we assume $j_1 = j_2 = j/2$. Then

$$\begin{aligned} &\left\|B_{m_-}(W_\alpha(t)f,\ W_\alpha(t)g)\right\|_{\widehat{L}^r_{t,\ x}} \\ &\lesssim \left\|\mathscr{F}(B_{m_-^j}(W_\alpha(t)f,\ W_\alpha(t)g))\right\|_{l^{r'}_j L^{r'}_{\tau,\ \xi}} \\ &\lesssim N_1^{-\alpha/r}N_3^{-1/r}\left\|\left\|\chi\left(\frac{jN_3/2+\xi}{N_3}\right)\widehat{f}\right\|_{L^{r'}_\xi}\left\|\chi\left(\frac{jN_3/2+\xi}{N_3}\right)\widehat{g}\right\|_{L^{r'}_\xi}\right\|_{l^{r'}_j} \\ &\lesssim N_1^{-\alpha/r}N_3^{-1/r}N_3^{2(1/r'-1/q)}(N_1/N_3)^{1/r'-2/q}\|\widehat{\Pi_k f}\|_{l^q_k L^{r'}_\xi}\|\widehat{\Pi_k g}\|_{l^q_k L^{r'}_\xi} \\ &\lesssim N_1^{-\alpha/r}N_1^{1/r'-2/q}N_3^{1/r'-1/r}\|\widehat{\Pi_k f}\|_{l^q_k L^{r'}_\xi}\|\widehat{\Pi_k g}\|_{l^q_k L^{r'}_\xi}, \end{aligned} \tag{37}$$

where the bilinear estimate (34) is used in the second step. By Lemma 1, the desired (35) with m_- is obtained.

We now turn to the case with m_+. The high frequency is decomposed into the intervals with length N_3.

$$\widehat{\varphi}(\tau)\mathscr{F}(B_{m_+}(W_\alpha(t)f,\ W_\alpha(t)g))$$

$$=\widehat{\varphi}(\tau)\mathscr{F}(P_{N_3}(W_\alpha(t)P_{N_1}f\cdot W_\alpha(t)P_{N_2}g))$$

$$=\sum_{|j_1+j_2|\leq 3}\widehat{\varphi}(\tau)\mathscr{F}(P_{N_3}(W_\alpha(t)P_{N_1}\Pi_{j_1}^{N_3}f\cdot W_\alpha(t)P_{N_2}\Pi_{j_2}^{N_3}g)).$$

Without loss of generality, we may assume $j_1=-j_2$. From the almost orthogonality property as follows, we see that

$$\begin{aligned}&\left\|\sum_j\widehat{\varphi}(\tau)\mathscr{F}(P_{N_3}(W_\alpha(t)P_{N_1}\Pi_j^{N_3}f\cdot W_\alpha(t)P_{N_2}\Pi_{-j}^{N_3}g))\right\|_{L^{r'}_{\tau,\ \xi}}\\&\lesssim\left\|\widehat{\varphi}(\tau)\mathscr{F}(P_{N_3}(W_\alpha(t)P_{N_1}\Pi_j^{N_3}f\cdot W_\alpha(t)P_{N_2}\Pi_{-j}^{N_3}g))\right\|_{l_j^{r'}L^{r'}_{\tau,\ \xi}}.\end{aligned}\tag{38}$$

Once we obtain (38), then the desired estimate with m_+ is proven. We now turn to (38). We observe that the support of $\widehat{\varphi}(\tau)\mathscr{F}[P_{N_3}(W_\alpha(t)P_{N_1}\Pi_j^{N_3}f\cdot W_\alpha(t)P_{N_2}\Pi_{-j}^{N_3}g)]$ lies in the set E_j where

$$E_j=\{(\tau,\ \xi):|\xi|\sim N_3,\ |\tau-(2+\alpha)|jN_3|^{1+\alpha}\xi|\lesssim N_1^\alpha N_3^2,\ |jN_3|\sim N_1\}.\tag{39}$$

It's easy to verify that $\{E_j\}$ is finitely overlapping. □

Remark 4 Lemma 4 implies that

$$\|B_{m_-}(u,\ v)\|_{\widehat{L}^r_{t,\ x}}+\|\varphi(t)B_{m_+}(u,\ v)\|_{\widehat{L}^r_{t,\ x}}\lesssim N_1^{-\alpha/r}N_1^{2/r'-2/q}N_3^{-1/r}\|u\|_{\widehat{X}^{0,\ b}_{r,\ q}}\|v\|_{\widehat{X}^{0,\ b}_{r,\ q}}.$$

However, with Lemma 5, we can gain a factor $N_1^{-1/r'}N_3^{1/r'}$, which is the key ingredient to control the case $N_3\ll N_1$. By Lemma 5, an improved L^4-estimates is obtained:

Corollary 2 Let $N\geq 1$. We have

$$\|\varphi(t)P_Nu\|_{L^4_{t,\ x}}\lesssim N^{-\alpha/8+}\|\varphi\|_{L^8_t\cap L^\infty_t}\|u\|_{\widehat{X}^{0,\ 1/2+}_{2,\ 4}}.\tag{40}$$

Furthermore, if $4/3<r\leq 2$, then for $b>1/r$

$$\|\varphi(t)P_Nu\|_{L^4_{t,\ x}}\lesssim N^{-\alpha(1/2r-1/8)+}\|u\|_{\widehat{X}^{0,\ b}_{r,\ 4}}.\tag{41}$$

Proof. We begin with (40). It suffices to prove

$$\|\varphi(t)W_\alpha(t)P_N^+ f\|_{L^4_{t,\,x}} \lesssim N^{-\alpha/8+}\|\varphi\|_{L^8_t\cap L^\infty_t}\|f\|_{\widehat{M}^0_{2,\,4}}. \tag{42}$$

Indeed,

$$\begin{aligned}\|\varphi(t)W_\alpha(t)P_N^+ f\|^2_{L^4_{t,\,x}} &= \|\varphi^2(t)W_\alpha(t)P_N^+ f\cdot W_\alpha(t)P_N^+ f\|_{L^2_{t,\,x}}\\ &\le \sum_{K\ge 1}\|\sigma_K(\varphi(t)W_\alpha(t)P_N^+ f,\ \varphi(t)W_\alpha(t)P_N^+ f)\|_{L^2_{t,\,x}}.\end{aligned}$$

Then for $K=1$, we have by Lemma 3.

$$\begin{aligned}&\|\sigma_1(\varphi(t)W_\alpha(t)P_N^+ f,\ \varphi(t)W_\alpha(t)P_N^+ f)\|_{L^2_{t,\,x}}\\ &\lesssim \|\sigma_1(\varphi(t)W_\alpha(t)P_N^+\Pi_k f,\ \varphi(t)W_\alpha(t)P_N^+\Pi_k f)\|_{\ell^2_k L^2_{t,\,x}}\\ &\lesssim \left\|\|\varphi(t)W_\alpha(t)P_N^+\Pi_k f\|^2_{L^4_{t,\,x}}\right\|_{\ell^2_k}\\ &\lesssim \|\varphi\|^2_{L^8_t}\left\|\|W_\alpha(t)P_N^+\Pi_k f\|^2_{L^8_t L^4_x}\right\|_{\ell^2_k}\\ &\lesssim N^{-\alpha/4}\|\varphi\|^2_{L^8_t}\left\|\|\Pi_k f\|_{\widehat{L}^2_x}\right\|^2_{\ell^4_k}\end{aligned}$$

where we applied (22) in the last step. Using Lemma 5, we can solve the case $K\ge 2$. Then (40) is obtained by Lemma 1.

We now turn to the proof of (41). Taking $r=4/3$ in (24), we have

$$\|\varphi(t)P_N u\|_{L^4_{t,\,x}} \lesssim N^{-\alpha/4-}\|u\|_{\widehat{X}^{0,\,3/4+}_{4/3,\,4}}. \tag{43}$$

On the other hand, from (40) we get

$$\|\varphi(t)P_N u\|_{L^4_{t,\,x}} \lesssim N^{-\alpha/8+}\|u\|_{\widehat{X}^{0,\,1/2+}_{2,\,4}}. \tag{44}$$

The interpolation of (43) with (44) yields (41). □

4. Proof of proposition 1

Define the resonant function as

$$\Phi(\xi_1,\ \xi_2,\ \xi_3) := \omega(\xi_1) + \omega(\xi_2) + \omega(\xi_3) - \omega(\xi_1 + \xi_2 + \xi_3). \tag{45}$$

We recall the following property.

Lemma 6 [Lemma 4.1, [2]] Assume that $\xi_1 + \xi_2 + \xi_3 = \xi$ and $|\xi_1| \geq |\xi_2| \geq |\xi_3|$. Then

$$|\Phi(\xi_1, \xi_2, \xi_3)| \sim |\xi_1|^{\alpha} |(\xi_1 + \xi_2)(\xi_2 + \xi_3)|. \tag{46}$$

Now we prove Proposition 1 using the estimates in the previous section and the non-resonant structure. For fixed $1 < r \leq 2$ and $0 < \varepsilon < \dfrac{\alpha(r-1)^2}{2^{10} r^2}$, we set $b = \dfrac{1}{r} + \varepsilon$ and $b' = -\dfrac{1}{r'} + 2\varepsilon$ without loss of generality. Using the duality argument, it is equivalent to show

$$\left| \int \widehat{\varphi}^4(t) u_1 u_2 u_3 v dxdt \right| \lesssim \prod_{j=1}^{3} \|u_j\|_{\widehat{X}^{s,\ 1/r+\varepsilon}_{r,\ q}} \|v\|_{\widehat{X}^{-1-s,\ 1/r'-2\varepsilon}_{r',\ q'}}. \tag{47}$$

To simplify notations, we write $u_{N_j} = \widehat{\varphi}(t) P_{N_j} u_j$ for $j = 1,\ 2,\ 3$, and $v_N = \widehat{\varphi}(t) P_N v$. By applying the Littlewood-Paley dyadic decomposition to each term in (47), it is sufficient to show that

$$\left| \int u_{N_1} u_{N_2} u_{N_3} v_N dxdt \right| \leq C(N_1,\ N_2,\ N_3,\ N) \prod_{j=1}^{3} \|u_{N_j}\|_{\widehat{X}^{s,\ 1/r+\varepsilon}_{r,\ q}} \|v_N\|_{\widehat{X}^{-1-s,\ 1/r'-2\varepsilon}_{r',\ q'}}, \tag{48}$$

where $C(N_1,\ N_2,\ N_3,\ N)$ is an appropriate constant that allows summation over all dyadic numbers. After a straightforward computation, we obtain

$$\int u_{N_1} u_{N_2} u_{N_3} v_N dxdt = \int_{\Gamma} \prod_{j=1}^{3} \widehat{u}_{N_j}(\xi_j,\ \tau_j) \widehat{v}_N(\xi,\ \tau) d\mu, \tag{49}$$

where $\Gamma = \{\xi_1 + \xi_2 + \xi_3 = \xi,\ \tau_1 + \tau_2 + \tau_3 = \tau\}$ and $d\mu$ represents the induced Lebesgue measure. For a function u defined on $\mathbb{R}^2$, we introduce the following transform:

$$\widetilde{u}(x,\ t) := \mathscr{F}^{-1} |\widehat{u}(\xi,\ \tau)|. \tag{50}$$

Note that $\|u\|_{\widehat{X}^{s,\ b}_{r,\ q}} = \|\widetilde{u}\|_{\widehat{X}^{s,\ b}_{r,\ q}}$ and

$$\int_{\Gamma} \prod_{j=1}^{3} |\widehat{u}_{N_j}(\xi_j,\ \tau_j)| \cdot |\widehat{v}_N(\xi,\ \tau)|\ d\mu \sim \left| \int \widetilde{u}_{N_1} \widetilde{u}_{N_2} \widetilde{u}_{N_3} \widetilde{v}_N\ dxdt \right|. \tag{51}$$

This allows us to make estimates on both the physical and frequency sides. To prove (48), it suffices to prove

$$\left|\int \widetilde{u}_{N_1}\widetilde{u}_{N_2}\widetilde{u}_{N_3}\widetilde{v}_N \; dxdt\right| \leq C(N_1,\, N_2,\, N_3,\, N)\prod_{j=1}^{3}\|u_{N_j}\|_{\widehat{X}^{s,\, 1/r+\varepsilon}_{r,\, q}}\|v_N\|_{\widehat{X}^{-1-s,\, 1/r'-2\varepsilon}_{r',\, q'}}. \tag{52}$$

By symmetry, we may assume $N_1 \geq N_2 \geq N_3$, which implies $N_1 \sim \max(N_2,\, N)$ due to the frequency support of the functions. Then the summation of dyadic numbers is dominated by

$$\begin{aligned}&\sum_{N_1,\, N_2,\, N_3,\, N: N_1\sim\max(N_2,\, N)} C(N_1,\, N_2,\, N_3,\, N)\prod_{j=1}^{3}\|u_{N_j}\|_{\widehat{X}^{s,\, 1/r+\varepsilon}_{r,\, q}}\|\widetilde{v}_N\|_{\widehat{X}^{-1-s,\, 1/r'-2\varepsilon}_{r',\, q'}}\\ &\lesssim \prod_{j=1}^{3}\|u_j\|_{\widehat{X}^{s,\, 1/r+\varepsilon}_{r,\, q}}\|v\|_{\widehat{X}^{-1-s,\, 1/r'-2\varepsilon}_{r',\, q'}}.\end{aligned} \tag{53}$$

Let σ_j and σ denote the modulations defined as

$$\sigma_j = \tau_j - \omega(\xi_j),\ \sigma = \tau - \omega(\xi).$$

Under the constraints $\xi = \xi_1 + \xi_2 + \xi_3$ and $\tau = \tau_1 + \tau_2 + \tau_3$, we obtain

$$\tau - \omega(\xi) = \tau_1 - \omega(\xi_1) + \tau_2 - \omega(\xi_2) + \tau_3 - \omega(\xi_3) + \Phi(\xi_1,\, \xi_2,\, \xi_3).$$

Let

$$\sigma_{max} = \max\{|\sigma_1|,\, |\sigma_2|,\, |\sigma_3|,\, |\sigma|\}.$$

From the given domain Γ in (49), we may assume that

$$\sigma_{max} \gtrsim |\Phi|. \tag{54}$$

It is time to prove (48) case by case.

Case 1: $N_1 \lesssim 1$ or $N_1 \gg 1$ and $\sigma_{max} \gg N_1^{20r'}$. These trivial cases can be directly treated.

• $N_1 \lesssim 1$, then $N_2,\, N_3,\, N \lesssim 1$ and we have

$$\begin{aligned}&\left|\int \widetilde{u}_{N_1}\widetilde{u}_{N_2}\widetilde{u}_{N_3}\widetilde{v}_N dxdt\right|\\ &\lesssim \|\widetilde{u}_{N_1}\|_{\widehat{L}^r_{t,\, x}}\|\widetilde{u}_{N_2}\|_{\widehat{L}^\infty_{t,\, x}}\|\widetilde{u}_{N_3}\|_{\widehat{L}^\infty_{t,\, x}}\|v_N\|_{\widehat{L}^{r'}_{t,\, x}}\\ &\lesssim \|\widetilde{u}_{N_1}\|_{\widehat{X}^{s,\, 1/r+}_{r,\, q}}\|\widetilde{u}_{N_2}\|_{\widehat{X}^{s,\, 1/r+}_{r,\, q}}\|\widetilde{u}_{N_3}\|_{\widehat{X}^{s,\, 1/r+}_{r,\, q}}\|\widetilde{v}_N\|_{\widehat{X}^{-1-s,\, 1/r'-2\varepsilon}_{r',\, q'}},\end{aligned}$$

which is summable.

• $N_1 \gg 1$ and $\sigma_{max} \gg N_1^{20r'}$. We consider $|\sigma| = \sigma_{max}$, as the other cases follow the same line. Using Hölder's inequality yields

$$\left|\int \widetilde{u}_{N_1}\widetilde{u}_{N_2}\widetilde{u}_{N_3}\widetilde{v}_N dxdt\right|$$

$$\lesssim \|\widetilde{u}_{N_1}\|_{\widehat{L}^r_{t,\,x}} \|\widetilde{u}_{N_2}\|_{\widehat{L}^\infty_{t,\,x}} \|\widetilde{u}_{N_3}\|_{\widehat{L}^\infty_{t,\,x}} \|v_N\|_{\widehat{L}^{r'}_{t,\,x}}$$

$$\lesssim N_1^{-2}\|\widetilde{u}_{N_1}\|_{\widehat{X}^{s,\,1/r+}_{r,\,q}} \|\widetilde{u}_{N_2}\|_{\widehat{X}^{s,\,1/r+}_{r,\,q}} \|\widetilde{u}_{N_3}\|_{\widehat{X}^{s,\,1/r+}_{r,\,q}} \|\widetilde{v}_N\|_{\widehat{X}^{-1-s,\,1/r'-2\varepsilon}_{r',\,q'}},$$

which is summable.

Therefore, we only need to consider

$$N_1 \gg 1 \quad \text{and} \quad \sigma_{max} \lesssim N_1^{20r'} \tag{55}$$

in the following.

Case 2: $N_1 \gg N_2 \geq N_3$.

In this case, we have $N_1 \sim N$ and $|\Phi| \sim N_1^{1+\alpha}|\xi_2+\xi_3|$. Assuming that $\sigma_{max} \gtrsim N_1^{1+\alpha}|\xi_2+\xi_3|$ in Γ, we obtain

$$\left|\int \widetilde{u}_{N_1}\widetilde{u}_{N_2}\widetilde{u}_{N_3}\widetilde{v}_N \;\; dxdt\right| \leq \sum_K \left|\int P_K(\widetilde{u}_{N_1}\widetilde{v}_N)\widetilde{u}_{N_2}\widetilde{u}_{N_3} dxdt\right|. \tag{56}$$

• $K \lesssim 1$ in (56). Since $|\xi_1+\xi| = |\xi_2+\xi_3| \lesssim 1$, we have

$$(56) \lesssim \sum_{|k_1+k|\lesssim 1} \int \left|(\Pi_{k_1}\widetilde{u}_{N_1})\widetilde{u}_{N_2}\widetilde{u}_{N_3}(\Pi_k \widetilde{v}_N)\right| dxdt. \tag{57}$$

Applying Hölder's inequality, bilinear estimate (33), and (16), we obtain

$$(57) \lesssim \sum_{|k_1+k|\lesssim 1} \|(\Pi_{k_1}\widetilde{u}_{N_1})\widetilde{u}_{N_2}\|_{\widehat{L}^r_{t,\,x}} \|\widetilde{u}_{N_3}(\Pi_k\widetilde{v}_N)\|_{\widehat{L}^{r'}_{t,\,x}}$$

$$\lesssim \sum_k N_1^{-(1+\alpha)/r} \|\Pi_{-k}\widetilde{u}_{N_1}\|_{\widehat{X}^{0,\,1/r+}_r} \|\widetilde{u}_{N_2}\|_{\widehat{X}^{0,\,1/r+}_r}$$

$$\cdot N_1^{-(1+\alpha)/r'} \|\widetilde{u}_{N_3}\|_{\widehat{X}^{0,\,1/r'+}_{r'}} \|\Pi_k\widetilde{v}_N\|_{\widehat{X}^{0,\,1/r'+}_{r'}}$$

$$\lesssim N_1^{-(1+\alpha)} N_2^{1/r'-1/q} N_3^{1/r-1/r'} N_3^{1/r'-1/q}$$

$$\cdot \|\widetilde{u}_{N_1}\|_{\widehat{X}^{0,\ 1/r+}_{r,\ q}} \|\widetilde{u}_{N_2}\|_{\widehat{X}^{0,\ 1/r+}_{r,\ q}} \|\widetilde{u}_{N_3}\|_{\widehat{X}^{0,\ 1/r+}_{r,\ q}} \|\widetilde{v}_N\|_{\widehat{X}^{0,\ 1/r'+}_{r',\ q'}}$$

$$\lesssim N_1^{-\alpha+\alpha/8r'+} N_2^{-s+1/r'-1/q} N_3^{-s+1/r-1/q} \tag{58}$$

$$\cdot \|u_{N_1}\|_{\widehat{X}^{s,\ 1/r+}_{r,\ q}} \|u_{N_2}\|_{\widehat{X}^{s,\ 1/r+}_{r,\ q}} \|u_{N_3}\|_{\widehat{X}^{s,\ 1/r+}_{r,\ q}} \|v_N\|_{\widehat{X}^{-1-s,\ 1/r'-2\varepsilon}_{r',\ q'}}.$$

In the last step, we applied the fact that

$$\|v_N\|_{\widehat{X}^{-1-s,\ 1/r'+}_{r',\ q'}} \lesssim N_1^{\frac{\alpha}{8r'}+} \|v_N\|_{\widehat{X}^{-1-s,\ 1/r'-2\varepsilon}_{r',\ q'}} \tag{59}$$

due to $\sigma_{max} \lesssim N_1^{20r'}$ and $0 < \varepsilon < \dfrac{\alpha(r-1)^2}{2^{10} r^2}$. If $-s+1/r-1/q < 0$, the above bound is summable. If $-s+1/r-1/q \geq 0$, then

$$(58) \lesssim N_1^{-\alpha+\alpha/8r'+} N_2^{-2s+1-2/q} \|u_{N_1}\|_{\widehat{X}^{s,\ 1/r+}_{r,\ q}} \|u_{N_2}\|_{\widehat{X}^{s,\ 1/r+}_{r,\ q}} \|u_{N_3}\|_{\widehat{X}^{s,\ 1/r+}_{r,\ q}} \|v_N\|_{\widehat{X}^{-1-s,\ 1/r'-2\varepsilon}_{r',\ q'}},$$

which is summable if $s > 1/2 - \alpha/2 - 1/q + \alpha/16r'$.

• $K \gtrsim 1$ in (56). Since $|\xi_1 + \xi| = |\xi_2 + \xi_3| \sim K \gtrsim 1$, we apply a unit decomposition with length K to the terms $\widetilde{u}_{N_1}$ and $\widetilde{v}_N$. Then we obtain

$$\begin{aligned} (56) &\lesssim \sum_K \sum_{|k_1+k|\lesssim 1} \int \left| (\Pi^K_{k_1} \widetilde{u}_{N_1}) \widetilde{u}_{N_2} \widetilde{u}_{N_3} (\sum_m \Pi_m \Pi^K_k \widetilde{v}_N) \right| dxdt \\ &\lesssim \sum_K \sum_k \int \left| (\Pi^K_{-k} \widetilde{u}_{N_1}) \widetilde{u}_{N_2} \widetilde{u}_{N_3} (\sum_m \Pi_m \Pi^K_k \widetilde{v}_N) \right| dxdt. \end{aligned} \tag{60}$$

If $\underline{|\sigma| = \sigma_{max}}$, we have

$$\lesssim \sum_K \sum_k \sum_m \left\| (\Pi^K_{-k} \widetilde{u}_{N_1}) \widetilde{u}_{N_2} \right\|_{\widehat{L}^r_{t,\ x}} \|\widetilde{u}_{N_3}\|_{\widehat{L}^{r'}_x \widehat{L}^\infty_t} \left\| \Pi_m \Pi^K_k \widetilde{v}_N \right\|_{\widehat{L}^\infty_x \widehat{L}^{r'}_t}$$

$$\lesssim \sum_K \sum_k K^{1/r'-1/q} N_1^{-(1+\alpha)/r} N_2^{1/r'-1/q} N_3^{1/r-1/q} (N_1^{1+\alpha} K)^{-1/r'-} K^{1/q}$$

$$\cdot \left\| \Pi^K_{-k} \widetilde{u}_{N_1} \right\|_{\widehat{X}^{0,\ 1/r+}_{r,\ q}} \|\widetilde{u}_{N_2}\|_{\widehat{X}^{0,\ 1/r+}_{r,\ q}} \|\widetilde{u}_{N_3}\|_{\widehat{X}^{0,\ 1/r+}_{r,\ q}} \|\Pi^K_k v_N\|_{\widehat{X}^{0,\ 1/r'+}_{r',\ q'}}$$

$$\lesssim N_1^{-\alpha+\alpha/8r'} N_2^{-s+1/r'-1/q} N_3^{-s+1/r-1/q}$$

$$(60)\cdot\|\widetilde{u}_{N_1}\|_{\widehat{X}^{s,\ 1/r+}_{r,\ q}}\|\widetilde{u}_{N_2}\|_{\widehat{X}^{s,\ 1/r+}_{r,\ q}}\|\widetilde{u}_{N_3}\|_{\widehat{X}^{s,\ 1/r+}_{r,\ q}}\|v_N\|_{\widehat{X}^{-1-s,\ 1/r'-2\varepsilon}_{r',\ q'}}. \tag{61}$$

The summation is convergent if $s > 1/2-\alpha/2-1/q+\alpha/16r'$.

If $\underline{|\sigma_3| = \sigma_{max}}$, then by the bilinear estimate (33), Hölder's inequality (16), and the bound $\sigma_{max} \gtrsim N_1^{1+\alpha}K$, we obtain

$$\begin{aligned}
(60) &\lesssim \sum_K\sum_k\sum_m \|(\Pi^K_{-k}\widetilde{u}_{N_1})\widetilde{u}_{N_2}\|_{\widehat{L}^r_{t,\ x}}\|\widetilde{u}_{N_3}\|_{\widehat{L}^{r'}_{t,\ x}}\|\Pi_m\Pi^K_k\widetilde{v}_N\|_{\widehat{L}^\infty_{t,\ x}} \\
&\lesssim \sum_K\sum_k N_1^{-(1+\alpha)/r}N_3^{1/r-1/r'}(N_1^{1+\alpha}K)^{-1/r-}K^{1/q} \\
&\quad\cdot\|\Pi^K_{-k}\widetilde{u}_{N_1}\|_{\widehat{X}^{0,\ 1/r+}_r}\|\widetilde{u}_{N_2}\|_{\widehat{X}^{0,\ 1/r+}_r}\|\widetilde{u}_{N_3}\|_{\widehat{X}^{0,\ 1/r+}_r}\|\Pi^K_k\widetilde{v}_N\|_{\widehat{X}^{0,\ 1/r'+}_{r',\ q'}} \\
&\lesssim N_1^{1-2(1+\alpha)/r+\alpha/8r'}N_2^{-s+1/r'-1/q}N_3^{-s+1/r-1/q} \\
&\quad\cdot\|\widetilde{u}_{N_1}\|_{\widehat{X}^{s,\ 1/r+}_{r,\ q}}\|\widetilde{u}_{N_2}\|_{\widehat{X}^{s,\ 1/r+}_{r,\ q}}\|\widetilde{u}_{N_3}\|_{\widehat{X}^{s,\ 1/r+}_{r,\ q}}\|v_N\|_{\widehat{X}^{-1-s,\ 1/r'-2\varepsilon}_{r',\ q'}}.
\end{aligned}\tag{62}$$

We can sum over all the dyadic numbers if $s > 1-(1+\alpha)/r-1/q+\alpha/16r'$.

If $\underline{|\sigma_2| = \sigma_{max}}$, following the proof of the subcase $|\sigma_3| = \sigma_{max}$, we get the same restriction for s and r.

If $\underline{|\sigma_1| = \sigma_{max}}$, we employ the bilinear estimate in the space $\widehat{L}^r_{t,\ x}$ due to $r \le r'$. We derive

$$\begin{aligned}
(60) &\lesssim \sum_K\sum_k\sum_m \|\Pi^K_{-k}\widetilde{u}_{N_1}\|_{\widehat{L}^r_{t,\ x}}\|\widetilde{u}_{N_2}(\Pi_m\Pi^K_k\widetilde{v}_N)\|_{\widehat{L}^{r'}_{t,\ x}}\|\widetilde{u}_{N_3}\|_{\widehat{L}^\infty_{t,\ x}} \\
&\lesssim \sum_K\sum_k\sum_m (N_1^{1+\alpha}K)^{-1/r-}N_1^{-(1+\alpha)/r'}N_3^{1/r} \\
&\quad\cdot\|\Pi^K_{-k}\widetilde{u}_{N_1}\|_{\widehat{X}^{0,\ 1/r+}_r}\|\widetilde{u}_{N_2}\|_{\widehat{X}^{0,\ 1/r'+}_{r'}}\|\widetilde{u}_{N_3}\|_{\widehat{X}^{0,\ 1/r+}_r}\|\Pi_m\Pi^K_k\widetilde{v}_N\|_{\widehat{X}^{0,\ 1/r'+}_{r'}} \\
&\lesssim \sum_k (N_1^{1+\alpha}K)^{-1/r-}K^{1/r'-1/q}N_1^{-(1+\alpha)/r'}N_2^{1/r-1/q}N_3^{1/r}N_3^{1/r'-1/q}K^{1/q} \\
&\quad\cdot\|\Pi^K_{-k}\widetilde{u}_{N_1}\|_{\widehat{X}^{0,\ 1/r+}_{r,\ q}}\|\widetilde{u}_{N_2}\|_{\widehat{X}^{0,\ 1/r+}_{r,\ q}}\|\widetilde{u}_{N_3}\|_{\widehat{X}^{0,\ 1/r+}_{r,\ q}}\|\Pi^K_k\widetilde{v}_N\|_{\widehat{X}^{0,\ 1/r'+}_{r',\ q'}} \\
&\lesssim N_1^{-\alpha+\alpha/8r'+}N_2^{-s+1/r-1/q}N_3^{-s+1-1/q} \\
&\quad\cdot\|\widetilde{u}_{N_1}\|_{\widehat{X}^{s,\ 1/r+}_{r,\ q}}\|\widetilde{u}_{N_2}\|_{\widehat{X}^{s,\ 1/r+}_{r,\ q}}\|\widetilde{u}_{N_3}\|_{\widehat{X}^{s,\ 1/r+}_{r,\ q}}\|v_N\|_{\widehat{X}^{-1-s,\ 1/r'-2\varepsilon}_{r',\ q'}}.
\end{aligned}\tag{63}$$

Without loss of generality, we may assume that $-s+1-1/q \geq 0$, as the result is trivial otherwise. Assume that $N_2 \sim N_1^{\theta}$ for $N_1 \gg N_2$ with $0<\theta \ll 1$. Then we have

$$(63) \lesssim N_1^{-\alpha+} N_1^{\theta(-2s+1+1/r-2/q)} \|\widetilde{u}_{N_1}\|_{\widehat{X}^{s,\,1/r+}_{r,\,q}} \|\widetilde{u}_{N_2}\|_{\widehat{X}^{s,\,1/r+}_{r,\,q}} \|\widetilde{u}_{N_3}\|_{\widehat{X}^{s,\,1/r+}_{r,\,q}} \|v_N\|_{\widehat{X}^{-1-s,\,1/r'-2\varepsilon}_{r',\,q'}}.$$

Hence the boundedness is fine provided that

$$-\alpha+\theta(-2s+1+1/r-2/q)<0. \tag{64}$$

When $|\sigma_1| = \sigma_{max}$, we have other choices for the bilinear estimate. We delve deeper into the connection between N_2 and N_3 in the following.

Subcase 2.1: $N_1 \gg N_2 \gg N_3$.

If $N_2 \gg N_3$, then $K \sim N_2$. (60) may also be bounded by

$$\begin{aligned}
(60) &\lesssim \sum_K \sum_k \sum_m \|\Pi^K_{-k}\widetilde{u}_{N_1}\|_{\widehat{L}^{r'}_{t,\,x}} \|\widetilde{u}_{N_2}\widetilde{u}_{N_3}\|_{\widehat{L}^{r}_{t,\,x}} \|\Pi_m \Pi^K_k \widetilde{v}_N\|_{\widehat{L}^{\infty}_{t,\,x}} \\
&\lesssim \sum_K \sum_k K^{1/r-1/r'} (N_1^{1+\alpha}K)^{-1/r-} N_2^{-(1+\alpha)/r} K^{1/q} \\
&\quad \cdot \|\Pi^K_{-k}\widetilde{u}_{N_1}\|_{\widehat{X}^{0,\,1/r+}_{r}} \|\widetilde{u}_{N_2}\|_{\widehat{X}^{0,\,1/r+}_{r}} \|\widetilde{u}_{N_3}\|_{\widehat{X}^{0,\,1/r+}_{r}} \|\Pi^K_k \widetilde{v}_N\|_{\widehat{X}^{0,\,1/r'+}_{r',\,q'}} \\
&\lesssim \sum_k N_1^{1-(1+\alpha)/r-} N_2^{-s-(1+\alpha)/r+1/r'-1/q} N_3^{-s+1/r'-1/q} \qquad (65) \\
&\quad \cdot \|\Pi^K_{-k} u_{N_1}\|_{\widehat{X}^{s,\,1/r+}_{r,\,q}} \|u_{N_2}\|_{\widehat{X}^{s,\,1/r+}_{r,\,q}} \|u_{N_3}\|_{\widehat{X}^{s,\,1/r+}_{r,\,q}} \|\Pi^K_k v_N\|_{\widehat{X}^{-1-s,\,1/r'+}_{r',\,q'}} \\
&\lesssim N_1^{1-(1+\alpha)/r+\alpha/8r'+} N_2^{-s-(1+\alpha)/r+1/r'-1/q} N_3^{-s+1/r'-1/q} \\
&\quad \cdot \|u_{N_1}\|_{\widehat{X}^{s,\,1/r+}_{r,\,q}} \|u_{N_2}\|_{\widehat{X}^{s,\,1/r+}_{r,\,q}} \|u_{N_3}\|_{\widehat{X}^{s,\,1/r+}_{r,\,q}} \|v_N\|_{\widehat{X}^{-1-s,\,1/r'-2\varepsilon}_{r',\,q'}}.
\end{aligned}$$

We are led to the condition

$$1-(1+\alpha)/r+\theta(-2s-(1+\alpha)/r+2/r'-2/q)<0. \tag{66}$$

Subcase 2.2: $N_1 \gg N_2 \sim N_3$.

We localize $|\xi_2+\xi_3| \sim K$ if ξ_2 and ξ_3 have opposite signs. Hence, (60) can also be bounded by

$$
\begin{aligned}
(60) &\lesssim \sum_K \sum_k \sum_m \|\Pi^K_{-k}\widetilde{u}_{N_1}\|_{\widehat{L}^{r'}_{t,\,x}} \|P_K(\widetilde{u}_{N_2}\widetilde{u}_{N_3})\|_{\widehat{L}^{r}_{t,\,x}} \|\Pi_m \Pi^K_k \widetilde{v}_N\|_{\widehat{L}^{\infty}_{t,\,x}} \\
&\lesssim \sum_K \sum_k K^{1/r-1/r'} K^{1/r'-1/q} (N_1^{1+\alpha}K)^{-1/r-} N_2^{-\alpha/r} N_2^{1/r'-2/q} K^{1/r'-1/r} K^{1/q} \\
&\quad \cdot N_1 N_2^{-2s} \|\Pi^K_{-k} u_{N_1}\|_{\widehat{X}^{s,\,1/r+}_{r,\,q}} \|u_{N_2}\|_{\widehat{X}^{s,\,1/r+}_{r,\,q}} \|u_{N_3}\|_{\widehat{X}^{s,\,1/r+}_{r,\,q}} \|\Pi^K_k v_N\|_{\widehat{X}^{-1-s,\,1/r'+}_{r',\,q'}} \qquad (67) \\
&\lesssim N_1^{1-(1+\alpha)/r+\alpha/8r'+} N_1^{\theta(-2s-(1+\alpha)/r+1-2/q)} \\
&\quad \cdot \|u_{N_1}\|_{\widehat{X}^{s,\,1/r+}_{r,\,q}} \|u_{N_2}\|_{\widehat{X}^{s,\,1/r+}_{r,\,q}} \|u_{N_3}\|_{\widehat{X}^{s,\,1/r+}_{r,\,q}} \|v_N\|_{\widehat{X}^{-1-s,\,1/r'-2\varepsilon}_{r',\,q'}},
\end{aligned}
$$

where we used the improved bilinear estimate (35) and (16) in the second inequality. We can sum over all the dyadic numbers provided that

$$
1-(1+\alpha)/r+\theta(-2s-(1+\alpha)/r+1-2/q)<0, \tag{68}
$$

which is stronger than (66). Therefore in the case $|\sigma_1| = \sigma_{max}$, (60) is bounded if either (64) or (68) is satisfied.

If $r \leq \alpha+1$, then it's easy to check that (68) is satisfied. In the case $r > \alpha+1$, substituting $s = 1/2-\alpha/2r$ in (68) and by direct computation we can check that either (64) or (68) is satisfied if $r' \leq q \leq \dfrac{2r(r-1)(1+\alpha)}{r-(1+\alpha)^2}$. In particular, if $\alpha > \sqrt{r}-1$, then for $r' \leq q \leq \infty$, we can sum over all the dyadic numbers.

Case 3: $N_1 \sim N_2 \gg N_3$.

In terms of Lemma 6, we may assume $\sigma_{max} \gtrsim N_1^{1+\alpha}|\xi_1+\xi_2| \sim N_1^{1+\alpha}K$. We have

$$
\left|\int \widetilde{u}_{N_1}\widetilde{u}_{N_2}\widetilde{u}_{N_3}\widetilde{v}_N dxdt\right| \leq \sum_K \left|\int P_K(\widetilde{u}_{N_1}\widetilde{u}_{N_2})\widetilde{u}_{N_3}\widetilde{v}_N dxdt\right|. \tag{69}
$$

• $K \lesssim 1$. We have $|\xi_3+\xi| \lesssim 1$. Hence,

$$
\begin{aligned}
(69) &\lesssim \sum_k N_1^{-(1+\alpha)/r} \|\widetilde{u}_{N_1}\|_{\widehat{X}^{0,\,1/r+}_{r}} \|\Pi_{-k}\widetilde{u}_{N_3}\|_{\widehat{X}^{0,\,1/r+}_{r}} \\
&\quad \cdot N_1^{-(1+\alpha)/r'} \|\widetilde{u}_{N_2}\|_{\widehat{X}^{0,\,1/r'+}_{r'}} \|\Pi_k v_N\|_{\widehat{X}^{0,\,1/r'+}_{r'}} \\
&\lesssim N_1^{-2s-\alpha-2/q+\alpha/8r'+} N_3^{-s} N^{1+s} \\
&\quad \cdot \|u_{N_1}\|_{\widehat{X}^{s,\,1/r+}_{r,\,q}} \|u_{N_2}\|_{\widehat{X}^{s,\,1/r+}_{r,\,q}} \|u_{N_3}\|_{\widehat{X}^{s,\,1/r+}_{r,\,q}} \|v_N\|_{\widehat{X}^{-1-s,\,1/r'-2\varepsilon}_{r',\,q'}}.
\end{aligned}
$$

We can sum over all the dyadic numbers provided that $s > 1/2-\alpha/2-1/q+\alpha/16r'$.

• $K\gtrsim 1$. We may assume $|\sigma_3| = \sigma_{max}$ as other cases are similar and easier. If $N_3 \sim N$, then we need to localize $|\xi_3+\xi| \sim K$ if ξ_3 and ξ have opposite signs. Therefore, whether $N_3 \gg N$, $N \gg N_3$ or $N_3 \sim N$, we can use the improved bilinear estimate (35) to obtain

$$\begin{aligned}
(69) &\lesssim \sum_K \sum_k \sum_m \left| \int \widetilde{u}_{N_1} \widetilde{u}_{N_2} (\Pi_{-k}^K \widetilde{u}_{N_3})(\Pi_m \Pi_k^K \widetilde{v}_N) dxdt \right| \\
&\lesssim \sum_K \sum_k \sum_m \|\widetilde{u}_{N_1} \widetilde{u}_{N_2}\|_{\widehat{L}^r_{t,\,x}} \|\Pi_{-k}^K \widetilde{u}_{N_3}\|_{\widehat{L}^{r'}_{t,\,x}} \|\Pi_m \Pi_k^K \widetilde{v}_N\|_{\widehat{L}^\infty_{t,\,x}} \\
&\lesssim \sum_K \sum_k N_1^{-\alpha/r} N_1^{1/r'-2/q} K^{1/r'-1/r} K^{1/r-1/r'} (N_1^{1+\alpha} K)^{-1/r-} K^{1/r'-1/q} K^{1/q} \\
&\quad \cdot \|\widetilde{u}_{N_1}\|_{\widehat{X}^{0,\,1/r+}_{r,\,q}} \|\widetilde{u}_{N_2}\|_{\widehat{X}^{0,\,1/r+}_{r,\,q}} \|\Pi_{-k}^K \widetilde{u}_{N_3}\|_{\widehat{X}^{0,\,1/r+}_{r,\,q}} \|\Pi_k^K v_N\|_{\widehat{X}^{0,\,1/r'+}_{r',\,q'}} \\
&\lesssim N_1^{-2s+1-2(1+\alpha)/r-2/q} N_3^{-s} N^{1+s} \\
&\quad \cdot \|\widetilde{u}_{N_1}\|_{\widehat{X}^{s,\,1/r+}_{r,\,q}} \|\widetilde{u}_{N_2}\|_{\widehat{X}^{s,\,1/r+}_{r,\,q}} \|\widetilde{u}_{N_3}\|_{\widehat{X}^{s,\,1/r+}_{r,\,q}} \|v_N\|_{\widehat{X}^{-1-s,\,1/r'-2\varepsilon}_{r',\,q'}}.
\end{aligned} \tag{70}$$

Hence, we end up with the condition $s > 1-(1+\alpha)/r-1/q$.

Case 4: $N_1 \sim N_2 \sim N_3$.

If $N_1 \sim N_2 \sim N_3 \gg N$, the situation is similar to Case 3. However, a more refined analysis is required when $N_1 \sim N_2 \sim N_3 \sim N$. Without loss of generality, by symmetry, we may assume that the integration domain is given by

$$\Gamma' = \Gamma \cap \{\sigma_{max} \gtrsim N_1 |(\xi_1+\xi)(\xi_3+\xi)|\} \cap \{|\xi_1+\xi| \le |\xi_3+\xi|\}. \tag{71}$$

Therefore, the integration turns to

$$\begin{aligned}
&\left| \int_{\Gamma'} \widehat{u}_{N_1} \widehat{u}_{N_2} \widehat{u}_{N_3} \widehat{v}_N d\mu \right| \\
&\le \left| \left(\int_{\Gamma',\,|\xi_3+\xi|\lesssim 1} + \int_{\Gamma',\,|\xi_1+\xi|\lesssim 1,\,|\xi_3+\xi|\gtrsim 1} + \int_{\Gamma',\,|\xi_1+\xi|\gtrsim 1,\,|\xi_3+\xi|\gtrsim 1} \right) \widehat{u}_{N_1} \widehat{u}_{N_2} \widehat{u}_{N_3} \widehat{v}_N d\mu \right| \\
&=: I + II + III.
\end{aligned} \tag{72}$$

• Term I. Using Corollary 4 in [14] and $|\xi_1+\xi| \le |\xi_3+\xi| \lesssim 1$ yields for $1 < r \le \infty$

$$
\begin{aligned}
I &\lesssim \sum_k \left| \int (\Pi_k \widetilde{u}_{N_1})(\Pi_{-k} \widetilde{u}_{N_2})(\Pi_k \widetilde{u}_{N_3})(\Pi_{-k} \widetilde{v}_N) dxdt \right| \\
&\lesssim \sum_k \|(\Pi_k \widetilde{u}_{N_1})(\Pi_{-k} \widetilde{u}_{N_2})(\Pi_k \widetilde{u}_{N_3})\|_{\widehat{L}^r_{t,\,x}} \|\Pi_{-k} \widetilde{v}_N\|_{\widehat{L}^{r'}_{t,\,x}} \\
&\lesssim \left(\prod_{j=1}^{3} N_j^{-\alpha/3r} \|\widetilde{u}_{N_j}\|_{\widehat{X}^{0,\,1/r+}_{r,\,q}} \right) \|\widetilde{v}_N\|_{\widehat{X}^{0,\,1/r'-2\varepsilon}_{r',\,q'}} \\
&\lesssim N_1^{-\alpha/r-2s+1} \prod_{j=1}^{3} \|u_{N_j}\|_{\widehat{X}^{s,\,1/r+}_{r,\,q}} \|v_N\|_{\widehat{X}^{-1-s,\,1/r'-2\varepsilon}_{r',\,q'}}.
\end{aligned}
\tag{73}
$$

The summation is convergent if $s \geq 1/2 - \alpha/2r$, which is the strongest condition in this paper.

• Term II. Since $|\xi_1 + \xi| \lesssim 1$ and $|\xi_3 + \xi| \gtrsim 1$, we may assume $|\xi_3 + \xi| = |\xi_1 + \xi_2| \sim K \gtrsim 1$. Then the term II implies

$$
II \lesssim \sum_k \sum_{K \gtrsim 1} \left| \int P_K(\Pi_k \widetilde{u}_{N_1} \widetilde{u}_{N_2}) P_K(\Pi_{-k} \widetilde{v}_N \widetilde{u}_{N_3}) dxdt \right|. \tag{74}
$$

Since $|\xi_1 + \xi_2| \sim K$ and $|\xi_1 - k| \leq 1$, ξ_2 is localized in an interval centered at $-k$ with length $\sim K$. Similarly, $|\xi_3 + \xi| \sim K$ and $|\xi + k| \leq 1$, implying that ξ_3 is localized in an interval centered at k with length $\sim K$. Therefore, it is equivalent to express $\widetilde{u}_{N_2}$ as $\Pi^K_{-k/K} \widetilde{u}_{N_2}$, and $\widetilde{u}_{N_3}$ as $\Pi^K_{k/K} \widetilde{u}_{N_3}$. By using (34) and (16), Term II (74) turns to

$$
\begin{aligned}
II &\lesssim \sum_k \sum_K \left| \int P_K(\Pi_k \widetilde{u}_{N_1} \Pi^K_{-k/K} \widetilde{u}_{N_2}) P_K(\Pi_{-k} \widetilde{v}_N \Pi^K_{k/K} \widetilde{u}_{N_3}) dxdt \right| \\
&\lesssim \sum_k \sum_K \|P_K(\Pi_k \widetilde{u}_{N_1} \Pi^K_{-k/K} \widetilde{u}_{N_2})\|_{\widehat{L}^r_{t,\,x}} \|P_K(\Pi_{-k} \widetilde{v}_N \Pi^K_{k/K} \widetilde{u}_{N_3})\|_{\widehat{L}^{r'}_{t,\,x}} \\
&\lesssim \sum_k \sum_K N_1^{-\alpha/r} K^{-1/r} N_1^{-2s+1} \|\Pi_k \widetilde{u}_{N_1}\|_{\widehat{X}^{s,\,1/r+}_r} \|\Pi^K_{-k/K} \widetilde{u}_{N_2}\|_{\widehat{X}^{s,\,1/r+}_r} \\
&\quad \cdot N_1^{-\alpha/r'} K^{-1/r'} \|\Pi^K_{k/K} \widetilde{u}_{N_3}\|_{\widehat{X}^{s,\,1/r'+}_{r'}} \|\Pi_{-k} \widetilde{v}_N\|_{\widehat{X}^{-1-s,\,1/r'+}_{r'}} \\
&\lesssim N_1^{-2s+1-\alpha+\alpha/8r'+} \|u_{N_1}\|_{\widehat{X}^{s,\,1/r+}_{r,\,q}} \|u_{N_2}\|_{\widehat{X}^{s,\,1/r+}_{r,\,q}} \|u_{N_3}\|_{\widehat{X}^{s,\,1/r+}_{r,\,q}} \|v_N\|_{\widehat{X}^{-1-s,\,1/r'-2\varepsilon}_{r',\,q'}}.
\end{aligned}
$$

This case is fine provided that $s > 1/2 - \alpha/2 + \alpha/16r'$.

• Term III. We localize $|\xi_1 + \xi| = |\xi_2 + \xi_3| \sim K_1$ and $|\xi_3 + \xi| = |\xi_1 + \xi_2| \sim K_2$ for dyadic numbers $1 \lesssim K_1 \leq K_2$, which implies $\sigma_{max} \gtrsim N_1^{\alpha} K_1 K_2$. We first discuss the case when $|\sigma_1| = \sigma_{max}$.

$|\sigma_1| = \sigma_{max}$. We have

$$III \lesssim \sum_k \sum_{1 \lesssim K_1} \left| \int \Pi_k^{K_1} \widetilde{u}_{N_1} P_{K_1} (\widetilde{u}_{N_2} \widetilde{u}_{N_3}) \Pi_{-k}^{K_1} \widetilde{v}_N dxdt \right|. \tag{75}$$

We observe that ξ_1 is located in an interval centred at kK_1 with length $\sim K_1$. Therefore, $|\xi_1 + \xi_2| \sim K_2$ implies that ξ_2 is located in an interval centred at $-kK_1$ with length $\sim K_2$. Similarly, since $|\xi_3 + \xi| \sim K_2$, then ξ_3 is located in an interval centred at kK_1 with length $\sim K_2$. Thus, $\widetilde{u}_{N_2}$ and $\widetilde{u}_{N_3}$ are rewritten as $\Pi_{-kK_1/K_2}^{K_2} \widetilde{u}_{N_2}$ and $\Pi_{kK_1/K_2}^{K_2} \widetilde{u}_{N_3}$, respectively. Hence, we have

$$\begin{aligned}
(75) &\lesssim \sum_k \sum_{K_1 \leq K_2} \left| \int (\Pi_k^{K_1} \widetilde{u}_{N_1}) P_{K_1} (\Pi_{-kK_1/K_2}^{K_2} \widetilde{u}_{N_2} \Pi_{kK_1/K_2}^{K_2} \widetilde{u}_{N_3}) (\Pi_{-k}^{K_1} \widetilde{v}_N) dxdt \right| \\
&\lesssim \sum_k \sum_{K_1 \leq K_2} \left\| \Pi_k^{K_1} \widetilde{u}_{N_1} \right\|_{\widehat{L}_{t,\,x}^{r'}} \left\| \Pi_{-kK_1/K_2}^{K_2} \widetilde{u}_{N_2} \Pi_{kK_1/K_2}^{K_2} \widetilde{u}_{N_3} \right\|_{\widehat{L}_{t,\,x}^{r}} \\
&\quad \cdot \left\| \sum_m \Pi_m \Pi_{-k}^{K_1} \widetilde{v}_N \right\|_{\widehat{L}_{t,\,x}^{\infty}} \\
&\lesssim \sum_k \sum_{K_1 \leq K_2} K_1^{1/r - 1/r'} (N_1^{\alpha} K_1 K_2)^{-1/r-} K_1^{1/r' - 1/q} \left\| \Pi_k^{K_1} \widetilde{u}_{N_1} \right\|_{\widehat{X}_{r,\,q}^{0,\,1/r+}} \\
&\quad \cdot N_1^{-\alpha/r} K_1^{1/r' - 1/r} \max(1,\, K_2^{1/r' - 2/q}) \left\| \Pi_{-kK_1/K_2}^{K_2} \widetilde{u}_{N_2} \right\|_{\widehat{X}_{r,\,q}^{0,\,1/r+}} \left\| \Pi_{kK_1/K_2}^{K_2} \widetilde{u}_{N_3} \right\|_{\widehat{X}_{r,\,q}^{0,\,1/r+}} \\
&\quad \cdot K_1^{1/q} \left\| \Pi_{-k}^{K_1} \widetilde{v}_N \right\|_{\widehat{X}_{r',\,q'}^{0,\,1/r'+}} \\
&\lesssim \sum_k \sum_{K_1 \leq K_2} N_1^{-2s+1-2\alpha/r+\alpha/8r'+} K_1^{1/r' - 1/r -} K_2^{1/r' - 1/r - 2/q -} \left\| \Pi_k^{K_1} \widetilde{u}_{N_1} \right\|_{\widehat{X}_{r,\,q}^{s,\,1/r+}} \\
&\quad \cdot \|\widetilde{u}_{N_2}\|_{\widehat{X}_{r,\,q}^{s,\,1/r+}} \|\widetilde{u}_{N_3}\|_{\widehat{X}_{r,\,q}^{s,\,1/r+}} \left\| \Pi_{-k}^{K_1} \widetilde{v}_N \right\|_{\widehat{X}_{r',\,q'}^{-1-s,\,1/r'-2\varepsilon}} \\
&\lesssim N_1^{-2s+1-2\alpha/r+\alpha/8r'+} \|u_{N_1}\|_{\widehat{X}_{r,\,q}^{s,\,1/r+}} \|u_{N_2}\|_{\widehat{X}_{r,\,q}^{s,\,1/r+}} \|u_{N_3}\|_{\widehat{X}_{r,\,q}^{s,\,1/r+}} \|v_N\|_{\widehat{X}_{r',\,q'}^{-1-s,\,1/r'-2\varepsilon}},
\end{aligned}$$

where, in the third step, we applied the improved bilinear estimate (35). We end up again with the condition $s > 1/2 - \alpha/r + \alpha/16r'$.

$|\sigma_2| = \sigma_{max}$. If $K_1 \leq K_2$, we need to localize $|\xi_1 + \xi_3| \sim K_3$ if ξ_1 and ξ_3 have different signs.

If $K_3 \leq K_1$, then ξ_2 and ξ_3 can be confined to intervals of size K_1. Specifically, since ξ_1 is contained in an interval centered at kK_1 with length $\sim K_1$ and satisfies $|\xi_1 + \xi_3| \sim K_3$, it follows that ξ_3 is located in an interval centered at $-kK_1$ with length $\sim K_1$. Furthermore, given that $|\xi_2 + \xi_3| \sim K_1$, we conclude that ξ_2 lies within an interval centered at kK_1 with length $\sim K_1$. Thus, we obtain

$$III \lesssim \sum_k \sum_{K_1 \le K_2} \left| \int (\Pi_k^{K_1} \widetilde{u}_{N_1})(\Pi_k^{K_1} \widetilde{u}_{N_2})(\Pi_{-k}^{K_1} \widetilde{u}_{N_3})(\Pi_{-k}^{K_1} \widetilde{v}_N) dxdt \right|. \tag{76}$$

First, for $1 \le r < \infty$, we apply Corollary 5 in [14] with $r = 1$ and derive

$$\begin{aligned}
(76) &\lesssim \sum_k \sum_{K_1 \le K_2} \left\| (\Pi_k^{K_1} \widetilde{u}_{N_1})(\Pi_k^{K_1} \widetilde{u}_{N_2})(\Pi_{-k}^{K_1} \widetilde{u}_{N_3}) \right\|_{\widehat{L}^1_{t,\,x}} \left\| \sum_m \Pi_m \Pi_{-k}^{K_1} \widetilde{v}_N \right\|_{\widehat{L}^\infty_{t,\,x}} \\
&\lesssim \sum_k \sum_{K_1 \le K_2} N_1^{-\alpha} \left\| \Pi_k^{K_1} \widetilde{u}_{N_1} \right\|_{\widehat{X}^{\delta,\,1+}_{1,\,\infty}} \left\| \Pi_k^{K_1} \widetilde{u}_{N_2} \right\|_{\widehat{X}^{0,\,1+}_{1,\,\infty}} \left\| \Pi_{-k}^{K_1} \widetilde{u}_{N_3} \right\|_{\widehat{X}^{0,\,1+}_{1,\,\infty}} \left\| \Pi_{-k}^{K_1} \widetilde{v}_N \right\|_{\widehat{X}^{0,\,0}_{\infty,\,1}} \\
&\lesssim N_1^{-2s+1-\alpha+} \|u_{N_1}\|_{\widehat{X}^{s,\,1+}_{1,\,\infty}} \|u_{N_2}\|_{\widehat{X}^{s,\,1+}_{1,\,\infty}} \|u_{N_3}\|_{\widehat{X}^{s,\,1+}_{1,\,\infty}} \|v_N\|_{\widehat{X}^{-1-s,\,0}_{\infty,\,1}}.
\end{aligned} \tag{77}$$

For $q \ge 2$, by (41), we have

$$\begin{aligned}
(76) &\lesssim \sum_k \sum_{K_1 \le K_2} \left\| (\Pi_k^{K_1} \widetilde{u}_{N_1})(\Pi_k^{K_1} \widetilde{u}_{N_2})(\Pi_{-k}^{K_1} \widetilde{u}_{N_3}) \right\|_{L^1_{t,\,x}} \left\| \sum_m \Pi_m \Pi_{-k}^{K_1} \widetilde{v}_N \right\|_{L^\infty_{t,\,x}} \\
&\lesssim \sum_k \sum_{K_1 \le K_2} \left\| \Pi_k^{K_1} \widetilde{u}_{N_1} \right\|_{L^4_{t,\,x}} \left\| \Pi_k^{K_1} \widetilde{u}_{N_2} \right\|_{L^2_{t,\,x}} \left\| \Pi_{-k}^{K_1} \widetilde{u}_{N_3} \right\|_{L^4_{t,\,x}} \left\| \sum_m \Pi_m \Pi_{-k}^{K_1} \widetilde{v}_N \right\|_{L^\infty_{t,\,x}} \\
&\lesssim \sum_k \sum_{K_1 \le K_2} N_1^{-\alpha/8+} \left\| \Pi_k^{K_1} \widetilde{u}_{N_1} \right\|_{\widehat{X}^{0,\,1/2+}_{2,\,4}} (N_1^\alpha K_1 K_2)^{-1/2-} \left\| \Pi_k^{K_1} \widetilde{u}_{N_2} \right\|_{\widehat{X}^{0,\,1/2+}_{2}} \\
&\quad \cdot N_1^{-\alpha/8+} \left\| \Pi_{-k}^{K_1} \widetilde{u}_{N_3} \right\|_{\widehat{X}^{0,\,1/2+}_{2,\,4}} K_1^{1/q} \left\| \Pi_{-k}^{K_1} \widetilde{v}_N \right\|_{\widehat{X}^{0,\,1/2+}_{2,\,q'}} \\
&\lesssim \sum_k \sum_{K_1 \le K_2} N_1^{-2s+1-3\alpha/4+} K_1^{1/q} (K_1 K_2)^{-1/2-} \max(1,\, K_1^{1/2-2/q}) K_1^{1/2-1/q} \\
&\quad \cdot \left\| \Pi_k^{K_1} \widetilde{u}_{N_1} \right\|_{\widehat{X}^{s,\,1/2+}_{2,\,q}} \left\| \Pi_k^{K_1} \widetilde{u}_{N_2} \right\|_{\widehat{X}^{s,\,1/2+}_{2,\,q}} \left\| \Pi_{-k}^{K_1} \widetilde{u}_{N_3} \right\|_{\widehat{X}^{s,\,1/2+}_{2,\,q}} \left\| \Pi_{-k}^{K_1} \widetilde{v}_N \right\|_{\widehat{X}^{-1-s,\,1/2+}_{2,\,q'}} \\
&\lesssim N_1^{-2s+1-3\alpha/4+} \|u_{N_1}\|_{\widehat{X}^{s,\,1/2+}_{2,\,q}} \|u_{N_2}\|_{\widehat{X}^{s,\,1/2+}_{2,\,q}} \|u_{N_3}\|_{\widehat{X}^{s,\,1/2+}_{2,\,q}} \|v_N\|_{\widehat{X}^{-1-s,\,1/2-2\varepsilon}_{2,\,q'}}.
\end{aligned} \tag{78}$$

Using the interpolation between (77) and (78), we see that for $1 \le r \le 2$.

$$(76) \lesssim N_1^{-2s+1-\alpha/2-\alpha/2r+} \|u_{N_1}\|_{\widehat{X}^{s,\,1/r+}_{r,\,q}} \|u_{N_2}\|_{\widehat{X}^{s,\,1/r+}_{r,\,q}} \|u_{N_3}\|_{\widehat{X}^{s,\,1/r+}_{r,\,q}} \|v_N\|_{\widehat{X}^{-1-s,\,1/r'-2\varepsilon}_{r',\,q'}}. \tag{79}$$

Hence, it is summable if $s > 1/2 - \alpha/4 - \alpha/4r$.

If $K_3 \geq K_1$, then we can localize ξ_2 and ξ_3 to intervals of size $\min(K_2,\ K_3)$. Without loss of generality, we assume $K_3 \leq K_2$. Indeed, since $|\xi_1 + \xi_3| \sim K_3$, and ξ_1 is located in an interval centred at kK_1 with length $\sim K_1$, then ξ_3 is located to an interval centred at $-kK_1$ with length $\sim K_3$. On the other hand, since $|\xi_2 + \xi_3| \sim K_1$, then ξ_2 is located to interval centred at kK_1 with length $\sim K_3$. $\widetilde{u}_{N_2}$ is rewritten as $\Pi^{K_3}_{kK_1/K_3}\widetilde{u}_{N_2}$ and $\widetilde{u}_{N_3}$ as $\Pi^{K_3}_{-kK_1/K_3}\widetilde{u}_{N_3}$ in this case. Thus, by the improved bilinear estimate (35) we have

$$
\begin{aligned}
III \lesssim & \sum_k \sum_{K_3 \gtrsim K_1} \sum_{K_1 \leq K_2} \left| \int (\Pi^{K_1}_k \widetilde{u}_{N_1})(\Pi^{K_3}_{kK_1/K_3}\widetilde{u}_{N_2})(\Pi^{K_3}_{-kK_1/K_3}\widetilde{u}_{N_3})(\Pi^{K_1}_{-k}\widetilde{v}_N)dxdt \right| \\
\lesssim & \sum_k \sum_{K_3 \gtrsim K_1} \sum_{K_1 \leq K_2} \left\| (\Pi^{K_1}_k \widetilde{u}_{N_1})(\Pi^{K_3}_{-kK_1/K_3}\widetilde{u}_{N_3}) \right\|_{\widehat{L}^r_{t,\ x}} \left\| \Pi^{K_3}_{kK_1/K_3}\widetilde{u}_{N_2} \right\|_{\widehat{L}^{r'}_{t,\ x}} \left\| \sum_m \Pi_m \Pi^{K_1}_{-k}\widetilde{v}_N \right\|_{\widehat{L}^\infty_{t,\ x}} \\
\lesssim & \sum_k \sum_{K_3 \gtrsim K_1} \sum_{K_1 \leq K_2} N_1^{-\alpha/r} K_3^{-1/r} K_1^{1/r'-1/q} K_3^{1/r'-1/q} \left\| \Pi^{K_1}_k u_{N_1} \right\|_{\widehat{X}^{0,\ 1/r+}_{r,\ q}} \left\| \Pi^{K_3}_{-kK_1/K_3} u_{N_3} \right\|_{\widehat{X}^{0,\ 1/r+}_{r,\ q}} \\
& \cdot K_3^{1/r-1/r'} (N_1^\alpha K_1 K_2)^{-1/r-} K_3^{1/r'-1/q} \left\| \Pi^{K_3}_{-kK_1/K_3} u_{N_2} \right\|_{\widehat{X}^{0,\ 1/r+}_{r,\ q}} \\
& \cdot K_1^{1/q} \left\| \Pi^{K_1}_{-k} v_N \right\|_{\widehat{X}^{0,\ 1/r'+}_{r',\ q'}} \\
\lesssim & N_1^{-2s+1-2\alpha/r+\alpha/8r'+} \|\widetilde{u}_{N_1}\|_{\widehat{X}^{s,\ 1/r+}_{r,\ q}} \|\widetilde{u}_{N_2}\|_{\widehat{X}^{s,\ 1/r+}_{r,\ q}} \|\widetilde{u}_{N_3}\|_{\widehat{X}^{s,\ 1/r+}_{r,\ q}} \|\widetilde{v}_N\|_{\widehat{X}^{-1-s,\ 1/r'-2\varepsilon}_{r',\ q'}}.
\end{aligned}
$$

We find the condition $s > 1/2 - \alpha/r + \alpha/16r'$. We complete the proof.

5. Conclusion

In conclusion, we investigate the Cauchy problem for the dispersion-generalized modified Benjamin-Ono equation with low-regularity initial data. To achieve this, we employ a function space, the generalized Fourier-Lebesgue space $\widehat{M}^s_{r,\ q}(\mathbb{R})$, which unifies the modulation and the Fourier-Lebesgue spaces. A crucial aspect of our approach is an improved bilinear estimate, which plays a key role in establishing well-posedness via perturbation arguments. Our results refine and extend previous studies and provide a broader framework for analyzing various type of dispersive equations in low-regularity settings.

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Conflict of interest

The authors declare no competing financial interest.

References

[1] Guo Z. Local well-posedness for dispersion generalized Benjamin-Ono equations in Sobolev spaces. *Journal of Differential Equations*. 2012; 252(3): 2053-2084.

[2] Guo Z, Huang C. Well-posedness of the modified dispersion-generalized Benjamin-Ono equations in modulation spaces. *Journal of Mathematical Analysis and Applications*. 2022; 509(1): 125933.

[3] Benjamin TB. Internal waves of permanent form in fluids of great depth. *Journal of Fluid Mechanics*. 1967; 29(3): 559-592.

[4] Ono H. Algebraic solitary waves in stratified fluids. *Journal of the Physical Society of Japan*. 1975; 39(4): 1082-1091.

[5] Guo Z. Local well-posedness and a priori bounds for the modified Benjamin-Ono equation. *Advances in Differential Equations*. 2011; 16(11/12): 1087-1137.

[6] Harrop-Griffiths B, Killip R, Vişan M. Sharp well-posedness for the cubic NLS and mKdV in $H^s(\mathbb{R})$. *Forum of Mathematics, Pi*. 2024; 12: e6.

[7] Kenig CE, Takaoka H. Global wellposedness of the modified Benjamin-Ono equation with initial data in $H^{1/2}$. *International Mathematics Research Notices*. 2006; 2006(9): 95702-95702.

[8] Martel Y, Pilod D. Construction of a minimal mass blow up solution of the modified Benjamin-Ono equation. *Mathematische Annalen*. 2017; 369: 153-245.

[9] Vivas-Cortez M, Yousif MA, Mohammed PO, Lupas AA, Ibrahim IS, Chorfi N. Hyperbolic non-polynomial spline approach for time-fractional coupled KdV equations: A computational investigation. *Symmetry*. 2024; 16(12): 1610.

[10] Vivas-Cortez M, Yousif MA, Mahmood BA, Mohammed PO, Chorfi N, Lupas AA. High-accuracy solutions to the time-fractional KdV-Burgers equation using rational non-polynomial splines. *Symmetry*. 2025; 17(1): 16.

[11] Kenig CE, Ponce G, Vega L. Well-posedness and scattering results for the generalized korteweg-de vries equation via the contraction principle. *Communications on Pure and Applied Mathematics*. 1993; 46(4): 527-620.

[12] Killip R, Laurens T, Vişan M. Sharp well-posedness for the Benjamin-Ono equation. *Inventiones mathematicae*. 2024; 236(3): 999-1054.

[13] Grünrock A. An improved local well-posedness result for the modified KdV equation. *International Mathematics Research Notices*. 2004; 2004(61): 3287-3308.

[14] Grünrock A, Vega L. Local well-posedness for the modified KdV equation in almost critical $\widehat{H}_s^r$ spaces. *Transactions of the American Mathematical Society*. 2009; 361(11): 5681-5694.

[15] Feichtinger HG. *Modulation Spaces on Locally Compact Abelian Groups*. University Park, PA, USA: Citeseer; 1983.

[16] Wang B, Huo Z, Hao C, Guo Z. *Harmonic Analysis Method for Nonlinear Evolution Equations, I*. Singapore: World Scientific; 2011.

[17] Guo S. On the 1D cubic nonlinear Schrödinger equation in an almost critical space. *Journal of Fourier Analysis and Applications*. 2017; 23: 91-124.

[18] Chen M, Guo B. Local well and ill posedness for the modified KdV equations in subcritical modulation spaces. *Communications in Mathematical Sciences*. 2020; 18(4): 909-946.

[19] Oh T, Wang Y. Global well-posedness of the one-dimensional cubic nonlinear Schrödinger equation in almost critical spaces. *Discrete and Continuous Dynamical Systems*. 2021; 41(6): 2971-2992.

[20] Feichtinger HG. Banach convolution algebras of Wiener type. In: *Proceedings of the Conference on Functions, Series, Operators*. Budapest, Hungary: János Bolyai Mathematical Society; 1980. p.509-524.

[21] Feichtinger HG. Modulation spaces: Looking back and ahead. *Sampling Theory in Signal and Image Processing*. 2006; 5: 109-140.

[22] Feichtinger HG. Choosing function spaces in harmonic analysis. *Excursions in Harmonic Analysis, The February Fourier Talks at the Norbert Wiener Center*. 2015; 4: 65-101.

[23] Forlano J, Okamoto M. A remark on norm inflation for nonlinear wave equations. *Dynamics of Partial Differential Equations*. 2020; 17(4): 361-381.

[24] Forlano JA. *On the Deterministic and Probabilistic Cauchy Problem of Nonlinear Dispersive Partial Differential Equations*. Doctoral Dissertation. Heriot-Watt University, Edinburgh, UK; 2020.

[25] Chen Z, Guo Z, Huang C. Complex-valued solutions of the mKdV equations in generalized Fourier-Lebesgue spaces. *Proceedings of the American Mathematical Society*. 2025; 153(4): 1621-1640.

[26] Tao T. Scattering for the quartic generalised Korteweg-de Vries equation. *Journal of Differential Equations*. 2007; 232(2): 623-651.